“Teach A Level Maths”

Vol. 2: A2 Core Modules

33: The equation $a\cos x + b\sin x = c$

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Module C3	Module C4
Edexcel	AQA
OCR	MEI/OCR

The equation $a\cos x + b\sin x = c$

Can you see why one of these equations is easy to solve and the other takes much more work ?

(a) $4\cos x + 3\sin x = 0$ (b) $4\cos x + 3\sin x = 2$

Both have 2 trig ratios but (a) can be solved by dividing by $\cos x$.

We get $$\frac{4\cos x}{\cos x} + \frac{3\sin x}{\cos x} = \frac{0}{\cos x}$$

$$\Rightarrow \qquad 4 + 3\tan x = 0$$

$$\Rightarrow \qquad \tan x = -\tfrac{4}{3}$$

This is a simple equation and can now be solved.

The equation $a\cos x + b\sin x = c$

Can you see why one of these equations is easy to solve and the other takes much more work ?

(a) $4\cos x + 3\sin x = 0$ (b) $4\cos x + 3\sin x = 2$

If we try the same method with (b), we get

$$\frac{4\cos x}{\cos x} + \frac{3\sin x}{\cos x} = \frac{2}{\cos x}$$

$$\Rightarrow \quad 4 + 3\tan x = 2\sec x$$

This is no better than the original equation as we still have 2 trig ratios.

However, we saw in the previous section that

$$4\cos x + 3\sin x \equiv 5\cos(x - 36\cdot 9)^\circ$$

so the equation $4\cos x + 3\sin x = 2$

can be written as $5\cos(x - 36\cdot 9)^\circ = 2$

Dividing by 5: $\cos(x - 36\cdot 9)^\circ = 0\cdot 4$

This is of the form $\cos\theta = 0\cdot 4$ where $\theta = x - 36\cdot 9^\circ$ so we can find solutions for θ and then find x by adding $36\cdot 9^\circ$ to each one.

The equation $a\cos x + b\sin x = c$

e.g. 1 Solve the following equation giving the solutions in the interval $0 \le x \le 360^\circ$ correct to 1 d.p.

$$5\cos x + 12\sin x = 10$$

Solution:

Let $5\cos x + 12\sin x \equiv R\cos(x - \alpha)$

$\Rightarrow \quad 5\cos x + 12\sin x \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$

Coef. of $\cos x$: $\quad 5 = R\cos\alpha \quad$ ------- (1)

Coef. of $\sin x$: $\quad 12 = R\sin\alpha \quad$ ------- (2)

$$\frac{(2)}{(1)} \Rightarrow \tan\alpha = \frac{12}{5} \Rightarrow \alpha = 67\cdot 4^\circ$$

$$R^2 = 5^2 + 12^2 \Rightarrow R = 13$$

The equation $a\cos x + b\sin x = c$

Substituting into the l.h.s. of the equation:

$$5\cos x + 12\sin x = 10 \quad \Rightarrow \quad 13\cos(x - 67\cdot 4^\circ) = 10$$

$$\Rightarrow \quad \cos(\underbrace{x - 67\cdot 4^\circ}_{\theta}) = \tfrac{10}{13} \approx 0\cdot 769$$

$$\Rightarrow \quad \theta = 39\cdot 7^\circ$$

Beware ! Don't find x at this stage.

We have $\cos\theta = 0\cdot 769$ NOT $\cancel{\cos x = 0\cdot 769}$

The 2nd solution will be wrong if we use the x value to try to find it.

At this stage we need to get all the solutions for θ.

So, $0 \le x \le 360^\circ \quad \Rightarrow \quad -67\cdot 4^\circ \le x - 67\cdot 4^\circ \le 292\cdot 6^\circ$

(Subtract $67\cdot 4^\circ$ from each part)

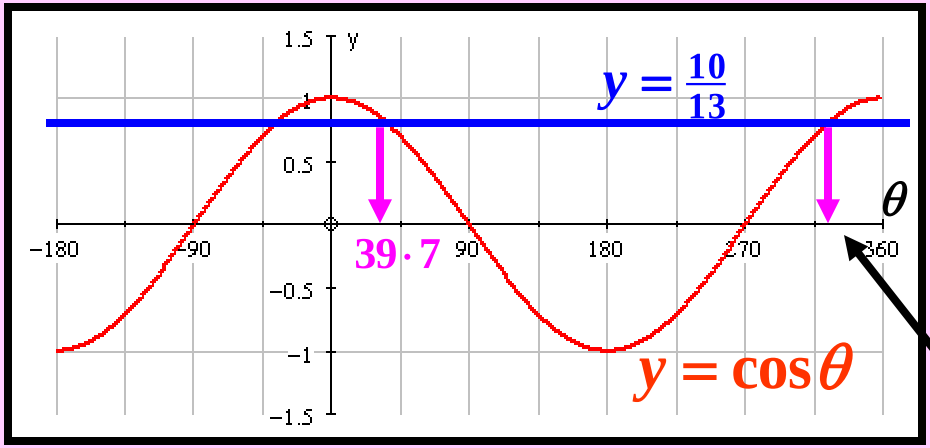
$$-67\cdot 4^\circ \le \theta^\circ \le 292\cdot 6^\circ$$

The equation $a\cos x + b\sin x = c$

$$\Rightarrow \cos(\underbrace{x - 67\cdot 4^\circ}_{\theta}) = \frac{10}{13} \approx 0\cdot 769, \qquad -67\cdot 4^\circ \le \theta^\circ \le 292\cdot 6^\circ$$

$$\Rightarrow \theta = 39\cdot 7^\circ$$

We sketch the usual cosine graph:



Outside the required interval

The equation $a\cos x + b\sin x = c$

$$\Rightarrow \cos(\underbrace{x - 67\cdot4^\circ}_{\theta}) = \frac{10}{13} \approx 0\cdot769, \qquad -67\cdot4^\circ \le \theta^\circ \le 292\cdot6^\circ$$

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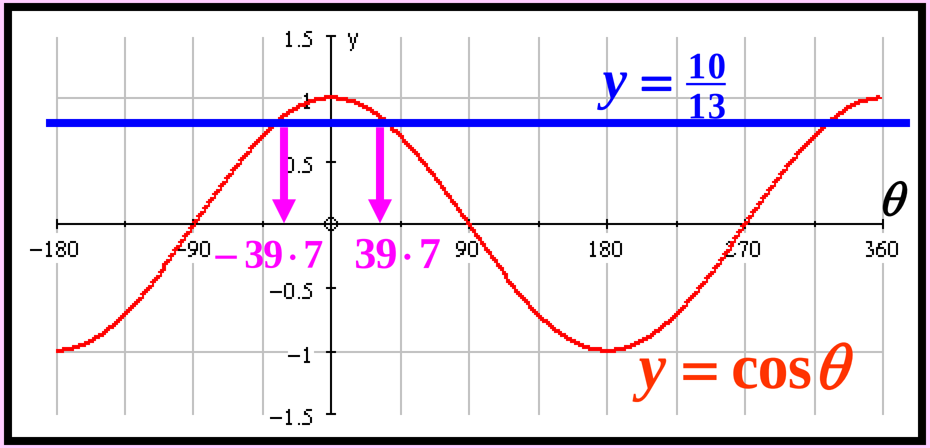
We sketch the usual cosine graph:



$$\Rightarrow \quad \theta = x - 67\cdot4^\circ = 39\cdot7^\circ, \; -39\cdot7^\circ$$

$\boxed{\text{Add}}$ $67\cdot4^\circ$: $\quad \Rightarrow \quad x = 39\cdot7^\circ + 67\cdot4^\circ = \boxed{107\cdot1^\circ}$

$$-39\cdot7^\circ + 67\cdot4^\circ = \boxed{27\cdot7^\circ}$$

ANS: x is $27\cdot7^\circ$ or $107\cdot1^\circ$ (1 d.p.)

The equation $a\cos x + b\sin x = c$

SUMMARY

To solve the equation $a\cos x + b\sin x = c, \quad c \neq 0$

- Write the *l.h.s.* in one of the forms

$$R\cos(x \pm \alpha), \quad R\sin(x \pm \alpha)$$

- Calculate the interval for θ using the one given for x, where $\theta = x + \alpha$ or $\theta = x - \alpha$.

- Solve the equation to find θ making sure you find all the solutions.

- Find the values of x.

N.B. for $\theta = x - \alpha$, add α and for $\theta = x + \alpha$, subtract α.

The equation $a\cos x + b\sin x = c$

Exercise

1(a) Write $\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where R and α are exact.

(b) Solve the equation

$$\cos x - \sin x = 1$$

for $0 \le x \le 360^\circ$.

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2. Solve the equation for $-180^\circ \le \theta \le 180^\circ$

$$\sin\theta + \sqrt{3}\cos\theta = 1$$

(Notice the different letter in the equation. You need to be able to cope with a switch of letters.)

The equation $a\cos x + b\sin x = c$

Solutions:

1(a) $\cos x - \sin x \equiv R\cos(x+\alpha)$

$$\equiv R\cos x\cos\alpha - R\sin x\sin\alpha$$

Coef. of $\cos x$: $1 = R\cos\alpha \quad ------- (1)$

Coef. of $\sin x$: $-1 = -R\sin\alpha$

$\Rightarrow \quad 1 = R\sin\alpha \quad ------- (2)$

$$\frac{(2)}{(1)} \Rightarrow \quad \tan\alpha = 1 \quad \Rightarrow \quad \alpha = 45^\circ$$

$$R^2 = 1^2 + 1^2 \quad \Rightarrow \quad R = \sqrt{2}$$

So, $\cos x - \sin x \equiv \sqrt{2}\cos(x + 45^\circ)$

The equation $a\cos x + b\sin x = c$

Solutions:

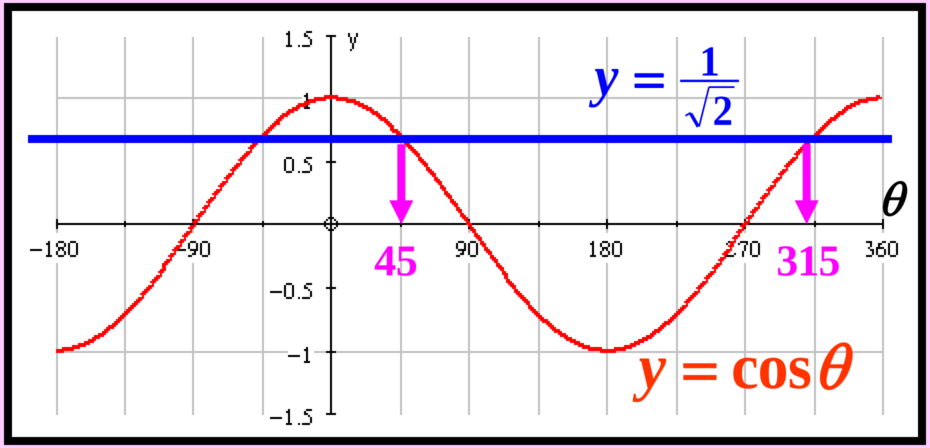
1(b) Solve $\cos x - \sin x = 1$ for $0 \le x \le 360^\circ$.

$$\cos x - \sin x \equiv \sqrt{2}\cos(x + 45^\circ)$$

so, the equation becomes

$$\sqrt{2}\cos(x + 45^\circ) = 1$$

$$\Rightarrow \quad \cos(\underbrace{x + 45^\circ}_{\theta}) = \frac{1}{\sqrt{2}}$$



$$0 \le x \le 360^\circ$$

$$\Rightarrow \quad 45^\circ \le \theta \le 405^\circ$$

$$x + 45^\circ = 45^\circ, \ 315^\circ, \ 405^\circ$$

$$\Rightarrow \quad x^\circ = 0, \ 270^\circ, \ 360^\circ$$

The equation $a\cos x + b\sin x = c$

Solutions:

2. Solve $\sin\theta + \sqrt{3}\cos\theta = 1$

Let $\sin\theta + \sqrt{3}\cos\theta \equiv R\sin(\theta + \alpha)$

$$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

Coef. of $\sin\theta$: $1 = R\cos\alpha$ ------- (1)

Coef. of $\cos\theta$: $\sqrt{3} = R\sin\alpha$ ------- (2)

$$\frac{(2)}{(1)} \Rightarrow \tan\alpha = \sqrt{3} \Rightarrow \alpha = 60^\circ$$

$$R^2 = 1^2 + (\sqrt{3})^2 \Rightarrow R = 2$$

So, $\sin\theta + \sqrt{3}\cos\theta \equiv 2\sin(\theta + 60^\circ)$

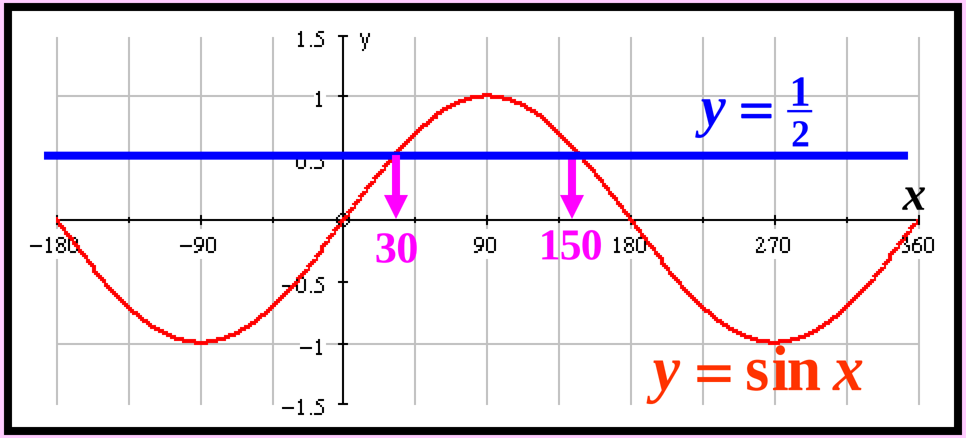
So, $\sin\theta + \sqrt{3}\cos\theta = 1$ becomes $2\sin(\theta + 60^\circ) = 1$

The equation $a\cos x + b\sin x = c$

Solutions:

Solve $2\sin(\theta + 60^\circ) = 1$ **for** $-180^\circ \le \theta \le 180^\circ$.

$$\Rightarrow \quad \sin(\underbrace{\theta + 60^\circ}_{x}) = \frac{1}{2}$$



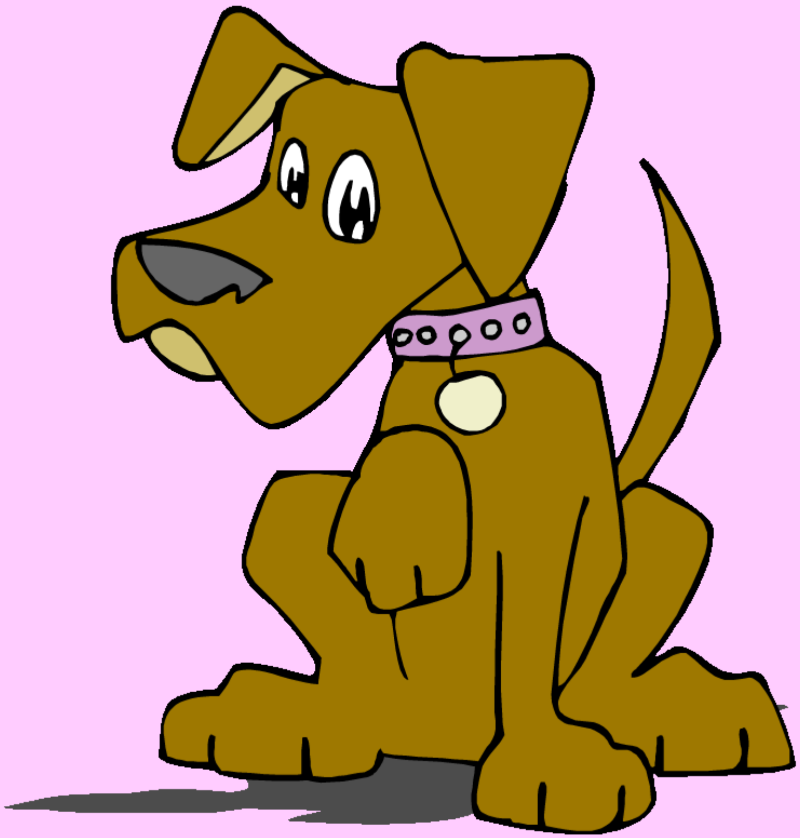
$$-180^\circ \le \theta \le 180^\circ$$

$$\Rightarrow \quad -120^\circ \le x \le 240^\circ$$

$$\theta + 60^\circ = 30^\circ, \quad 150^\circ$$

$$\Rightarrow \quad \theta^\circ = -30^\circ, \quad 90^\circ$$

The equation $a\cos x + b\sin x = c$



The following slides contain repeats of information on earlier slides, shown without colour, so that they can be printed and photocopied.

For most purposes the slides can be printed as “Handouts” with up to 6 slides per sheet.

The equation $a\cos x + b\sin x = c$

Think about these 2 equations.

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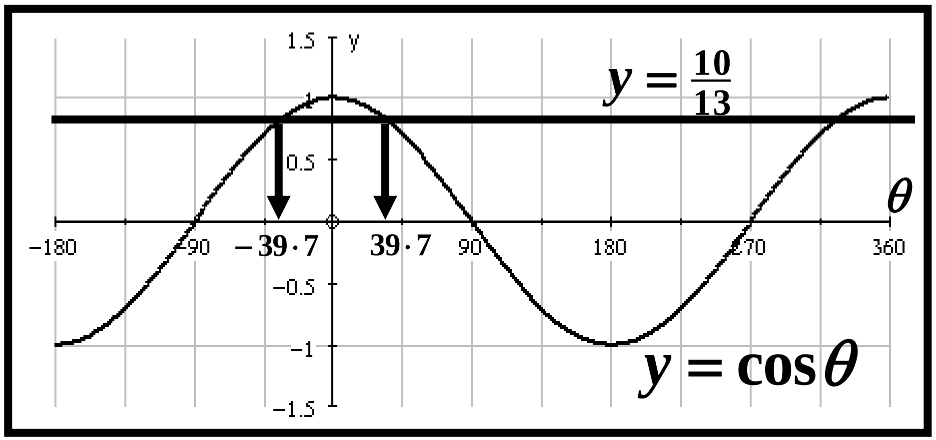

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