APPLIED PARTIAL DIFFERENTIAL EQUATIONS

by

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Preface

This text evolved, as have so many others, from notes used to teach partial differential equations to advanced undergraduate mathematics and physics students and graduate engineering students. Major emphasis is placed on techniques for solving partial differential equations found in physics and engineering, but discussions on existence and uniqueness of solutions are also included. Every opportunity is taken to show that there may be more than one way to solve a particular problem and to discuss the advantages of each solution relative to the others. In addition, physical interpretations of mathematical solutions are stressed whenever possible.

In Chapter 1, we use the method of characteristics to solve first-order quasilinear and general nonlinear equations. Applications included are the one-dimensional wave equation, the eikonal equation from geometric optics, and traffic flow problems.

Section 2.1 introduces second-order equations and describes how initial boundary value problems are associated with such equations. To distinguish between physical assumptions leading to various models of heat conduction, vibration, and potential problems, and the mathematical techniques to solve these problems, models are developed in Sections 2.2–2.6 with no attempt at solutions. At this stage, the reader concentrates only on how mathematics describes physical phenomena. Once these ideas are firmly entrenched, it is then reasonable to proceed to various solution techniques. It has been our experience that confusion often arises when new mathematical techniques are prematurely applied to unfamiliar problems. In this chapter, we also classify second-order PDEs in two variables as being hyperbolic, parabolic, or elliptic, with the wave equation, the heat conduction equation, and Laplace's equation being their canonical forms. The wave equation, together with d'Alembert's solution and its extension to nonhomogenoues problems, is given special consideration. We are careful to point out, however, that such representations of solutions of initial boundary value problems are not possible for parabolic and elliptic equations.

One of the most fundamental classical techniques for solving partial differential equations is that of separation of variables, which leads, in the simplest of examples to trigonometric Fourier series. Chapter 3 develops the theory of Fourier series to the point where it is easily accessible to separation of variables in Chapter 4. The method of *variation of constants* is introduced in order to deal with nonhomogeneities. The examples in Chapter 4 also suggest the possibility of expansions other than trigonometric Fourier series, and these are discussed in detail through Sturm-Liouville systems in Chapter 5. They are then used in Chapter 6 to solve homogeneous problems in one, two, and three space variables. In this chapter, we also illustrate how to verify series solutions of initial boundary value problems, and we discuss distinguishing properties of parabolic, elliptic, and hyperbolic partial differential equations. In Chapter 7, finite Fourier transforms are presented as an alternative to variation of constants for nonhomogeneous problems. They are particularly useful for multi-dimensional problems.

Chapters 8 and 9 essentially repeat material in Chapters 5 and 6, but in polar, cylindrical, and spherical coordinates.

In Chapters 10 and 11, we introduce three further transforms for solving partial differential equations, Laplace, Fourier and Hankel. Chapter 10 contains a thorough presentation of the theory of Laplace transforms, particularly as it pertains to solving ordinary and partial differential equations. The transform is applied to PDEs on finite and infinite spatial domains. Fourier transforms, and Fourier sine and cosine transforms, in Chapter 11 are developed from Fourier integrals. They are then applied to problems on infinite and semi-infinite domains. Hankel transforms are applied to problems in polar and cylindrical coordinates.

Green's functions for ordinary differential equations and partial differential equations are discussed in Chapters 12 and 13. Chapter 13 utilizes separation techniques from Chapter 6, Section 9.1, and Chapter 12.

Chapters 14, 15, and 16 provide an introduction to numerical techniques for approximating solutions to PDEs, namely finite differences, weighted residuals, and finite elements.

To work through most sections of the book, students require a first course in ordinary differential equations and an introduction to advanced calculus. Sections 10.3–10.5, and Chapter 11 assume a working knowledge of the theory of complex functions.

There are six appendices of material. Appendices A and B give proofs of convergence theorems for Fourier series and Fourier integrals. Appendix C reviews ever so briefly those aspects of vector integral calculus that are used throughout the book. Appendix D contains discussions on parts of the theory of complex functions necessary in the chapters on Laplace and Fourier transforms. Appendix E contains numerical answers to all exercises. Appendix F is a reference to examples and exercises in Chapters 2-13 that contain physical applications of PDEs. Hopefully, it will help readers locate a problem in which they have a particular interest.

We believe that exercises are of the utmost importance to a student's learning. There must be straightforward problems to reinforce fundamentals and more difficult problems to challenge enterprising students. We have attempted to provide more than enough of each type. Problems in each set of exercises are graded from easy to difficult, and numerical answers to all exercises are provided in Appendix E. Exercise sets in sixteen sections (4.2, 4.3, 6.2, 6.3, 6.4, 7.2, 7.3, 9.1, 9.2, 10.2, 10.4, 10.5, 11.4, 11.6, 11.7, and 12.4) stress applications. They have been divided into four parts:

Part A — Heat Conduction
Part B — Vibrations
Part C — Potential, Steady-state Heat Conduction, Static Deflections of Membranes
Part D — General Results

Students interested in heat conduction could concentrate on problems from Part A. Students interested in mechanical vibrations will find problems in Part B particularly appropriate. All students can profit from problems in Part C, since every problem therein, although stated in terms of one of the three applications, is easily interpretable in terms of the other two. We recommend the exercises in Part D to all students.

A solutions manual containing solutions to all exercises is available.

CHAPTER 1 FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

§1.1 Quasilinear First-order Partial Differential Equations

A **partial differential equation** (PDE) is an equation that must be solved for an unknown function of at least two independent variables when the equation contains partial derivatives of the unknown function. The **order** of a PDE is the highest-order partial derivative contained therein. In this chapter we discuss first-order partial differential equations, equations that contain only first-order partial derivatives of the unknown function. We use the *method of characteristics* to solve *quasilinear* equations in this section and general nonlinear equations in Section 1.2. Some physical applications of first-order equations are discussed in Section 1.3. In many respects, first-order PDEs are different from second and higher order equations which arise in the many physical problems developed in the remainder of this text. If the reader is interested in only these particular applications, he/she can, without too much difficulty, omit this chapter and proceed directly to Chapter 2. (The one major exception is the derivation of a general solution of the wave equation in Section 2.7.) In addition, a number of concepts introduced in later chapters have their origin in first-order equations, and it can therefore be beneficial to at least give this chapter a cursory reading.

In the method of characteristics, integration of the initial-value problem associated with a first-order PDE is reduced to integration of the initial value problem for a system of ordinary differential equations. For simplicity in calculations and for geometric visualizations, we consider problems in two independent variables, but extensions to higher numbers of independent variables are algebraically straightforward. We begin with the **quasilinear** PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{1.1}$$

for u as a function of x and y. We assume that coefficient functions $a(x,y,u)$, $b(x,y,u)$, and $c(x,y,u)$ are continuous in some region of xyu-space, and we seek solutions of the PDE that have continuous first partial derivatives u_x and u_y. If $c(x,y,u)$ is of the form $c(x,y)u+d(x,y)$, and $a(x,y,u)$ and $b(x,y,u)$ are independent of u, the equation is said to be **linear**. Quasilinear equations are linear in u_x and u_y, but not in u itself. Theory for linear equations is the same as that given here for more general quasilinear equations, but simplifications in calculations occur in the linear case. This is illustrated in some of our examples and in a short discussion at the end of this section.

If $u(x,y)$ is a solution of PDE 1.1, a normal to the surface $u = u(x,y)$ is $\nabla[u(x,y) - u] = \langle u_x, u_y, -1\rangle$. Since the PDE can be expressed in the form

$$0 = au_x + bu_y - c = \langle u_x, u_y, -1\rangle \cdot \langle a, b, c\rangle,$$

the PDE demands that at each point on a solution surface, the vector $\langle a, b, c\rangle$ must be normal to the vector $\langle u_x, u_y, -1\rangle$, and hence lie in the tangent plane to the surface at that point. Thus, the PDE defines a direction field $\langle a(x,y,u), b(x,y,u), c(x,y,u)\rangle$, called the **characteristic directions**, such that $u = u(x,y)$ is a solution surface if, and only if, at each point $(x,y,u(x,y))$ on the surface, the tangent plane to the surface contains the characteristic direction (Figure 1.1). In Figure 1.2, we have shown a number of these tangent vectors at various points on the solution surface.

A curve that begins at a point on the surface, lies in the surface, and remains tangent to the characteristic direction at every point is called a **characteristic curve** (C-curve, for short). The solution surface can be thought of as being comprised of C-curves; in fact a one-parameter family of C-curves.

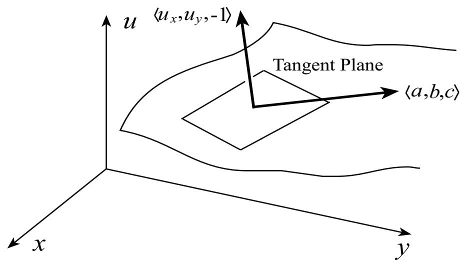

Figure 1.1

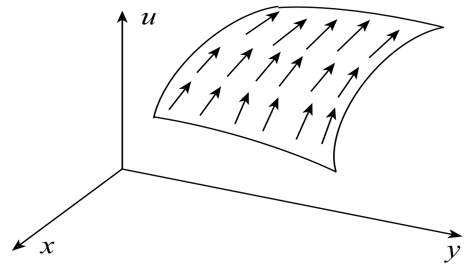

Figure 1.2

This suggests that solution surfaces to PDE 1.1 can be obtained by finding all C-curves and extracting from them one-parameter families. C-curves are defined by the ordinary differential equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}, \tag{1.2}$$

called the **characteristic equations**. Solving these equations gives a two-parameter family of C-curves that can be expressed in the form

$$F(x, y, u, \alpha, \beta) = 0, \qquad G(x, y, u, \alpha, \beta) = 0. \tag{1.3}$$

Through each point (x, y, u) in space there is a unique C-curve and a tangent vector to a C-curve at every such point is $\langle a, b, c\rangle$ (Figure 1.3). Any smooth surface composed of C-curves is a solution of PDE 1.1. Such surfaces can be found analytically by specifying β as a function of α, $\beta = \beta(\alpha)$. This creates a one-parameter family of C-curves, a surface,

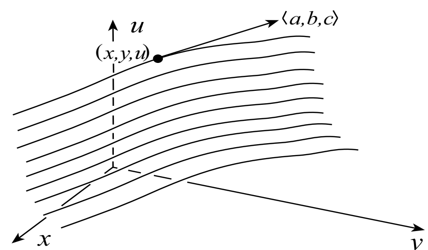

Figure 1.3

$$F[x, y, u, \alpha, \beta(\alpha)] = 0, \qquad G[x, y, u, \alpha, \beta(\alpha)] = 0. \tag{1.4}$$

The equation of the surface is found implicitly or explicitly by eliminating α between these equations. Here is an example to illustrate these ideas.

Example 1.1 Find a solution surface for the quasilinear first-order partial differential equation

$$xu_x + yu_y = xyu^2.$$

Solution The C-equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xyu^2}.$$

Left and middle terms integrate to $y = \alpha x$. Substitution of this into the first and third terms gives

$$\frac{du}{u^2} = \alpha x\, dx, \qquad \text{from which} \qquad -\frac{1}{u} = \frac{\alpha x^2}{2} + \beta.$$

Thus,

$$y = \alpha x, \quad -\frac{1}{u} = \frac{\alpha x^2}{2} + \beta$$

is a 2-parameter family of C-curves. Specifying β as a function of α defines a 1-parameter family of C-curves, a solution surface. Suppose we set $\beta = \alpha + 3$, in which case

$$y = \alpha x, \quad -\frac{1}{u} = \frac{\alpha x^2}{2} + \alpha + 3.$$

The solution surface for this particular choice of $\beta(\alpha)$ can be obtained implicitly by eliminating α between these equations,

$$-\frac{1}{u} = \frac{x^2}{2}\left(\frac{y}{x}\right) + \frac{y}{x} + 3 = \frac{xy}{2} + \frac{y}{x} + 3 = \frac{x^2 y + 2y + 6x}{2x}.$$

An explicit definition of this surface is

$$u = u(x, y) = \frac{-2x}{x^2 y + 2y + 6x}.\bullet$$

We summarize these ideas in the following theorem.

Theorem 1.1 Every one-parameter family of characteristic curves generates a solution surface to PDE 1.1. Conversely, every solution surface may be considered as a one-parameter family of characteristic curves.

Seldom are PDEs solved in isolation. Usually we must find the solution surface to PDE 1.1 that also contains some specified initial curve. This is called an **initial-value problem** or **Cauchy problem**. In view of Theorem 1.1, we can solve the Cauchy problem by finding all C-curves that pass though the initial curve; together they constitute the solution surface. Specification of the initial curve can be thought of in two equivalent ways. First, we can think of the initial curve C being given in xyu-space as shown in Figure 1.4a. In this case we seek a function $u(x, y)$ satisfying PDE 1.1 so that the surface $u = u(x, y)$ contains the initial curve C. Alternatively, we can think of values of u being specified along some curve C' in the xy-plane (Figure 1.4b). We now seek a function $u(x, y)$ satisfying PDE 1.1 that takes on the prescribed values along C'. From either point of view, the solution surface is the surface containing all C-curves passing though C.

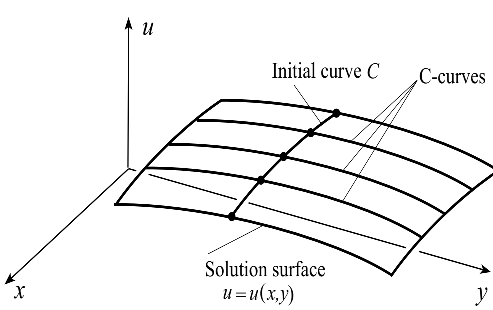

Figure 1.4a

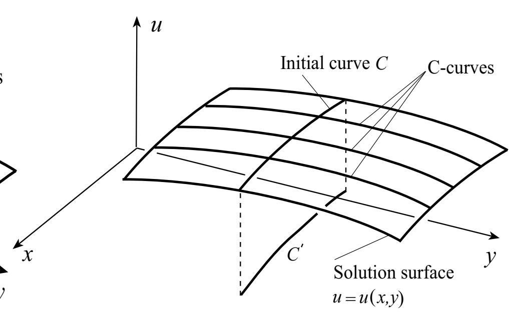

Figure 1.4b

To solve Cauchy's problem associated with PDE 1.1, we must determine the function $\beta(\alpha)$ in equations 1.4 so that the solution surface contains the initial curve.

To do this, suppose equations specifying the initial curve are written in the form

$$y = y(x), \qquad u = u(x).$$

(Other forms may be more convenient in some problems.) If C-curves are to pass through the initial curve for some particular choice of α, then

$$F[x, y(x), u(x), \alpha, \beta(\alpha)] = 0, \qquad G[x, y(x), u(x), \alpha, \beta(\alpha)] = 0. \tag{1.5}$$

When x is eliminated between these equations, the function $\beta(\alpha)$ is determined, in which case equations 1.4 determine the one-parameter family of C-curves that constitutes the solution surface. If α can be eliminated between the equations, an implicit or explicit definition of the solution surface is obtained.

The above procedure is called the **method of characteristics** for solving initial-value problems associated with quasilinear first-order PDEs. To review it quickly, we solve the C-equations for a two-parameter family of C-curves. From this two-parameter family of C-curves, we determine the one-parameter family that passes through the initial curve. These C-curves constitute the solution surface for the Cauchy problem.

Not only can the method of characteristics be used to solve Cauchy problems, it can also be used to show which Cauchy problems have solutions, and which have unique solutions. In these discussions, we deal with **base C-curves**; they are the projections of C-curves in the xy-plane. Theory will show that existence and uniqueness of solutions to Cauchy problems is intimately connected to whether the projection of the initial curve is a base C-curve. We illustrate this in the following examples before stating general results. It is perhaps worthwhile to note at this juncture that if functions a and b in PDE 1.1 are independent of u (and this would be the case for a linear PDE), then the equation $dx/a = dy/b$ can be integrated for a one-parameter family of base C-curves (as opposed to a two-parameter family).

Example 1.2 Find the solution surface for the linear PDE $3u_x + 4u_y = 10$ that contains the line $2x = y = 5u$. Show that the projection of the initial curve in the xy-plane is nowhere tangent to a base C-curve.

Solution Characteristic equations 1.2 for the PDE are

$$\frac{dx}{3} = \frac{dy}{4} = \frac{du}{10}.$$

Integration of these gives C-curves

$$y = \frac{4x}{3} + \alpha, \qquad u = \frac{10x}{3} + \beta.$$

Specifying β as a function of α gives a 1-parameter family of C-curves, a solution surface,

$$y = \frac{4x}{3} + \alpha, \qquad u = \frac{10x}{3} + \beta(\alpha).$$

To find $\beta(\alpha)$ so that the solution surface contains the initial curve, we substitute the initial curve into these equations,

$$2x = \frac{4x}{3} + \alpha, \qquad \frac{2x}{5} = \frac{10x}{3} + \beta(\alpha).$$

These imply that $\beta(\alpha) = -22\alpha/5$. C-curves generating the solution surface are therefore

$$y = \frac{4x}{3} + \alpha, \qquad u = \frac{10x}{3} - \frac{22\alpha}{5}.$$

When α is eliminated between these equations, the result is

$$u = \frac{138x}{15} - \frac{22y}{5}.$$

The solution surface is a plane defined for all x and y. Base C-curves are straight lines $y = 4x/3 + \alpha$ with slope 4/3. Since the projection of the initial curve in the xy-plane is the line $y = 2x$ with slope 2, it is nowhere tangent to a base C-curve.●

Example 1.3 Find the solution surface for the linear PDE $y^2(x-y)u_x + x^2(y-x)u_y = (x^2+y^2)u$ that contains the hyperbola $xu = 1$, $y = 0$, $(x > 0)$. Show that the projection of the initial curve in the xy-plane is nowhere tangent to a base C-curve.

Solution Characteristic equations 1.2 for the PDE are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{du}{u(x^2+y^2)}.$$

The first two terms give

$$y^2\,dy = -x^2\,dx \qquad \Longrightarrow \qquad x^3 + y^3 = \alpha.$$

On the other hand, when we subtract the following equations

$$\frac{dx}{du} = \frac{y^2(x-y)}{u(x^2+y^2)}, \qquad \frac{dy}{du} = \frac{x^2(y-x)}{u(x^2+y^2)},$$

we obtain

$$\frac{dx}{du} - \frac{dy}{du} = \frac{x-y}{u} \qquad \Longrightarrow \qquad \frac{dx-dy}{x-y} = \frac{du}{u} \qquad \Longrightarrow \qquad x - y = \beta u.$$

By specifying β as a function of α, we obtain a one-parameter family of C-curves,

$$x^3 + y^3 = \alpha, \qquad x - y = \beta(\alpha)u.$$

For the solution surface defined by these C-curves to pass through the initial curve, we set

$$x^3 = \alpha, \qquad x = \beta(\alpha)\frac{1}{x}.$$

These give $\beta(\alpha) = \alpha^{2/3}$, and therefore the one-parameter family of C-curves generating the solution surface is

$$x^3 + y^3 = \alpha, \qquad x - y = \alpha^{2/3}u.$$

When we eliminate α, we obtain an explicit definition of the solution

$$x - y = (x^3+y^3)^{2/3}u \quad \Longrightarrow \quad u = \frac{x-y}{(x^3+y^3)^{2/3}}.$$

Base C-curves $x^3 + y^3 = \alpha$ that arise from C-curves passing through the initial curve are shown in Figure 1.5. They intersect the projection of the initial curve, namely the positive x-axis, at right-angles. The solution is only defined for $y > -x$ and becomes unbounded as (x, y) approaches this line. This is a consequence of the fact that u becomes unbounded as $x \to 0$ along the initial curve and the base C-curve through the origin is $y = -x$.•

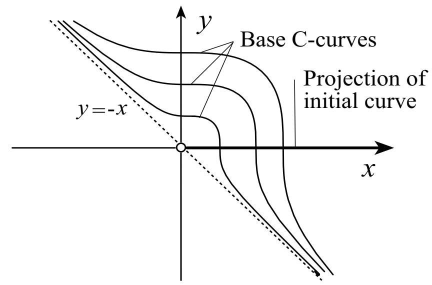

Figure 1.5

Example 1.4 Discuss the solution for the PDE in Example 1.3 if the initial curve is $x + y = 2$, $u = 1$.

Solution Integration of the C-equations as in Example 1.3 leads to the one-parameter family of C-curves $x^3 + y^3 = \alpha$, $x - y = \beta(\alpha)u$. For the solution to pass through the initial curve, we set

$$x^3 + (2 - x)^3 = \alpha, \qquad x - (2 - x) = \beta(\alpha).$$

The first of these can be written in the form $6x^2 - 12x + (8 - \alpha) = 0$, and when it is solved for x, the result is $x = (6 \pm \sqrt{6\alpha - 12})/6$. Substitution of this into the second equation gives $\beta(\alpha) = \pm(1/3)\sqrt{6\alpha - 12}$. Consequently, the one-parameter family of C-curves generating the solution surface is

$$x^3 + y^3 = \alpha, \qquad x - y = \frac{\pm\sqrt{6\alpha - 12}}{3}u.$$

When α is eliminated between these equations, the explicit solution is

$$u(x, y) = \sqrt{\frac{3(x - y)^2}{2(x^3 + y^3 - 2)}}.$$

This solution becomes unbounded as (x, y) approaches any point on the base C-curve $x^3 + y^3 = 2$ through $(1, 1)$. Coincidentally, notice that the projection of the initial curve in the xy-plane is tangent to the base curve at $(1, 1)$.•

The following theorem describes conditions that guarantee a unique solution of the Cauchy problem.

Theorem 1.2 When curve C' in Figure 1.4b is nowhere tangent to a base C-curve, the Cauchy problem associated with PDE 1.1 has a unique solution. When C' is a base C-curve, PDE 1.1 does not have a solution unless the initial curve is a C-curve, in which case there is an infinity of solutions.

We shall not prove this result, but we can see it geometrically. When C' is nowhere tangent to a base C-curve, we have the situation in Figure 1.4b. The solution surface consists of the C-curves through the initial curve C, and as such, it is unique. When C' is a base C-curve (Figure 1.6a), there is at least one C-curve projecting onto C'. If C is not a C-curve, itself, then it is impossible to have a surface $u = u(x, y)$ that contains both C and the C-curve. On the other hand, if C is a C-curve, take any other curve C'' intersecting C in a unique point (Figure 1.6b). The solution of the PDE containing C'' contains the point of intersection of C and C'', and therefore the C-curve through that point, namely C. In other words, there is an infinity of solutions of the PDE containing C.

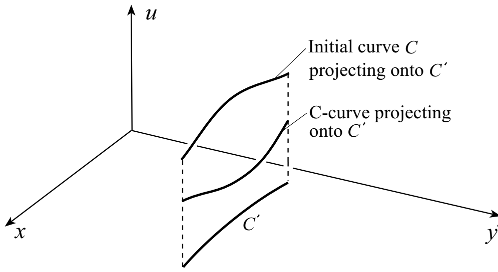

Figure 1.6a

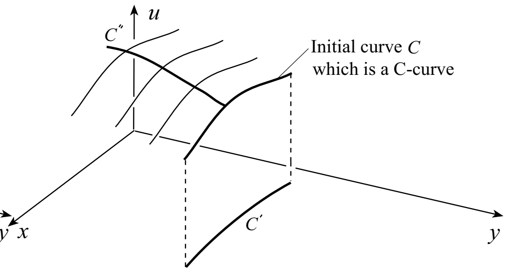

Figure 1.6b

Theorem 1.2 does not discuss ramifications when the projection of the initial curve is tangent to a base C-curve at isolated points. We saw one such situation in Example 1.4; we shall see others.

Example 1.5 Find, if possible, a solution surface for the quasilinear PDE $u\,u_x + u_y = 1$ that also contains each of the following curves:

(a) $C: \; x = y = u$ (b) $C: \; x = y^2/2, \, u = y$ (c) $C: \; 4x = y^2, \, u = y/2$

Solution Characteristic equations for the PDE are

$$\frac{dx}{u} = dy = du.$$

Integration of these gives C-curves

$$u = y + \alpha, \qquad x = \frac{u^2}{2} + \beta.$$

Specifying β as a function of α gives a 1-parameter family of C-curves, a solution surface,

$$u = y + \alpha, \qquad x = \frac{u^2}{2} + \beta(\alpha).$$

(a) For C-curves to pass through the initial curve C: $x = y = u$, we set

$$u = u + \alpha, \quad u = \frac{u^2}{2} + \beta(\alpha).$$

These do not determine $\beta(\alpha)$, but notice that the first equation suggests that $\alpha = 0$, in which case $u = y$. This is a solution surface that contains the initial curve. Base C-curves are parabolas

$$x = \frac{1}{2}(y + \alpha)^2 + \beta,$$

a 2-parameter family of them, some of which are shown in Figure 1.7. This is due to the fact that the PDE is quasilinear, not linear as in the last three examples. When the PDE is linear, the first two terms in C-equations 1.2 are independent of u, and can be integrated for a 1-parameter family of base C-curves. When functions a and b in equations 1.2 depend

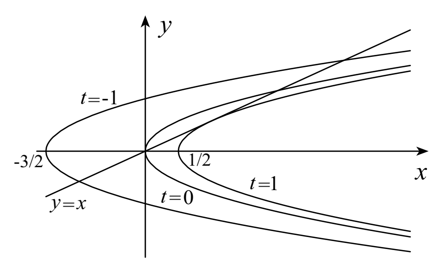

Figure 1.7

on u (as in this example), this cannot be done. Base C-curves depend not only on x and y, but also on u, and a 2-parameter family of base C-curves results. The same solution $u = y$ results for any noncharacteristic initial curve in the plane $u = y$.

(b) Since this initial curve is once again in the plane $u = y$, we can conclude that the function $u = y$ is a solution of the PDE containing the initial curve. In this case, however, the initial curve is a C-curve. Consequently, $u = y$ is one of an infinite number of solutions of the PDE containing C. For instance, suppose we find the solution surface that contains the initial curve C': $x = y = 2u$ (which intersects the original curve C: $x = y^2/2$, $u = y$ at the origin), but does not lie in the plane $u = y$. For the 1-parameter family of C-curves $u = y + \alpha$, $x = u^2/2 + \beta(\alpha)$ to contain this curve, we set

$$u = 2u + \alpha, \quad 2u = \frac{u^2}{2} + \beta(\alpha).$$

These imply that $\beta(\alpha) = -2\alpha - \alpha^2/2$, and therefore the 1-parameter family of C-curve defining the solution surface is

$$u = y + \alpha, \quad x = \frac{u^2}{2} - 2\alpha - \frac{\alpha^2}{2}.$$

Elimination of α leads to the explicit solution

$$u = \frac{2x + y^2 - 4y}{2(y - 2)}.$$

(c) For C-curves to pass through the initial curve C: $4x = y^2$, $u = y/2$, we must have

$$\frac{y}{2} = y + \alpha, \quad \frac{y^2}{4} = \frac{y^2}{8} + \beta(\alpha).$$

These imply that $\beta(\alpha) = \alpha^2/2$, and therefore the 1-parameter family of C-curve generating the solution surface is

$$u = y + \alpha, \quad x = \frac{u^2}{2} + \frac{\alpha^2}{2}.$$

Substitution of $\alpha = u - y$ into the second of theses gives

$$x = \frac{u^2}{2} + \frac{1}{2}(u - y)^2,$$

and when this quadratic equation is solved for u, the result is

$$u(x, y) = \frac{y}{2} \pm \sqrt{x - \frac{y^2}{4}}.$$

Solutions are only defined inside and on the initial parabola $x = y^2/4$, but because partial derivatives are unbounded on the parabola, the solution must be rejected. This can be attributed to the projection $x = y^2/4$ of the initial curve being tangent to the base C-curve $x = y^2/2$ at $(0, 0)$.•

Discontinuities in $u(x,y)$ and/or its partial derivatives at a point on the initial curve C are propagated along the C-curve through that point. For instance, suppose a solution of a quasilinear PDE is to contain the non-characteristic curve C in Figure 1.8 which has a discontinuity in u at the point (x_0,y_0). The solution surface $u=u(x,y)$ maintains the discontinuity along the C-curve through the point $(x_0,y_0,u(x_0,y_0))$, unless the solution breaks down with two C-curves intersecting. This is illustrated in the following example.

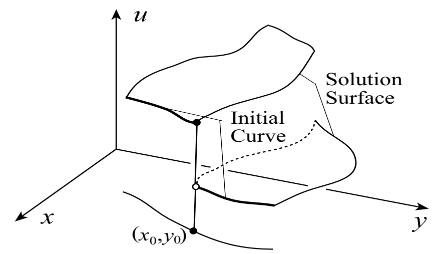

Figure 1.8

Example 1.6 Find the solution of the linear PDE $u_x+u_y=0$ that takes on the values $u(0,y)=\begin{cases}-y/3, & y\le 0\\ 2y+1, & y>0,\end{cases}$ (see Figure 1.9).

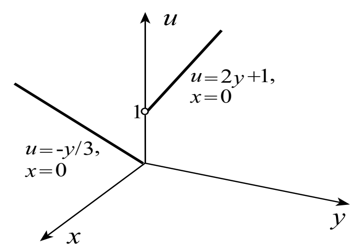

Figure 1.9

Solution We cannot write full characteristic equations 1.2 for the PDE (since $c=0$). We replace them with

$$dx=dy,\qquad du=0.$$

Integration gives

$$y=x+\alpha,\qquad u=\beta,$$

so that u is constant along C-curves; that is, C-curves are horizontal. This does not necessarily imply that the solution of the initial value problem is $u=$ constant. Indeed, the initial curve in Figure 1.9 indicates that this cannot be the case. For C-curves to pass through $u=-y/3$ when $x=0$ and $y\le 0$, we set

$$y=\alpha,\qquad -\frac{y}{3}=\beta(\alpha).$$

Thus, C-curves through this part of the initial curve are

$$y=x+\alpha,\quad u=-\frac{\alpha}{3}\qquad\Longrightarrow\qquad u(x,y)=-\frac{1}{3}(y-x)=\frac{1}{3}(x-y).$$

For C-curves to pass through $u=2y+1$ when $x=0$ and $y>0$, we set

$$y=\alpha,\qquad 2y+1=\beta(\alpha).$$

Thus, C-curves through this part of the initial curve are

$$y=x+\alpha,\quad u=2\alpha+1\qquad\Longrightarrow\qquad u(x,y)=2(y-x)+1.$$

The solution surface is composed of two planes, and to determine regions in the xy-plane onto which these planes project, we draw base C-curve. They are the lines $y=x+\alpha$ shown in Figure 1.10a. Below the C-curve $y=x$ are C-curves along which $u=-y/3$; along C-curves above $y=x$, $u=2y+1$. The solution surface in Figure 1.10b, consists of two planes above the regions corresponding to these two sets of C-curves. It is discontinuous along the base C-curve $y=x$ through the point $(0,0)$ where the initial data is discontinuous.●

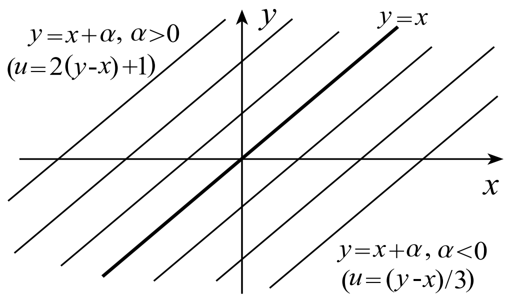

Figure 1.10a

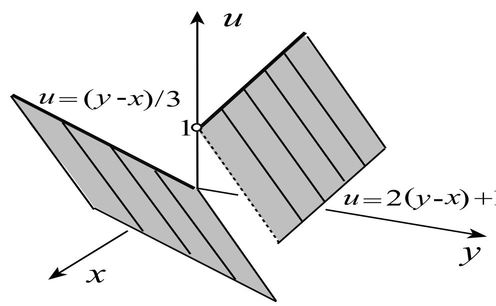

Figure 1.10b

In the next example, we illustrate that discontinuities in partial derivatives of a solution are also propagated along C-curves even when the function itself is continuous.

Example 1.7 Find the solution of the linear PDE $yu_x + xu_y = u$ in the first quadrant that takes on values x^3 along the positive x-axis and values y^3 along the positive y-axis.

Solution Characteristic equations 1.2 for the PDE are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}.$$

The first two of these give

$$y\,dy = x\,dx \qquad \Longrightarrow \qquad y^2 - x^2 = \alpha.$$

In addition, when we add the equations $dx = \dfrac{y\,du}{u}$ and $dy = \dfrac{x\,du}{u}$, we obtain

$$dx + dy = \frac{y\,du}{u} + \frac{x\,du}{u} = (x+y)\frac{du}{u} \qquad \Longrightarrow \qquad \frac{dx+dy}{x+y} = \frac{du}{u} \qquad \Longrightarrow \qquad u = \beta(x+y).$$

Base C-curves in the first quadrant are the hyperbolas $y^2 - x^2 = \alpha$ in Figure 1.11. The base C-curve $y = x$ $(\alpha = 0)$ separates the first quadrant into two regions R_1 and R_2 corresponding to base C-curves that have $\alpha < 0$ and $\alpha > 0$, respectively. Solution surfaces to the PDE are obtained by specifying β as a function of α,

$$y^2 - x^2 = \alpha, \quad u = \beta(\alpha)(x+y).$$

This one-parameter family of C-curves takes on the values $u(x,0) = x^3$ if

$$-x^2 = \alpha, \quad x^3 = \beta(\alpha)x \quad \Longrightarrow \quad \beta(\alpha) = -\alpha.$$

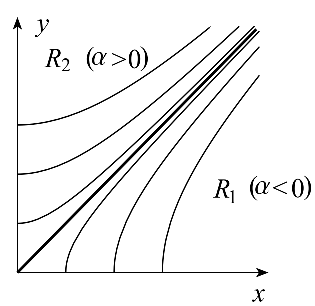

Figure 1.11

In region R_1 then, the solution surface is given by

$$y^2 - x^2 = \alpha, \quad u = -\alpha(x+y) \qquad \Longrightarrow \qquad u = (x+y)(x^2 - y^2).$$

The one-parameter family of C-curves takes on the values $u(0,y) = y^3$ if

$$y^2 = \alpha, \quad y^3 = \beta(\alpha)y \qquad \Longrightarrow \qquad \beta(\alpha) = \alpha.$$

In region R_2 then, the solution surface is given by

$$y^2 - x^2 = \alpha, \quad u = \alpha(x+y) \qquad \Longrightarrow \qquad u = (x+y)(y^2 - x^2).$$

The solution is continuous for all x and y, even across the base C-curve $y = x$ that separates regions R_1 and R_2. However, in region R_1,

$$\begin{aligned} u_x &= (x^2 - y^2) + 2x(x+y) = (x+y)(3x - y), \\ u_y &= (x^2 - y^2) - 2y(x+y) = (x+y)(x - 3y), \end{aligned}$$

whereas in region R_2,

$$\begin{aligned} u_x &= (y^2 - x^2) - 2x(x+y) = (x+y)(y - 3x), \\ u_y &= (y^2 - x^2) + 2y(x+y) = (x+y)(3y - x). \end{aligned}$$

If points (x, y) on $y = x$ are approached from region R_1,

$$u_x \quad \rightarrow \quad 2x(2x) = 4x^2, \qquad u_y \quad \rightarrow \quad 2x(-2x) = -4x^2,$$

whereas if approached from region R_2,

$$u_x \quad \rightarrow \quad 2x(-2x) = -4x^2, \qquad u_y \quad \rightarrow \quad 2x(2x) = 4x^2.$$

Thus, although $u(x, y)$ itself is continuous, its derivatives u_x and u_y are discontinuous across the base C-curve $y = x$ through $(0, 0)$.•

The next example illustrates that the method of characteristics, as outlined, may not give a solution for all values of x and y, but it may be possible to extend such a solution to encompass all values of the independent variables. This will be an important consideration in traffic flow applications in Section 1.3.

Example 1.8 Find the solution of the quasilinear PDE $u_x + uu_y = 0$, (for $x > 0$) that takes on the values $u(0, y) = \begin{cases} k_1, & y \leq 0 \\ k_2, & y > 0. \end{cases}$

Solution Characteristic equations for the PDE are

$$dx = \frac{dy}{u}, \qquad du = 0.$$

It then follows that $u = \beta$ so that u is constant along C-curves, and $y = \beta x + \alpha$. A one-parameter family of C-curves is

$$y = \beta(\alpha)x + \alpha, \qquad u = \beta(\alpha).$$

For C-curves to take on the values $u = k_1$ when $x = 0$ and $y \leq 0$, we set

$$y = \alpha, \qquad k_1 = \beta(\alpha).$$

Thus, C-curves through this part of the initial curve ($y \leq 0$) are

$$y = k_1 x + \alpha \qquad u = k_1.$$

Similarly, C-curves passing through the other half of the initial curve ($y > 0$) are

$$y = k_2 x + \alpha, \qquad u = k_2.$$

The solution always has value $u = k_1$ or $u = k_2$. To determine regions of the xy-plane where these values of u are taken on, we draw base C-curves. Base C-curves along which $u = k_1$ are straight lines $y = k_1 x + \alpha$ ($\alpha \le 0$) defining region R_1 in Figure 1.12a. Base C-curves along which $u = k_2$ are straight lines $y = k_2 x + \alpha$ ($\alpha > 0$) defining region R_2 in Figure 1.12a. We have drawn base C-curves only for $x > 0$ as specified in the original problem. This leaves the solution undefined in the wedge R_3 between the lines $y = k_1 x$ and $y = k_2 x$. To obtain a solution in R_3, imagine a fan of straight lines $y = mx$ eminating from $(0, 0)$ into R_3. Since u takes on constant values along the C-curves in R_1 and R_2, suppose we let $u = m$ along $y = mx$. In other words, let equations for the lines in the fan be $y = ux$, $k_1 < u < k_2$, and let the value of the solution $u(x, y)$ along each line in the fan be the slope u of the line. It is straightforward to show that when we invert this equation, the function $u = y/x$ satisfies the PDE in R_3. It also joins the planes $u = k_1$ and $u = k_2$ in R_1 and R_2 to create a continuous solution (Figure 1.12b). The straight lines of the fan in R_3 satisfy the C-equations, and are called **fanlike** base C-curves.●

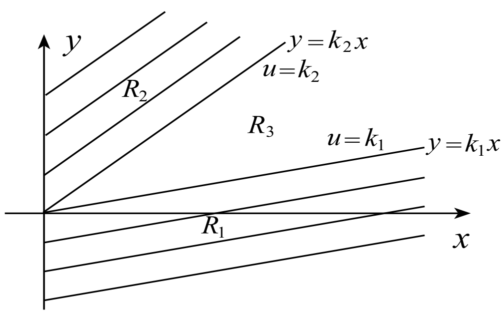

Figure 1.12a

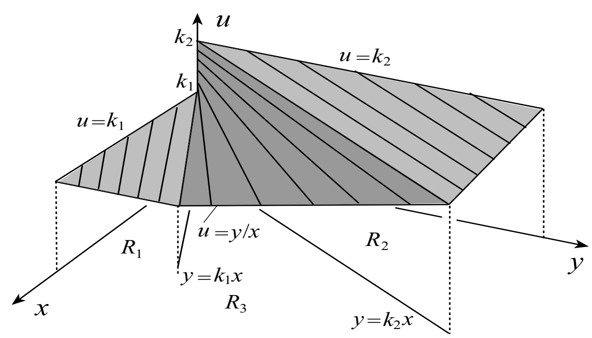

Figure 1.12b

Linear First-order Equations

Because linear first-order PDEs are quasilinear, foregoing results also apply to linear equations, equations of the form

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} = c(x, y)u + d(x, y). \tag{1.6}$$

Because such equations are linear in u as well as u_x and u_y, they possess special attributes that are useful for comparison purposes when we discuss C-curves for linear higher order PDEs in subsequent chapters. For linear first-order PDEs, C-equations 1.2 are

$$\frac{dx}{a(x, y)} = \frac{dy}{b(x, y)} = \frac{du}{c(x, y)u + d(x, y)}.$$

As noted earlier, because u is absent from the first two terms, this ODE can be solved independently of u to give a 1-parameter family of base C-curves. The base C-curve through a point (x, y, u) is the same for all values of u; C-curves will vary with u, but they all project onto the same base C-curve.

EXERCISES 1.1

1. Find solutions to the PDE in Example 1.1 corresponding to the following choices for the function $\beta(\alpha)$: (a) $3\alpha + 4$ (b) $\sin\alpha$ (c) $\alpha e^{-2\alpha}$

In Exercises 2–8, find, if possible, the solution surface to the PDE that also contains the given curve. Plot the projection of the initial curve in the xy-plane and base C-curves to determine whether the projection is a base C-curve. Verify the same result algebraically.

2. $x\,u_x + y\,u_y = u, \quad C\colon\ y = x^2,\ u = x$

3. $x^2u_x + y^2u_y = 4, \quad C\colon\ xy = x + y,\ u = 1\ (x > 1)$

4. $u_x + u\,u_y = 4u, \quad C\colon\ x = y = u$

5. $u_x - u\,u_y = u, \quad C\colon\ y = 2x,\ u = 3x$

6. $2y(u-3)u_x + (2x-u)u_y = y(2x-3), \quad C\colon\ x^2 + y^2 = 2x,\ u = 0$

7. $x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u, \quad C\colon\ x + y = 0,\ u = 1$

8. $u(x+y)u_x + u(x-y)u_y = x^2 + y^2, \quad C\colon\ y = 2x,\ u = 0$

9. Find the solution of the PDE $xu_x + yuu_y = -xy$ containing the following two initial curves:
(a) $C\colon\ xy = 1,\ u = c,$ where $c > 0$ is a constant
(b) $C\colon\ x = 0,\ u = c,$ where $c > 0$ is a constant
(c) Can you use the technique of parts (a) and (b) when the initial curve is $C\colon\ x = 1,\ u = c$?

10. Find the solution to the PDE $2u_x + u\,u_y = 2$ containing the curve $y^2 = -4x$, $u = -y$. What is its domain of definition?

11. Suppose initial values for the unknown function $u(x,y)$ are specified along a straight line $y = mx$ for the linear PDE $xu_x + yu_y = u$. When will there be solutions of the Cauchy problem?

12. Discuss the initial value problem of solving the PDE $yu_x - xu_y = 0$ containing:
(a) $C\colon\ x^2 + y^2 = a^2,\ u = y$
(b) $C\colon\ x + y = a,\ u = y$

13. Discuss the initial value problem of solving the PDE $xu_x + yu_y = 4$ containing:
(a) $C\colon\ x = t,\ y = t+1,\ u = t^2, \quad t \geq 1$
(b) $C\colon\ x = t,\ y = 3t,\ u = t^2, \quad t \geq 1$
(c) $C\colon\ x = t,\ y = 3t,\ u = 4\ln t, \quad t \geq 1$

14. Find all solution surfaces of the PDE

$$(ny - mu)u_x + (lu - nx)u_y = mx - ly$$

where l, m, and n are constants. Interpret the result geometrically.

15. Show that the solution of the quasilinear PDE $u_x + u_y = u^2$ passing through the initial curve $u = x = -y$ becomes infinite along the hyperbola $x^2 - y^2 = 4$.

16. Find the solution of the quasilinear PDE $u_x + u_y = u^2$ that contains the initial curve $y = -x$, $u = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$. Determine its domain of validity and show that the solution is discontinuous along the base C-curve passing through the point where $u(x,y)$ is initially discontinuous.

17. Find the solution of the quasilinear PDE $u_x + u_y = u^2 + 1$ that contains the initial curve $x = 0$, $u = f(y)$.

18. Explain why there are no solutions of the PDE $u_x + u_y = u$ passing through the line $y = x$, $u = 1$.

19. Find the solution of the PDE $uu_x + u_y = 0$ passing through the initial curve $y = 0$, $u = f(x)$, where

$$f(x) = \begin{cases} 1, & x \leq 0 \\ 1 - x, & 0 < x < 1 \\ 0, & x > 1. \end{cases}$$

Indicate domains of definition for various parts of the solution.

20. Find the solution of the quasilinear PDE $u_x + 2uu_y = 1$ passing through the initial curves:
(a) $y = x$, $u = x$
(b) $y = x$, $u = x^2$

21. Find the solution of the quasilinear PDE $(u + y)u_x + yu_y = x - y$ passing through the initial curve $x = y - 1$, $u = y^2 + 1$.

22. Show that if a C-curve has one point in common with a solution surface to PDE 1.1, then the C-curve lies entirely within the surface.

23. Show that if two solution surfaces of PDE 1.1 intersect in a point, then they intersect in the entire C-curve through that point.

24. If the solution in Example 1.7 takes on the values $u(0, y) = y$ and $u(x, 0) = x$, are u_x and u_y still discontinuous across the base C-curve $y = x$?

25. Show that when a and b in PDE 1.1 are constants, base C-curves are straight lines with slope b/a.

26. Show that when a, b, and d in linear PDE 1.6 are constants and $c = 0$, and the initial curve is a straight line, the solution surface is a plane. Describe the plane.

§1.2 General Nonlinear First-order Partial Differential Equations

We now turn our attention to the general nonlinear first-order PDE

$$F(x, y, u, p, q) = 0, \tag{1.7}$$

where $p = u_x$ and $q = u_y$. Quasilinear PDEs in Section 1.1 defined a unique tangent vector at each point in space (C-direction), and we used these vectors to derive a two-parameter family of C-curves. The one-parameter subfamily of C-curves through the initial curve constituted the solution surface of the initial value problem. PDE 1.7 does not define a unique tangent vector at each point on a solution surface. It defines, as we shall see, a one-parameter family of possible tangent vectors, and these will lead to *characteristic strips* rather than charactersistic curves.

Suppose that $u(x, y)$ is a solution of PDE 1.7 and (x_0, y_0, u_0) is any point on the surface $u = u(x, y)$ defined by this solution. The vector $\langle p, q, -1\rangle$ where p and q are evaluated at (x_0, y_0, u_0), is normal to the tangent plane to the surface, and the equation of the tangent plane is

$$u - u_0 = p(x - x_0) + q(y - y_0). \tag{1.8}$$

Since $u(x, y)$ is a solution of equation 1.7, the relation

$$F(x_0, y_0, u_0, p, q) = 0, \tag{1.9}$$

is also valid at the point. Conversely, equation 1.8 can be the tangent plane to a solution surface $u = u(x, y)$ at a point (x_0, y_0, u_0) only if p and q satisfy equation 1.9. Since we could regard equation 1.7 as defining q as a function p at each point of space, the PDE defines a one-parameter family of planes that can be tangent to a solution surface. A function $u(x, y)$ satisfies PDE 1.7 when its tangent plane at each point belongs to this one-parameter family of permissible tangent planes. The envelope of this family is called the **Monge cone** at each point. Thus, $u(x, y)$ is a solution of PDE 1.7 if, and only if, at each point on the surface $u = u(x, y)$, its tangent plane is also tangent to the Monge cone at that point.

It may not be clear from equations 1.8 and 1.9 why the envelope of the one-parameter family of planes is called a *cone*. To see why, we discuss the PDE

$$p^2 + q^2 = 1. \tag{1.10}$$

When we solve this equation for q in terms of p, and substitute into equation 1.8, we obtain

$$u - u_0 = p(x - x_0) \pm \sqrt{1 - p^2}(y - y_0). \tag{1.11}$$

These are the possible tangent planes at a point (x_0, y_0, u_0) to a solution surface containing the point. Each tangent plane determines a line through (x_0, y_0, u_0) as the limiting position of the line of intersection of the plane with neigbouring planes in the family. The accumulation of all such lines is the Monge cone; it is the envelope of the possible tangent planes at (x_0, y_0, u_0). We find its equation to show that it is indeed a cone. If p is changed by a small amount h, the equation of the neighbouring tangent plane is

$$u - u_0 = (p + h)(x - x_0) \pm \sqrt{1 - (p + h)^2}\,(y - y_0). \tag{1.12}$$

Together, equations 1.11 and 1.12 define the line of intersection of the planes. If we subtract the equations, then the equations

$$u - u_0 = p(x - x_0) \pm \sqrt{1-p^2}\,(y - y_0), \tag{1.13a}$$
$$0 = h(x - x_0) \pm [\sqrt{1-(p+h)^2} - \sqrt{1-p^2}](y - y_0) \tag{1.13b}$$

also define the line of intersection. To find the line on the Monge cone determined by plane 1.11 for a particular value of p, we take the limit as h approaches zero. If we divide equation 1.13b by h, and take the limit, we obtain

$$u - u_0 = p(x - x_0) \pm \sqrt{1-p^2}(y - y_0),$$
$$0 = \lim_{h\to 0}\left\{x - x_0 \pm \left[\frac{\sqrt{1-(p+h)^2} - \sqrt{1-p^2}}{h}\right](y - y_0)\right\}.$$

Since $\lim_{h\to 0} \dfrac{\sqrt{1-(p+h)^2} - \sqrt{1-p^2}}{h}$ is the derivative of $\sqrt{1-p^2}$, we have

$$u - u_0 = p(x - x_0) \pm \sqrt{1-p^2}\,(y - y_0), \quad 0 = x - x_0 \pm \left(\frac{-p}{\sqrt{1-p^2}}\right)(y - y_0).$$

This is the line on the Monge cone determined by the plane with parameter value p. If we eliminate p, we obtain the equation of the Monge cone itself. The second equation can be solved for

$$p = \frac{\pm(x - x_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}},$$

and when this is substituted into the first equation,

$$u - u_0 = \frac{\pm(x-x_0)^2}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \pm \sqrt{1 - \frac{(x-x_0)^2}{(x-x_0)^2 + (y-y_0)^2}}(y - y_0).$$

This reduces to $(u-u_0)^2 = (x-x_0)^2 + (y-y_0)^2$, a pair of right circular cones through (x_0, y_0, u_0) with vertical axis of symmetry. Let us take this one step further before returning to our general discussion of PDE 1.7. Suppose we require the solution of PDE 1.10 that contains the line $y = x$, $u = 0$ in the xy-plane. We have drawn the upper Monge cones at a few points on the line in Figure 1.13. They are the same at every point in space. Geometrically, we can see that planes containing the line $y = x$, $u = 0$ and inclined at $\pi/4$ radians to the xy-plane would be tangent to the Monge cones along the line and at every point on the planes. In other words, these planes should be solution surfaces to the initial value problem. Indeed, equations of the planes are $u = \pm(x - y)/\sqrt{2}$, and it is a mental calculation to show that they satisfy $p^2 + q^2 = 1$. We will discuss this PDE further in Example 1.10, but for now, we at least see the Monge cones associated with the PDE, and how solution surfaces are tangent to the Monge cone at each point.

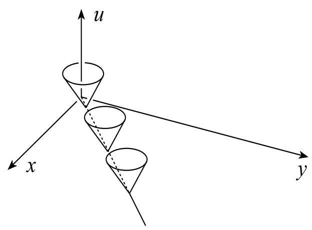

Figure 1.13

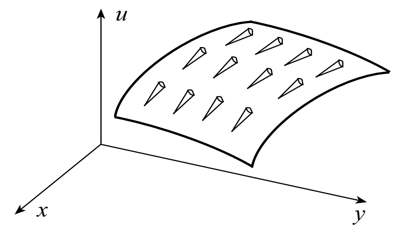

Figure 1.14

We now return to our discussion of PDE 1.7. The PDE defines a one-parameter family of possible tangent planes to a solution surface at each point in space. At each point, the envelope of this family is the Monge cone. A solution surface shares a tangent plane with the Monge cone. We have shown a solution surface in Figure 1.14 with Monge cones at a number of points on the surface. At each point, one of the generating lines of the Monge cone is tangent to the solution surface. This suggests that we derive generating lines for the Monge cone. To do this, we repeat the above discussion but with general PDE 1.7 rather than the specific example $p^2 + q^2 = 1$. Equation 1.8 defines the tangent plane to a surface $u = u(x, y)$ at a point (x_0, y_0, u_0) on the surface. This is a tangent plane to a solution surface of PDE 1.7 only if p and q satisfy equation 1.9. In other words, PDE 1.7 defines a 1-parameter family of possible tangent planes to a solution surface at each point in space. Suppose we take p as the parameter defining the tangent planes and q as a function of p defined implicitly by PDE 1.7. Tangent planes at (x_0, y_0, u_0) corresponding to parameter values p and $p + h$ are

$$u - u_0 = p(x - x_0) + q(p)(y - y_0), \qquad u - u_0 = (p + h)(x - x_0) + q(p + h)(y - y_0).$$

Together these equations define the line of intersection of the planes. When these equations are subtracted, the line of intersection is also defined by the equations

$$u - u_0 = p(x - x_0) + q(p)(y - y_0), \quad h(x - x_0) + (y - y_0)[q(p + h) - q(p)] = 0.$$

If we divide the second of these by h, and let h approach zero, we obtain the generating line of the Monge cone corresponding to the tangent plane with parameter value p,

$$u - u_0 = p(x - x_0) + q(p)(y - y_0), \qquad (x - x_0) + (y - y_0)q'(p) = 0. \quad (1.14)$$

(Elimination of p between these equations would lead to the equation for the Monge cone at the point (x_0, y_0, u_0).) To obtain differential equations that can be solved for all generating lines of the Monge cone, we eliminate $q'(p)$ between these equations. To do this, we note that because

$$F[x_0, y_0, u_0, p, q(p)] = 0$$

is an identity in p, differentiation gives

$$F_p + F_q\, q'(p) = 0.$$

When this equation and the second equation in 1.14 are solved for $q'(p)$, and the expressions are equated, the result is

$$\frac{x - x_0}{F_p} = \frac{y - y_0}{F_q}.$$

This is the equation for the projections in the xy-plane for the generators of the Monge cone. By replacing $x-x_0$ and $y-y_0$ with dx and dy, and dropping evaluation of arguments of F_p and F_q at x_0, y_0, and u_0, we obtain a differential equation for projections in the xy-plane of generators of Monge cones,

$$\frac{dx}{F_p} = \frac{dy}{F_q}.$$

To find the generators themselves, we must add an equation for the u-coordinate. Because

$$du = u_x\,dx + u_y\,dy = p\,dx + q\,dy = p\,dx + q\left(\frac{F_q}{F_p}dx\right) = \frac{pF_p + qF_q}{F_p}dx,$$

it follows that we may expand the above differential equation to the system

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q}. \tag{1.15}$$

These equations define the generator along the Monge cone for a specified value of p (and q as determined by the function $q(p)$). Remember that a Monge cone has an infinite number of generators, one corresponding to each plane tangent to the cone. In other words, if we solve system 1.15, equations of the generators will be functions of p. In general, this is impossible since it requires that the PDE $F(x, y, u, p, q) = 0$ be solved for $q(p)$. The alternative is to expand the system to include equations for p and q. To do this, we differentiate $F(x, y, u, p, q) = 0$ with respect to x, and substitute $q_x = p_y$, and for F_q from system 1.15,

$$\begin{aligned} 0 &= F_x + F_u p + F_p\,p_x + F_q\,q_x = F_x + pF_u + F_p p_x + F_p\frac{dy}{dx}p_y \\ &= F_x + pF_u + \frac{F_p}{dx}(p_x\,dx + p_y\,dy) = F_x + pF_u + \frac{F_p\,dp}{dx}. \end{aligned}$$

This can be rewritten in the form

$$\frac{dx}{F_p} = \frac{-dp}{F_x + pF_u}.$$

Similarly,

$$\frac{dy}{F_q} = \frac{-dq}{F_y + qF_u}.$$

When these are added to system 1.15, the result is

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{-dp}{F_x + pF_u} = \frac{-dq}{F_y + qF_u}. \tag{1.16}$$

Equations 1.16 are called **characteristic equations** (C-equations) for PDE 1.7. They yield a four-parameter family of solutions, called **characteristic strips** (C-strips), but the PDE reduces it to a three-parameter family. In practice, we often use equations 1.16 to obtain four of the dependent quantities, and the PDE to find the fifth. One-parameter subsets of C-strips generate solution surfaces of the PDE. We have characteristic curves, as in the quasilinear case, but at each point on a characteristic curve is added pairs of numbers (p, q), that determine the possible orientations of the tangent plane to a solution surface. In the quasilinear case, the Monge cone reduces to a single line, the line containing the characteristic direction at the point (see Exercise 14).

The initial value problem for PDE 1.7 consists in solving the PDE for a solution surface $u = u(x, y)$ passing through an initial curve C which is best written in parametric form $x = x(t)$, $y = y(t)$, $u = u(t)$. Quantities p and q cannot be specified along C; they are dictated by C itself and the PDE. Functions $p(t)$ and $q(t)$ must satisfy the following conditions:

$$\frac{du}{dt} = p(t)\frac{dx}{dt} + q(t)\frac{dy}{dt}, \tag{1.17a}$$

$$0 = F[x(t), y(t), u(t), p(t), q(t)]. \tag{1.17b}$$

The first of these, called the **strip condition**, ensures that $p(t)$ and $q(t)$ define a plane at each point of C which is tangent to C. The second guarantees that initial values of x, y, u, p, and q satisfy the PDE.

When the three-parameter family of C-strips is made to pass through C, the resulting one-parameter family of C-strips generates the solution of the PDE.

Example 1.9 Determine the solution of the PDE $pq = u$ passing through the curve $x = 0$, $u = y^2$.

Solution Characteristic equations for the PDE are

$$\frac{dx}{q} = \frac{dy}{p} = \frac{du}{2pq} = \frac{dp}{p} = \frac{dq}{q}.$$

The last equation gives

$$q = \alpha p.$$

When this is substituted into the first equation, integration gives

$$x = \alpha y + \beta.$$

The second and fourth terms give

$$p = y + \gamma.$$

Instead of integrating to get u, we use the PDE, $u = pq = \alpha p^2$. Thus, a three-parameter family of C-strips is

$$x = \alpha y + \beta, \quad u = \alpha p^2 = \alpha(y + \gamma)^2, \quad p = y + \gamma, \quad q = \alpha(y + \gamma).$$

To obtain initial values for p and q, we parametrize the initial curve $C: \ x = 0, \ y = t, \ u = t^2$, and invoke conditions 1.17,

$$2t = 0 \cdot p + 1 \cdot q, \qquad pq = t^2.$$

These give $p = t/2$ and $q = 2t$. To find the C-strips that pass through the initial curve, we set

$$0 = \alpha t + \beta, \quad t^2 = \alpha(t + \gamma)^2, \quad \frac{t}{2} = t + \gamma, \quad 2t = \alpha(t + \gamma).$$

These give

$$\alpha = 4, \quad \beta = -4t, \quad \gamma = -\frac{t}{2}.$$

The one-parameter family of C-strips generating the solution surface is

$$x = 4y - 4t, \quad u = 4\left(y - \frac{t}{2}\right)^2, \quad p = y - \frac{t}{2}, \quad q = 4\left(y - \frac{t}{2}\right).$$

Substitution of $t = (4y - x)/4$ into the second gives the explicit solution

$$u = 4\left(y - \frac{4y - x}{8}\right)^2 = \frac{1}{16}(x + 4y)^2.\bullet$$

The following theorem is the analogue for PDE 1.7 of Theorem 1.2 for PDE 1.1.

Theorem 1.3 When curve C' in Figure 1.4b is nowhere tangent to a base C-curve, the Cauchy problem associated with PDE 1.7 has a unique solution. When C' is a base C-curve, PDE 1.7 does not have a solution unless the initial strip is a C-strip, in which case there is an infinity of solutions.

It is important to note, however, that uniqueness results only after initial values for p and q have been obtained from equations 1.17. There may be more than one set of values for p and q satisfying these equations, and for each set, there is a unique solution of the Cauchy problem. We illustrate this in the following example.

Example 1.10 We demonstrated earlier that Monge cones for the PDE $p^2 + q^2 = 1$ are right circular cones with vertical axis of symmetry. We also indicated that $u = \pm(x - y)/\sqrt{2}$ are solution surfaces containing the line $y = x$, $u = 0$. (a) Verify that these are the only solutions by solving the C-equations. (b) Show that there are two solutions of the PDE that contain the curve C: $y = x^2$, $u = y$, one of which is a plane.

Solution (a) Characteristic equations for the PDE are

$$\frac{dx}{2p} = \frac{dy}{2q} = \frac{du}{2p^2 + 2q^2}, \quad dp = 0, \quad dq = 0.$$

These imply that

$$p = \alpha, \quad q = \beta, \quad \alpha y = \beta x + \gamma, \quad \beta u = y + \delta.$$

This is a four-parameter family of C-strips, but the PDE requires $\alpha^2 + \beta^2 = 1$, so that only three of the parameters are independent. Along the initial curve C: $x = t$, $y = t$, $u = 0$, initial values for p and q are determined by equations 1.17,

$$0 = p \cdot 1 + q \cdot 1, \quad p^2 + q^2 = 1 \qquad \Longrightarrow \qquad p = \pm\frac{1}{\sqrt{2}}, \quad q = \mp\frac{1}{\sqrt{2}}.$$

There are two possible initial strips,

$$x = t, \quad y = t, \quad u = 0, \quad p = \pm\frac{1}{\sqrt{2}}, \quad q = \mp\frac{1}{\sqrt{2}}.$$

For the above family of C-strips to contain these initial strips, we set

$$p = \alpha = \pm\frac{1}{\sqrt{2}}, \quad q = \beta = \mp\frac{1}{\sqrt{2}}, \quad \alpha t = \beta t + \gamma, \quad 0 = t + \delta.$$

Consequently, $\gamma = \pm t/\sqrt{2}$ and $\delta = -t$, and the one-parameter family of C-strips generating the solution surface is

$$\pm\frac{y}{\sqrt{2}} = \mp\frac{x}{\sqrt{2}} \pm \sqrt{2}t, \quad \mp\frac{u}{\sqrt{2}} = y - t, \quad p = \pm\frac{1}{\sqrt{2}}, \quad q = \mp\frac{1}{\sqrt{2}}.$$

Explicit solutions are

$$u = \mp\sqrt{2}(y-t) = \mp\sqrt{2}\left(y - \frac{y+x}{2}\right) = \pm\frac{1}{\sqrt{2}}(x-y).$$

(b) Along the initial curve C: $x = t$, $y = t^2$, $u = t^2$, values for p and q are determined by

$$2t = p \cdot 1 + q \cdot 2t, \quad p^2 + q^2 = 1.$$

There are two solutions to these equations,

$$p = 0, \quad q = 1 \qquad \text{and} \qquad p = \frac{4t}{4t^2+1}, \quad q = \frac{4t^2-1}{4t^2+1}.$$

We show that $p = 0$ and $q = 1$ leads to a planar solution. For C-strips to contain the initial strip $x = t$, $y = t^2$, $u = t^2$, $p = 0$, and $q = 1$, we set

$$0 = p = \alpha, \quad 1 = q = \beta, \quad \alpha t^2 = \beta t + \gamma, \quad \beta t^2 = t^2 + \delta.$$

Consequently, $\gamma = -t$ and $\delta = 0$, and the one-parameter family of C-strips generating the solution surface is

$$0 = x - t, \quad u = y;$$

that is, the solution is $u = y$.•

Example 1.11 Partial differential equations of the form $u = px + qy + f(p,q)$ are called **Clairaut equations**. Show that C-strips are always straight lines along which p and q are constant.

Solution Characteristic equations 1.16 for the Clairaut equation are

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{du}{xp+yq+pf_p+qf_q}, \quad dp = 0, \quad dq = 0.$$

The last two obviously imply that p and q are constant along C-strips, $p = \alpha$ and $q = \beta$. The first equation gives

$$y + f_q = \gamma(x + f_p) \qquad \Longrightarrow \qquad y = \gamma x + (\gamma f_p - f_q).$$

The PDE then yields $u = \alpha x + \beta y + f(\alpha,\beta)$. Characteristic strips are therefore straight lines.•

EXERCISES 1.2

In Exercises 1–5, find, if possible, the solution surface to the PDE containing the given curve.

1. $pq = 4, \quad C: \ y = 0, \ u = x$

2. $u = p^2 - q^2, \quad C: \ 4u + x^2 = 0, \ y = 0$

3. $p^2 + q^2 = 4u, \quad C: \ u = x^2 + 1, \ y = 0$

4. $p + q^2 + u = 0, \quad C: \ u = y, \ x = 0$

5. $u = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y), \quad C: \; y = u = 0$

6. Find, if possible, solution surfaces to the PDE $pu = q$ containing the following curves:
(a) C: $y = x^3$, $u = 1$
(b) C: $u = x$, $y = 0$
(c) C: $y = x^3$, $u = x$
(d) C: $u = y = x$, $x \geq 0$

7. (a) Show that when a PDE is explicitly independent of x, y, and u, then p and q are constant along C-strips and u is a linear function of x and y.
(b) Verify that p and q must be constant over the entire solution surface of $pq^2 = 4$ if the solution must contain the line C: $u = x$, $y = 0$, and that the problem has two solutions.
(c) Verify that p and q must be constant over the entire solution surface of $pq^2 = 4$ if the solution must contain the line C: $u = y$, $x = 0$, and that the problem has one solution.
(d) Verify that p and q are not constant over the entire solution surface of $pq^2 = 4$ if the solution must contain the curve C: $u = x^2$, $y = 0$.

8. Find the solution of the PDE $(1 + q^2)u = px$ which passes through the curve $y = 0$, $u = x^2/2$.

9. Find the Monge cone for the PDE $pq = 1$ at any point (x_0, y_0, u_0).

10. Find the Monge cone for the PDE $p^2 + q = u$ at any point (x_0, y_0, u_0).

11. (a) Find the Monge cone for the PDE $pq = u$ at any point (x_0, y_0, u_0).
(b) Find, if possible, the solution of the PDE containing each of the following curves:

(i) $y = x, \; u = 1$ (ii) $y = x, \; u = -1$ (iii) $u = x, \; y = 0$

12. (a) Find values of k so that the Clairaut PDE $u = px + qy + (p^2 + q^2)/2$ with initial curve $x^2 + y^2 = a^2$, $u = k$ does not have solutions.
(b) Solve the initial value problem when $k = 0$.

13. Repeat Exercise 12 for the PDE $u = px + qy + \sqrt{1 + p^2 + q^2}$.

14. Show that when a PDE is quasilinear, equations 1.14 for the Monge cone at a point (x_0, y_0, u_0) reduce to a single line through the point in the C-direction.

§1.3 Applications of First-order Partial Differential Equations

Unidirectional Wave Motion

Suppose a function $f(x)$ is defined either along the entire x-axis, or some portion of it. We call $f(x)$ the initial wave form, because it is to be propagated along the x-axis starting at time $t = 0$. We let $u(x, t)$ represent the shape of the wave at any time $t > 0$. The shape of the wave is determined by the equation governing propagation. We examine motion determined by the first-order PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \tag{1.18}$$

called the **unidirectional wave equation**. We compare motions for three values of coefficient c: (i) c is a constant, in which case the PDE is linear; (ii) $c = u$, in which case the PDE is a very simple quasilinear one; (iii) $c = c(u)$, leading to a more complicated quasilinear equation.

Case (i) - $c > 0$ a constant

We solve PDE 1.18 for $t > 0$ subject to the Cauchy data $u(x, 0) = f(x)$. The C-equations are

$$\frac{dx}{c} = dt, \qquad du = 0,$$

with a two-parameter family of solutions

$$x = ct + \alpha, \qquad u = \beta.$$

For C-curves to pass through the initial curve $u(x, 0) = f(x)$, we set $\beta = \beta(\alpha)$, and

$$x = \alpha, \qquad f(x) = \beta(\alpha).$$

Consequently, $\beta(\alpha) = f(\alpha)$, and C-curves generating the solution surface are

$$x = ct + \alpha, \qquad u = f(\alpha).$$

Elimination of α gives the explicit solution $u = f(x - ct)$. We have shown the solution surface in Figure 1.15b for the initial triangular wave form in Figure 1.15a. It is generated by C-curves that are horizontal, projecting onto base C-curves that are straight lines with slope c. Along any given base C-curve $x = ct + \alpha$, the value of u has constant value $f(\alpha)$.

The initial wave form $f(x)$ has a discontinuous first derivative at $x = 0$, $x = 1$, and $x = 2a$. These discontinuities are propagated along the base C-curves $x = ct$, $x = ct + a$, and $x = ct + 2a$; partial derivatives u_x and u_t are discontinuous across each of these lines.

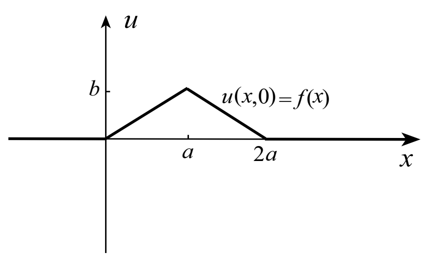

Figure 1.15a

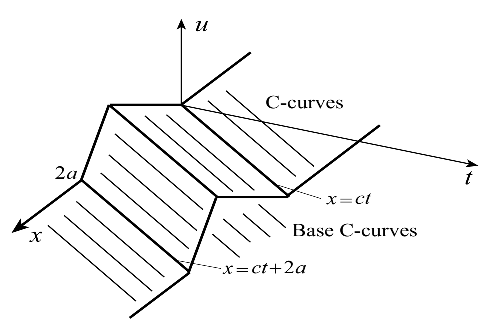

Figure 1.15b

The shape of the wave form at any given time is the intersection of this surface with a constant t-plane. For PDE 1.18, with constant c, the initial wave form $f(x)$ remains unchanged (Figure 1.15c). In time t, the wave moves ct units in the x-direction all points on the initial wave form move with the same velocity $dx/dt = c$.

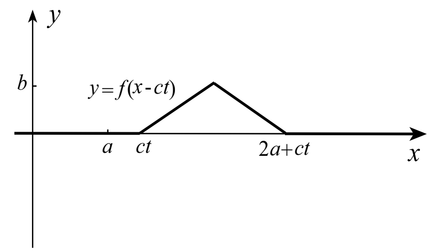

Figure 1.15c

The discussion of equation 1.18 is the same when constant c is negative. The initial wave moves to the left with speed c rather than to the right.

Case (ii) - $c = u$

C-equations for the quasilinear PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{1.19}$$

are

$$\frac{dx}{u} = dt, \qquad du = 0,$$

with solutions

$$x = \beta t + \alpha, \qquad u = \beta.$$

Suppose the initial wave form is $u(x, 0) = f(x)$ (not necessarily that in Figure 1.15a). For C-curves to pass through this initial curve, we set $\beta = \beta(\alpha)$ and

$$x = \alpha, \qquad f(x) = f(\beta).$$

Consequently, $\beta(\alpha) = f(\alpha)$, and C-curves generating the solution surface are

$$x = f(\alpha)t + \alpha, \qquad u = f(\alpha).$$

We obtain an implicit definition of the solution surface by writing

$$u = f(x - tf(\alpha)) = f(x - tu).$$

Unlike Case (i), the velocity of the wave at any point x and time t is not constant; it depends on the height of the wave ($dx/dt = u$). When $f(x) > 0$, the corresponding point on the wave moves to the right; when $f(x) < 0$, motion is to the left; and when $f(x) = 0$, the point is at rest. In general then, motion of the initial wave form as described by PDE 1.19 is not unidirectional. Since higher points on the wave have greater velocity than lower ones, it follows that when points to the left of others on the initial wave form are higher than those to the right, the higher points may overtake the lower ones, at which time the solution breaks down, becoming multi-valued thereafter. (It is possible in such situations to extend the validity of the solution by introducing *shock waves*, a topic that is beyond the scope of these notes.) For example, suppose $f(x) = \sin x$. Figure 1.16a shows the initial wave form on the interval $0 \le x \le 2\pi$, along with the wave at two additional values of t before breaking occurs. Figure 1.16b shows the situation at the instant the solution breaks down; the tangent line is vertical at $x = 2\pi$.

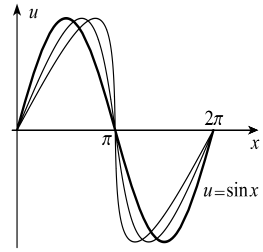

Figure 1.16a

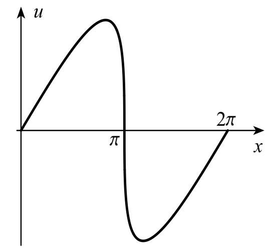

Figure 1.16b

Case (iii) - $c = c(u)$

C-equations for the quasilinear equation

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0, \tag{1.20}$$

are

$$\frac{dx}{c(u)} = dt, \quad du = 0,$$

with solutions

$$x = c(\beta)t + \alpha, \qquad u = \beta.$$

If the initial wave form is $u(x,0) = f(x)$, then for C-curves to pass through the initial curve, we set $\beta = \beta(\alpha)$, and

$$x = \alpha, \qquad f(x) = \beta(\alpha).$$

Consequently, $\beta(\alpha) = f(\alpha)$, and C-curves generating the solution surface are

$$x = c[f(\alpha)]t + \alpha, \qquad u = f(\alpha).$$

These imply that $x = c(u)t + \alpha$ from which $\alpha = x - tc(u)$, and the solution of the Cauchy problem associated with PDE 1.20 is defined implicitly by $u = f\big(x - tc(u)\big)$. The qualitative behaviour of this solution is the same as in Case (ii), complicated by the presence of $c(u)$.

Consider now solving unidirectional wave equation 1.18, (with constant c), on the interval $x > 0$, subject to the initial condition $u(0,t) = f(t)$. Think of this as the transmission of a signal $f(t)$ that is emitted in the positive x-direction at $x = 0$. For C-curves $x = ct + \alpha$, $u = \beta$ to pass through the initial curve $x = 0$, $u = f(t)$, we set $\beta = \beta(\alpha)$, and

$$0 = ct + \alpha, \qquad f(t) = \beta(\alpha).$$

Hence, $\beta(\alpha) = f(-\alpha/c)$, and C-curves generating the solution surface are

$$x = ct + \alpha, \qquad u = f(-\alpha/c).$$

Since $\alpha = x - ct$, the solution surface has equation $u = f(t - x/c)$. We have shown the solution surface in Figure 1.17b for the piecewise linear input signal in Figure 1.17a. It is generated by C-curves that are horizontal, projecting onto base C-curves that are straight lines with slope c. Along any given base C-curve $x = c(t - \tau)$, the value of u has constant value $f(\tau)$. The input signal is transmitted at speed c, arriving at point x in time $t = x/c$.

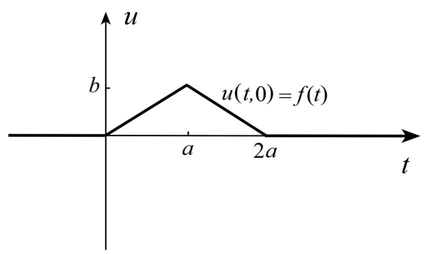

Figure 1.17a

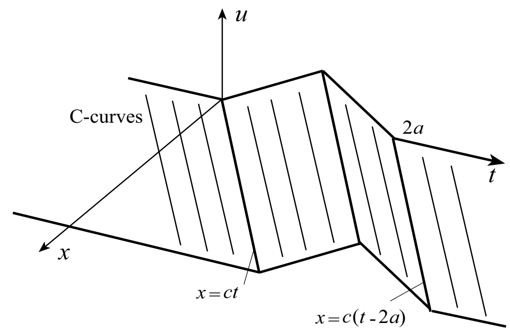

Figure 1.17b

The Eikonal Equation

The eikonal equation of geometric optics in two dimensions is the nonlinear PDE

$$p^2 + q^2 = n^2. \tag{1.21}$$

It describes the propagation of waves in optics, acoustics, and elasticity. In optics, $n = c_0/c$, called the index of refraction, c_0 is the speed of light in a vacuum, and c is the velocity of light in the optical medium. For the Cauchy problem associated with this equation, we take an initial curve in the form $x = x(t)$, $y = y(t)$, $u = u(t)$. C-equations for the PDE are

$$\frac{dx}{2p} = \frac{dy}{2q} = \frac{du}{2n^2}, \quad dp = 0, \quad dq = 0.$$

Integration of the last two of these, and then the first two, gives

$$\delta x = \gamma y + \alpha, \quad \gamma u = n^2 x + \beta, \quad p = \gamma, \quad q = \delta, \tag{1.22}$$

provided n is constant. Although we have four integration constants, the PDE demands that $\gamma^2 + \delta^2 = n^2$, so that we have a three-parameter family of C-strips. Because light rays travel along C-curves, the first two of these show that light travels along straight lines. In addition, because wave fronts are given by the level curve $u(x, y) =$ constant, the vector $\nabla u = \langle p, q \rangle$ is normal to wave fronts. Since the equation $\delta x = \gamma y + \alpha$ implies that the vector $\langle \gamma, \delta \rangle$ is tangent to C-curves, it follows that light rays are normal to wave fronts.

Along the initial curve, p and q are determined by equations 1.17,

$$\frac{du}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt}, \quad p^2 + q^2 = n^2.$$

Let the solutions be $p(t)$ and $q(t)$. Along the initial curve, then

$$x = x(t), \quad y = y(t), \quad u = u(t), \quad p = p(t), \quad q = q(t).$$

When these are substituted into equations 1.22,

$$\delta x(t) = \gamma y(t) + \alpha, \quad \gamma u(t) = n^2 x(t) + \beta, \quad p(t) = \gamma, \quad q(t) = \delta.$$

These give $\alpha = q(t)x(t) - p(t)y(t)$ and $\beta = p(t)u(t) - n^2 x(t)$, and therefore the C-strips generating the solution surface are

$$q(t)x = p(t)y + q(t)x(t) - p(t)y(t), \quad p(t)u = n^2 x + p(t)u(t) - n^2 x(t), \quad p = p(t), \quad q = q(t).$$

They can also be written in the form

$$q(t)[x - x(t)] = p(t)[y - y(t)], \quad p(t)[u - u(t)] = n^2[x - x(t)], \quad p = p(t), \quad q = q(t).$$

Using the fact that $p^2 + q^2 = n^2$, these can be combined to give the following implicit definition of the solution

$$[u - u(t)]^2 = n^2\{[x - x(t)]^2 + [y - y(t)]^2\}. \tag{1.23}$$

Both surfaces defined by this equation satisfy eikonal equation 1.21 and pass through the initial curve. The solution is singular, however, since u_x and u_y are undefined along the initial curve.

Solutions are now derived for two specific initial situations.

Case 1 - $x(t) = 0$, $y(t) = 0$, $u(t) = 0$, a point

In this case the disturbance is initiated at the origin. The positive root in equation 1.23 reduces to the conical, singular solution

$$u = n\sqrt{x^2 + y^2}.$$

Wave fronts $u(x, y)$ = constant are cylinders about the u-axis and light rays are horizontal straight lines through the u-axis.

Case 2 - $x = t$, $y = at$, $u = bt$, where a and b are constants, a line

Initial values of p and q can be obtained from equations 1.17

$$b = p + aq, \qquad p^2 + q^2 = n^2.$$

To avoid the radicals if we solve these directly, we notice that

$$\left(\frac{p}{n}\right)^2 + \left(\frac{q}{n}\right)^2 = 1.$$

This implies that the vector $\langle p/n, q/n\rangle$ has length unity, and we may therefore set

$$p = n\cos\theta, \qquad q = n\sin\theta,$$

where θ is defined by

$$n\cos\theta + an\sin\theta = b \qquad \Longrightarrow \qquad \cos\theta + a\sin\theta = \frac{b}{n}.$$

Since the left side can always be expressed in the form

$$\sqrt{1 + a^2}\sin(\theta + \phi) = \frac{b}{n}$$

for some ϕ, it follows that solutions can only exist when $\dfrac{b}{n} \le \sqrt{1 + a^2}$, and this implies that $b \le n\sqrt{1 + a^2}$. Along the initial curve, then

$$x = t, \quad y = at, \quad u = bt, \quad p = n\cos\theta, \quad q = n\sin\theta.$$

Substitution of these into equations 1.22 gives

$$\delta t = a\gamma t + \alpha, \quad b\gamma t = n^2 t + \beta, \quad \gamma = n\cos\theta, \quad \delta = n\sin\theta.$$

These imply that

$$\alpha = (\delta - a\gamma)t = n(\sin\theta - a\cos\theta)t, \quad \beta = (b\gamma - n^2)t = n(b\cos\theta - n)t.$$

When these are substituted into equation 1.22 and a factor n is removed, C-strips generating the solution surface are given by

$$\sin\theta x = \cos\theta y + (\sin\theta - a\cos\theta)t, \quad \cos\theta u = nx + (b\cos\theta - n)t, \quad p = n\cos\theta, \quad q = n\sin\theta.$$

When t is eliminated between the first two equations, an explicit definition of the solution results,

$$u(x, y) = n(x\cos\theta + y\sin\theta).$$

For instance, when $a = 1$ and $b = n$, angle θ is defined by

$$\cos\theta + \sin\theta = 1 \qquad \Longrightarrow \qquad \theta = 0 \quad \text{or} \quad \theta = \pi/2.$$

Thus, there are two solutions to this problem, $u = nx$ and $u = ny$. They represent plane wave solutions.

Traffic Flow

First order PDEs can be used to model traffic flow on a single-lane highway represented by the x-axis with flow to the right. Two important quantities in the analysis are density $\rho(x, t)$ of vehicles on the highway (number of vehicles per unit length) at position x and time t, and speed $v(x, t)$ of vehicles. The integral

$$\int_a^x \rho(\zeta, t)\, d\zeta$$

represents the number of vehicles on that part of the highway between a fixed point $x = a$ and any other point $x > a$. Its time-derivative is the rate of change of the number of vehicles on this part of the highway. It must be equal to the rate at which vehicles enter this part of the highway at $x = a$ less the rate at which they leave at x. Since ρv represents the number of vehicles passing point x on the highway per unit time, we may write that

$$\frac{\partial}{\partial t}\int_a^x \rho(\zeta, t)\, d\zeta = \rho(a, t)v(a, t) - \rho(x, t)v(x, t).$$

When we differentiate this equation with respect to x and interchange order of operations on the left,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v) \qquad \Longrightarrow \qquad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0. \tag{1.24}$$

This is the fundamental equation of traffic flow; it relates the density and speed of the flow. It is necessary to specify a functional relationship between ρ and v in order to solve this PDE for both quantities. We assume that v is a function of ρ only, $v = v(\rho)$ which logically should be a decreasing function $dv/d\rho < 0$, velocity decreasing as density increases. Equation 1.24 then becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}[\rho\, v(\rho)] = 0 \qquad \Longrightarrow \qquad \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{dv}{d\rho}\frac{\partial \rho}{\partial x} = 0.$$

Replacing equation 1.24 is

$$\frac{\partial \rho}{\partial t} + \left(v + \rho\frac{dv}{d\rho}\right)\frac{\partial \rho}{\partial x} = 0. \tag{1.25}$$

Once $v(\rho)$ is specified, this is a quasilinear first-order PDE for $\rho(x,t)$.

The simplest conceivable relationship would be $v(\rho) = v_m\left(1 - \dfrac{\rho}{\rho_m}\right)$, a linear function with maximum $v = v_m$ when $\rho = 0$, and minimum $v = 0$ when $\rho = \rho_m$ is a maximum. In this case,

$$v + \rho\frac{dv}{d\rho} = v_m\left(1 - \frac{\rho}{\rho_m}\right) + \rho\left(-\frac{v_m}{\rho_m}\right) = v_m\left(1 - \frac{2\rho}{\rho_m}\right),$$

and PDE 1.25 becomes

$$\frac{\partial\rho}{\partial t} + v_m\left(1 - \frac{2\rho}{\rho_m}\right)\frac{\partial\rho}{\partial x} = 0. \tag{1.26}$$

C-equations for this first-order quasilinear PDE are

$$\frac{dx}{v_m\left(1 - \frac{2\rho}{\rho_m}\right)} = dt, \qquad d\rho = 0.$$

C-curves are therefore

$$x = v_m\left(1 - \frac{2\beta}{\rho_m}\right)t + \alpha, \qquad \rho = \beta.$$

They are straight lines along which density is constant, but the value of ρ varies from C-curve to C-curve.

To proceed further, we must specify the initial density of vehicles on the road. We do so in the following example.

Example 1.12 Find the density and velocity of traffic when a traffic light at $x = 0$ turns green at time $t = 0$. Assume that initially there is no traffic to the right of the light and traffic to the left is stationary at maximum density ρ_m.

Solution The initial data is

$$\rho(x,0) = \begin{cases} \rho_m, & x < 0 \\ 0, & x > 0. \end{cases}$$

For C-curves to pass through this curve when $x < 0$, we set $\beta = \beta(\alpha)$ and

$$x = \alpha, \qquad \rho_m = \beta(\alpha).$$

C-curves for $x < 0$ are therefore

$$x = v_m\left(1 - \frac{2\rho_m}{\rho_m}\right)t + \alpha = -v_m t + \alpha, \quad \rho = \rho_m.$$

For C-curves to pass through the initial curve when $x > 0$, we again set $\beta = \beta(\alpha)$, and

$$x = \alpha, \qquad 0 = \beta(\alpha).$$

C-curves for $x > 0$ are therefore

$$x = v_m t + \alpha, \quad \rho = 0.$$

Base C-curves (Figure 1.18a) are two sets of parallel lines. For any point x and time t in region R_1, the density of traffic flow is a maximum, meaning that motion has not yet commenced. On the other hand, for any point and time in region R_2, density is a minimum and traffic would flow at maximum velocity except for the fact that no cars have reached this value for x at this time.

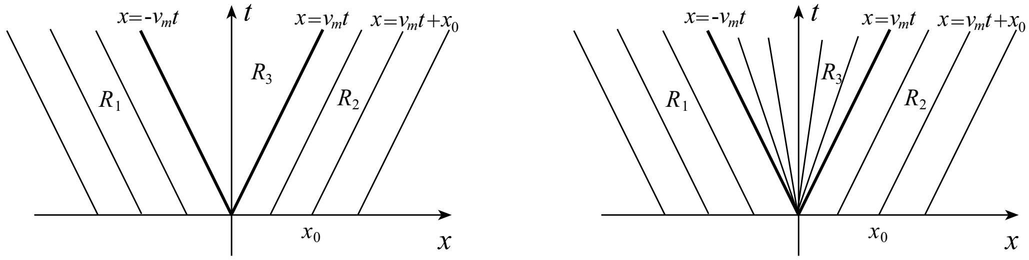

Figure 1.18a **Figure 1.18b**

What is missing is the transition from zero velocity to maximum velocity, and this corresponds to the fact that the solution is as yet undefined in region R_3. To remedy this, we follow the lead of Example 1.8 and introduce fanlike base C-curves in R_3. They are straight lines through the origin, and in order that the definition of $\rho(x,t)$ in this region satisfy PDE 1.26, equations for these lines are specified in the form

$$x = v_m\left(1 - \frac{2\rho}{\rho_m}\right)t, \qquad -v_m < v_m\left(1 - \frac{2\rho}{\rho_m}\right) < v_m.$$

When we solve this equation for ρ, the result is

$$\rho(x,t) = \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right), \qquad -v_m t < x < v_m t.$$

This is the transition traffic density in region R_3, and it does indeed satisfy PDE 1.26. Substitution into $v = v_m(1 - \rho/\rho_m)$ gives the transition flow velocity,

$$v(x,t) = \frac{1}{2}\left(v_m + \frac{x}{t}\right), \qquad -v_m t < x < v_m t.$$

Complete specifications of $\rho(x,t)$ and $v(x,t)$ are

$$\rho(x,t) = \begin{cases} \rho_m, & x < -v_m t \\ \dfrac{\rho_m}{2}\left(1 - \dfrac{x}{v_m t}\right), & -v_m t < x < v_m t \\ 0 & x > v_m t \end{cases} \qquad v(x,t) = \begin{cases} 0, & x < -v_m t \\ \dfrac{1}{2}\left(v_m + \dfrac{x}{t}\right), & -v_m t < x < v_m t \\ v_m, & x > v_m t. \end{cases}$$

It is interesting, informative, and surprising to plot these as functions of x for fixed t. Plots are shown in Figures 1.19.

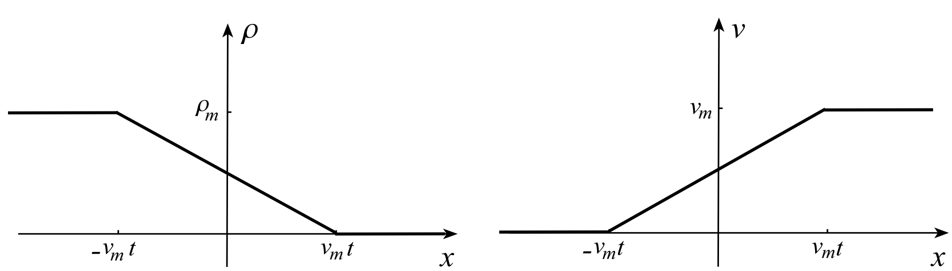

Figure 1.19a **Figure 1.19b**

As time increases, the slanted portions of each of these graphs increase in length and become more horizontal. The point $x = -v_m t$ on the x-axis of the velocity graph moves to the left with velocity $-v_m$. This indicates that when the light turns green, all cars do not begin to move; there is a time delay for all but the first car. For a car at position x on the negative x-axis, the time delay is $t = -x/v_m$. Cars further back in line experience longer delays. It's as if a signal to move propagates back through the line of stationary cars at velocity v_m.

In Figures 1.20a,b, we have plotted velocity as a function of time for fixed $x < 0$ and $x > 0$, respectively. Figure 1.20a confirms what we saw in Figure 1.19b. The car at position x in the line of stationary cars experiences a time delay $t = -x/v_m$ before it begins to move. Thereafter, velocities of cars at this position gradually increase, ultimately approaching $v_m/2$. On the other hand, for $x > 0$, Figure 1.19b indicates velocity v_m at position x until $t = x/v_m$. It is not that cars move at this velocity at this position for these times because until time x/v_m no cars will have reached position x. The lead car in line travels with velocity v_m, reaching position x at time x/v_m. Thereafter, velocities of cars at this position gradually decrease, approaching $v_m/2$.

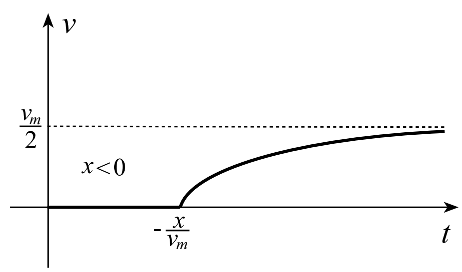

Figure 1.20a

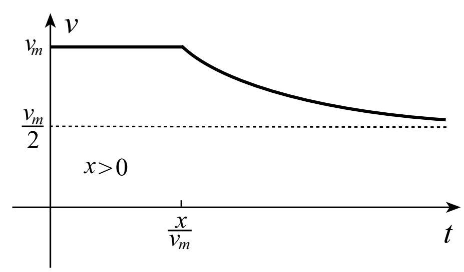

Figure 1.20b

What may seem strange at first is that for very large t, all cars move with velocity $v_m/2$. This is a result of the fact that the number of cars on the highway must be preserved; the initial number of cars to the left of the traffic light with density ρ_m eventually spreads out over the entire highway at density $\rho_m/2$.•

EXERCISES 1.3

1. Show that when $f(x) = mx$, m a positive constant, the solution of PDE 1.19 never breaks. Draw the wave form for $t = 0, 1, 2, 3$.

2. Show that when $f(x) = -mx$, m a postive constant, the solution of PDE 1.19 breaks. Determine the time for breaking to occur. Draw the wave form when $m = 1/4$ for $t = 0, 3, 3.5, 3.75$.

3. (a) Find an explicit solution of PDE 1.19 when the initial wave form $f(x) = 1 - x^2$. Determine when breaking occurs.
(b) Repeat part (a) if $f(x) = 1 - x^2$ for $-1 \le x \le 1$.

4. Show that when the initial wave form for PDE 1.19 is $f(x)$, breaking time occurs at the smallest value of t satisfying the equation

$$f'(x - tu) = -\frac{1}{t}.$$

5. Use the result of Exercise 4 to verify the breaking times in (a) Exercise 1, (b) Exercise 2, and (c) Exercise 3.

6. Use the result of Exercise 4 to show that if the initial wave form $f(x)$ for PDE 1.19 is such that $f'(x) > 0$ for all x, breaking will not occur.

7. Values of the solution to PDE 1.19 generally vary from base C-curve to base C-curve. As a result, the breaking time for a solution is the earliest positive time at which two base C-curves intersect. Illustrate this in (a) Exercise 1, (b) Exercise 2, and (c) Exercise 3.

8. Use the technique of Exercise 7 to find the breaking time for the solution of PDE 1.19 when $f(x) = 1 - x^2$ for $-1 \leq x \leq a$, where $a > 1$.

9. Solve the unidirectional wave equation 1.18 (with $c > 0$) for $x > 0$ and $t > 0$ with an initial wave form $u(x, 0) = f(x)$ and an input signal $u(0, t) = g(t)$.

10. Solve the damped unidirectional wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \beta u = 0, \qquad u(x, 0) = f(x),$$

where c and $\beta > 0$ are constants.

11. (a) Find, in integral form, the solution to the following nonhomogeneous unidirectional wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = F(x, t), \qquad u(x, 0) = f(x),$$

where c is constant.

(b) Simplify the solution in part (a) when $F(x, t) = x + t$.

12. Show that the singular solution in Case 1 for the eikonal equation is the Monge cone at the origin.

13. Find the solution of eikonal equation 1.21 when the initial curve is the straight line $x = x_0 + t$, $y = y_0 + at$, $u = u_0 + bt$.

CHAPTER 2 DERIVATION OF PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

§2.1 Introduction

In this and the remaining chapters of the text, we concentrate on PDEs that contain derivatives of second and higher order. For example, in Figure 2.1 we picture a circular rod of length L that at some initial time (say $t = 0$) has constant temperature 10°C. Suppose that at this time, the lateral side of the rod is perfectly insulated and the ends are suddenly heated to 100°C and maintained at this temperature thereafter. In Section 2.2 it is shown that the temperature U at points in the rod is a function of x and t only, $U = U(x, t)$, and that this function must satisfy the PDE

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \tag{2.1a}$$

where k is a constant (the thermal diffusivity of the material in the rod). This second-order PDE is called the **one-dimensional heat conduction equation**; it states that the temperature function $U(x, t)$ must have a first partial derivative with respect to t identical to k times its second partial derivative with respect to x.

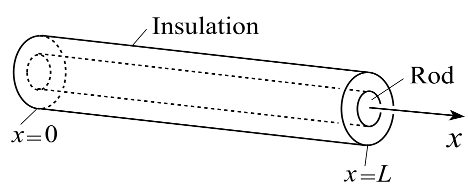

Figure 2.1

Other PDEs that we shall consider include the one-dimensional wave equation for displacement $y(x, t)$ of a vibrating string (Section 2.3),

$$\frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\rho}\frac{\partial^2 y}{\partial x^2}; \tag{2.2}$$

the three-dimensional Poisson equation for potential $V(x, y, z)$ (Section 2.6)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = F(x, y, z); \tag{2.3}$$

and the beam-vibration equation for displacement $y(x, t)$ (Section 2.5),

$$\frac{w}{g}\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = F(x, t). \tag{2.4}$$

Equations 2.2 and 2.3 are second order, and equation 2.4 is fourth order.

In the study of ODEs, it is customary to solve a certain class of equations and thereafter to deal with applications involving equations in this class. For example, a general solution of the second-order linear ODE $p\dfrac{d^2y}{dt^2} + q\dfrac{dy}{dt} + ry = 0$ is $y(t) = Ay_1(t) + By_2(t)$, where A and B are arbitrary constants and $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions of the equation. Once $y_1(t)$ and $y_2(t)$ are known, every solution of the equation is of the form $Ay_1(t) + By_2(t)$ for some A and B. When such an equation is found in an application, say a vibrating mass-spring system or an LCR-circuit, it is accompanied by two initial conditions that the solution $y(t)$ must also satisfy. These conditions determine the values for A and B. What we are saying is that in applications, ODEs are often solved by first finding general solutions and then using subsidiary conditions to determine arbitrary constants. (Using Laplace transforms is an exception to this method.)

It is very unusual to approach PDEs in this way, principally because arbitrary constants in general solutions of ODEs are replaced by arbitrary functions in PDEs, and determination of these arbitrary functions using subsidiary conditions is usually impossible. In other words, general solutions of PDEs are of limited utility in solving PDEs. (The one major exception is wave equation 2.2, and this particular situation is discussed in Section 2.7.) In general, then, it is necessary to consider a PDE and any extra conditions that accompany the equation simultaneously. We must proceed directly to a solution of the PDE and subsidiary conditions, as opposed to PDE first and subsidiary conditions later.

Subsidiary conditions that accompany PDEs are called *initial* and *boundary conditions*. For example, it is clear that the temperature function $U(x,t)$ for the rod in Figure 2.1 must also satisfy the **boundary conditions**

$$U(0,t) = 100, \tag{2.1b}$$

$$U(L,t) = 100, \tag{2.1c}$$

since the ends of the rod, $x = 0$ and $x = L$, are held at temperature 100°C. In addition, $U(x,t)$ must satisfy the **initial condition**

$$U(x,0) = 10, \tag{2.1d}$$

since its temperature at time $t = 0$ is 10°C throughout.

Partial differential equation 2.1a, boundary conditions 2.1b,c, and initial condition 2.1d constitute the complete **initial boundary value problem** for temperature in the rod. It is more precise to describe the problem as follows:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{2.5a}$$

$$U(0,t) = 100, \quad t > 0, \tag{2.5b}$$

$$U(L,t) = 100, \quad t > 0, \tag{2.5c}$$

$$U(x,0) = 10, \quad 0 < x < L. \tag{2.5d}$$

All that we have done is affix intervals on which conditions 2.1 must be satisfied, but, perhaps unexpectedly, these intervals are all open. To see why this is the case, consider first PDE 2.5a. Physically, $U(x,t)$ is a function of one space variable x and the time variable t, but mathematically, it is simply a function of two independent variables x and t. It must satisfy PDE 2.5a in some region of the xt-plane, and we take

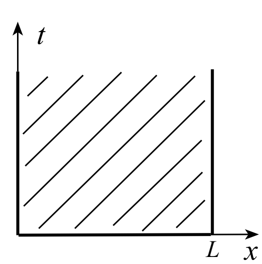

Figure 2.2

this region to be described by the inequalities $0 < x < L$ and $t > 0$ (Figure 2.2). By keeping these intervals open, we avoid discussing the PDE on the boundary of the region. Otherwise it would be necessary to consider one-sided derivatives with respect to x along $x = 0$ and $x = L$, one-sided derivatives with respect to t along $t = 0$, and both types of one-sided derivatives at $(0, 0)$ and $(L, 0)$. We take as a general principle that partial differential equations are always considered on open regions. *

Replacement of $t > 0$ and $0 < x < L$ in equations 2.5b–d with $t \geq 0$ and $0 \leq x \leq L$ would lead to contradictions. Conditions 2.5b,c would then require $U(x,t)$ to have values $U(0, 0) = U(L, 0) = 100$, whereas condition 2.5d would demand that $U(0, 0) = U(L, 0) = 10$. By imposing boundary and initial conditions on open intervals, we eliminate such mathematical contradictions. Realize, however, that although conditions 2.5b,c,d contain no mathematical contradictions, it is physically impossible to change the temperature of the ends of the rod instantaneously from 10°C to 100°C, and yet problem 2.5 does demand this. We must therefore anticipate some type of anomaly in the solution to problem 2.5 near positions $x = 0$ and $x = L$ at times close to $t = 0$.

It is not always necessary to use open intervals for boundary and initial conditions. If the initial temperature in the rod were not constant but varied with x according to, say, $f(x) = 400x(L - x) + 100$, it would not be necessary to heat the ends of the rod suddenly to 100°C at time $t = 0$; they would already be at that temperature, since $f(0) = f(L) = 100$. It would be necessary only to maintain them at 100°C thereafter. In this case, it would be quite acceptable to replace the open intervals in conditions 2.5b,c,d with

$$\begin{aligned} U(0,t) &= 100, \quad t \geq 0, \\ U(L,t) &= 100, \quad t \geq 0, \\ U(x,0) &= 400x(L-x) + 100, \quad 0 \leq x \leq L. \end{aligned}$$

It will be our practice to state initial and boundary conditions on open intervals even when closed intervals are acceptable.

Example 2.1 The ends of a violin string of length L are fixed on the x-axis at positions $x = 0$ and $x = L$. When the middle of the string is elevated to the position in Figure 2.3 and then released from rest (at time $t = 0$), subsequent displacements of points of the string must satisfy PDE 2.2, where τ is the tension in the string and ρ is its linear density. What are the boundary and initial conditions for $y(x,t)$?

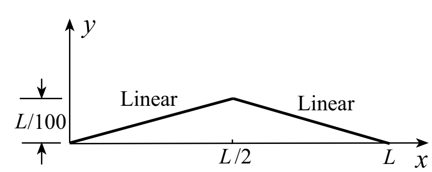

Figure 2.3

Solution Since the ends of the string are fixed on the x-axis, boundary conditions are

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* A region of the xy-plane is said to be open if about every point in the region there can be drawn a circle such that its interior contains only points of the region. A region in space is open if about every point in the region there can be drawn a sphere such that its interior contains only points of the region.

$$y(0,t) = 0, \quad t > 0,$$
$$y(L,t) = 0, \quad t > 0.$$

Because the string has the position in Figure 2.3 at time $t = 0$, $y(x,t)$ must satisfy the initial condition

$$y(x,0) = \begin{cases} x/50, & 0 < x \le L/2 \\ (L-x)/50, & L/2 < x < L. \end{cases}$$

In addition, the fact that the string is released from rest indicates that its velocity at time $t = 0$ is equal to zero. Since velocity is the time rate of change of displacement, the second initial condition is

$$\frac{\partial y(x,0)}{\partial t} = 0, \quad 0 < x < L.$$

There would be no conflict in replacing each of the open intervals in these four conditions with closed intervals.•

In problem 2.5, boundary conditions 2.5b,c specify the temperature of the rod at its ends, $x = 0$ and $x = L$. Likewise, in Example 2.1, the boundary conditions specify the displacement of the string at its ends. These are examples of what are called *Dirichlet* boundary conditions. A **Dirichlet boundary condition** specifies the value of the unknown function on a physical boundary. As another example, consider the two-dimensional version of Poisson's equation 2.3,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x,y), \quad (x,y) \text{ in } R, \tag{2.6a}$$

for the region R in Figure 2.4. ($F(x,y)$ is a given function.) In compliance with our previous remarks, R is the open region consisting of all points interior to the bounding curve $\beta(R)$ but not including $\beta(R)$ itself. A Dirichlet boundary condition specifies the value of $V(x,y)$ on $\beta(R)$:

$$V(x,y) = G(x,y), \quad (x,y) \text{ on } \beta(R), \tag{2.6b}$$

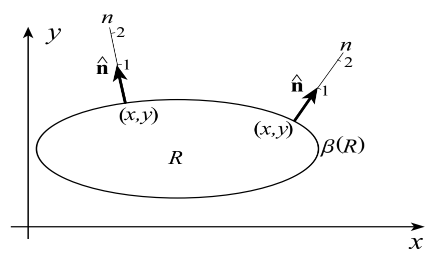


Figure 2.4

where $G(x,y)$ is some given function. Poisson's equation 2.6a together with boundary condition 2.6b is called a **boundary value problem**.

Two other types of boundary conditions arise frequently in applications — *Neumann* and *Robin*. A **Neumann boundary condition** for equation 2.6a specifies the rate of change of $V(x,y)$ at points on $\beta(R)$ in a direction outwardly normal (perpendicular) to $\beta(R)$. We express this in the form

$$\frac{\partial V}{\partial n} = G(x,y), \quad (x,y) \text{ on } \beta(R), \tag{2.7a}$$

where n is understood to be a measure of distance at (x,y) in a direction perpendicular to $\beta(R)$ (Figure 2.4). Because $\partial V/\partial n$ is the directional derivative of V along the outward normal to $\beta(R)$, equation 2.7a may be expressed in the equivalent form

$$\nabla V \cdot \hat{\mathbf{n}} = G(x, y), \quad (x, y) \text{ on } \beta(R), \tag{2.7b}$$

where ∇V is the gradient of V at (x, y) and $\hat{\mathbf{n}}$ is the unit outward normal vector to $\beta(R)$ at (x, y).

A **Robin boundary condition** is a linear combination of a Dirichlet and a Neumann condition. For equation 2.6a, it takes the form

$$l\frac{\partial V}{\partial n} + hV = G(x, y), \quad (x, y) \text{ on } \beta(R), \tag{2.8a}$$

where l and h are nonzero constants. What is important is not the individual values of l and h but their ratio, l/h or h/l; division of condition 2.8a by l and h leads to boundary conditions

$$\frac{\partial V}{\partial n} + \left(\frac{h}{l}\right) V = \frac{G(x, y)}{l}, \quad (x, y) \text{ on } \beta(R), \tag{2.8b}$$

and

$$\left(\frac{l}{h}\right) \frac{\partial V}{\partial n} + V = \frac{G(x, y)}{h}, \quad (x, y) \text{ on } \beta(R), \tag{2.8c}$$

both of which are equivalent to 2.8a. The advantage of condition 2.8a, however, is that solutions of problems with Dirichlet and Neumann boundary conditions can be obtained from those with Robin conditions by specifying $l = 0$, $h = 1$ and $h = 0$, $l = 1$, respectively. Boundary conditions 2.6b, 2.7, and 2.8 are said to be **homogeneous** if $G(x, y) \equiv 0$; otherwise, they are said to be **nonhomogeneous**. Physical interpretations of Neumann and Robin boundary conditions are discussed in the context of applications in Sections 2.2–2.6.

Example 2.2 What form do Robin boundary conditions take for the heat conduction problem described by equations 2.5a–d?

Solution At the end $x = L$ of the rod, the outward normal is in the positive x-direction. Consequently, at $x = L$, we have $\partial U/\partial n = \partial U/\partial x$, and a Robin boundary condition there is

$$l_2 \frac{\partial U(L, t)}{\partial x} + h_2 U(L, t) = G_2(t), \quad t > 0.$$

With the outward normal at $x = 0$ in the negative x-direction, it follows that the normal derivative there is $\dfrac{\partial U(0, t)}{\partial n} = -\dfrac{\partial U(0, t)}{\partial x}$, and a Robin condition there takes the form

$$-l_1 \frac{\partial U(0, t)}{\partial x} + h_1 U(0, t) = G_1(t), \quad t > 0. \bullet$$

In order that an (initial) boundary value problem adequately represent a physical situation, its solution should have certain properties. First, there should be a solution to the problem. Second, this solution should be unique; that is, the problem should not have more than one solution. For example, if problem 2.5 had more than one solution, how could it possibly be an adequate description of the temperature in the rod? Solutions should also have one further property, which we explain through problem 2.6. The solution of this problem depends on the functions

$F(x, y)$ and $G(x, y)$. In practice, these quantities may not be known exactly; they may, for instance, be obtained from physical measurements. It would be reasonable to expect that small changes in either $F(x, y)$ or $G(x, y)$ should not appreciably affect $V(x, y)$. These three conditions lead to what is called a *well-posed* problem. An (initial) boundary value problem is said to be **well-posed** if:

(1) the problem has a solution;
(2) the solution is unique;
(3) the solutions depends continuously on source terms and initial and boundary data (that is, small changes in source terms and initial and boundary data produce small changes in the solution).

All stable physical situations should be modelled by well-posed problems. The situation in Figure 2.5 indicates why we have added the adjective "stable". Pictured is a vertical rod that can pivot about its lower end. If this is the initial position of the rod, then it remains at this position forever. If, however, this initial position is changed ever so slightly, then the rod experiences oscillations. These oscillations may persist forever if friction and damping are neglected, or die out if friction or damping is taken into account. In either case, a small change in the initial conditions has resulted in large changes in the solution, contrary to item (3) above. But clearly, the vertical position of the rod is an unstable situation. Hence, we cannot expect unstable physical situations to be modelled by well-posed problems.

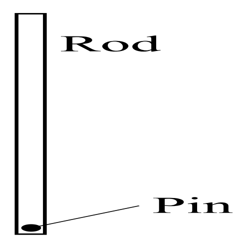


Figure 2.5

In this book we discuss only existence and uniqueness of solutions; continuous dependence of solutions on source terms and subsidiary data is beyond our scope. Existence of solutions can be approached in two ways. One might be interested in knowing whether a particular initial boundary value problem has a solution but might not be at all interested in what the solution is. This is "existence" in its purest sense. Our approach is more pragmatic. We discuss different ways to solve initial boundary value problems, and if one of these methods succeeds in giving a solution to a problem, then clearly "existence" of a solution has been established. It is important to know that a problem has only one solution, however, since then, and only then, may we conclude that once *a* solution has been found, it must be *the* solution to the problem. Uniqueness is discussed in Sections 6.6–6.8.

In Sections 2.2–2.6 we derive partial differential equations that arise in physics and engineering. Each section is self-contained and may therefore be read independently of the others. This means that readers interested in heat conduction could study Section 2.2 and omit Sections 2.3–2.6 without fear of missing any central ideas concerning PDEs. Likewise, readers interested in mechanical vibrations could omit Sections 2.2 and 2.6 and concentrate on Sections 2.3–2.5.

Arising in many of these applications is the *Laplacian* of a function. The **Laplacian** of a function $V(x, y)$ in Cartesian coordinates x and y is defined as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}, \tag{2.9a}$$

and if $V(x, y, z)$ is a function of three variables, as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \tag{2.9b}$$

When a function is expressed in polar, cylindrical, or spherical coordinates, its Laplacian is more complicated to calculate. We list the formulae here, leaving verification to Exercises 10 and 11. In polar coordinates (r, θ) (Figure 2.6),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}; \tag{2.10a}$$

in cylindrical coordinates (r, θ, z),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}; \tag{2.10b}$$

and in spherical coordinates (r, ϕ, θ) (Figure 2.7),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial V}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}. \tag{2.10c}$$

The PDE obtained by setting the Laplacian of a function equal to zero,

$$\nabla^2 V = 0, \tag{2.11}$$

is called **Laplace's equation**. When set equal to a nonzero function,

$$\nabla^2 V = F, \tag{2.12}$$

it is Poisson's equation.

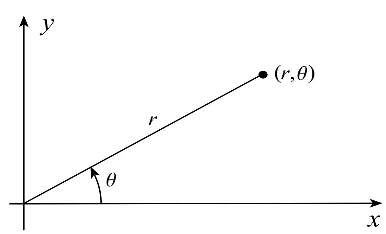


Figure 2.6

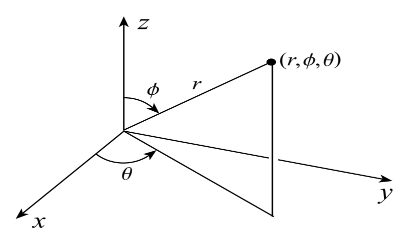


Figure 2.7

Dirac-delta Functions

Dirac-delta functions (or delta functions, for short) have become an accepted way to model point "sources" in physical systems (point charges in electrostatics, point masses in physics, point sources in heat conduction and fluid flow, and concentrated forces on beams and membranes, to name a few). They are essential to our discussions of Green's functions in Chapters 12 and 13, and we shall discuss them in detail in these chapters. But to make use of delta functions only in this context would neglect a valuable tool for solving many problems. For the first eleven chapters, we need only what might be called the fundamental property of delta functions. When c is a constant, the delta function $\delta(x - c)$ operates on a functions $f(x)$ that is continuous at $x = c$ in the following way

$$\int_a^b f(x)\,\delta(x-c)\,dx = \begin{cases} f(c), & \text{when } a<c<b \\ 0, & \text{otherwise,} \end{cases} \tag{2.13}$$

where limits a and b may be finite or infinite. For instance,

$$\int_0^5 (2x-1)\,\delta(x-3)\,dx = 5, \qquad \int_{-\infty}^{\infty} x^2\,\delta(x-2)\,dx = 4, \qquad \int_{-5}^{-2} (x^2+1)\,\delta(x)\,dx = 0.$$

Delta functions are not ordinary functions in the sense that we can ask for the value of $\delta(x-c)$ at say $x=5$. They are examples of what are called "generalized functions", and they are characterized by their effect on other functions through integral 2.13. It follows from this property that when $c>a$,

$$\int_a^x \delta(t-c)\,dt = \begin{cases} 1, & x>c \\ 0, & x<c. \end{cases} \tag{2.14}$$

This function is called the (Heaviside) **unit step function**, which we denote by

$$h(x-c) = \begin{cases} 0, & x<c \\ 1, & x>c. \end{cases} \tag{2.15}$$

Its graph is shown in Figure 2.8. Since the left side of equation 2.14 is essentially an antiderivative of $\delta(x-c)$, we can write that

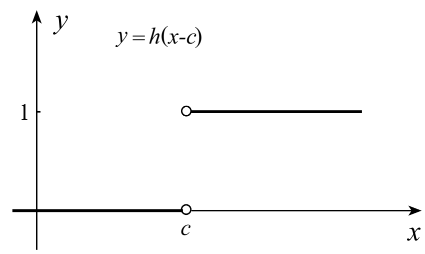


Figure 2.8

$$\int \delta(x-c)\,dx = h(x-c) + C \quad\Longleftrightarrow\quad h'(x-c) = \delta(x-c), \tag{2.16}$$

where C is an arbitrary constant. Antiderivatives of the unit step function are discussed in Exercise 8. Detailed discussions of delta functions can be found in Sections 12.1 and 13.1.

EXERCISES 2.1

On the region in Exercises 1–7 what form do (a) Dirichlet, (b) Neumann, and (c) Robin boundary conditions take for the PDE?

1. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = F(x,y), \quad 0<x<L, \quad 0<y<L'$

2. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} + V = F(x,y,z), \quad 0<x<L, \quad y>0, \quad z>0$

3. $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} = F(r,\theta), \quad 0<r<r_0, \quad -\pi<\theta\le\pi$

4. $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} = F(r,\theta), \quad 0<r<r_0, \quad 0<\theta<\pi$

5. $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} + \dfrac{\partial^2 V}{\partial z^2} = F(r,\theta,z), \quad 0<r<r_0, \quad -\pi<\theta\le\pi, \quad z>0$

6. $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2\sin\phi}\dfrac{\partial}{\partial\phi}\left(\sin\phi\dfrac{\partial V}{\partial\phi}\right) + \dfrac{1}{r^2\sin^2\phi}\dfrac{\partial^2 V}{\partial\theta^2} = F(r,\phi,\theta), \quad 0<r<r_0,$

$$-\pi < \theta \le \pi, \quad 0 < \phi < \pi$$

7. Use the same PDE as in the previous exercise but on the region $0 < r < r_0$, $-\pi < \theta \le \pi$, $0 < \phi < \pi/2$.

8. (a) Show that continuous antiderivatives of $h(x-c)$ are

$$\int h(x-c)\,dx = (x-c)h(x-c) + C,$$

provided the right side is given value 0 at $x = c$.

(b) Extend the result in part (a) to verify the following continuous antiderivatives

$$\int (x-c)^n h(x-c)\,dx = \frac{1}{n+1}(x-c)^{n+1}h(x-c) + C,$$

provided once again that the right side is given value 0 at $x = c$.

9. When a boundary value problem (but not an initial boundary value problem) has a Neumann boundary condition on all parts of its boundary, nonhomogeneities must satisfy a consistency condition. In this exercise we derive this condition for two- and three-dimensional problems.

(a) Use Green's theorem in the plane to show that if $V(x,y)$ is a solution of Poisson's equation 2.6a in some region R of the xy-plane with boundary $\beta(R)$, then

$$\iint_R F(x,y)\,dA = \oint_{\beta(R)} \frac{\partial V}{\partial n}\,ds. \tag{2.17}$$

(Green's theorem is stated in Appendix C.) If the boundary condition for the PDE is Neumann (condition 2.7a), show that

$$\iint_R F(x,y)\,dA = \oint_{\beta(R)} G(x,y)\,ds. \tag{2.18}$$

The right side of this equation is the line integral of $G(x,y)$ around the boundary $\beta(R)$, and the left side is the double integral of $F(x,y)$ over R. Thus, the *source term* $F(x,y)$ in 2.6a and the boundary data $G(x,y)$ in 2.7a cannot be specified independently; they must satisfy this consistency condition. Physical interpretations of this condition will be given later (see, for example, Exercise 24 in Section 2.2). This condition is also sufficient for existence of a solution, but this is more difficult to prove. When the boundary condition is Dirichlet (equation 2.6a), equation 2.17 is not a consistency condition, since $\partial V/\partial n$ is not specified. It can be used as a check on the acceptability of a proposed solution to the boundary value problem. Likewise, when the boundary condition is Robin (2.8a), equation 2.17 can be written in the form

$$\iint_R F(x,y)\,dA = \oint_{\beta(R)} \frac{1}{l}[G(x,y) - hU(x,u)]\,ds,$$

which again serves as a check on solutions, but not a consistency condition.

(b) Show that the analogue of the consistency condition in part (a) for the three-dimensional boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = F(x,y,z), \quad (x,y,z) \text{ in } V,$$

$$\frac{\partial V}{\partial n} = G(x,y,z), \quad (x,y,z) \text{ on } \beta(V),$$

is

$$\oiint_{\beta(V)} G(x,y,z)\,dS = \iiint_V F(x,y,z)\,dV.$$

(You will need the divergence theorem from Appendix C.)

10. In this exercise we verify expression 2.10a for the Laplacian in polar coordinates. Formula 2.10b is then obvious.

(a) Verify that when a function $V(x,y)$ is expressed in polar coordinates r and θ, its Cartesian derivatives $\partial V/\partial x$ and $\partial V/\partial y$ may be calculated according to

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x}, \qquad \frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

(b) Obtain formulas for $\partial r/\partial x$, $\partial r/\partial y$, $\partial\theta/\partial x$, and $\partial\theta/\partial y$ from the equations $x = r\cos\theta$ and $y = r\sin\theta$ between polar and Cartesian coordinates, and use them to show that

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}, \quad \frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}.$$

(c) Use the result in part (b) to calculate the following expressions for second partial derivatives of V with respect to x and y:

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial \theta},$$

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial \theta}.$$

(d) Finally, add the results in part (c) to obtain expression 2.10a.

11. Use the technique in the previous exercise to obtain expression 2.10c.

§2.2 Heat Conduction

In this section we develop the mathematics necessary to describe conductive heat flow in various physical bodies — rods, plates, and three-dimensional regions. We could begin with one-dimensional flow, such as that in the rod of Figure 2.1, and generalize later to plates and volumes. Alternatively, we could begin with three-dimensional heat flow and specialize later to plates and rods. We find the latter approach more satisfactory; it does not require special physical apparatus to ensure heat flow in only one or two directions. Furthermore, the mathematical and physical quantities that describe heat flow have units that are more natural in a three-dimensional setting.

When we consider temperature at various points in some object (a nuclear reactor, say), seldom is it constant; temperature normally varies from point to point and changes with time. Experience has shown that when temperature does vary, heat flows by *conduction*. Heat can flow by other means as well, namely by *convection* and by *radiation*. Heat received by the earth from the sun is due to **radiation**. We do not consider heat transfer by radiation in this book. The engine of a car illustrates the difference between convective and conductive heat flow. In order to keep the engine cool, water carries heat from the engine to the radiator through hoses; it is the motion of the water that transfers heat from engine to radiator. This is called **convective** heat transfer. Heat will also pass through the walls of the engine to be dissipated into the air. The process by which heat is moved from molecule to molecule in the engine wall is called heat transfer by **conduction**; it is due to vibrations of molecules, the vibrations increasing with higher and higher temperatures. In this book we discuss only heat transfer by conduction. To describe conductive heat flow in a medium, and ultimately obtain a PDE that determines temperature in the medium, we introduce the heat flux vector.

Definition 2.1 The **heat flux vector** $\mathbf{q}(\mathbf{r}, t)$ is a vector function of position $\mathbf{r}$ and time t. Its direction corresponds to the direction of heat flow at position $\mathbf{r}$ and time t, and its magnitude is equal to the amount of heat per unit time crossing unit area normal to the direction of $\mathbf{q}$.

This vector, which has units of watts per square metre (W/m^2), is defined at every point in a conducting medium except possibly at sources or sinks of heat (Figure 2.9).

A medium is said to be **isotropic** if, when any point within it is heated, heat spreads out equally in all directions. In other words, isotropic media have no preferred directions for heat flow. It has been shown experimentally that in an isotropic medium, heat flows in the direction in which temperature decreases most rapidly, and the amount of heat flowing in that direction is proportional to the rate of change of temperature in that direction. This is called **Fourier's law of heat conduction**.

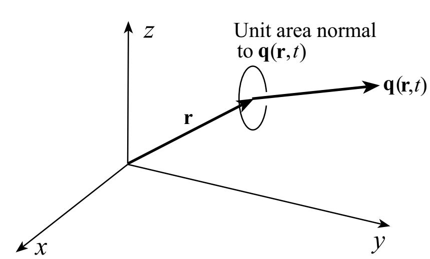


Figure 2.9

Mathematically, if $U(\mathbf{r}, t)$ is the temperature distribution in the medium, then the negative of its gradient $-\nabla U$ points in the direction in which the function U decreases most rapidly and $-|\nabla U|$ is the maximum rate of decrease. Consequently, Fourier's law of heat conduction in an isotropic medium can be stated vectorially as

$$\mathbf{q}(\mathbf{r}, t) = -\kappa \nabla U, \tag{2.19}$$

where $\kappa > 0$ is the "constant" of proportionality called the **thermal conductivity** of the medium. It has units of watts per metre per degree Kelvin or Celsius (W/mK). In general, thermal conductivity may depend both on the temperature of and the position in the medium. If, however, the range of temperature is "limited" (and we shall consider only this case), the variation of κ with temperature is negligible, and κ becomes a function of position only, $\kappa = \kappa(\mathbf{r})$. The medium is said to be **homogeneous** if κ is independent of position, in which case κ becomes a numerical constant. Rough values for thermal conductivities of various homogeneous materials are given in Table 2.1. The larger the value of κ, the more readily the material conducts heat. Other thermal properties are also included; they will be introduced shortly.

Thermal Properties of Some Materials

Material	Density (kg/m^3)	Specific Heat (Ws/kgK)	Thermal Conductivity (W/mK) at 273K	Thermal Diffusivity (m^2/s)
Copper	8950	381	390	114×10^{-6}
Mild Steel	7884	460	45	12.4×10^{-6}
Pyrex Glass	2413	837	1.18	0.584×10^{-6}
Water	1000	1000	0.600	0.600×10^{-6}
Asbestos	579	1047	0.15	0.247×10^{-6}

Table 2.1

To obtain a PDE governing temperature in a medium, we consider an imaginary surface $\beta(R)$ bounding a portion R of the medium (Figure 2.10). Heat is added to (or removed from) R in two ways — across $\beta(R)$ by conduction and by internal heat sources or sinks. When $g(\mathbf{r}, t)$ is the amount of heat generated (or removed) per unit time per unit volume at position $\mathbf{r}$ and time t, the total heat generation per unit time within R is expressed by the triple integral

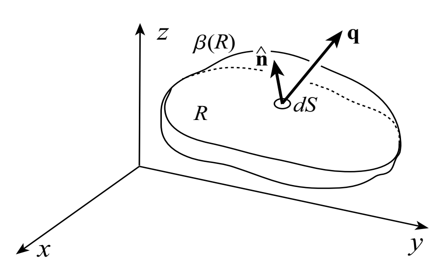


Figure 2.10

$$\iiint_R g(\mathbf{r}, t)\, dV. \tag{2.20}$$

The amount of heat flowing into R through $\beta(R)$ per unit time is given by the surface integral on the left side of the equation

$$\iint_{\beta(R)} \mathbf{q} \cdot (-\hat{\mathbf{n}})\, dS = \iint_{\beta(R)} \kappa \nabla U \cdot \hat{\mathbf{n}}\, dS, \tag{2.21}$$

where $\hat{\mathbf{n}}$ is the unit outward-pointing normal to $\beta(R)$. Fourier's law 2.19 has been used to obtain the integral on the right. The total heat represented by expressions 2.20 and 2.21 changes the temperature of points (x, y, z) in R by an amount $\partial U/\partial t$ in unit time. The heat requirement for this change is

$$\iiint_R \frac{\partial U}{\partial t} s\rho\, dV, \tag{2.22}$$

where ρ and s are the density and specific heat of the medium. (**Specific heat** is the amount of heat required to produce unit temperature change in unit mass.) Energy balance requires that expression 2.22 be equal to the sum of expressions 2.20 and 2.21:

$$\iiint_R \frac{\partial U}{\partial t} s\rho\, dV = \iiint_R g(\mathbf{r}, t)\, dV + \iint_{\beta(R)} \kappa \nabla U \cdot \hat{\mathbf{n}}\, dS, \tag{2.23}$$

and when the divergence theorem (see Appendix C) is applied to the surface integral, the result is

$$\iiint_R \left[\rho s \frac{\partial U}{\partial t} - g(\mathbf{r}, t) - \nabla \cdot (\kappa \nabla U)\right] dV = 0. \tag{2.24}$$

For this integral to vanish for an arbitrary volume R, in particular for an arbitrarily small volume, the integrand must vanish at each point of R; that is, U must satisfy the PDE

$$\rho s \frac{\partial U}{\partial t} - g(\mathbf{r}, t) - \nabla \cdot (\kappa \nabla U) = 0. \tag{2.25}$$

In actual fact, this conclusion is correct only when we know that the integrand in equation 2.24 is a continuous function throughout R. When this is not the case, equation 2.25 may not be valid at every point of R. It will, however, be true in each subregion in which the integrand is continuous. Since equation 2.24 must be valid even when its integrand is discontinuous, it is a more general statement of energy balance than equation 2.25.

Equation 2.25 is the PDE for heat conduction in an isotropic medium. If the medium is also homogeneous, we define $k = \kappa/(s\rho)$ as the **thermal diffusivity** of the medium, in which case equation 2.25 reduces to

$$\frac{\partial U}{\partial t} = k \left[\nabla^2 U + \frac{g(\mathbf{r}, t)}{\kappa}\right]. \tag{2.26}$$

The units of k are metres squared per second; typical values are given in Table 2.1.

Accompanying the PDE of heat conduction will be initial and/or boundary conditions. If R now represents the entire region in which $U(\mathbf{r}, t)$ is to be considered, rather than a particular part of it as in the foregoing discussion, an initial condition describes temperature in R at some initial time (usually $t = 0$):

$$U(\mathbf{r}, 0) = f(\mathbf{r}), \quad \mathbf{r} \text{ in } R, \tag{2.27}$$

where $f(\mathbf{r})$ is some given function of position.

The three types of boundary conditions that we consider are those introduced in Section 2.1 — Dirichlet, Neumann, and Robin. A Dirichlet condition prescribes temperature on the boundary $\beta(R)$ of R:

$$U(\mathbf{r},t) = F(\mathbf{r},t), \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0, \tag{2.28}$$

where $F(\mathbf{r},t)$ is a given function.

Sometimes in applications we know the heat flux vector $\mathbf{q}$ on $\beta(R)$ (Figure 2.10). Suppose we represent $\mathbf{q}$ on $\beta(R)$ in terms of its tangential and normal components to $\beta(R)$,

$$\mathbf{q} = q_n(\mathbf{r},t)\hat{\mathbf{n}} + q_T(\mathbf{r},t)\hat{\mathbf{T}}.$$

Component $q_n(\mathbf{r},t)$ is negative when heat is added to R and positive when heat is extracted. Fourier's law 2.19 on $\beta(R)$ yields

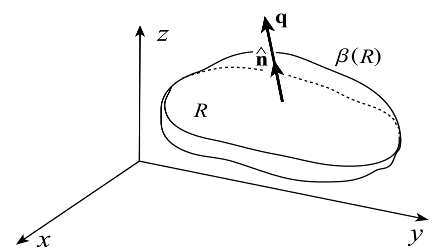


Figure 2.10

$$q_n\hat{\mathbf{n}} = -\kappa\nabla U, \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0, \tag{2.29}$$

and scalar products with $\hat{\mathbf{n}}$ give

$$\frac{\partial U}{\partial n} = -\frac{q_n(\mathbf{r},t)}{\kappa}, \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0. \tag{2.30}$$

In other words, specification of heat flow on $\beta(R)$ leads to a Neumann boundary condition. In particular, if a bounding surface is perfectly insulated, the heat flux vector thereon vanishes and consequently that surface satisfies a homogeneous Neumann boundary condition

$$\frac{\partial U}{\partial n} = 0, \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0. \tag{2.31}$$

A Robin boundary condition is a linear combination of a Dirichlet and a Neumann condition:

$$l\frac{\partial U}{\partial n} + hU = F(\mathbf{r},t), \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0. \tag{2.32}$$

Dirichlet and Neumann boundary conditions are obtained by setting l and h equal to zero, respectively. To show that Robin boundary conditions are physically realistic, suppose the conducting medium transfers heat to or from a surrounding medium according to Newton's law of cooling (heat transfer proportional to temperature difference). Then, on $\beta(R)$,

$$\mathbf{q}(\mathbf{r},t) = \mu(U - U_m)\hat{\mathbf{n}},$$

where $\mu > 0$ is a constant, called the **surface heat transfer coefficient**, and U_m is the temperature of the surrounding medium. When we substitute this into Fourier's law 2.19,

$$\mu(U - U_m)\hat{\mathbf{n}} = -\kappa\nabla U, \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0.$$

Scalar products with $\hat{\mathbf{n}}$ give

$$-\kappa\frac{\partial U}{\partial n} = \mu(U - U_m), \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0. \tag{2.33a}$$

Alternatively,

$$\kappa\frac{\partial U}{\partial n} + \mu U = \mu U_m = F(\mathbf{r}, t), \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0. \tag{2.33b}$$

This is clearly a Robin condition. Homogeneous Robin conditions

$$\kappa\frac{\partial U}{\partial n} + \mu U = 0, \quad \mathbf{r} \text{ on } \beta(R), \quad t > 0, \tag{2.34}$$

describe heat transfer according to Newton's law of cooling to (or from) media at temperature zero.

The initial boundary value problem of heat conduction in a homogeneous, isotropic medium can thus be stated as

$$\frac{\partial U}{\partial t} = k\left[\nabla^2 U + \frac{g(\mathbf{r}, t)}{\kappa}\right], \quad \mathbf{r} \text{ in } R, \quad t > 0, \tag{2.35a}$$

Boundary conditions, if applicable, (2.35b)

Initial condition $U(\mathbf{r}, 0) = f(\mathbf{r})$, $\quad \mathbf{r}$ in R, if applicable. (2.35c)

If boundary conditions 2.35b and heat sources $g(\mathbf{r}, t)$ in 2.35a are independent of time, there may exist solutions of 2.35a,b that are also independent of time. Such solutions are called **steady-state** solutions; they satisfy

$$\nabla^2 U = -\frac{g(\mathbf{r})}{\kappa}, \quad \mathbf{r} \text{ in } R, \tag{2.36a}$$

Boundary conditions, if applicable. (2.36b)

For example, suppose a conducting sphere of radius a (Figure 2.11) has at time $t = 0$ some temperature distribution $f(r, \phi, \theta)$, where r, ϕ and θ are spherical coordinates shown in Figure 2.7. If the sphere is suddenly packed on the outside with perfect insulation, and no heat generation occurs within the sphere, the temperature distribution $U(r, \phi, \theta)$ thereafter must satisfy the initial boundary value problem

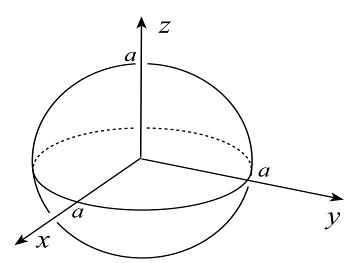


Figure 2.11

$$\frac{\partial U}{\partial t} = k\nabla^2 U, \quad 0 < r < a, \quad 0 < \phi < \pi, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{2.37a}$$

$$\frac{\partial U(a, \phi, \theta, t)}{\partial r} = 0, \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{2.37b}$$

$$U(r, \phi, \theta, 0) = f(r, \phi, \theta), \quad 0 \le r < a, \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi. \tag{2.37c}$$

Steady-state solutions $U(r, \phi, \theta)$ for this problem, if there are any, must satisfy

$$\nabla^2 U = 0, \quad 0 < r < a, \quad 0 < \phi < \pi, \quad -\pi < \theta \le \pi, \tag{2.38a}$$

$$\frac{\partial U(a, \phi, \theta)}{\partial r} = 0, \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi. \tag{2.38b}$$

Obviously a solution of problem 2.38 is $U = C$, where C is any constant whatsoever. Thus, constant functions are steady-state solutions for problem 2.37. We can realize the physical significance of steady-state solutions and determine a useful value for C if we return to initial boundary value problem 2.37. Physically it is clear that because no heat can enter or leave the sphere, heat will eventually redistribute itself until the temperature at every point in the sphere becomes the same constant value. In Section 9.1 we prove that the value of this constant is the average value $\tilde{U}$ of $f(r, \phi, \theta)$ over the sphere. In other words, the steady-state solution will be $U = \tilde{U}$. Later we shall see that the solution of problem 2.37 contains two parts. One is the steady-state (time-independent) part $U = \tilde{U}$; the other is a transient (time-dependent) part that describes the transition from initial temperature $f(r, \phi, \theta)$ to final temperature $\tilde{U}$.

When $g(\mathbf{r})$ in Poisson's equation 2.36 is identically zero (i.e., no internal heat generation occurs within R), the PDE reduces to **Laplace's** equation. Problem 2.36 then reads

$$\nabla^2 U = 0, \quad \mathbf{r} \text{ in } R, \tag{2.39a}$$

$$\text{Boundary conditions, if applicable.} \tag{2.39b}$$

Problems 2.36 and 2.39 are called boundary value problems rather than initial boundary value problems, since no initial conditions are present.

Example 2.3 Formulate the initial boundary value problem for the temperature in a cylindrical rod with insulated sides and with flat ends at $x = 0$ and $x = L$. The end at $x = 0$ is kept at temperature 60°C; the end at $x = L$ is insulated; and at time $t = 0$ the temperature distribution is $f(x)$, $0 \le x \le L$. Assume no internal heat generation. Are there steady-state solutions for this problem?

Solution Notwithstanding the fact that the rod is three-dimensional, we note that because all cross sections are identical, the sides are insulated, and the initial temperature distribution is a function of x alone, heat flows only in the x-direction. In other words, the heat conduction problem is one-dimensional, namely,

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\
U(0,t) &= 60, \quad t > 0, \\
\frac{\partial U(L,t)}{\partial x} &= 0, \quad t > 0, \\
U(x,0) &= f(x), \quad 0 < x < L.
\end{aligned}$$

Steady-state solutions $\psi(x)$ for this problem must satisfy

$$\begin{aligned}
\frac{d^2\psi}{dx^2} &= 0, \quad 0 < x < L, \\
\psi(0) = 60, &\quad \psi'(L) = 0.
\end{aligned}$$

A general solution of this ODE is $\psi(x) = Ax + B$, and the boundary conditions require

$$60 = B, \qquad 0 = A;$$

that is, $\psi(x) = 60$. After a very long time, the temperature in the rod will become 60°C throughout.●

Example 2.4 The top and bottom of a horizontal, semicircular plate $0 \le r \le r_0$, $0 \le \theta \le \pi$ are insulated. At time $t = 0$, its temperature is $f(r, \theta)$. For $t > 0$, the curved edge of the plate is insulated. That part of the x-axis for which $0 < x < r_0$ is held at temperature 5°C, and along the remaining part of the x-axis $-r_0 < x < 0$, heat is added at a constant rate $q > 0$ W/m^2. Formulate the initial boundary value problem for temperature in the plate.

Solution Temperature $U(r, \theta, t)$ satisfies the PDE

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right), \quad 0 < r < r_0, \quad 0 < \theta < \pi, \quad t > 0,$$

and the initial condition

$$U(r, \theta, 0) = f(r, \theta), \quad 0 < r < r_0, \quad 0 < \theta < \pi.$$

Since the curved edge is insulated, $U(r, \theta, t)$ satisfies a homogeneous, Neumann condition thereon. Because coordinate r is perpendicular to the semi-circle, this condition is

$$U_r(r_0, \theta, t) = 0, \quad 0 < \theta < \pi, \quad t > 0.$$

Since the positive part of the x-axis is held at temperature 5°C, the boundary condition along this edge is Dirichlet,

$$U(r, 0, t) = 5, \quad 0 < r < r_0, \quad t > 0.$$

We could use equation 2.30 to find the boundary condition along the negative x-axis, but it is often easier to keep signs straight if we return to Fourier's law $\mathbf{q} = -\kappa\nabla U$. Along $y = 0$ (and $x < 0$), $\mathbf{q} = q\hat{\mathbf{j}}$, and therefore

$$q\hat{\mathbf{j}} = -\kappa\nabla U.$$

Scalar products with $\hat{\mathbf{j}}$ gives

$$q = -\kappa\frac{\partial U}{\partial y}, \quad \text{or} \quad \frac{\partial U(x, 0, t)}{\partial y} = -\frac{q}{\kappa}.$$

But from Exercise 4 in Section 2.1, $\partial U/\partial y$ can be expressed in polar coordinates as follows

$$\frac{\partial U}{\partial y} = \sin\theta\frac{\partial U}{\partial r} + \frac{\cos\theta}{r}\frac{\partial U}{\partial \theta}.$$

Since $\theta = \pi$ along $y = 0$ (and $x < 0$), we can write that

$$-\frac{1}{r}\frac{\partial U(r, \pi, t)}{\partial \theta} = -\frac{q}{\kappa} \quad \text{from which} \quad \frac{1}{r}\frac{\partial U(r, \pi, t)}{\partial \theta} = \frac{q}{\kappa}, \quad 0 < r < r_0, \quad t > 0.●$$

Heat equation 2.26 is often called the diffusion equation because it also describes the diffusion of other quantities, chemicals for example. When a chemical diffuses through a medium, we define the chemical flux vector $\mathbf{q}(\mathbf{r}, t)$ as the amount of chemical flowing though unit area perpendicular to $\mathbf{r}$ per unit time t. Many chemicals diffuse according to Fick's Law which states that $\mathbf{q}$ is proportional to the gradient of the density or concentration $U(\mathbf{r}, t)$ of the chemical,

$$\mathbf{q} = -k\nabla U. \tag{2.40}$$

This is Fourier's law 2.19 in a different setting. As a result, chemical density $U(\mathbf{r}, t)$ must satisfy PDE 2.26, now called the one-dimensional diffusion equation.

EXERCISES 2.2

1. (a) A cylindrical, homogeneous, isotropic rod has flat ends at $x = 0$ and $x = L$ and insulated sides. Initially the temperature distribution in the rod is a function of x only, and heat generation at points x in the rod takes place uniformly over the cross section at x. Apply an energy balance to a segment of the rod from a fixed point $x = a$ to an arbitrary value of x to show that the PDE governing temperature $U(x, t)$ in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x, t),$$

where $g(x, t)$ is the amount of heat per unit volume per unit time generated at position x and time t.

(b) What form do Robin boundary conditions take at $x = 0$ and $x = L$?

In Exercises 2–18, set up, but do not solve, an initial boundary value problem for the required temperature. Assume that the medium is isotropic and homogeneous.

2. A cylindrical rod has flat ends at $x = 0$ and $x = L$ and insulated sides. At time $t = 0$ its temperature is a function $f(x)$, $0 \le x \le L$, of x only. For $t > 0$, both ends are kept at 100°C.

3. Repeat Exercise 2 except that the end at $x = 0$ is insulated.

4. Repeat Exercise 2 except that the temperature at end $x = L$ is changed from 0°C to 100°C at a constant rate over a period of T seconds and maintained at 100°C thereafter.

5. Repeat Exercise 2 except that heat is transferred according to Newton's law of cooling from the ends $x = 0$ and $x = L$ into media at temperatures U_0 and U_L, respectively.

6. Repeat Exercise 2 except that both ends are insulated and at each point in the rod heat is generated at a rate $g(x, t)$ per unit volume per unit time. What is $g(x, t)$ if heat generation is q calories per cubic centimetre per minute over that part of the rod between $x = L/4$ and $x = 3L/4$ and is zero otherwise?

7. Repeat Exercise 2 except that heat is added to the end $x = 0$ at a constant rate $Q_0 > 0$ W/m^2 uniformly over the end and is removed at a variable rate $Q_L(t) > 0$ W at $x = L$ uniformly over the end.

8. The top and bottom of a horizontal rectangular plate $0 \le x \le L$, $0 \le y \le L'$ are insulated. At time $t = 0$ its temperature is a function $f(x, y)$ of x and y only. The edges $x = 0$ and $y = L'$ are kept at 50°C for $t > 0$, and the edges $y = 0$ and $x = L$ are insulated.

9. Repeat Exercise 8 except that along $y = 0$ heat is transferred according to Newton's law of cooling into a medium with temperature $f_1(t)$, and heat is generated at a rate of $e^{\alpha t}$ joules per cubic metre per second at every point in the plate for the first T seconds.

10. The top and bottom of a horizontal circular plate $0 \le r \le r_0$, $-\pi < \theta \le \pi$ are insulated. At time $t = 0$ its temperature is a function $f(r, \theta)$ of polar coordinates r and θ only. For $t > 0$, heat is transferred along its edge according to Newton's law of cooling into a medium at temperature zero, and heat is generated at constant rate q W/m^3 inside the ring $0 < r_1 < r < r_2 < r_0$.

11. A right circular cylinder of length L and radius r_0 has its axis along the z-axis with flat faces in the planes $z = 0$ and $z = L$. At time $t = 0$ its temperature is a function $f(r, \theta)$ of r and θ only. For $t > 0$, faces $z = 0$ and $z = L$ are insulated, and $r = r_0$ is kept at temperature $f_1(\theta, t)$.

12. Repeat Exercise 11 except that $f(r, \theta)$ is replaced by $f(r, \theta, z)$.

13. Repeat Exercise 11 except that the ends $z = 0$ and $z = L$ are kept at 100°C for $t > 0$ and the cylindrical side is insulated.

14. Repeat Exercise 11 except that heat is transferred according to Newton's law of cooling from the top and cylindrical faces into air at temperature 20°C. Initially, temperature is a function $f(r)$ of r only.

15. Repeat Exercise 11 except that the initial temperature is a function $f(r)$ of r only and $r = r_0$ is kept at temperature $f_1(t)$.

16. The top and bottom of a horizontal semicircular plate $0 \le r \le r_0$, $0 \le \theta \le \pi$ are insulated. At time $t = 0$, its temperature is $f(r, \theta)$. For $t > 0$, the curved edge of the plate is insulated, but along the straight edge, heat is added at a constant rate $q > 0$ W/m^2.

17. Repeat Exercise 16 except that along $r = r_0$, heat is extracted at a constant rate $q > 0$ W/m^2 and along the straight edge, heat is exchanged according to Newton's law of cooling with an environment at constant temperature U_0.

18. A sphere of radius r_0 has an initial temperature ($t = 0$) of 100°C. For $t > 0$, heat is transfered according to Newton's law of cooling to an environment at constant temperature 10°C.

19. A hemisphere of radius r_0 above the xy-plane has flat face in the xy-plane. The curved face of the sphere is insulated. If the heat flux vector on the face $z = 0$ is $\mathbf{q} = f(r, \theta)\hat{\mathbf{k}}$, formulate the boundary value problem for steady-state temperature in the hemisphere. Can $f(r, \theta)$ be arbitrarily specified? (See Exercise 9(b) in Section 2.1 and Exercise 24 below.)

20. A homogeneous, isotropic rod with insulated sides has its ends $x = 0$ and $x = L$ held at temperatures U_0 and U_L, respectively. If no heat is generated in the rod, can there be a steady-state temperature distribution in the rod?

21. Heat is added at the end $x = 0$ of a homogeneous, isotropic rod with insulated sides at a constant rates $q_0 > 0$. It is extracted from end $x = L$ at a constant rate $q_L > 0$. Can there be a steady-state temperature distribution in the rod when there is no internal heat generation?

22. Discuss each of the following statements for temperature in a homogeneous, isotropic rod with insulated sides:
(a) If temperature at points in the rod changes in time, heat must flow in the rod.
(b) If heat flows in the rod, temperature at points in the rod must change in time.

23. Heat is transferred at the ends $x = 0$ and $x = L$ of a homogeneous, isotropic rod with insulated sides to surrounding media at constant, but different temperatures, according to Newton's law of cooling. If no heat is generated in the rod, can there be a steady-state temperature distribution in the rod?

24. (a) Suppose there is a steady-state temperature distribution in a region R of the xy-plane that satisfies Poisson's equation $\nabla^2 U = -g(x, y)/\kappa$. Use the result of Exercise 9 in Section 2.1 (or Green's theorem) to show that the solution must satisfy the equation

$$\iint_R g(x,y)\,dA = \oint_{\beta(R)} -\kappa \frac{\partial U}{\partial n}\,ds.$$

Interpret this equation physically.

(b) Suppose now that the boundary condition on the boundary is of Neumann type, $\partial U/\partial n = f(x,y)$ for (x,y) on $\beta(R)$. Show that $f(x,y)$ and $g(x,y)$ must satisfy the consistency condition

$$\iint_R g(x,y)\,dA = \oint_{\beta(R)} -\kappa f(x,y)\,ds.$$

(c) What is the three-dimensional analogue of the result in part (b)?

25. A steady-state temperature distribution in the rectangle $R: 0 \le x \le L$, $0 \le y \le L'$ of the xy-plane must satisfy Poisson's equation $\nabla^2 U = -g(x,y)/\kappa$. Suppose that boundary conditions are of Neumann type, specified in the form

$$\frac{\partial U(0,y)}{\partial x} = f_1(y), \quad 0 < y < L', \qquad \frac{\partial U(L,y)}{\partial x} = f_2(y), \quad 0 < y < L',$$
$$\frac{\partial U(x,0)}{\partial y} = f_3(x), \quad 0 < x < L, \qquad \frac{\partial U(x,L')}{\partial y} = f_4(x), \quad 0 < x < L.$$

Use the result of Exercise 24 to determine the appropriate consistency condition for the problem.

In Exercises 26–31 we discuss steady-state temperature in spheres and hollow spheres where heat flow is radial. In such situations, temperature U is only a function of the radial coordinate r in spherical coordinates, $U = U(r)$. Make this assumption in each of the exercises.

26. Show that if a sphere is in a steady-state temperature situation, and its temperature depends only on distance from the centre of the sphere, then temperature must be constant throughout the sphere. Assume no heat generation within the sphere.

27. Show that the result of Exercise 26 is not true if there is heat generation in the sphere. Assume that heat is generated at the rate $g(r)$ so that heat flows radially. Find the temperature $U(r)$ within the sphere when the boundary condition on the surface $r = b$ of the sphere is:
(a) $U(b) = U_b$; (b) $U'(b) = -Q/\kappa$; (c) $\kappa U'(b) + \mu U(b) = \mu U_m$.
In each case simplify the solution when $g(r) = G$, a constant.

28. A hollow sphere has inner radius a and outer radius b. Find the steady-state temperature if inner and outer surface temperatures are U_a and U_b. Assume no internal heat generation.

29. Repeat Exercise 28 if the inner surface has temperature U_a and heat is added uniformly at the outer surface at rate Q.

30. Repeat Exercise 28 if the temperature of the inner surface is U_a and heat is exchanged at the outer surface according to Newton's law of cooling with a medium at temperature U_m.

31. Repeat Exercise 28 if heat is added uniformly at the inner surface at rate Q and heat is exchanged at the outer surface according to Newton's law of cooling with a medium at temperature U_m.

32. In Exercise 1 we developed the one-dimensional heat conduction equation based on energy balance for a small segment of the rod. In this exercise we use the PDE to discuss energy balance for the entire rod. Multiply the PDE in Exercise 1 by $A\kappa/k$ (A is the cross-sectional area of the rod), integrate with respect to x over the length $0 \le x \le L$ of the rod, and integrate with respect to t from $t = 0$ to an arbitrary value of t, to obtain the following result:

$$\int_0^L A\rho s U(x,t)\,dx - \int_0^L A\rho s U(x,0)\,dx = \int_0^t A\kappa \frac{\partial U(L,t)}{\partial x} dt - \int_0^t A\kappa \frac{\partial U(0,t)}{\partial x} dt + \int_0^t \int_0^L Ag(x,t)\,dx\,dt.$$

Interpret each term in this equation physically, and hence deduce that the equation is a statement of energy balance for the rod.

33. Repeat Exercise 32 to obtain energy balance for a volume R using PDE 2.26.

34. Consider the following heat conduction problem in a rod with insulated sides

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}xt, \quad 0 < x < L, \quad t > 0,$$
$$U_x(0,t) = 10e^{-t}, \quad t > 0,$$
$$U_x(L,t) = -5, \quad t > 0,$$
$$U(x,0) = f(x), \quad 0 < x < L.$$

(a) Use the result of Exercise 32 to determine the amount of thermal energy that has been added to the rod from time $t = 0$ to an arbitrary time t.
(b) Could you determine the thermal energy in part (a) if either boundary condition were Dirichlet?

35. Repeat part (a) of Exercise 34 for the following problem in a rectangle R with insulated top and bottom

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{kt}{\kappa}\sin\frac{2\pi x}{L}, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,$$
$$U_x(0,y,t) = 10e^t, \quad 0 < y < L', \quad t > 0,$$
$$U_x(L,y,t) = 10e^t, \quad 0 < y < L', \quad t > 0,$$
$$U_y(x,0,t) = 5t, \quad 0 < x < L, \quad t > 0,$$
$$U_y(x,L',t) = 5t, \quad 0 < x < L, \quad t > 0,$$
$$U(x,y,0) = f(x,y), \quad 0 < x < L, \quad 0 < y < L'.$$

Hint: See Exercises 32 and 33.

36. (a) The inside temperature of a flat wall is a constant U_{in}°C and the outside temperature is a constant temperature U_{out}°C. If the wall is considered as part of an infinite slab that is in a steady-state temperature situation, find an expression for the amount of heat lost through an area A of the wall per unit time. Is this expression inversely proportional to the thickness of the wall?
(b) Evaluate the result in part (a) if A is 15 m^2, the thickness of the wall is 10 cm, the thermal conductivity of the material in the wall is 0.11 W/mK, $U_{\text{out}} = -20$°C, and $U_{\text{in}} = 20$°C.

37. (a) Steam is passed through a pipe with inner radius r_{in} and outer radius r_{out}. The temperature of the inner wall is a constant U_{in}°C and that on the outer wall is a constant U_{out}°C. If the pipe is considered part of an infinitely long pipe that is in a steady-state temperature situation, find an expression for the amount of heat per unit area per unit time flowing radially outward.

(b) How much heat (per second) is lost at the outer surface of the pipe in a section 2 m long if $r_{\text{in}} = 3.75$ cm, $r_{\text{out}} = 5.0$ cm, $U_{\text{in}} = 205°$C, $U_{\text{out}} = 195°$C, and $\kappa = 54$ W/mK?

(c) Illustrate that the same amount of heat is transferred through the inner wall of the section. Must this be the case?

38. A homogeneous, isotropic rod with insulated sides has temperature $\sin(n\pi x/L)$, n a positive integer, at time $t = 0$. For time $t > 0$ its ends at $x = 0$ and $x = L$ are held at temperature 0°C.

(a) Find the initial boundary value problem for temperature in the rod and verify that a solution is

$$U(x,t) = e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}.$$

(b) Find the rate of heat flow across cross sections of the rod at $x = 0$, $x = L/2$, and $x = L$ by calculating

$$\lim_{x\to 0^+} q(x,t), \qquad q(L/2,t), \qquad \lim_{x\to L^-} q(x,t).$$

(c) Calculate limits of the heat flows in part (b) as $t \to 0^+$ and $t \to \infty$.

39. (a) When two media with different thermal conductivities κ_1 and κ_2 are brought into intimate contact, heat flows from the hotter to the cooler medium. Assuming that heat transfer follows Newton's law of cooling, show that the following boundary conditions must be satisfied by the temperatures in the media at the interface:

$$-\kappa_1\frac{\partial U(0-)}{\partial n} = \mu[U(0-) - U(0+)], \qquad \kappa_2\frac{\partial U(0+)}{\partial n} = \mu[U(0+) - U(0-)],$$
$$-\kappa_1\frac{\partial U(0-)}{\partial n} = -\kappa_2\frac{\partial U(0+)}{\partial n},$$

where n is a coordinate perpendicular to the interface with positive direction from medium 1 into medium 2. Are these conditions independent?

(b) What do these conditions become in the event that μ is so high that there is essentially no resistance to heat flow across the interface?

40. (a) A homogeneous, isotropic sphere of radius R is heated uniformly from heat sources within at the rate of Q watts per cubic metre. Heat is transferred to a surrounding medium at constant temperature U_m according to Newton's law of cooling until a steady-state situation is achieved. Find the steady-state temperature distribution in the sphere.

(b) What is the initial boundary value problem for temperature in the sphere for $t > 0$ if the heat sources are turned off at time $t = 0$ and the steady-state situation has been achieved?

41. A thin wire of uniform cross section radiates heat from its sides (not ends) at a rate per unit area per unit time that is proportional to the difference between the temperature of the wire on its surface and that of its surroundings. It follows that variations in temperature should occur over cross sections of the wire. In many applications, these variations are sufficiently small that they may be considered negligible. In such a case, temperature at points in the wire is a function of time t and only one space variable along the wire, which we take as x, $U = U(x,t)$. Temperature problems of this type are called **thin-wire problems**. By considering heat flow into, and out of, the segment of the wire from a fixed point $x = a$ to an arbitrary x, show that the PDE for thin-wire problems is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} - h(U - U_m) + \frac{k}{\kappa}g(x,t),$$

where $h > 0$ is a constant and U_m is the temperature of the medium surrounding the wire.

42. Heat generation within a rod can be effected by passing an electric current along the length of the rod. Show that when the current is I,

$$g(x,t) = \frac{I^2}{A^2\sigma},$$

where σ is the electrical conductivity of the material of the rod and A is its cross-sectional area.

43. A cylindrical pipe of inner and outer radii a and b is sufficiently long that end effects may be neglected. The temperature of the inner wall is a constant U_a, and heat is transferred at the outer wall to a medium at constant temperature $U_m < U_a$ with surface heat transfer coefficient μ.

(a) Find U as a function of r when the steady-state situation has been achieved.

(b) Show that the amount of heat flowing radially through unit length of the pipe at any radius $a < r < b$ is

$$\frac{2\pi\mu\kappa b(U_a - U_m)}{\kappa + \mu b \ln(b/a)}.$$

44. A long straight wire of circular cross section has thermal conductivity κ and carries a current I. Surrounding the wire is insulation with thermal conductivity κ^*, $b - a$ units thick. If r is a radial coordinate measured from the centre of the wire, the wire occupies the region $0 < r < a$, and the insulation, $a < r < b$. Heat transfer takes place at $r = b$ into a medium at constant temperature U_m with surface heat transfer coefficient μ^*. Find the steady-state temperature $U(r)$ in the wire and insulation under the assumption that $U(r)$ must be continuous at $r = a$. (Hint: See Exercise 42 for $g(r)$ and Exercise 39 for the additional boundary condition at the wire-insulation interface.)

45. Repeat Exercise 44 except that continuity of $U(r)$ at $r = a$ is replaced by the condition that heat transfer from the wire to the insulation occurs according to Newton's law of cooling with surface heat transfer coefficient μ.

§2.3 Transverse Vibrations of Strings; Longitudinal and Angular Vibrations of Bars

In this section we discuss three vibration problems that all give rise to the same mathematical representation.

Transverse Vibrations of Strings

A perfectly flexible string (such as, perhaps, a violin string) is stretched tightly between two fixed points $x = 0$ and $x = L$ on the x-axis (Figure 2.12). Suppose the string is somehow set into motion in the xy-plane (perhaps by pulling vertically on the midpoint of the string and then releasing it). Our objective is to study the subsequent motion of the string. When the string is very taut and displacements are small, horizontal displacements of particles of the string are negligible compared with vertical displacements; that is, displacements may be taken as purely transverse, representable in the form $y(x,t)$.

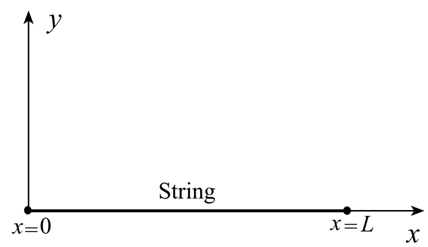


Figure 2.12

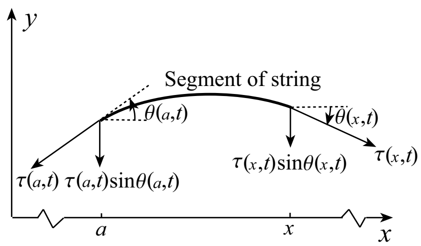


Figure 2.13

To find a PDE for $y(x,t)$, we analyze the forces on a segment of the string from a fixed position $x = a$ to an arbitrary position x (Figure 2.13). We denote by $\tau(x,t)$ the magnitude of the tension in the string at position x and time t. Because the string is perfectly flexible, tension in the string is always in the tangential direction of the string. If we let $\theta(x,t)$ be the angle of inclination of the tangent line of the curve to the positive x-axis, then the y-component of tension on the right end of the string segment is $\tau(x,t)\sin\theta(x,t)$. Since the y-component of tension on the left end of the segment is $-\tau(a,t)\sin\theta(a,t)$, the sum of these two forces is

$$\tau(x,t)\sin\theta(x,t) - \tau(a,t)\sin\theta(a,t).$$

We group all other forces acting on the segment into one function by letting $F(x,t)$ be the y-component of the sum of all external forces acting on the string per unit length in the x-direction. The total of all external forces acting on the segment then has y-component

$$\int_a^x F(\zeta,t)\,d\zeta.$$

Newton's second law states that the time rate of change of the momentum of the segment of the string must be equal to the resultant force thereon:

$$\frac{\partial}{\partial t}\left[\int_a^x \frac{\partial y(\zeta,t)}{\partial t}\rho(\zeta,t)\sqrt{1+\left(\frac{\partial y(\zeta,t)}{\partial x}\right)^2}\,d\zeta\right]$$
$$= \tau(x,t)\sin\theta(x,t) - \tau(a,t)\sin\theta(a,t) + \int_a^x F(\zeta,t)\,d\zeta, \qquad (2.41)$$

where $\rho(x,t)$ is the linear density of the string (mass per unit x-length). The quantity $\sqrt{1+[\partial y(\zeta,t)/\partial x]^2}\,d\zeta$ is the length of string that projects onto a length $d\zeta$ along the x-axis. Multiplication by $\rho(\zeta,t)\partial y(\zeta,t)/\partial t$ gives the momentum of this infinitesimal length of the string, and integration yields the momentum of that segment of the string from $x=a$ to an arbitrary position x. If we differentiate this equation with respect to x, we obtain

$$\frac{\partial}{\partial t}\left[\rho\frac{\partial y}{\partial t}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\right] = \frac{\partial}{\partial x}(\tau\sin\theta) + F(x,t). \tag{2.42}$$

When vibrations of the string are such that the slope of the displaced string, $\partial y/\partial x$, is very much less than unity (and this is the only case that we consider), the radical may be dropped from the equation and $\sin\theta$ approximated by $\tan\theta = \partial y/\partial x$. The resulting PDE for $y(x,t)$ is

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial x}\left(\tau\frac{\partial y}{\partial x}\right) + F(x,t). \tag{2.43}$$

For most applications, both the density of and the tension in the string may be taken as constant, in which case equation 2.43 reduces to

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \qquad c^2 = \tau/\rho. \tag{2.44}$$

This is the mathematical model for small transverse vibrations of a taut string; it is called the **one-dimensional wave equation**. In its derivation we have assumed that the slope of the string at every point is always very much less than 1 and that tension and density are constant.

When the only external force acting on the string is gravity, $F(x,t)$ takes the form

$$F = -\rho g, \quad g > 0. \tag{2.45}$$

Other possibilities include a damping force proportional to velocity,

$$F = -\beta\frac{\partial y}{\partial t}, \quad \beta > 0; \tag{2.46}$$

and a restoring force proportional to displacement,

$$F = -ky, \quad k > 0. \tag{2.47}$$

Accompanying the wave equation will be initial and/or boundary conditions. Initial conditions describe the displacement and velocity of the string at some initial time (usually $t=0$):

$$y(x,0) = f(x), \quad x \text{ in } I, \tag{2.48a}$$

$$\frac{\partial y(x,0)}{\partial t} = y_t(x,0) = g(x), \quad x \text{ in } I, \tag{2.48b}$$

where I is the interval over which the string is stretched. In Figure 2.12, I is $0<x<L$, but other intervals are also possible. Interval I also dictates the number of boundary conditions. There are three possibilities, depending upon whether the

string is of finite length, of *semi-infinite* length, or of *infinite* length. If the string is of finite length, the interval I is customarily taken as $0 < x < L$ and two boundary conditions result, one at each end. The string is said to be of **semi-infinite length**, or the **problem is semi-infinite**, if the string has only one end that satisfies some prescribed condition. The interval I in this case is always chosen as $0 < x < \infty$, and the one boundary condition is at $x = 0$. The string is said to be of **infinite length**, or the **problem is infinite**, if the string has no ends. In this case interval I becomes $-\infty < x < \infty$ and there are no boundary conditions.

It might be argued that there is no such thing as a semi-infinitely long or infinitely long string, and we must agree. There are, however, situations in which the model of a semi-infinite or infinite string is advantageous. For example, suppose a fairly long string (with ends at $x = 0$ and $x = L$) is initially at rest along the x-axis. Suddenly, something disturbs the string at its midpoint, $x = L/2$ (perhaps it is struck by an object). The effect of this disturbance travels along the string in both directions toward $x = 0$ and $x = L$. Before the disturbance reaches $x = 0$ and $x = L$, the string reacts exactly as if it had no ends whatsoever. If we are interested only in these initial disturbances, and consideration of the infinite problem provides straightforward explanations, it is an advantage to analyze the infinite problem rather than the finite one.

We consider only three types of boundary conditions at an end of the string — Dirichlet, Neumann, and Robin. When the string has an end at $x = 0$, a Dirichlet boundary condition takes the form

$$y(0,t) = f_1(t), \quad t > 0. \tag{2.49a}$$

It states that the end $x = 0$ of the string is caused by some external mechanism to perform the vertical motion described by $f_1(t)$. Similarly, if the string has an end at $x = L$, a Dirichlet condition

$$y(L,t) = f_2(t), \quad t > 0 \tag{2.49b}$$

indicates that this end has a vertical displacement described by $f_2(t)$. For the string in Figure 2.12, $f_1(t) = f_2(t) = 0$.

Instead of prescribing the motion of the end $x = 0$ of the string, suppose that this end is restricted to move vertically along the y-axis (Figure 2.14). The vertical component of the tension of the string acting on the end is $\tau(0,t)\sin\theta(0,t)$, which for small slopes can be approximated by

$$\tau(0,t)\sin\theta(0,t) \approx \tau(0,t)\tan\theta(0,t) = \tau(0,t)\frac{\partial y(0,t)}{\partial x}. \tag{2.50}$$

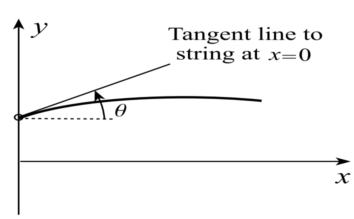


Figure 2.14

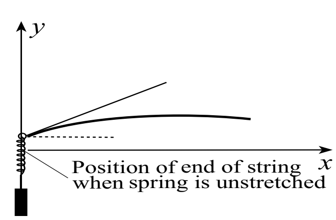


Figure 2.15

Consequently, when Newton's second law is applied to the motion of the end of the string, and the end is taken to be massless,

$$0 = \tau(0,t)\frac{\partial y(0,t)}{\partial x} + f_1(t), \quad t > 0, \tag{2.51}$$

where $f_1(t)$ represents the y-component of all other forces acting on the end. Thus, the motion of the end of the string must satisfy

$$\frac{\partial y(0,t)}{\partial x} = -\frac{1}{\tau(0,t)} f_1(t), \quad t > 0, \tag{2.52}$$

a Neumann boundary condition. In particular, if the massless end of the string is free to slide vertically with no force acting on it except tension in the string, it satisfies a homogeneous Neumann condition

$$\frac{\partial y(0,t)}{\partial x} = 0, \quad t > 0. \tag{2.53}$$

What this equation says is that when the end of a taut string is free of external forces, the slope of the string there will always be zero.

Similarly, if the string has a massless end at $x = L$ that is subjected to a vertical force with component $f_2(t)$, the boundary condition there is once again Neumann:

$$\frac{\partial y(L,t)}{\partial x} = \frac{1}{\tau(L,t)} f_2(t), \quad t > 0. \tag{2.54}$$

What we have shown, then, is that Neumann boundary conditions result when the ends of the string, taken as massless, move vertically under the influence of forces that are specified as functions of time.

Robin boundary conditions, which are linear combinations of Dirichlet and Neumann conditions, arise when the ends of the string are attached to springs that are unstretched on the x-axis (Figure 2.15). When this is the case at $x = 0$, equation 2.51 becomes

$$0 = \tau(0,t)\frac{\partial y(0,t)}{\partial x} - ky(0,t) + f_1(t), \tag{2.55}$$

where $f_1(t)$ now represents all external forces acting on the end of the string other than the spring and tension in the string. For constant tension τ, equation 2.55 takes the form

$$-\tau\frac{\partial y}{\partial x} + ky = f_1(t), \quad x = 0, \quad t > 0. \tag{2.56a}$$

Similarly, attaching the end $x = L$ to a spring gives the Robin condition

$$\tau\frac{\partial y}{\partial x} + ky = f_2(t), \quad x = L, \quad t > 0. \tag{2.56b}$$

The initial boundary value problem for the vibrating string consists of the one-dimensional wave equation together with two initial conditions and/or zero, one, or two boundary conditions:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \quad x \text{ in } I, \quad t > 0, \tag{2.57a}$$

$$\text{Boundary conditions, if applicable,} \tag{2.57b}$$

$$y(x,0) = f(x), \quad x \text{ in } I, \quad \text{if applicable,} \tag{2.57c}$$

$$y_t(x,0) = g(x), \quad x \text{ in } I, \quad \text{if applicable.} \tag{2.57d}$$

When the boundary conditions and external force F are independent of time, there may exist solutions of problem 2.57a,b that are also independent of time. Such solutions, called **static deflections**, satisfy the boundary value problem

$$\frac{d^2 y}{dx^2} = -\frac{F(x)}{\tau}, \quad x \text{ in } I, \tag{2.58a}$$

$$\text{Boundary conditions.} \tag{2.58b}$$

No vibrations occur; the string remains in static equilibrium under the forces present. We shall see that, in such cases, the solution of problem 2.57 divides into two parts, the static deflection part plus a second part that represents vibrations about the static solution.

Example 2.5 Formulate the initial boundary value problem for transverse vibrations of a string stretched tightly along the x-axis between $x = 0$ and $x = L$. The end $x = 0$ is free to move without friction along a vertical support, and the end $x = L$ is fixed on the x-axis. Initially, the string is released from rest at a position described by the function $f(x)$, $0 \le x \le L$. Take gravity into account. Are there static deflections for this problem?

Solution The initial boundary value problem for displacements $y(x,t)$ of points in the string is

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2} - 9.81, \quad 0 < x < L, \quad t > 0,\\
\frac{\partial y(0,t)}{\partial x} &= 0, \quad t > 0,\\
y(L,t) &= 0, \quad t > 0,\\
y(x,0) &= f(x), \quad 0 < x < L,\\
\frac{\partial y(x,0)}{\partial t} &= 0, \quad 0 < x < L.
\end{aligned}$$

The PDE is a result of equations 2.44 and 2.45, and the boundary condition at $x = 0$ is equation 2.53. Static deflections must satisfy

$$0 = c^2 \frac{d^2 y}{dx^2} - 9.81, \quad 0 < x < L,$$

$$y'(0) = 0, \quad y(L) = 0,$$

the solution of which is

$$y(x) = \frac{9.81}{2c^2}(x^2 - L^2)$$

(Figure 2.16). This is the position that the string would occupy were it to hang motionless under gravity. Notice, in particular, that the parabola has zero slope at its free end $x = 0$.●

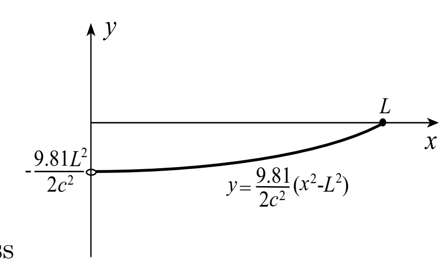


Figure 2.16

It is a standard example in ODEs to find the shape of a string that hangs between two points under the influence of gravity. The solution, called a **catenary**, is a hyperbolic cosine function, not a parabola as derived in Example 2.5. The difference lies in the assumptions leading to the ODEs describing the two situations. In Example 2.5, it is assumed that tension τ in the taut string is constant, and deflections are small. This leads to the differential equation $d^2y/dx^2 = 9.81\rho/\tau$ for static deflections. For the catenary problem, the string is not sufficiently taut that tension is constant, and deflections are not necessarily small. This leads to the differential equation $d^2y/dx^2 = (9.81\rho/\tau)\sqrt{1+(dy/dx)^2}$, where τ is tension at only the lowest point in the string.

Longitudinal Vibrations of Bars

In Figure 2.17 we show a cylindrical bar of natural length L lying along the x-axis. Suppose that the end $x = 0$ is clamped at that position and the end $x = L$ is struck with a hammer. This will set up longitudinal vibrations in the bar. We show that the one-dimensional wave equation, which describes transverse vibrations of a taut string, also describes these longitudinal vibrations of the bar. Although we have drawn the bar in a horizontal position, it could equally well be vertical. We denote by x the positions of cross sections of the bar when the bar is in an unstrained state, and we denote by $y(x, t)$ the positions of cross sections relative to their unstrained positions (Figure 2.18). It is assumed that cross sections remain plane during vibrations.

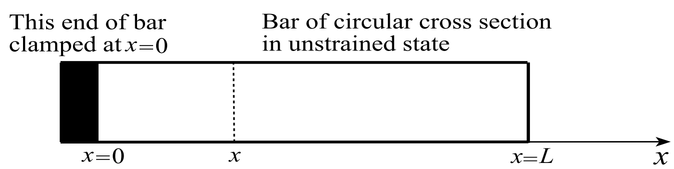


Figure 2.17

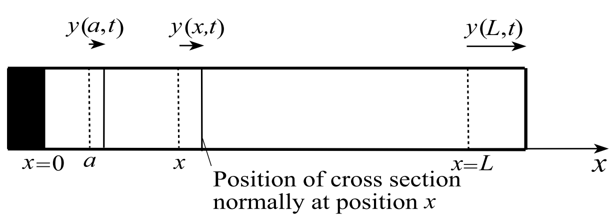


Figure 2.18

Consider the segment of the bar that in an unstrained state occupies the region between $x = a$ (a some fixed number) and an arbitrary position x. At time t, this segment is stretched an amount $y(x, t) - y(a, t)$. Hooke's law states that the force exerted across the segment due to this extension (or compression) is given by

$$AE\left[\frac{y(x,t) - y(a,t)}{x - a}\right], \tag{2.59}$$

where A is the cross-sectional area of the bar and E is Young's modulus of elasticity of the material in tension and compression. It follows (by limits as $x \to a$) that the internal force exerted on the face at $x = a$ by that part of the bar to its right at time t has component

$$AE\frac{\partial y(a,t)}{\partial x}. \tag{2.60}$$

(The internal force on the face at $x = a$ due to that part of the bar to its left has component $-AE\partial y(a,t)/\partial x$.)

We now apply Newton's second law to the motion of the above segment of the bar:

$$AE\frac{\partial y(x,t)}{\partial x} - AE\frac{\partial y(a,t)}{\partial x} + \int_a^x F(\zeta,t)A\,d\zeta = \frac{\partial}{\partial t}\left[\int_a^x \frac{\partial y(\zeta,t)}{\partial t}\rho(\zeta,t)A\,d\zeta\right], \tag{2.61}$$

where $\rho(x,t)$ is the density of the bar (mass per unit volume) and $F(x,t)$ is the x-component of all external forces acting on the bar per unit volume. It is assumed that these external forces are constant over each cross section of the bar. Differentiation of this equation with respect to x and division by A give

$$E\frac{\partial^2 y}{\partial x^2} + F(x,t) = \frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right). \tag{2.62}$$

In most applications, ρ can be taken as constant, in which case PDE 2.62 reduces to the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \qquad c^2 = E/\rho. \tag{2.63}$$

Initial conditions that accompany PDE 2.63 describe the displacement and velocity of cross sections of the bar at some initial time, usually $t = 0$ (see equations 2.48). Boundary conditions must also be specified. When the bar is of finite length ($0 < x < L$), two boundary conditions occur, one at each end. If the bar is of semi-infinte length ($0 < x < \infty$), only one end $x = 0$ satisfies a boundary condition; and when the bar is of infinite length, no boundary conditions are present. Dirichlet boundary conditions are of form 2.49a,b; they specify displacements $y(0,t)$ and $y(L,t)$ of the ends of the bar. Neumann boundary conditions result when longitudinal forces that are prescribed functions of time are applied to the faces of the bar. To see this, note that the force exerted on the face $x = 0$ by the bar (to the right) is $AE\partial y(0,t)/\partial x$. Consequently, if the end $x = 0$ of the bar is subjected to an external force with x-component $f_1(t)$, then taking the face as massless, Newton's second law for the face gives

$$AE\frac{\partial y(0,t)}{\partial x} + f_1(t) = 0 \quad \Longrightarrow \quad \frac{\partial y(0,t)}{\partial x} = -\frac{1}{AE}f_1(t), \quad t > 0, \tag{2.64}$$

a Neumann condition. Similarly, if the bar has an end $x = L$ with external force $f_2(t)$, the Neumann boundary condition there is

$$\frac{\partial y(L,t)}{\partial x} = \frac{1}{AE}f_2(t), \quad t > 0. \tag{2.65}$$

Homogeneous Neumann boundary conditions describe free ends.

Were we to attach the end of the bar at $x = 0$ to a spring (of constant $k > 0$) so that the spring is unstretched when the end of the bar is at $x = 0$ (Figure 2.19), boundary condition 2.64 would be replaced by

$$AE\frac{\partial y(0,t)}{\partial x} - ky(0,t) = 0 \quad \Longrightarrow \quad -AE\frac{\partial y(0,t)}{\partial x} + ky(0,t) = 0, \quad t > 0. \quad (2.66a)$$

This is a homogeneous Robin condition. Similarly, when end $x = L$ is attached to a spring, the resulting boundary condition is the homogeneous Robin condition

$$AE\frac{\partial y(L,t)}{\partial x} + ky(L,t) = 0, \quad t > 0. \quad (2.66b)$$

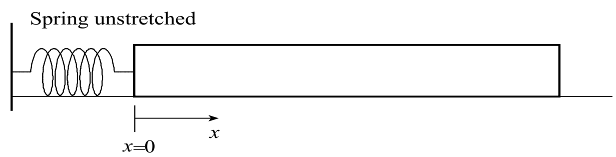


Figure 2.19

The initial boundary value problem for longitudinal displacements in the bar consists of the one-dimensional wave equation 2.63 together with two initial conditions and zero, one, or two boundary conditions, a problem identical to that for the string.

Angular Vibrations of Bars

Angular vibrations of a bar also give rise to the above mathematical problem. Let x denote distance from some fixed reference point to cross sections of a cylindrical elastic bar (Figure 2.20). At time t, the angular displacement of the section labeled x from its torque-free position is denoted by $y(x,t)$, where it is assumed that in each cross section, lines that are radial in the bar before torque is applied remain straight after the bar is twisted. At this time, the segment of the bar between a and x has its right face twisted relative to its left face by an amount $y(x,t) - y(a,t)$. The torque exerted across the element is then

$$IE\left[\frac{y(x,t) - y(a,t)}{x - a}\right], \quad (2.67)$$

where I is the moment of inertia of the cross-sectional area about the axis of the bar and E is Young's modulus of elasticity of the material in shear. It follows (by limits) that the internal torque exerted on the face at $x = a$ by that part of the bar to its right at time t is

$$IE\frac{\partial y(a,t)}{\partial x}. \quad (2.68)$$

(The internal torque on the face at $x = a$ due to that part of the bar to its left is $-IE\partial y(a,t)/\partial x$.)

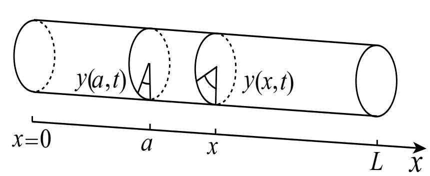


Figure 2.20

If, in addition, an external torque per unit length $\tau(x,t)$ acts, and $\rho(x,t)$ is the density (mass per unit volume) of the bar, then the PDE for angular vibrations of the bar can be obtained from Newton's second law applied to the element between a and x:

$$IE\frac{\partial y(x,t)}{\partial x} - IE\frac{\partial y(a,t)}{\partial x} + \int_a^x \tau(\zeta,t)\,d\zeta = \frac{\partial}{\partial t}\left[\int_a^x I\rho(\zeta,t)\frac{\partial y(\zeta,t)}{\partial t}d\zeta\right]. \quad (2.69)$$

Differentiation of this equation with respect to x and division by I give

$$E\frac{\partial^2 y}{\partial x^2} + \frac{\tau(x,t)}{I} = \frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right). \quad (2.70)$$

When ρ is constant, this reduces to the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{\tau(x,t)}{\rho I}, \quad c^2 = E/\rho. \quad (2.71)$$

Accompanying this PDE will be two initial conditions and/or zero, one, or two boundary conditions.

EXERCISES 2.3

In Exercise 1–8, set up, but do not solve, an (initial) boundary value problem for the required displacement. Assume that density of and tension in the string are constant (or that Young's modulus and density are constant in the bar).

1. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial displacement at $t = 0$ of $f(x)$, $0 \le x \le L$ and initial velocity $g(x)$, $0 \le x \le L$. Formulate the initial boundary value problem for displacement $y(x,t)$ of the string for $0 < x < L$ and $t > 0$.

2. Repeat Exercise 1 except that the end at $x = L$ is free to slide without friction along a vertical support and gravity on the string is taken into account.

3. Repeat Exercise 1 where oscillations take place in a medium that creates a damping force proportional to velocity and the ends of the string are elastically connected to the x-axis. Furthermore, do not neglect the weight of the string.

4. Repeat Exercise 1 except that a vertical force $F(t) = \cos\omega t$, $t > 0$, acts on the massless end $x = 0$ of the string which is looped around a vertical support. The string is initially at rest along the x-axis.

5. Repeat Exercise 3 except that the force $F(t)$ in Exercise 4 also acts on the end $x = 0$.

6. A horizontal cylindrical bar is originally at rest and unstrained along the x-axis between $x = 0$ and $x = L$. For time $t > 0$, the left end is fixed and the right end is subjected to a constant elongating force per unit area F parallel to the bar. Formulate the initial boundary value problem for displacements $y(x,t)$ of cross sections of the bar.

7. A bar of unstrained length L is clamped along its length, turned to the vertical position and hung from its end $x = 0$. At time $t > 0$, the clamp is removed and gravity is therefore permitted to act on the bar. Formulate an initial boundary value problem for displacements $y(x,t)$ of cross sections of the bar.

8. Repeat Exercise 7 except that the top of the bar is attached to a spring with constant k. Let $x = 0$ correspond to the top of the bar when the spring is in the unstretched position at $t = 0$.

9. Robin condition 2.56a applies to the massless left end of the string in Figure 2.15. Assuming $f_1(t) = 0$, show that when the end of the string is above the x-axis, the slope of the string at $x = 0$ must be positive (as shown in the figure). Furthermore, when the end of the string is below the x-axis, the slope must be negative. Is this also true at the right end of the string? Illustrate your results graphically.

10. Suppose the left end of the spring in Figure 2.19 is attached to a nonstationary support. Its horizontal displacement relative to its position in Figure 2.19 is $g(t)$. Formulate the boundary condition at $x = 0$ for motion of the bar.

11. The ends of a taut string are fixed at $x = 0$ and $x = L$ on the x-axis. The string is initially at rest along the x-axis, and is then allowed to drop under its own weight. Formulate an initial boundary value problem for displacements of the string. What are the static deflections for this string?

12. Repeat Exercise 11 except that motion takes place in a medium that creates a damping force proportional to velocity.

13. Repeat Exercise 11 except that the end of the string at $x = L$ is looped around a smooth vertical support and a constant vertical force F_L acts on this loop.

14. An unstrained elastic bar falls vertically under gravity with its axis vertical. When its velocity is v (which we take at time $t = 0$), it strikes a solid object and remains in contact with it thereafter. Formulate an initial boundary value problem for displacements of cross sections of the bar.

15. A cylindrical bar has unstrained length L. If it is hung vertically from one end so that no oscillations occur, what is its length?

16. The bar in Exercise 15 is hung from a spring with constant $k > 0$. How far below $x = 0$ (the position of the lower end of the spring in the unstretched position) will the lower end of the bar lie?

17. Verify that Robin and Neumann conditions at $x = L$ take the forms 2.56b and 2.54 for massless ends.

18. The end $x = 0$ of a horizontal bar of length L is kept fixed, and the other end has a mass m attached to it. The mass m is then subjected to a horizontal periodic force $F = F_0 \sin \omega t$. If the bar is initially unstrained and at rest, set up the initial boundary value problem for longitudinal displacements in the bar.

19. The one-dimensional wave equation 2.44 for vibrations of a taut string was derived by applying Newton's second law to a segment of the string. In this exercise, we use the PDE to discuss energy balance for the entire string (assumed finite in length).

(a) Multiply the PDE by $\partial y/\partial t$, integrate the result with respect to x over the length of the string $0 \leq x \leq L$, and use integration by parts to obtain

$$\frac{1}{2}\int_0^L \left[\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)^2 + c^2 \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right)^2\right] dx = c^2 \left\{\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right\}_0^L + \int_0^L \frac{F(x,t)}{\rho}\frac{\partial y}{\partial t} dx.$$

(b) Integrate the result in part (a) with respect to time from $t = 0$ to an arbitrary t to show that

$$\int_0^L \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 dx + \int_0^L \frac{\tau}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = \int_0^L \frac{\rho}{2}\left[\frac{\partial y(x,0)}{\partial t}\right]^2 dx + \int_0^L \frac{\tau}{2}\left[\frac{\partial y(x,0)}{\partial x}\right]^2 dx$$

$$+\int_0^t \tau\frac{\partial y(L,t)}{\partial x}\frac{\partial y(L,t)}{\partial t}\,dt + \int_0^t\left[-\tau\frac{\partial y(0,t)}{\partial x}\right]\frac{\partial y(0,t)}{\partial t}\,dt$$
$$+\int_0^t\int_0^L F(x,t)\frac{\partial y}{\partial t}\,dx\,dt.$$

Interpret each of these terms physically and thereby conclude that the equation is a statement of work-energy balance. It is often called the **energy equation** for the string.

20. (a) Use Exercise 19 to show that if $F(x,t)=0$ in equation 2.44, then the sum of the kinetic and strain energies of the string is constant in time if boundary conditions are homogeneous Dirichlet or Neumann.
(b) Describe the situation when one or both of the boundary conditions is homogeneous Robin.

21. (a) Evaluate the last integral in the energy equation of Exercise 19 if $F(x,t)$ is due only to gravity (see equation 2.45)?
(b) Are the results of Exercise 20 still valid?

22. Repeat Exercise 21 if the only force in $F(x,t)$ is a damping force proportional to velocity (see equation 2.46)?

23. Repeat Exercise 21 if the only force in $F(x,t)$ is a restoring force proportional to displacement (see equation 2.47)?

24. Show that when the cross-sectional area of the bar in Figure 2.17 varies with position, equation 2.63 is replaced by

$$\frac{\partial^2 y}{\partial t^2}=\frac{c^2}{A(x)}\frac{\partial}{\partial x}\left[A(x)\frac{\partial y}{\partial x}\right]+\frac{F(x,t)}{\rho},\qquad c^2=E/\rho,$$

provided expression 2.60 still gives forces across cross sections of the bar.

25. A bar of unstrained length L is clamped at end $x=0$. For time $t<0$, it is at rest, subjected to a force with x-component F distributed uniformly over the other end. If the force is removed at time $t=0$, formulate the initial boundary value problem for subsequent displacements in the bar.

26. In this exercise we derive the PDE for small vibrations of a suspended heavy cable. Consider a heavy cable of uniform density ρ (mass/length) and length L suspended vertically from one end. Take the origin of coordinates at the position of equilibrium of the lower end of the cable and the positive x-axis along the cable. Denote by $y(x,t)$ small horizontal deflections of points in the cable from equilibrium.
(a) Apply Newton's second law to a segment of the cable to obtain the PDE for small deflections

$$\frac{\partial^2 y}{\partial t^2}=g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right)+\frac{F}{\rho},$$

where $g>0$ is the acceleration due to gravity and F is the y-component of all external horizontal forces per unit length in the x-direction.
(b) What boundary condition must $y(x,t)$ satisfy at $x=L$?

27. (a) The current $I(x,t)$ and potential $V(x,t)$ in a long insulated cable must satisfy the pair of first-order partial differential equations

$$\frac{\partial V}{\partial x}+RI+L\frac{\partial I}{\partial t}=0,\qquad \frac{\partial I}{\partial x}+GV+C\frac{\partial V}{\partial t}=0,$$

where R, L, C, and G are respectively the resistance, inductance, capacitance, and conductance per unit length of the cable. Show that I and V must satisfy the same second order PDE, which for I is

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC + LG)\frac{\partial I}{\partial t} + RG\, I,$$

the one-dimensional wave equation 2.44 with terms of the form 2.46 and 2.47. It is often called the **telegraph equation**.

(b) Verify that when leakage to the ground is small so that L and G can be neglected, the PDE becomes the one-dimensional heat conduction equation with $k = 1/(RC)$.

(c) Verify that in the high-frequency case when R and G can be neglected, the PDE becomes the one-dimensional wave equation 2.44 with $c^2 = 1/(LC)$ and $F = 0$.

§2.4 Transverse Vibrations of Membranes

In this section we study vibrations of perfectly flexible membranes stretched over regions of the xy-plane (Figure 2.21). When the membrane is very taut and displacements are small, the horizontal components of these displacements are negligible compared with vertical components; that is, displacements may be taken as purely transverse, representable in the form $z(x, y, t)$.

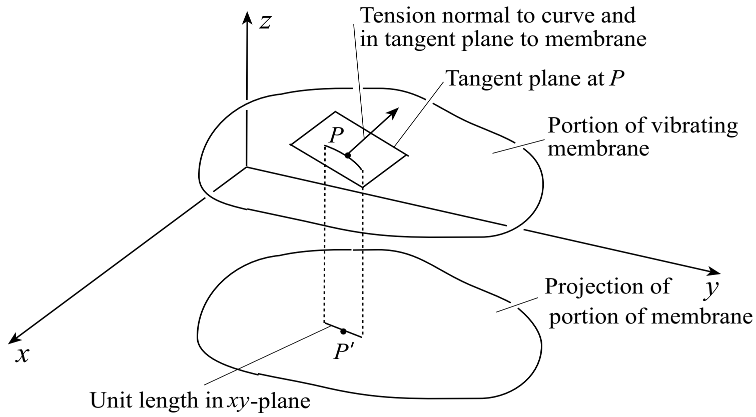


Figure 2.21

In discussing transverse vibrations of strings, tension played an integral role. No less important is the tension in a membrane. Suppose a line of unit length is drawn in any direction at a point P' in the xy-plane and projected onto a curve on the membrane (Figure 2.21). The material on one side of the curve exerts a force on the material on the other side, the force acting normal to the curve and in the tangent plane of the surface at P. This force is called the **tension** τ of the membrane.

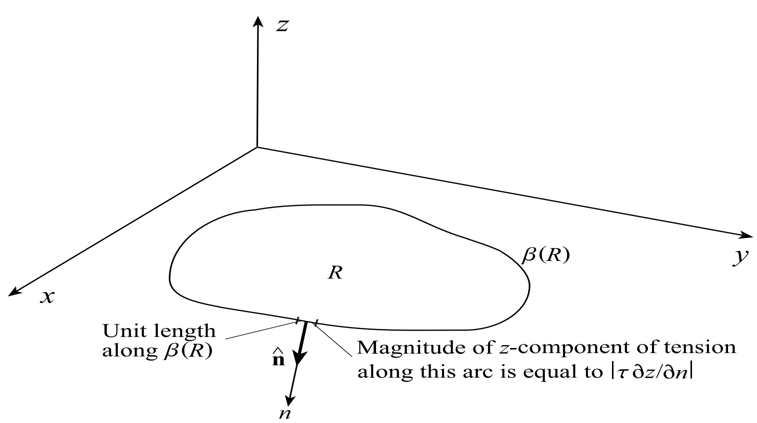


Figure 2.22

To obtain a PDE for displacements $z(x, y, t)$ of the membrane, we examine forces acting on a portion S of the membrane that projects onto an area A in the xy-plane (Figure 2.22). The vertical component of the tension on S is obtained by taking vertical components of the tension on the boundary $\beta(S)$. Tension on a small element ds' along $\beta(S)$ acts in the tangent plane to S at ds' and normal to $\beta(S)$. If parametric equations for $\beta(A)$, the projection of $\beta(S)$ in the xy-plane, are $x = x(s)$, $y = y(s)$, where s is arc length along $\beta(A)$, then a vector normal to $\beta(A)$ is $\frac{dy}{ds}\hat{\mathbf{i}} - \frac{dx}{ds}\hat{\mathbf{j}}$. It follows that the vector $\frac{dy}{ds}\hat{\mathbf{i}} - \frac{dx}{ds}\hat{\mathbf{j}} + a\hat{\mathbf{k}}$ is normal to $\beta(S)$ for any constant a. Since a vector normal to S is $\nabla[z(x,y,t) - z] = \frac{\partial z}{\partial x}\hat{\mathbf{i}} + \frac{\partial z}{\partial y}\hat{\mathbf{j}} - \hat{\mathbf{k}}$, it follows that

$$0 = \left(\frac{dy}{ds}\hat{\mathbf{i}} - \frac{dx}{ds}\hat{\mathbf{j}} + a\hat{\mathbf{k}}\right) \cdot \left(\frac{\partial z}{\partial x}\hat{\mathbf{i}} + \frac{\partial z}{\partial y}\hat{\mathbf{j}} - \hat{\mathbf{k}}\right).$$

This implies that $a = \frac{\partial z}{\partial x}\frac{dy}{ds} - \frac{\partial z}{\partial y}\frac{dx}{ds}$, and therefore a vector normal to ds' is

$$\mathbf{n} = \frac{dy}{ds}\hat{\mathbf{i}} - \frac{dx}{ds}\hat{\mathbf{j}} + \left(\frac{\partial z}{\partial x}\frac{dy}{ds} - \frac{\partial z}{\partial y}\frac{dx}{ds}\right)\hat{\mathbf{k}}.$$

A unit normal in this direction is

$$\hat{\mathbf{n}} = \frac{\frac{dy}{ds}\hat{\mathbf{i}} - \frac{dx}{ds}\hat{\mathbf{j}} + \left(\frac{\partial z}{\partial x}\frac{dy}{ds} - \frac{\partial z}{\partial y}\frac{dx}{ds}\right)\hat{\mathbf{k}}}{\sqrt{\left(\frac{dy}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2 + \left(\frac{\partial z}{\partial x}\frac{dy}{ds} - \frac{\partial z}{\partial y}\frac{dx}{ds}\right)^2}}.$$

If we assume that vibrations of the membrane are very small, then the third term in the denominator is negligible compared to the first two terms. With this assumption, the denominator is now unity (since $ds^2 = dx^2 + dy^2$), and a unit tangent vector is

$$\hat{\mathbf{n}} = \frac{dy}{ds}\hat{\mathbf{i}} - \frac{dx}{ds}\hat{\mathbf{j}} + \left(\frac{\partial z}{\partial x}\frac{dy}{ds} - \frac{\partial z}{\partial y}\frac{dx}{ds}\right)\hat{\mathbf{k}}.$$

Since the vertical component of tension acting on ds' is $\tau(x,y,t)\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\,ds'$, the vertical component of tension on $\beta(S)$ is

$$\oint_{\beta(S)} \tau(x,y,t)\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\,ds' = \oint_{\beta(S)} \tau(x,y,t)\left(\frac{\partial z}{\partial x}\frac{dy}{ds} - \frac{\partial z}{\partial y}\frac{dx}{ds}\right)ds'.$$

With the assumption on small oscillations, we can also say that $ds' = ds$, where ds is an element of arc length along $\beta(A)$, and hence, the line integral along $\beta(S)$ can be converted to a line integral along $\beta(A)$; that is, the vertical component of tension on $\beta(S)$ is

$$\oint_{\beta(A)} \tau(x,y,t)\left(\frac{\partial z}{\partial x}\frac{dy}{ds} - \frac{\partial z}{\partial y}\frac{dx}{ds}\right)ds = \oint_{\beta(A)} \tau(x,y,t)\left(-\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right).$$

If we apply Green's theorem (see Appendix C), this becomes

$$\iint_A \left[\frac{\partial}{\partial x}\left(\tau\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\tau\frac{\partial z}{\partial y}\right)\right]dA.$$

We group all other forces acting on the membrane into one function by letting $F(x,y,t)$ be the z-component of the sum of all external forces acting on the membrane per unit area in the xy-plane. The total of all external forces acting on element S of the membrane then has z-component

$$\iint_A F(x,y,t)\,dA.$$

We now apply Newton's second law (force equals time rate of change of momentum) to element S of the membrane,

$$\iint_A \left[\frac{\partial}{\partial x}\left(\tau\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\tau\frac{\partial z}{\partial y}\right)\right]dA+\iint_A F(x,y,t)\,dA=\frac{\partial}{\partial t}\iint_A \frac{\partial z}{\partial t}\rho\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA,$$

where ρ is the mass per unit in the xy-plane of the membrane. Once again, with small oscillations, we can remove the squared terms under the radical, and write

$$\iint_A \left[\frac{\partial}{\partial x}\left(\tau\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\tau\frac{\partial z}{\partial y}\right)+F(x,y,t)-\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\right)\right]dA=0.$$

For this integral to vanish for an arbitrary area A, in particular, for an arbitrarily small area, the integrand must vanish at each point of A; that is, $z(x,y,t)$ must satisfy the PDE

$$\frac{\partial}{\partial x}\left(\tau\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\tau\frac{\partial z}{\partial y}\right)+F(x,y,t)-\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\right)=0,$$

or,

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\right)=\frac{\partial}{\partial x}\left(\tau\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\tau\frac{\partial z}{\partial y}\right)+F. \tag{2.72}$$

For most applications, both the mass per unit area of and the tension in the membrane may be taken as constant, in which case equation 2.72 reduces to

$$\frac{\partial^2 z}{\partial t^2}=c^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)+\frac{F}{\rho},\quad c^2=\tau/\rho. \tag{2.73}$$

This is the PDE for transverse vibrations of the membrane, called the **two-dimensional wave equation**.

For an external force due only to gravity,

$$F=-\rho g,\quad g>0; \tag{2.74}$$

for a damping force proportional to velocity,

$$F=-\beta\frac{\partial z}{\partial t},\quad \beta>0; \tag{2.75}$$

and for a restoring force proportional to displacement,

$$F=-kz,\quad k>0. \tag{2.76}$$

Initial conditions that accompany PDE 2.73 describe the displacement and velocity of the membrane at some initial time (usually $t=0$):

$$z(x,y,0)=f(x,y),\quad (x,y)\text{ in } R, \tag{2.77a}$$

$$\frac{\partial z(x,y,0)}{\partial t}=g(x,y),\quad (x,y)\text{ in } R, \tag{2.77b}$$

where R is the open region in the xy-plane onto which the membrane projects. A Dirichlet boundary condition for PDE 2.73 prescribes the displacement $z(x,y,t)$ on the boundary $\beta(R)$ of R,

$$z(x,y,t) = f(x,y,t), \quad (x,y) \text{ on } \beta(R), \quad t > 0, \tag{2.78}$$

where $f(x,y,t)$ is some given function.

Suppose instead that the edge of the membrane can move vertically and that it is subjected to an external vertical force per unit length $f(x,y,t)$. The edge is also acted on by the tension in the membrane, and the magnitude of the z-component of the tension acting across a unit length along $\beta(R)$ is $|\tau \partial z/\partial n|$, where n is a coordinate measuring distance in the xy-plane normal to $\beta(R)$ (Figure 2.23). Consequently, if we take the edge of the membrane as massless, Newton's second law for vertical components of forces on an element ds of $\beta(R)$ gives

$$-\left(\tau \frac{\partial z}{\partial n}\right)_{|\beta(R)} ds + f(x,y,t)ds = 0 \tag{2.79a}$$

or,

$$\frac{\partial z}{\partial n} = \frac{1}{\tau} f(x,y,t), \quad (x,y) \text{ on } \beta(R), \quad t > 0. \tag{2.79b}$$

This is a nonhomogeneous Neumann boundary condition. When the only force acting on the edge of the membrane is that due to tension, $z(x,y,t)$ must satisfy a homogeneous Neumann condition,

$$\frac{\partial z}{\partial n} = 0, \quad (x,y) \text{ on } \beta(R), \quad t > 0. \tag{2.80}$$

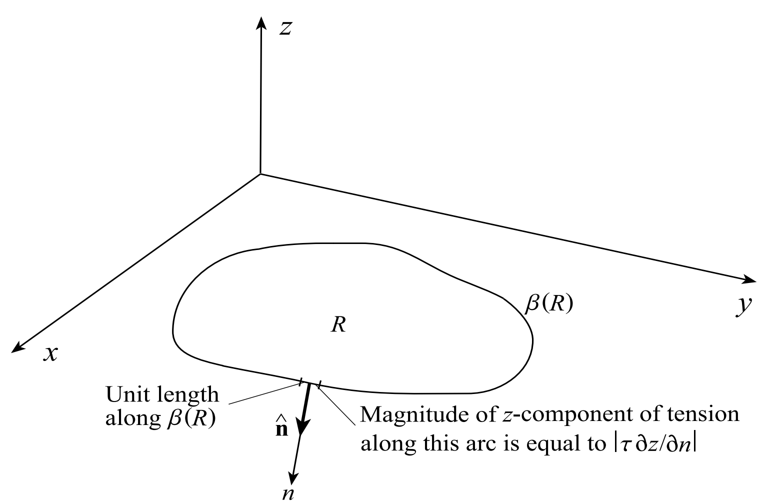


Figure 2.23

Another possibility is to have the edge of the membrane elastically attached to the xy-plane in such a way that the restoring force per unit length along $\beta(R)$ is proportional to displacement. Then, according to condition 2.79a,

$$-\left(\tau \frac{\partial z}{\partial n}\right) ds + [-kz + f(x,y,t)]ds = 0, \quad (x,y) \text{ on } \beta(R), \quad t > 0, \tag{2.81a}$$

where $k > 0$, and $f(x, y, t)$ now represents all external forces acting on $\beta(R)$ other than tension and the restoring force. This equation can be written in the equivalent form

$$\tau \frac{\partial z}{\partial n} + kz = f(x, y, t), \quad (x, y) \text{ on } \beta(R), \quad t > 0, \tag{2.81b}$$

a Robin condition.

The initial boundary value problem for displacements of the membrane is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) + \frac{F(x, y, t)}{\rho}, \quad (x, y) \text{ in } R, \quad t > 0, \tag{2.82a}$$

$$\text{Boundary conditions,} \tag{2.82b}$$

$$z(x, y, 0) = f(x, y), \quad (x, y) \text{ in } R, \tag{2.82c}$$

$$z_t(x, y, 0) = g(x, y), \quad (x, y) \text{ in } R. \tag{2.82d}$$

If boundary conditions 2.82b and external force $F(x, y, t)$ are independent of time, there may exist solutions of 2.82a,b that are also independent of time. Such solutions, called **static deflections**, satisfy Poisson's equation

$$\nabla^2 z = -\frac{F(x, y)}{\tau}, \quad (x, y) \text{ in } R, \tag{2.83}$$

and the appropriate boundary conditions. If, in addition, no external forces are present, the PDE reduces to Laplace's equation

$$\nabla^2 z = 0, \quad (x, y) \text{ in } R. \tag{2.84}$$

An important technique in solving two-dimensional wave equation 2.82a is the method of separation of variables, a method we shall deal with at length in Section 4.2. In this method it is assumed that displacement can be separated into a function of x and y multiplied by a function of time t, $z(x, y, t) = u(x, y)T(t)$. Substitution of this into equation 2.82a when $F = 0$ gives

$$u(x, y) \frac{d^2 T}{dt^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} T(t) + \frac{\partial^2 u}{\partial y^2} T(t) \right]$$

or,

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{u} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Because the left side of this equation is a function of only t and the right side is a function of x and y, it follows that each must be equal to a constant, say $-k$. Then $u(x, y)$ must satisfy

$$\frac{1}{u} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = -k,$$

or

$$\nabla^2 u + ku = 0. \tag{2.85}$$

This equation is called the two-dimensional **Helmholtz** equation. In the present context, it is also called the **reduced wave equation**. In essence, it describes the amplitude of the oscillations of each point in the membrane; $T(t)$ contains the time dependence of the vibrations. Boundary conditions for the wave equation will yield boundary conditions for the Helmholtz equation.

EXERCISES 2.4

In Exercises 1–7, set up, but do not solve, an (initial) boundary value problem for the required displacement. Assume that mass per unit area of and tension in the membrane are constant.

1. A vibrating circular membrane of radius r_1 is given initial displacement $f(r,\theta)$ and zero initial velocity. If its edge $r = r_1$ is fixed on the xy-plane, formulate an initial value problem for subsequent displacements of the membane. Assume that no external forces act on the membrane.

2. Repeat Exercise 1 except that $f(r,\theta)$ is replaced by $f(r)$.

3. A circular membrane of radius r_1 is in a static position with radial lines $\theta = 0$ and $\theta = \alpha$ clamped on the xy-plane. If the displacement of the edge $r = r_1$ is $f(\theta)$ for $0 \le \theta \le \alpha$, formulate the boundary value problem for displacement in the sector $0 < \theta < \alpha$. Would there be any restriction on $f(\theta)$?

4. Repeat Exercise 3 except that gravity acts on the membrane.

5. Repeat Exercise 1 except that gravity and a damping force proportional to velocity act on the membrane.

6. A rectangular membrane is initially ($t = 0$) at rest over the region $0 \le x \le L$, $0 \le y \le L'$ in the xy-plane. For time $t > 0$, a periodic force per unit area $\cos\omega t$ acts at all points in the membrane. If the edge of the membrane is fixed on the xy-plane, formulate an initial boundary value problem for displacements of the membrane.

7. Repeat Exercise 6 except that the boundaries $x = 0$ and $x = L$ are elastically connected to the xy-plane and the boundaries $y = 0$ and $y = L'$ are forced to exhibit motion described by $f_1(x,t)$ and $f_2(x,t)$, respectively.

8. Suppose a membrane over the region R of the xy-plane is in a static position with displacement $z(x,y)$ satisfying Poisson's equation $\nabla^2 z = -F(x,y)/\tau$. Suppose further that the boundary condition on the boundary $\beta(R)$ of R is of Neumann type, $\partial z/\partial n = f(x,y)/\tau$ for (x,y) on $\beta(R)$. Use the result of Exercise 9 in Section 2.1 (or Green's theorem) to show that $F(x,y)$ and $f(x,y)$ must satisfy the consistency condition

$$\oint_{\beta(R)} f(x,y)\,ds = \iint_R -F(x,y)\,dA.$$

What is the physical significance of this requirement?

9. A circular membrane of radius r_2 has its edge $r = r_2$ fixed on the xy-plane. If gravity and tension are the only forces acting on the membrane, what are the static deflections of points of the membrane?

10. In this exercise we replace gravity in Exercise 9 with an arbitrary (but continuous) function $f(r)$; that is, assume that the only forces acting on the membrane are tension and a force per unit area with z-component $f(r)$.
(a) What is the boundary value problem for static deflections $z(r)$ of the membrane?
(b) Show that $z'(r)$ must be of the form

$$z'(r) = -\frac{1}{\tau r}\int r f(r)\,dr.$$

(c) Express the antiderivative in part (b) as a definite integral

$$z'(r) = -\frac{1}{\tau r}\int_0^r u f(u)\,du$$

and integrate once more to find $z(r)$ in the form

$$z(r) = \frac{1}{\tau}\left[\int_0^{r_2}\int_0^v \frac{u}{v} f(u)\,du\,dv - \int_0^r\int_0^v \frac{u}{v} f(u)\,du\,dv\right].$$

(d) Interchange orders of integration to obtain

$$z(r) = \frac{1}{\tau}\left[\int_0^{r_2} u f(u)\ln\left(\frac{r_2}{u}\right)du - \int_0^r u f(u)\ln\left(\frac{r}{u}\right)du\right].$$

(e) Verify that the result in part (d) yields the solution to Exercise 9 when $f(r) = -\rho g$.
(f) Find deflections when $f(r) = k(r - r_2)$, $k > 0$ a constant.

11. The two-dimensional wave equation 2.73 was derived by applying Newton's second law to a segment of the membrane. In this exercise we use the PDE to discuss energy balance for the entire membrane.

(a) Multiply equation 2.73 by $\partial z/\partial t$ and integrate over the region R in the xy-plane onto which the membrane projects to show that

$$\iint_R \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial t}\right)^2 dA = c^2\iint_R \frac{\partial z}{\partial t}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)dA + \iint_R \frac{F(x,y,t)}{\rho}\frac{\partial z}{\partial t}dA.$$

(b) Verify that for z a function of x, y, and t,

$$\frac{\partial z}{\partial t}\nabla^2 z = \nabla\cdot\left(\frac{\partial z}{\partial t}\nabla z\right) - \frac{1}{2}\frac{\partial}{\partial t}|\nabla z|^2,$$

and use this identity together with Green's theorem to rewrite the result in part (a) in the form

$$\frac{1}{2}\iint_R\left[\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)^2 + c^2\frac{\partial}{\partial t}|\nabla z|^2\right]dA = c^2\oint_{\beta(R)}\frac{\partial z}{\partial t}\frac{\partial z}{\partial n}ds + \iint_R \frac{F(x,y,t)}{\rho}\frac{\partial z}{\partial t}dA.$$

(c) Integrate the result in part (b) with respect to time from $t = 0$ to an arbitrary t to obtain

$$\iint_R\left[\frac{\rho}{2}\left(\frac{\partial z}{\partial t}\right)^2 + \frac{\tau}{2}|\nabla z|^2\right]dA = \iint_R\left[\frac{\rho}{2}\left(\frac{\partial z(x,y,0)}{\partial t}\right)^2 + \frac{\tau}{2}|\nabla z(x,y,0)|^2\right]dA$$
$$+\int_0^t\oint_{\beta(R)}\left(\tau\frac{\partial z}{\partial n}\right)\frac{\partial z}{\partial t}ds\,dt + \int_0^t\iint_R F(x,y,t)\frac{\partial z}{\partial t}dA\,dt.$$

Interpret each term in this result physically, and hence obtain a physical interpretation of the equation as a whole. It is often called the **energy equation** for the membrane.

§2.5 Transverse Vibrations of Beams

In this section we study vertical oscillations of horizontal beams (Figure 2.24). It is assumed that the beam is symmetric about the xy-plane and that all cross sections (which would be plane in the absence of any loading) remain plane during vibrations. Displacements are then described by the position $y(x,t)$ of the neutral axis.

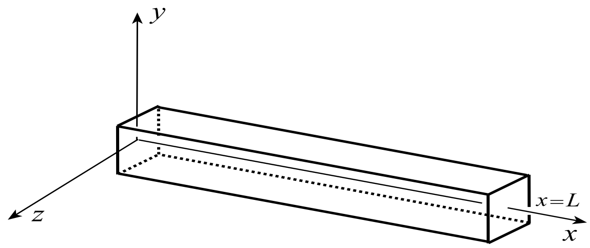


Figure 2.24

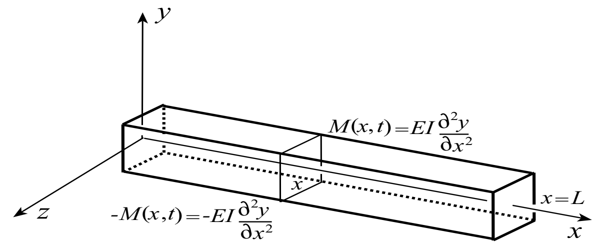


Figure 2.25

Stretches and compressions in various parts of the beam lead to internal forces and moments. It has been shown experimentally that the bending moment $M(x,t)$ on the right face of the cross section of the beam at position x due to the rest of the beam to its right is related to the signed curvature $\kappa(x,t)$ of the neutral axis by the equation

$$M = EI\kappa, \tag{2.86}$$

where $E = E(x) > 0$ is Young's modulus of elasticity (depending on the material in the beam) and $I = I(x)$ is the moment of inertia of the cross section of the beam (Figure 2.25). It is shown in elementary calculus that

$$\kappa = \frac{\partial^2 y/\partial x^2}{\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{3/2}}, \tag{2.87}$$

but if we assume that vibrations produce only small slopes, then $\partial y/\partial x << 1$, and we may take

$$\kappa = \frac{\partial^2 y}{\partial x^2}. \tag{2.88}$$

Consequently, for vibrations producing small slopes, bending moments are related to curvature by

$$M(x,t) = EI\frac{\partial^2 y}{\partial x^2}. \tag{2.89}$$

Since $\partial^2 y/\partial x^2$ is positive when the beam is concave upward (as in Figure 2.25), it follows that M must be positive on the right face for the direction shown. The moment on the left face of the same cross section due to the material in the beam to its left is therefore $-M(x,t) = -EI\partial^2 y/\partial x^2$.

Shear forces also act on any cross section. We denote by $Q(x,t)$ the y-component of the shear force acting on the right face of the cross section at position x due to that part of the beam to its right. Then, $-Q(x,t)$ is the shear force acting on the left face. Shear and bending moments are related by the equation

$$Q(x,t) = -\frac{\partial M(x,t)}{\partial x}. \tag{2.90}$$

Vibrations of the beam are determined by the interactions of the internal bending moments and shear forces with the exterior loading $w(x,t)$ per unit x-length (including the weight of the beam) and all external forces $F(x,t)$ per unit x-length (including loading). The function $w(x,t)$ is the load per unit length which we take as positive, while $F(x,t)$, the y-component of all external forces, may be positive, negative, or zero. To describe these interactions, we apply Newton's second law to the vertical translational motion of the segment of the beam in Figure 2.26:

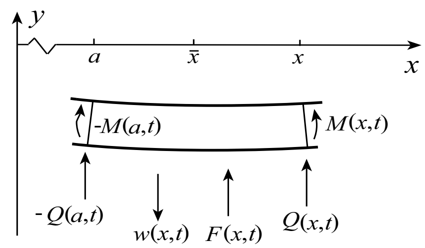


Figure 2.26

$$\frac{\partial}{\partial t}\left[\int_a^x \frac{\partial y(\zeta,t)}{\partial t}\frac{w}{g}d\zeta\right] = \int_a^x F(\zeta,t)\,d\zeta + Q(x,t) - Q(a,t), \quad (g>0). \tag{2.91}$$

The integral on the left is the momentum of the segment; $wd\zeta/g$ is the mass on an element of the beam of length $d\zeta$ along the x-axis, and multiplication by velocity $\partial y(\zeta,t)/\partial t$ gives momentum. The integral on the right is the sum of all external forces on the segment, and $Q(x,t)$ and $Q(a,t)$ are the shear forces on the faces at x and a, respectively. Differentiation of this equation with respect to x gives

$$\frac{\partial}{\partial t}\left(\frac{w}{g}\frac{\partial y}{\partial t}\right) = F(x,t) + \frac{\partial Q}{\partial x}. \tag{2.92}$$

Substitutions for $\partial Q/\partial x$ and M from equations 2.90 and 2.89 yield the PDE satisfied by transverse vibrations of the beam:

$$\frac{\partial}{\partial t}\left(\frac{w}{g}\frac{\partial y}{\partial t}\right) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = F(x,t). \tag{2.93}$$

When E and I are independent of x, and $w(x,t)$ is independent of t, the PDE can be written in the simplified form

$$\frac{w}{EIg}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = \frac{F}{EI}. \tag{2.94}$$

In many applications, the internal forces in the beam are so large that the effect of F is negligible. In such cases, PDE 2.94 may be replaced by the *homogeneous** equation

$$\frac{w}{EIg}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = 0. \tag{2.95}$$

* A general definition for a homogeneous PDE is given in Section 4.1.

This is illustrated in Exercise 5, where it is shown that when $F(x)$ is due only to the weight of the beam itself, static deflections are small.

Accompanying PDEs 2.94 or 2.95 will be two initial conditions that describe the displacement and velocity of the beam at some initial time (usually $t = 0$):

$$y(x,0) = f(x), \quad 0 < x < L, \tag{2.96a}$$
$$y_t(x,0) = g(x), \quad 0 < x < L. \tag{2.96b}$$

Various types of boundary conditions may exist at each end of the beam. If the end $x = 0$ is **simply-supported** (Figure 2.27), displacement and curvature (moment) there are both zero:

$$y(0,t) = 0, \qquad \frac{\partial^2 y(0,t)}{\partial x^2} = 0. \tag{2.97}$$

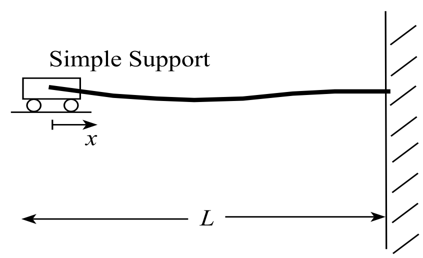


Figure 2.27

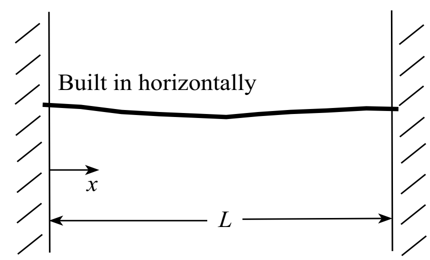


Figure 2.28

If this end is **built in horizontally** (Figure 2.28), displacement and slope vanish:

$$y(0,t) = 0, \qquad \frac{\partial y(0,t)}{\partial x} = 0. \tag{2.98}$$

Finally, if this end is **free** or **cantilevered** (Figure 2.29), curvature and shear are both zero:

$$\frac{\partial^2 y(0,t)}{\partial x^2} = 0, \qquad \frac{\partial^3 y(0,t)}{\partial x^3} = 0. \tag{2.99}$$

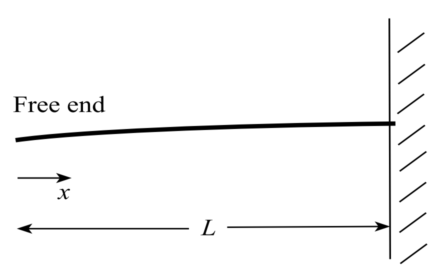


Figure 2.29

Similar conditions exist at the end $x = L$.

When $F(x,t)$ and the boundary conditions are independent of time, there may exist solutions of the initial boundary value problem that are also independent of time, called **static deflections**. They satisfy the ordinary differential equation

$$\frac{d^4 y}{dx^4} = \frac{F(x)}{EI}, \tag{2.100}$$

and the appropriate boundary conditions. When the beam lies on an elastic foundation, and the only other force acting on the beam is gravity, then $F = w - ky$, and this differential equation becomes

$$\frac{d^4y}{dx^4} = \frac{1}{EI}(w - ky). \tag{2.101}$$

EXERCISES 2.5

In Exercises 1–4, set up, but do not solve, an (initial) boundary value problem for the required displacement. Assume that Young's modulus E and the moment of inertia I of the cross section of the beam are constant.

1. A horizontal beam of length L has flat ends at $x = 0$ and $x = L$. At time $t = 0$, it is at rest but its neutral axis is deflected according to the function $f(x)$, $0 \le x \le L$. It is then released from this position. The left end of the beam is built in horizontally, and the right end is free.

2. Repeat Exercise 1 except that both ends are simply supported on the x-axis.

3. Repeat Exercise 1 except that a mass m is distributed uniformly along the beam and both ends are built in horizontally.

4. A beam of length L is clamped horizontally at $x = 0$ and is cantilevered at $x = L$. For time $t < 0$, it is deflected, but motionless, under a downward force of magnitude F at $x = L$ and its own weight. At time $t = 0$, this force is removed. (Hint: In the static situation, the boundary conditions at $x = L$ are $y''(L) = 0$ and $y'''(L) = F/(EI)$.)

5. In this exercise we discuss deflections and forces for a typical static beam.

(a) What is the boundary value problem for static deflections of a beam of length L, simply supported at both ends? Solve this problem when the external force is constant.

(b) Suppose now that F is due only to the weight of the beam itself. Find the maximum deflection of the beam using the following data: $E = 2.1 \times 10^{11}$ N/m^2, $\rho = 7.85 \times 10^3$ kg/m^3, $L = 5$ m, $I = 6.5$ kg·m^2, Cross-sectional area $= 0.02$ m^2.

(c) What constant force (per unit x-length) over the beam would create a maximum deflection of 1 cm? How large is this compared with the weight per unit length of the beam?

6. Show that when the ends of the beam in Exercise 5 are clamped horizontally, the maximum deflection is only one-fifth that for the simply supported beam.

7. Solve equation 2.101 for static deflections of a beam of length L on an elastic foundation when both ends are simply-supported.

§2.6 Electrostatic Potential

When two positive point charges Q and q are r units apart in free space, Coulomb's law states that each repels the other with a force whose magnitude is

$$F = \frac{qQ}{4\pi\epsilon_0 r^2}, \tag{2.102}$$

where ϵ_0 is the permittivity of free space. The force on unit charge q due to Q is called the **electric field intensity**

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^3}\mathbf{r}, \tag{2.103}$$

where $\mathbf{r}$ is the vector from Q to $q = 1$ (Figure 2.30). It is straightforward to show that the curl and divergence of this vector field vanish:

$$\nabla \times \mathbf{E} = \mathbf{0}, \tag{2.104a}$$
$$\nabla \cdot \mathbf{E} = 0. \tag{2.104b}$$

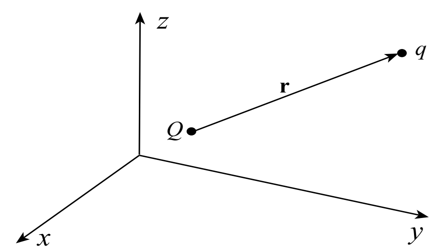


Figure 2.30

A vanishing curl implies the existence (in a suitably defined domain not containing Q) of a potential function V satisfying

$$\mathbf{E} = -\nabla V. \tag{2.105}$$

Combine this with property 2.104b, and we find that V must satisfy Laplace's equation

$$\nabla^2 V = 0. \tag{2.106}$$

For such a simple charge distribution, it is easily shown (from 2.105) that to an additive constant,

$$V = \frac{Q}{4\pi\epsilon_0 r}. \tag{2.107}$$

When Q is replaced by a distribution of charge with density σ in some region of free space (or other medium), determination of a potential function is more complex. In this case we appeal to Maxwell's equations, which govern all electromagnetic fields. In a static situation, Maxwell's equations still require the electric field intensity equation $\mathbf{E}$ to satisfy 2.104a, in which case the potential function V associated with the field is once again defined by 2.105. Unfortunately, however, we do not know $\mathbf{E}$ (as we did for the point charge) and therefore cannot solve equation 2.105 for V. To find an equation determining V that does not contain $\mathbf{E}$, we use another of Maxwell's equations that requires the electric field displacement $\mathbf{D}$ to satisfy

$$\nabla \cdot \mathbf{D} = \sigma \tag{2.108}$$

at each point in the medium. When the medium is isotropic with constant permittivity ϵ, then $\mathbf{D}$ and $\mathbf{E}$ are related by

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{2.109}$$

and hence $\mathbf{E}$ must satisfy

$$\nabla \cdot \mathbf{E} = \frac{\sigma}{\epsilon}. \tag{2.110}$$

Between equations 2.105 and 2.110 we may eliminate $\mathbf{E}$, the result being Poisson's equation

$$\nabla^2 V = -\frac{\sigma}{\epsilon}. \tag{2.111}$$

In other words, the electrostatic potential function V associated with an electrostatic field $\mathbf{E}$ must satisfy Poisson's equation 2.111 at every point interior to the charge distribution. At points outside the charge distribution, σ vanishes and V satisfies Laplace's equation 2.106.

Partial differential equations 2.106 and 2.111 are not, by themselves, sufficient to determine V. It is necessary to specify boundary conditions as well. A Dirichlet boundary condition specifies $V(x, y, z)$ on the bounding surface $\beta(R)$ of the medium:

$$V(x, y, z) = f(x, y, z), \quad (x, y, z) \text{ on } \beta(R), \tag{2.112}$$

where $f(x, y, z)$ is a given function. A Neumann boundary condition prescribes the directional derivative of $V(x, y, z)$ normal to the bounding surface:

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}} = f(x, y, z), \quad (x, y, z) \text{ on } \beta(R), \tag{2.113}$$

where $\hat{\mathbf{n}}$ is the unit outward normal to $\beta(R)$. Since $\nabla V = -\mathbf{E}$, it follows that specification of the electrostatic force on a bounding surface yields a Neumann boundary condition. If a bounding surface is free of electrostatic forces, it satisfies a homogeneous Neumann boundary condition.

A Robin boundary condition is a linear combination of a Dirichlet and a Neumann condition:

$$l\frac{\partial V}{\partial n} + hV = f(x, y, z), \quad (x, y, z) \text{ on } \beta(R). \tag{2.114}$$

Dirichlet and Neumann boundary conditions are obtained by setting l and h equal to zero, respectively.

EXERCISES 2.6

In Exercises 1–2, set up, but do not solve, a boundary value problem for the required potential.

1. Region R in space is bounded by the planes $x = 0$, $y = 0$, $x = L$, and $y = L'$. If the planes $y = 0$ and $x = 0$ are held at zero potential, whereas $x = L$ and $y = L'$ are maintained at a potential of 100, what is the boundary value problem for potential in R?

2. Repeat Exercise 1 except that a uniform charge (with density σ) is spread over the volume $L/4 \le x \le 3L/4$, $L'/4 \le y \le 3L'/4$.

3. A region R of space has a subregion $\overline{R}$ occupied by charge with density $\sigma(x, y, z)$ coulombs per cubic metre, assumed continuous (figure below). Consider the function $V(x, y, z)$ defined by

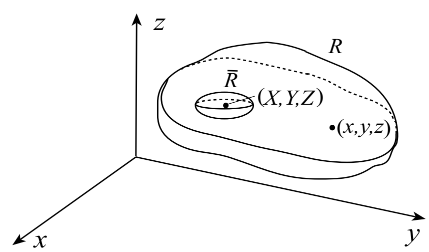
$$V(x, y, z) = \iiint_{\overline{R}} \frac{\sigma(X, Y, Z)}{4\pi\epsilon_0 \sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}} dZ\, dY\, dX.$$

Coordinates (X, Y, Z) identify points in $\overline{R}$.

(a) When (x, y, z) is in R but not in $\overline{R}$, $V(x, y, z)$ is clearly well defined. By using spherical coordinates originating at (x, y, z) for integration variables, show that when (x, y, z) is in $\overline{R}$, the improper integral converges. In other words, $V(x, y, z)$ is well defined throughout all R.

(b) By interchanging the order of differentiations with respect to x, y, and z and integrations with respect to X, Y, and Z, show that when (x, y, z) is in R, but not in $\overline{R}$, $V(x, y, z)$ satisfies Laplace's equation 2.106.

To prove that $V(x, y, z)$ satisfies Poisson's equation 2.111 when (x, y, z) is in $\overline{R}$ requires the theory of *generalized functions*. Parts of this theory are introduced in Chapters 12 and 13, but the development is not carried far enough to permit verification of the integral as a solution to Poisson's equation. This is not really a problem, however, because the integral representation of $V(x, y, z)$ is of limited utility anyway. Seldom can the integral be evaluated in closed form. In addition, the integral does not take into account any boundary conditions that may be present, and there is no obvious way to modify the integral in order to encompass boundary conditions.



§2.7 General Solutions of Partial Differential Equations

When boundary and/or initial conditions accompany an ODE, we often find a general solution and then use the subsidiary conditions to determine the arbitrary constants. This procedure seldom works for PDEs. Arbitrary constants in ODEs are replaced by arbitrary functions in PDEs, and to use initial and/or boundary conditions to determine these functions is usually impossible. We give one very simple, but important, example to illustrate the direction the analysis might take in using a general solution for a PDE to solve an initial boundary value problem. The one-dimensional vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.115a}$$

$$y(x,0) = f(x), \quad -\infty < x < \infty, \tag{2.115b}$$

$$y_t(x,0) = g(x), \quad -\infty < x < \infty, \tag{2.115c}$$

describes free oscillations of an infinitely long taut string with initial displacement $f(x)$ and initial velocity $g(x)$. By changing independent variables according to $\nu = x + ct$ and $\eta = x - ct$ and denoting $y[x(\nu,\eta), t(\nu,\eta)]$ by $w(\nu,\eta)$, wave equation 2.115a is replaced by

$$\frac{\partial^2 w}{\partial \nu \partial \eta} = 0 \tag{2.116}$$

(see Exercise 1 for details). A general solution of this PDE is

$$w(\nu,\eta) = F(\nu) + G(\eta), \tag{2.117}$$

where F and G are arbitrary but continuous functions with continuous first derivatives. As a result, a general solution of PDE 2.115a is

$$y(x,t) = F(x+ct) + G(x-ct). \tag{2.118}$$

It now remains to determine the exact form of these functions. Application of initial conditions 2.115b,c requires

$$f(x) = F(x) + G(x), \quad -\infty < x < \infty,$$
$$g(x) = cF'(x) - cG'(x), \quad -\infty < x < \infty.$$

When the first of these is differentiated with respect to x and combined with the second,

$$F'(x) = \frac{1}{2c}[cf'(x) + g(x)]$$

and therefore

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\zeta)\,d\zeta + D, \qquad G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\zeta)\,d\zeta - D,$$

where D is an arbitrary constant. When x is replaced by $x + ct$ in $F(x)$ and by $x - ct$ in $G(x)$, we obtain

$$y(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\zeta)\,d\zeta + D$$
$$+ \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(\zeta)\,d\zeta - D$$
$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta. \qquad (2.119)$$

This is called **d'Alembert's solution** of initial value problem 2.115. As was stated earlier, this is a particularly simple example, and analyses of this type are not usually possible. For this reason, it is unusual to solve initial boundary value problems by finding a general solution for the PDE and attempting to use initial and/or boundary conditions to determine the arbitrary functions. More direct methods must be devised.

Notwithstanding the fact that general solutions of PDEs are seldom of use in solving initial boundary value problems, d'Alembert's solution 2.119 of problem 2.115 provides considerable insight into the behaviour of vibrating strings that are free of external forces. Consider first a taut string that at time $t = 0$ is released from rest ($g(x) = 0$) from the position in Figure 2.31a ($f(x) = 0$ for $|x - 1/2| \geq 1/16$). This is not a realistic initial displacement in view of the assumptions in Section 2.3 that displacements and slopes must be small. But because our discussion is independent of $f(x)$, we have purposely exaggerated the initial shape in order that our graphical representations be unmistakable. According to d'Alembert's solution 2.119, subsequent displacements of the string are defined by

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)], \qquad (2.120)$$

and it is quite simple to obtain a pictorial history of the string using this function. For any given time t, the graph of $f(x+ct)/2$ is one-half that of $f(x)$ translated ct units to the left; $f(x-ct)/2$ is one-half of $f(x)$ shifted ct units to the right. The position of the string at this particular time is the sum of these two graphs. We have shown this procedure for the times $t = 1/(64c)$, $1/(32c)$, $3/(64c)$, $1/(16c)$, and $1/(8c)$ in Figures 2.31b–f, respectively. The dotted curves represent $f(x+ct)/2$ and $f(x-ct)/2$, and the solid curve $y(x,t)$.

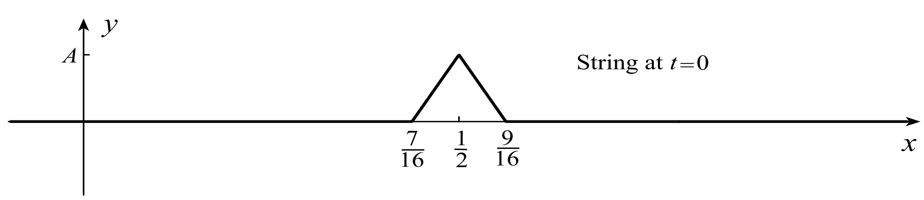


Figure 2.31a

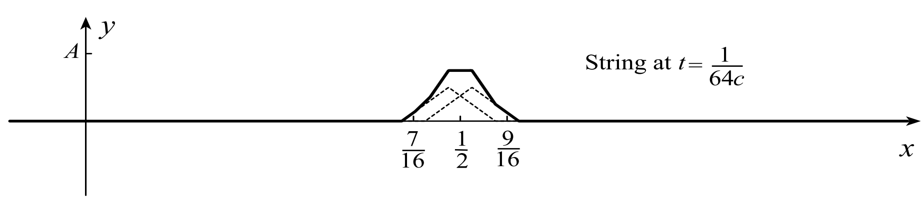


Figure 2.31b

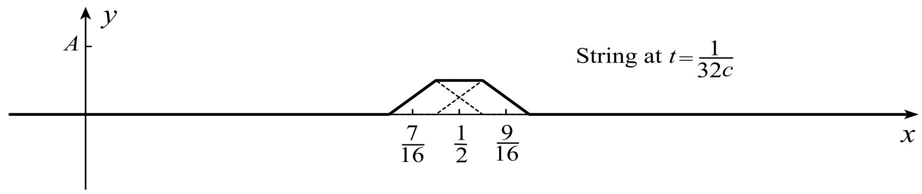


Figure 2.31c

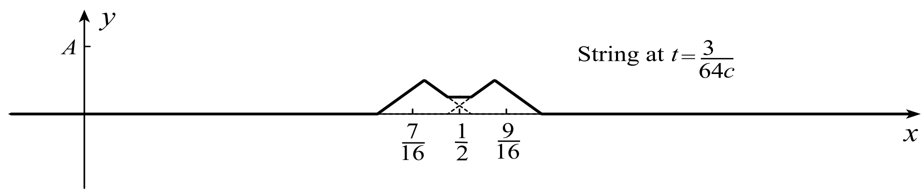


Figure 2.31d

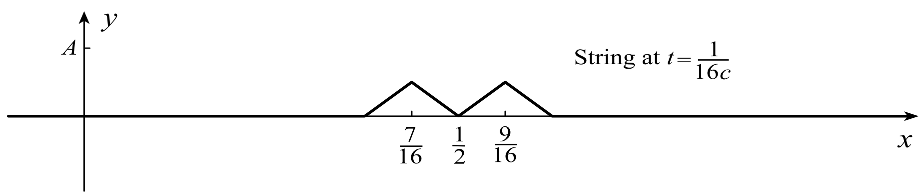


Figure 2.31e

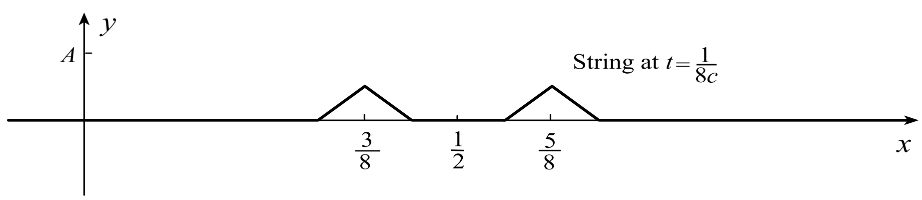


Figure 2.31f

What is most interesting is that these graphs suggest the following physical description for the motion of the string. The initial deflection $f(x)$ of the string divides into two parts, each equal to one-half of $f(x)$, one traveling to the left with velocity $-c$ and the other traveling to the right with velocity c. At first they interfere with each another, but at time $t = 1/(16c)$, they separate and travel in opposite directions along the string.

Consider now the situation in which the string is given a nonzero initial velocity $g(x)$, but no initial displacement, $f(x) = 0$. In this case, equation 2.119 yields

$$y(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(\zeta)\, d\zeta \qquad (2.121a)$$

as the displacement of the string at position x and time t. Suppose, for example, that

$$g(x) = \begin{cases} 0, & x \le 7/16, \\ k, & 7/16 < x < 9/16, \\ 0, & x \ge 9/16 \end{cases}$$

where $k > 0$ is a constant (Figure 2.52). (Think of only that part of the string $7/16 < x < 9/16$ being struck by a hammer.)

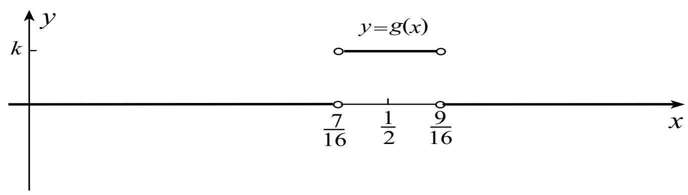


Figure 2.52

Suppose we denote by $G(x)$ the antiderivative

$$G(x) = \frac{1}{2c}\int_0^x g(\zeta)\,d\zeta,$$

the graph of which is shown in Figure 2.33.

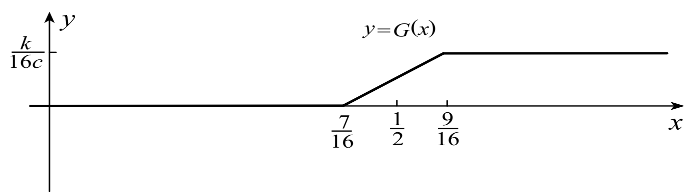


Figure 2.33

Displacement of the string can be expressed in the form

$$y(x,t) = G(x+ct) - G(x-ct). \tag{2.121b}$$

Because of the negative sign, it is the destructive combination of the left-traveling wave $G(x+ct)$ and the right traveling wave $G(x-ct)$. Results are shown for various times in Figures 2.34a–e.

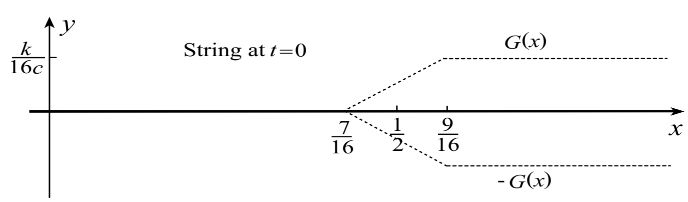


Figure 2.34a

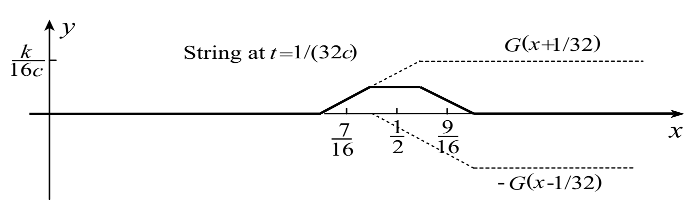


Figure 2.34b

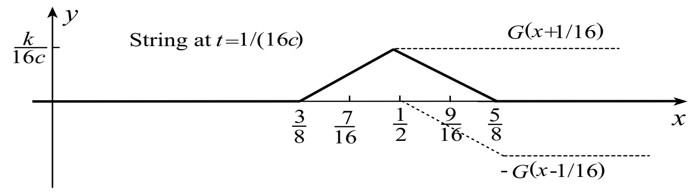


Figure 2.34c

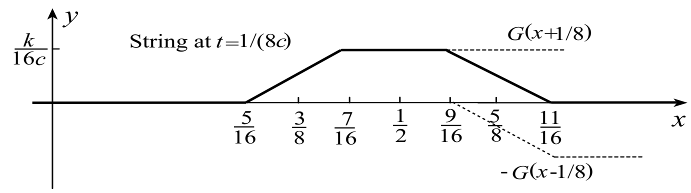


Figure 2.34d

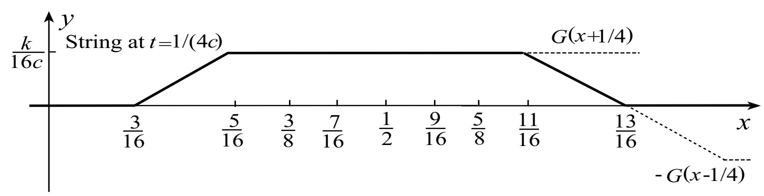


Figure 2.34e

Each point moves from its initial position on the x-axis to a stationary position $k/(16c)$ units above the x-axis.

When a string has both an initial displacement $f(x)$ and an initial velocity $g(x)$, graphical techniques may still be used to determine the solution of problem 2.115. We express $y(x,t)$ in the form $y(x,t) = u(x,t) + v(x,t)$, where $u(x,y)$ and $v(x,t)$ satisfy the problems

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, & \qquad \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial x^2}, \\ u(x,0) &= f(x), & v(x,0) &= 0, \\ u_t(x,0) &= 0; & v_t(x,0) &= g(x). \end{aligned}$$

We began this section suggesting how discussions might ensue were we to attempt to find a general solution of a PDE and then use initial and boundary data to determine functions in the general solution. It is important to point out that what we have done here with the one-dimensional wave equation is not possible with other PDEs such as the heat equation and Laplace's equation; we should not expect to repeat similar discussions in other applications. On the other hand, these discussions of the wave equation have shed light on some of the properties of vibrations problems, and with the material from Section 2.8, we will be able to continue discussions in Sections 2.9–2.11, and obtain further insight into solutions of the wave equation. In particular, we shall see how to modify the above discussion in the case of a single boundary condition for a semi-infinite string and a pair of boundary conditions for a finite string.

EXERCISES 2.7

1. Show that the transformation of independent variables $\nu = x + ct$ and $\eta = x - ct$ replaces wave equation 2.115a with 2.116.

2. Use the graphical techniques of this section to determine the displacements of an infinite string with zero initial velocity and initial displacement

$$f(x) = \begin{cases} 0, & x < 0 \\ x/8, & 0 \le x \le 1/2 \\ (1-x)/8, & 1/2 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

at the times (a) $t = 1/(8c)$, (b) $t = 1/(4c)$, (c) $t = 3/(8c)$, (d) $t = 1/(2c)$, (e) $t = 5/(8c)$.

3. Repeat Exercise 2 with $f(x) = \begin{cases} 0, & x < 0 \\ \sin(2\pi x), & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$.

4. Use the graphical techniques of this section to determine the displacements of an infinite string with zero initial displacement and initial velocity

$$g(x) = \begin{cases} 0, & x < 1/4 \\ 1, & 1/4 < x < 3/4 \\ 0, & x > 3/4. \end{cases}$$

for the times in Exercise 2.

5. Repeat Exercise 4 with

$$g(x) = \begin{cases} 0, & x < 1/8 \\ 1, & 1/8 < x < 3/8 \\ 0, & 3/8 < x < 5/8 \\ 1, & 5/8 < x < 7/8 \\ 0, & x > 7/8. \end{cases}$$

§2.8 Classification of Second-Order Partial Differential Equations

The material in this section is not essential at this point in our discussions. It can be considered at any time, since it is neither a prerequisite for subsequent discussions nor dependent on them. We include it here because it provides justification for the approach that we take in the remainder of the book. We intend solving the initial boundary value problems in Sections 2.2–2.6 using the techniques of separation of variables; Fourier transforms, finite and infinite; Laplace transforms; and Green's functions. Second-order PDEs play a prominent role in these problems; the only application we have seen so far that gives rise to a PDE that is not second order is that for beam vibrations. What we illustrate here is that all *linear* second-order PDEs (we define this term shortly) are basically of three types. These types correspond generally to Poisson's equation, the wave equation, and the heat conduction equation. Consequently, once we learn how to apply the above techniques to these three equations, we have essentially learned how to handle all second-order linear equations.

For purposes of classification, it is not necessary to restrict consideration to *linear* equations, although it is worth noting that a second-order PDE in $u(x,y)$ is **linear** if it can be represented in the form

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y). \quad (2.122)$$

The classification that we develop here is also valid for more general equations; in particular, equations of the form

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y,u,u_x,u_y), \quad (2.123)$$

where f is any function of its arguments. Such equations are linear in the second derivatives only. It is assumed that a, b, and c have continuous first partial deriatives in some domain D and that these coefficient functions do not all vanish simultaneously. We classify such PDEs into one of three types — *elliptic*, *parabolic*, or *hyperbolic* — and each type of PDE displays characterisitics quite distinct from the others. This classification is stated as follows.

Definition 2.2 Partial differential equation 2.123 is said to be **hyperbolic** at a point (x,y) if

$$b^2 - 4ac > 0; \quad (2.124a)$$

parabolic if

$$b^2 - 4ac = 0; \quad (2.124b)$$

and **elliptic** if

$$b^2 - 4ac < 0. \quad (2.124c)$$

This classification is a pointwise one so that a PDE may change its type from point to point. The one-dimensional wave equation 2.44 is hyperbolic at all points; the one-dimensional heat conduction equation in Example 2.3 in Section 2.2 is parabolic; and Poisson's equation 2.6a is elliptic.

Example 2.6 Determine points at which the PDE

$$x^2\, u_{xx} - xy\, u_{xy} + y\, u_{yy} = xyu^2 + 3u_x$$

is hyperbolic, parabolic, and elliptic.

Solution We calculate
$b^2 - 4ac = (-xy)^2 - 4(x^2)(y) = x^2y(y-4).$
The PDE is therefore hyperbolic in the region $y > 4$, $x > 0$, the region $y > 4$, $x < 0$, the region $y < 0$, $x > 0$, and the region $y < 0$, $x < 0$. It is elliptic for $0 < y < 4$, $x > 0$ and $0 < y < 4$, $x < 0$. It is parabolic on the lines $x = 0$, $y = 0$ and $y = 4$ (Figure 2.35).•

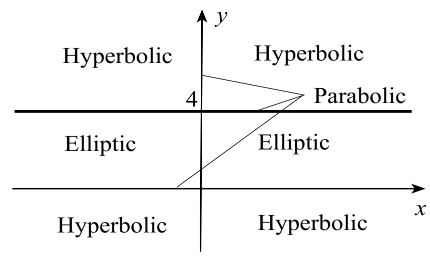


Figure 2.35

When PDE 2.123 is of the same type at every point in a domain D, we show that by means of a change of independent variables

$$\nu = \nu(x,y), \qquad \eta = \eta(x,y), \tag{2.125}$$

the PDE can be transformed into a simpler form. We require that functions $\nu(x,y)$ and $\eta(x,y)$ have continuous second partial derivatives in D and that the Jacobian (determinant)

$$\frac{\partial(\nu,\eta)}{\partial(x,y)} = \begin{vmatrix} \nu_x & \nu_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \tag{2.126}$$

in order that the original variables x and y be retrievable:

$$x = x(\nu,\eta), \qquad y = y(\nu,\eta). \tag{2.127}$$

When we replace x and y by ν and η, we denote the dependent variable by $w(\nu,\eta) = u[x(\nu,\eta), y(\nu,\eta)]$. It follows, then, that $u(x,y) = w[\nu(x,y), \eta(x,y)]$, and chain rules for partial deriatives permit us to express derivatives of $u(x,y)$ with respect to x and y in terms of derivatives of w with respect to ν and η:

$$u_x = w_\nu \nu_x + w_\eta \eta_x, \qquad u_y = w_\nu \nu_y + w_\eta \eta_y,$$

and

$$\begin{aligned}
u_{xx} &= (w_{\nu\nu}\nu_x + w_{\nu\eta}\eta_x)\nu_x + w_\nu \nu_{xx} + (w_{\eta\nu}\nu_x + w_{\eta\eta}\eta_x)\eta_x + w_\eta \eta_{xx},\\
u_{xy} &= (w_{\nu\nu}\nu_y + w_{\nu\eta}\eta_y)\nu_x + w_\nu \nu_{xy} + (w_{\eta\nu}\nu_y + w_{\eta\eta}\eta_y)\eta_x + w_\eta \eta_{xy},\\
u_{yy} &= (w_{\nu\nu}\nu_y + w_{\nu\eta}\eta_y)\nu_y + w_\nu \nu_{yy} + (w_{\eta\nu}\nu_y + w_{\eta\eta}\eta_y)\eta_y + w_\eta \eta_{yy}.
\end{aligned}$$

The PDE in w as a function of ν and η equivalent to equation 2.123 is therefore

$$\begin{aligned}
&(a{\nu_x}^2 + b\nu_x\nu_y + c{\nu_y}^2)w_{\nu\nu} + [2a\nu_x\eta_x + b(\nu_x\eta_y + \nu_y\eta_x) + 2c\nu_y\eta_y]w_{\nu\eta}\\
&\quad + (a{\eta_x}^2 + b\eta_x\eta_y + c{\eta_y}^2)w_{\eta\eta} + (a\nu_{xx} + b\nu_{xy} + c\nu_{yy})w_\nu\\
&\quad + (a\eta_{xx} + b\eta_{xy} + c\eta_{yy})w_\eta = f[x(\nu,\eta), y(\nu,\eta), w, w_\nu\nu_x + w_\eta\eta_x, w_\nu\nu_y + w_\eta\eta_y]
\end{aligned}$$

or

$$\alpha w_{\nu\nu} + \beta w_{\nu\eta} + \gamma w_{\eta\eta} = F(\nu, \eta, w, w_\nu, w_\eta), \tag{2.128a}$$

where

$$\begin{aligned} \alpha &= a{\nu_x}^2 + b\nu_x\nu_y + c{\nu_y}^2, \\ \beta &= 2a\nu_x\eta_x + b(\nu_x\eta_y + \nu_y\eta_x) + 2c\nu_y\eta_y, \\ \gamma &= a{\eta_x}^2 + b\eta_x\eta_y + c{\eta_y}^2, \end{aligned} \tag{2.128b}$$

and

$$\begin{aligned} F(\nu, \eta, w, w_\nu, w_\eta) = {} & f[x(\nu, \eta), y(\nu, \eta), w, w_\nu\nu_x + w_\eta\eta_x, w_\nu\nu_y + w_\eta\eta_y] \\ & - (a\nu_{xx} + b\nu_{xy} + c\nu_{yy}w_x)w_\nu - (a\eta_{xx} + b\eta_{xy} + c\eta_{yy}w_\eta)w_\eta. \end{aligned} \tag{2.128c}$$

It is a simple exercise to show that

$$\beta^2 - 4\alpha\gamma = (b^2 - 4ac)\left[\frac{\partial(\nu, \eta)}{\partial(x, y)}\right]^2, \tag{2.129}$$

a result that proves that our classification of PDEs is invariant under a real transformation of independent variables.

We now show that when PDE 2.123 is hyperbolic at every point (x, y) in D, then it can be transformed into the **canonical form**

$$w_{\nu\eta} = F(\nu, \eta, w, w_\nu, w_\eta); \tag{2.130a}$$

when it is parabolic at every point of D, it can be transformed into the canonical form

$$w_{\nu\nu} = F(\nu, \eta, w, w_\nu, w_\eta); \tag{2.130b}$$

and when it is elliptic, it can be transformed into the canonical form

$$w_{\nu\nu} + w_{\eta\eta} = F(\nu, \eta, w, w_\nu, w_\eta). \tag{2.130c}$$

Hyperbolic Equations

For hyperbolic PDEs, we claim the existence of a transformation 2.125 that reduces the PDE to canonical form 2.130a. This is possible if functions $\nu(x, y)$ and $\eta(x, y)$ can be found to satisfy

$$0 = \alpha = a{\nu_x}^2 + b\nu_x\nu_y + c{\nu_y}^2, \qquad 0 = \gamma = a{\eta_x}^2 + b\eta_x\eta_y + c{\eta_y}^2, \tag{2.131a}$$

or,

$$0 = a\left(\frac{\nu_x}{\nu_y}\right)^2 + b\left(\frac{\nu_x}{\nu_y}\right) + c, \qquad 0 = a\left(\frac{\eta_x}{\eta_y}\right)^2 + b\left(\frac{\eta_x}{\eta_y}\right) + c; \tag{2.131b}$$

that is; the ratios ν_x/ν_y and η_x/η_y must satisfy the equation

$$a\lambda^2 + b\lambda + c = 0. \tag{2.132}$$

Since $b^2-4ac>0$, there are two distinct solutions, $\lambda_1=\lambda_1(x,y)$ and $\lambda_2=\lambda_2(x,y)$, of this quadratic equation. Consequently, when function $\nu(x,y)$ and $\eta(x,y)$ satisfy the first-order PDEs

$$\nu_x=\lambda_1(x,y)\nu_y, \qquad \eta_x=\lambda_2(x,y)\eta_y, \tag{2.133}$$

the PDE in w as a function of ν and η is reduced to the form

$$\beta w_{\nu\eta}=F(\nu,\eta,w,w_\nu,w_\eta). \tag{2.134}$$

With $\alpha=\gamma=0$, it follows that $\beta^2-4\alpha\gamma=(b^2-4ac)[\partial(\nu,\eta)/\partial(x,y)]^2\neq 0$. Consequently, we may divide by β and obtain the canonical form 2.130a for a hyperbolic PDE.

Because of the form of PDEs 2.133, solutions can be obtained with ODEs. Indeed, suppose the ordinary differential equation

$$\frac{dy}{dx}=-\lambda_1(x,y) \tag{2.135}$$

has a solution defined implicitly by

$$\nu(x,y)=C_1. \tag{2.136}$$

Then each curve in this one-parameter family has slope defined by

$$\nu_x+\nu_y\frac{dy}{dx}=0 \quad\Longrightarrow\quad \frac{dy}{dx}=-\frac{\nu_x}{\nu_y}. \tag{2.137}$$

Thus, when ODE 2.135 is solved in form 2.136, function $\nu(x,y)$ satisfies the PDE

$$\frac{\nu_x}{\nu_y}=\lambda_1(x,y). \tag{2.138}$$

The curves defined implicitly by equation 2.136 are called **characteristic curves** for the hyperbolic PDE; they are determined by the coefficients a, b, and c in the equation.

(For those who have read Chapter 1, equation 2.135 follows naturally. Characteristic equations for the first of PDEs 2.133 are

$$dx=\frac{dy}{\lambda_1(x,y)}, \quad d\nu=0, \qquad\Longrightarrow\qquad \frac{dy}{dx}=-\lambda_1(x,y).$$

Characteristic curves here are base C-curves in Chapter 1.)

Similarly, the ODE

$$\frac{dy}{dx}=-\lambda_2(x,y) \tag{2.139}$$

defines another family of characteristic curves

$$\eta(x,y)=C_2, \tag{2.140}$$

and $\eta(x,y)$ is a solution of $\eta_x=\lambda_2(x,y)\eta_y$.

Each of the families $\nu(x,y)=C_1$ and $\eta(x,y)=C_2$ forms a covering of the domain of the xy-plane in which the PDE is hyperbolic (Figure 2.36). At no point can the particular curves from each family share a

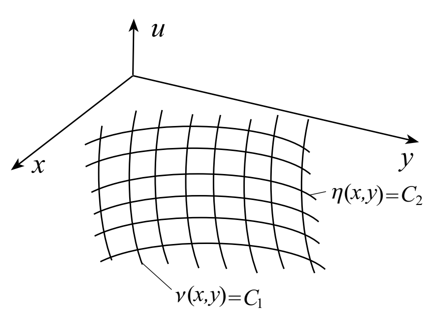


Figure 2.36

common tangent line (since at no point can $\lambda_1 = \lambda_2$).

Under the transformation $\nu = \nu(x, y)$, $\eta = \eta(x, y)$, regarded as a mapping from the xy-plane to the $\nu\eta$-plane, curves along which ν and η are constant in the xy-plane become coordinates lines in the $\nu\eta$-plane. Since these are precisely the characteristic curves, we conclude that when a hyperbolic PDE is in canonical form, coordinate lines are characteristic curves for the PDE. In other words, characteristic curves of a hyperbolic PDE are those curves to which the PDE must be referred as coordinate curves in order that it take on canonical form.

Example 2.7 Show that the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = \frac{1}{\rho} F(x, t, y, y_x, y_t)$$

is hyperbolic, and find an equivalent canonical form.

Solution When we note that coefficient c in PDE 2.123 is equal to $-c^2$ in the wave equation, we calculate that $b^2 - 4ac = 4c^2 > 0$, and therefore the PDE is hyperbolic. Characteristic curves can be found by solving the ODE

$$\frac{dx}{dt} = -\lambda(x, t),$$

where $\lambda(x, t)$ satisfies $\lambda^2 - c^2 = 0$. From the equations

$$\frac{dx}{dt} = -\lambda_1 = -c \qquad \text{and} \qquad \frac{dx}{dt} = -\lambda_2 = c,$$

we obtain the characteristic curves

$$x = -ct + C_1, \qquad x = ct + C_2.$$

It follows, then, that the transformation

$$\nu = x + ct, \quad \eta = x - ct$$

reduces the wave equation to canonical form in $w(\nu, \eta)$:

$$\frac{\partial^2 w}{\partial \nu \partial \eta} = \frac{-1}{4\tau} F\left(x(\nu, \eta), t(\nu, \eta), w, w_\nu + w_\eta, c w_\nu - c w_\eta\right). \bullet$$

Notice that when $F = 0$ in this example, the canonical form for the one-dimensional wave equation $y_{tt} - c^2 y_{xx} = 0$ is

$$\frac{\partial^2 w}{\partial \nu \partial \eta} = 0.$$

This is equation 2.116 of Section 2.7, but now we see the origin of the transformation $\nu = x + ct$ and $\eta = x - ct$.

If $\psi(\nu)$ and $\phi(\eta)$ are any two (twice continuously differentiable) functions of the canonical variables ν and η, then

$$\psi_x = \psi'(\nu)\nu_x = \lambda_1(x, y)\nu_y \psi'(\nu) = \lambda_1(x, y)\psi_y,$$
$$\phi_x = \phi'(\eta)\eta_x = \lambda_2(x, y)\eta_y \phi'(\eta) = \lambda_2(x, y)\phi_y.$$

Thus, $\psi[\nu(x, y)]$ and $\phi[\eta(x, y)]$ also satisfy equation 2.133, and it follows that any transformation of the form

$$\psi = \psi[\nu(x,y)], \quad \phi = \phi[\eta(x,y)] \tag{2.141}$$

also reduces the PDE to canonical form (in ψ and ϕ).

Finally, notice that if we set $r = \nu + \eta$, $s = \nu - \eta$, and $f(r,s) = w[\nu(r,s), \eta(r,s)]$, then

$$\begin{aligned} w_\nu &= f_r r_\nu + f_s s_\nu = f_r + f_s, \\ w_{\nu\eta} &= f_{rr} r_\eta + f_{rs} s_\eta + f_{sr} r_\eta + f_{ss} s_\eta = f_{rr} - f_{rs} + f_{sr} - f_{ss} = f_{rr} - f_{ss}. \end{aligned}$$

Consequently, the PDE in $f(r,s)$ corresponding to form 2.130a is

$$f_{rr} - f_{ss} = F[\nu(r,s), \eta(r,s), f, f_r + f_s, f_r - f_s]; \tag{2.142}$$

this is sometimes used as a canonical form for hyperbolic equations.

Parabolic Equations

Parabolic equations can be transformed into canonical form 2.130b by transformation 2.125 if functions $\nu(x,y)$ and $\eta(x,y)$ can be found to satisfy

$$0 = \beta = 2a\nu_x\eta_x + b(\nu_x\eta_y + \nu_y\eta_x) + 2c\nu_y\eta_y, \quad 0 = \gamma = a{\eta_x}^2 + b\eta_x\eta_y + c{\eta_y}^2. \tag{2.143}$$

The second equation can be written in the form

$$a\left(\frac{\eta_x}{\eta_y}\right)^2 + b\left(\frac{\eta_x}{\eta_y}\right) + c = 0 \tag{2.144}$$

so that η_x/η_y must satisfy

$$a\lambda^2 + b\lambda + c = 0. \tag{2.145}$$

Since $b^2 - 4ac = 0$, there is exactly one solution $\lambda = \lambda(x,y)$ of this quadratic, and $\eta(x,y)$ must therefore satisfy the first-order PDE

$$\eta_x = \lambda(x,y)\eta_y. \tag{2.146}$$

When $\eta(x,y)$ is so defined, $\gamma = 0$ and, from identity 2.129,

$$0 = (b^2 - 4ac)\left[\frac{\partial(\nu,\eta)}{\partial(x,y)}\right]^2 = \beta^2 - 4\alpha\gamma = \beta^2.$$

Thus, β must also vanish, and PDE 2.128a in the parabolic case reduces to

$$\alpha w_{\nu\nu} = F(\nu, \eta, w, w_\nu, w_\eta). \tag{2.147}$$

Since $\alpha \neq 0$ (why?), we may divide to obtain the canonical form 2.130b for a paraboic PDE.

We may obtain $\eta(x,y)$ by writing the solutions of the ODE

$$\frac{dy}{dx} = -\lambda(x,y) \tag{2.148}$$

in the form

$$\eta(x,y) = C. \tag{2.149}$$

The curves in this one-parameter family are called characteristic curves for the parabolic PDE. Parabolic PDEs therefore have only one family of characteristic curves. Notice that no mention of ν has been made throughout the discussion. It follows that the canonical form for parabolic PDEs is obtained for arbitrary $\nu(x, y)$, except that $\nu(x, y)$ must be chosen to yield a nonvanishing Jacobian 2.126 in whatever domain is under consideration.

Example 2.8 Is the one-dimensional heat conduction equation

$$k\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} - \frac{k}{\kappa}g(x,t)$$

parabolic? What are its characteristic curves?

Solution The equation is already in canonical form for a parabolic PDE. If we replace y with t in equation 2.148, characteristic curves are defined by

$$\frac{dt}{dx} = -\lambda(x,t),$$

where $\lambda(x,t)$ must satisfy $k\lambda^2 = 0$. Consequently, $\lambda = 0$, and characteristic curves are $t =$ constant.•

Example 2.9 Show that the PDE

$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} = x^2 + u_y$$

is parabolic, and find an equivalent PDE in canonical form.

Solution Because $b^2 - 4ac = (-2xy)^2 - 4(x^2)(y^2) = 0$, the PDE is everywhere parabolic. Characteristic curves can be found by solving

$$\frac{dy}{dx} = -\lambda(x,y)$$

where $\lambda(x,y)$ satisfies

$$0 = x^2\lambda^2 - 2xy\lambda + y^2 = (x\lambda - y)^2.$$

Consequently, we solve

$$\frac{dy}{dx} = -\frac{y}{x},$$

the solution of which can be written in the form $xy = C$. We choose therefore $\eta(x,y) = xy$, and $\nu(x,y)$ is arbitrary except that the Jacobian $\partial(\nu,\eta)/\partial(x,y) \neq 0$. If we choose $\nu(x,y) = y$, then

$$\frac{\partial(\nu,\eta)}{\partial(x,y)} = \begin{vmatrix} 0 & 1 \\ y & x \end{vmatrix} = -y \neq 0 \quad \text{(except along the } x\text{-axis)}.$$

Instead of using equation 2.128 to write the PDE in $w(\nu,\eta)$ equivalent to the original equation in $u(x,y)$, let us perform the transformation. To do this, we require the following partial derivatives:

$$u_x = w_\nu \nu_x + w_\eta \eta_x = yw_\eta, \qquad u_y = w_\nu \nu_y + w_\eta \eta_y = w_\nu + xw_\eta,$$
$$u_{xx} = y(w_{\eta\nu}\nu_x + w_{\eta\eta}\eta_x) = y^2 w_{\eta\eta},$$
$$u_{xy} = w_\eta + y(w_{\eta\nu}\nu_y + w_{\eta\eta}\eta_y) = w_\eta + yw_{\eta\nu} + xyw_{\eta\eta},$$

$$u_{yy} = w_{\nu\nu}\nu_y + w_{\nu\eta}\eta_y + x(w_{\eta\nu}\nu_y + w_{\eta\eta}\eta_y) = w_{\nu\nu} + 2xw_{\nu\eta} + x^2 w_{\eta\eta}.$$

Substitution of these into the PDE for $u(x,y)$ along with $x = \eta/\nu$ and $y = \nu$ gives

$$\frac{\eta^2}{\nu^2}\nu^2 w_{\eta\eta} - 2\frac{\eta}{\nu}\nu\left(w_\eta + \nu w_{\eta\nu} + \frac{\eta}{\nu}\nu w_{\eta\eta}\right) + \nu^2\left(w_{\nu\nu} + 2\frac{\eta}{\nu}w_{\eta\nu} + \frac{\eta^2}{\nu^2}w_{\eta\eta}\right)$$
$$= \frac{\eta^2}{\nu^2} + w_\nu + \frac{\eta}{\nu}w_\eta.$$

Thus, the PDE equivalent to the given equation is

$$w_{\nu\nu} = \frac{1}{\nu^4}[\eta^2 + \nu^2 w_\nu + \eta\nu(1+2\nu)w_\eta],$$

valid in any domain not containing points on the x-axis (for which $\nu = 0$).●

Elliptic Equations

Transformation 2.125 reduces an elliptic PDE to canonical form 2.130c if functions $\nu(x,y)$ and $\eta(x,y)$ can be found to satisfy

$$0 = 2a\nu_x\eta_x + b(\nu_x\eta_y + \nu_y\eta_x) + 2c\nu_y\eta_y, \tag{2.150a}$$
$$0 = a({\nu_x}^2 - {\eta_x}^2) + b(\nu_x\nu_y - \eta_x\eta_y) + c({\nu_y}^2 - {\eta_y}^2). \tag{2.150b}$$

For hyperbolic PDEs, $\nu(x,y)$ and $\eta(x,y)$ satisfied first-order PDEs that were separated one from the other. Similarly, $\eta(x,y)$ in the parabolic case satisfied a first-order equation that was independent of $\nu(x,y)$. Unfortunately, equations 2.150 for $\nu(x,y)$ and $\eta(x,y)$ are coupled; both unknowns appear in both equations. In an attempt to separate them, we multiply the first by the complex number i and add to the second to give

$$a(\nu_x + i\eta_x)^2 + b(\nu_x + i\eta_x)(\nu_y + i\eta_y) + c(\nu_y + i\eta_y)^2 = 0.$$

This equation can be solved for two possible values of the ratio

$$\frac{\nu_x + i\eta_x}{\nu_y + i\eta_y} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}, \tag{2.151}$$

since $b^2 - 4ac$ is known to be negative. Real and imaginary parts of this equation give

$$\nu_x = \frac{-b\nu_y \mp \eta_y\sqrt{4ac - b^2}}{2a}, \qquad \eta_x = \frac{-b\eta_y \pm \nu_y\sqrt{4ac - b^2}}{2a} \tag{2.152a}$$

or

$$2a\nu_x + b\nu_y = \mp\eta_y\sqrt{4ac - b^2}, \qquad \pm\nu_y\sqrt{4ac - b^2} = 2a\eta_x + b\eta_y. \tag{2.152b}$$

These are linear equations in ν_x and ν_y that have the following solutions in terms of the partial derivatives η_x and η_y:

$$\nu_x = -\frac{2c\eta_y + b\eta_x}{\pm\sqrt{4ac - b^2}}, \qquad \nu_y = \frac{2a\eta_x + b\eta_y}{\pm\sqrt{4ac - b^2}}. \tag{2.153}$$

These equations (called the Beltrami equations) are equivalent to conditions 2.150, but they still form a coupled set of equations in the sense that ν and η appear in both. A second-order PDE for $\eta(x,y)$ is evidently

$$\frac{\partial}{\partial x}\left(\frac{2a\eta_x + b\eta_y}{\sqrt{4ac-b^2}}\right) = \frac{\partial}{\partial y}\left(-\frac{2a\eta_y + b\eta_x}{\sqrt{4ac-b^2}}\right). \tag{2.154}$$

If this equation is solved for $\eta(x,y)$ and then used to determine $\nu(x,y)$, the original PDE in u is transformed to the form

$$\alpha w_{\nu\nu} + \alpha w_{\eta\eta} = F(\nu, \eta, w, w_\nu, w_\eta). \tag{2.155}$$

Since $0 < (b^2 - 4ac)[\partial(\nu,\eta)/\partial(x,y)]^2 = \beta^2 - 4\alpha\gamma = -4\alpha^2$, it follows that $\alpha \neq 0$, and the elliptic PDE can be obtained in canonical form 2.130c.

The only difficulty with this procedure is that in general, PDE 2.154 for $\eta(x,y)$ may not be significantly easier to solve than the original PDE in $u(x,y)$. Instead, notice that the form of equation 2.151 suggests that we define a complex function $\phi(x,y)$ of two real variables x and y,

$$\phi(x,y) = \nu(x,y) + i\eta(x,y), \tag{2.156}$$

in which case $\nu(x,y)$ and $\eta(x,y)$ can be retrieved as the real and imaginary parts of $\phi(x,y)$. It is clear that $\phi(x,y)$ must satisfy one of the equations

$$\frac{\phi_x}{\phi_y} = \frac{-b \pm i\sqrt{4ac-b^2}}{2a}. \tag{2.157}$$

To solve either one of these complex PDEs for $\phi(x,y)$, we employ the same technique used for hyperbolic and parabolic PDEs: we consider the ordinary differential equation

$$\frac{dy}{dx} = \frac{b + i\sqrt{4ac-b^2}}{2a} \tag{2.158}$$

[or $dy/dx = (b - i\sqrt{4ac-b^2})/(2a)$] for y as a function of x. Because the right side is complex, we must (temporarily) regard x and y as complex variables. If we obtain a solution in the form

$$\phi(x,y) = C, \tag{2.159}$$

then

$$\frac{dy}{dx} = -\frac{\phi_x}{\phi_y} = \frac{b + i\sqrt{4ac-b^2}}{2a}, \tag{2.160}$$

clearly indicating that $\phi(x,y)$ is the required function. Real and imaginary parts of ϕ (once again regarding x and y as real) give $\nu(x,y)$ and $\eta(x,y)$.

Because x and y in equation 2.158 are considered complex, elliptic PDEs do not have real characteristic curves.

Example 2.10 Find regions in which

$$u_{xx} + x^2 u_{yy} = y u_y$$

is elliptic, and find an equivalent PDE in canonical form.

Solution Since $b^2 - 4ac = -4x^2$, the PDE is elliptic in any region that does not contain points on the y-axis. To find a transformation that will reduce the PDE to canonical form, we set

$$\frac{dy}{dx} = -\lambda(x, y),$$

where $\lambda(x, y)$ is one of the complex solutions of $\lambda^2 + x^2 = 0$. If we choose $\lambda = -ix$, then

$$\frac{dy}{dx} = ix$$

and $y = ix^2/2 + C$. The transformation functions ν and η are the real and imaginary parts of $y - ix^2/2$,

$$\nu(x, y) = y, \qquad \eta(x, y) = \frac{-x^2}{2}.$$

With this transformation,

$$\begin{aligned} u_x &= w_\nu \nu_x + w_\eta \eta_x = -x w_\eta, \qquad u_y = w_\nu \nu_y + w_\eta \eta_y = w_\nu, \\ u_{xx} &= -w_\eta - x(w_{\nu\eta}\nu_x + w_{\eta\eta}\eta_x) = -w_\eta + x^2 w_{\eta\eta}, \\ u_{yy} &= w_{\nu\nu}\nu_y + w_{\nu\eta}\eta_y = w_{\nu\nu}. \end{aligned}$$

Substitution of these into the original PDE gives

$$w_{\nu\nu} + w_{\eta\eta} = \frac{-1}{2\eta}(w_\eta + \nu w_\nu).$$

Had we chosen to set $dy/dx = -ix$, the transformation would have been $\nu(x, y) = y$, $\eta(x, y) = x^2/2$, and the equivalent PDE would have been

$$w_{\nu\nu} + w_{\eta\eta} = \frac{1}{2\eta}(w_\eta + \nu w_\nu).\bullet$$

To summarize our results, all second-order PDEs in two independent variables that are linear in their second derivatives can be classified as hyperbolic, parabolic, or elliptic. The one-dimensional wave equation is hyperbolic, the one-dimensional heat equation is parabolic, and the two-dimensional Poisson equation is elliptic. We can therefore discover properties of all second-order PDEs in two independent variables that are linear in second derivatives by analyzing strings, heat conduction in rods, and two-dimensional electrostatic problems. Each type of equation has properties distinct from the others; properties of hyperbolic equations differ from those of parabolic equations, and these in turn differ from those of elliptic equations. For instance, in Section 2.7 we saw that a disturbance (more generally, information) is transmitted by the wave equation (a hyperbolic equation) at finite speed. Information (in the form of heat) is transmitted infinitely fast by the heat equation (see Section 6.6). Elliptic equations represent static or steady-state situations. Other properties of hyperbolic, parabolic, and elliptic equations are discussed throughout the book, particularly in Sections 6.6–6.8. Problems associated with these equations are even characterized differently; all three are accompanied by boundary conditions, but the wave equation has two initial conditions, the heat equation has one, and Poisson's equation has none.

Second-order PDEs in more than two independent variables can also be classified into types, including parabolic, elliptic, and hyperbolic. However, it is not usually possible to reduce such equations to simple canonical forms. One instance in which a canonical form is possible is for PDEs with constant coefficients. We shall not discuss the classification and canonical forms here, but we should point out that in this classification, the three-dimensional Laplace equation is elliptic, the multidimensional wave equation is hyperbolic, and the multidimensional heat equation is parabolic.

EXERCISES 2.8

1. Show that characteristic curves of PDE 2.123, when it is hyperbolic or parabolic, have slopes defined by the ordinary differential equation

$$\frac{dy}{dx} = \lambda(x, y),$$

where $\lambda(x, y)$ is a solution of the equation

$$a\lambda^2 - b\lambda + c = 0.$$

In Exercises 2–8 determine where the PDE is hyperbolic, parabolic, and elliptic. Illustrate each region graphically in the xy-plane.

2. $u_{xx} + 2yu_{xy} + 5u_{yy} = 15x + 2y$

3. $x^2u_{xx} + 4yu_{yy} = u$

4. $x^2yu_{xx} + xyu_{xy} - y^2u_{yy} = 0$

5. $xyu_{xx} - xu_{xy} + u_{yy} = uu_x + 3$

6. $(\sin x)u_{xx} + (2\cos x)u_{xy} + (\sin x)u_{yy} = 0$

7. $(x \ln y)u_{xx} + 4u_{yy} = u_x - 3xyu$

8. $u_{xx} + xu_{xy} + yu_{yy} = 0$

In Exercises 9–12 classify the PDE as hyperbolic, parabolic, or elliptic and find an equivalent PDE in canonical form.

9. $u_{xx} + 2u_{xy} + u_{yy} = u_x - xu_y$

10. $u_{xx} + 2u_{xy} + 5u_{yy} = 3u_x - yu$

11. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$

12. $u_{xx} + 6u_{xy} + u_{yy} = 4uu_x$

13. Find a PDE in canonical form equivalent to the PDE in Example 2.9 that is valid in regions not containing points on the y-axis.

14. (a) Show that the Tricomi PDE $yu_{xx} + u_{yy} = 0$ is hyperbolic when $y < 0$, parabolic when $y = 0$, and elliptic when $y > 0$.
(b) Find an equivalent PDE in canonical form when $y < 0$.
(c) Find an equivalent PDE in canonical form when $y > 0$.
(d) Find an equivalent PDE in canonical form when $y = 0$.

15. Find regions in which the PDE $x^2u_{xx} + 4u_{yy} = u$ is hyperbolic, parabolic, and elliptic. In each region, find an equivalent PDE in canonical form.

16. Show that the PDE $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = 0$ is everywhere parabolic. Find an equivalent PDE in canonical form valid in regions not containing points on the x-axis.

17. Show that the PDE $u_{xx} + x^2u_{xy} - (x^2/2 + 1/4)u_{yy} = 0$ is hyperbolic in the entire xy-plane. Find its characteristic curves and illustrate them geometrically.

18. Show that the PDE $xu_{xy} + yu_{yy} = 0$ is hyperbolic when $x \neq 0$. Find an equivalent PDE in canonical form.

19. (a) In this exercise we examine the extent to which canonical forms for linear PDEs with constant coefficients can be simplified. Show that when coefficients of linear PDE 2.122 are constants, and the PDE is transformed to canonical form, these forms remain linear with constant coefficients:

$$\begin{aligned} w_{\nu\eta} + pw_\nu + qw_\eta + rw &= G \quad \text{(hyperbolic)}, \\ w_{\nu\nu} + pw_\nu + qw_\eta + rw &= G \quad \text{(parabolic)}, \\ w_{\nu\nu} + w_{\eta\eta} + pw_\nu + qw_\eta + rw &= G \quad \text{(elliptic)}. \end{aligned}$$

(b) Prove that in the case of a hyperbolic equation, a change of dependent variable

$$z(\nu, \eta) = e^{\epsilon\nu + \rho\eta} w(\nu, \eta)$$

can, for appropriate constants ϵ and ρ, be used to eliminate the first-derivative terms z_η and z_ν.

(c) Verify that the transformation in part (b) can be used to eliminate the first derivatives for elliptic equations also.

(d) Show that the transformation in part (b) can be used to eliminate z_ν and z for a parabolic equation when $q \neq 0$, and to eliminate z_ν and z_η when $q = 0$.

In Exercises 20–22 use the results of Exercise 19 to find a simplified canonical representation for the linear PDE.

20. $u_{xx} + 2u_{xy} + 5u_{yy} = 3u_x$

21. $u_{xx} + 6u_{xy} + u_{yy} = 4u_x$

22. $u_{xx} + 2u_{xy} + u_{yy} = u_x - u_y$

§2.9 The Cauchy Problem on Infinite Intervals

The initial value problem associated with partial differential equation 2.123 is also known as the **Cauchy problem**. It provides insight into an important property of characteristic curves for hyperbolic and parabolic PDEs. The problem is to solve equation

$$au_{xx} + bu_{xy} + cu_{yy} = f(x, y, u, p, q), \tag{2.161a}$$

where p and q represent u_x and u_y, for a function $u(x, y)$ such that the surface $u = u(x, y)$ takes on prescribed values along some curve C': $x = x(\tau)$, $y = y(\tau)$ in the xy-plane (Figure 2.37a). In other words, $u(x, y)$ must satisfy

$$u(x, y) = u(\tau), \quad \text{when} \quad x = x(\tau), \quad y = y(\tau). \tag{2.161b}$$

These define a curve C in space through which the solution surface must pass (Figure 2.37b). If we use a dot $\cdot$ above a variable to indicate its derivative with respect to the parameter τ, then the functions $x(\tau)$ and $y(\tau)$ must satisfy the condition that $\dot{x}^2 + \dot{y}^2 \neq 0$ in order that $(\dot{x}, \dot{y})$ define a tangent vector to C' at each point on C'.

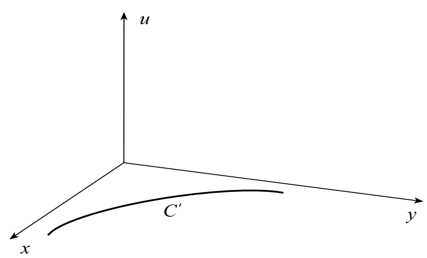


Figure 2.37a

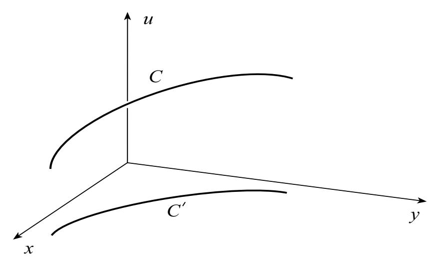


Figure 2.37b

There would be many surfaces satisfying PDE 2.161a that contain curve C. A unique surface is obtained if we also require the surface to have a given orientation along C. We do this by specifying values of $p = u_x$ and $q = u_y$ along C',

$$p = p(\tau), \quad q = q(\tau). \tag{2.161c}$$

Since the vector $\langle u_x, u_y, -1\rangle$ is perpendicular to the surface $u = u(x, y)$, it is also perpendicular to the tangent plane to the surface. By prescribing $p = u_x$ and $q = u_y$ along the curve in Figure 2.37b, we are specifying the tangent plane to the solution surface along this curve. We have shown this as an infinitesimal strip in Figure 2.38. The solution surface must pass through the curve and be tangent to the strip. The functions $p(\tau)$ and $q(\tau)$ cannot be specified independently, however. Chain rules for derivatives require

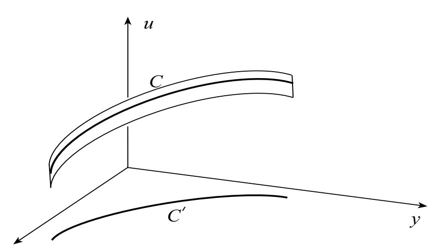


Figure 2.38

$$\frac{du}{d\tau} = \frac{\partial u}{\partial x}\frac{dx}{d\tau} + \frac{\partial u}{\partial y}\frac{dy}{d\tau}.$$

Consequently, the initial functions $p(\tau)$, $q(\tau)$ and $u(\tau)$ must satisfy the **strip condition**

$$\dot{u} = p\dot{x} + q\dot{y}. \tag{2.162}$$

(Although u, p and q cannot be specified independently along C', u and its normal derivative to C' can be specified independently.) Recall that in our discussion of nonlinear first-order PDEs in Section 1.2, the function $p(t)$ and $q(t)$ were both determined. This was so because they had to satisfy not only the strip condition, but the PDE $F(x, y, u, p, q) = 0$. For second-order PDEs, the strip condition remains, but the PDE is not a restriction on p and q.

With the function u, and its first partial derivatives specified along the initial curve C', suppose we attempt to find higher order derivatives of u along C', thus generating a Taylor series for $u(x, y)$ in some neighbourhood of C'. Second order partial derivatives must satisfy PDE 2.161a. In addition, differentiation of $p(\tau) = u_x[(x(\tau), y(\tau)]$ and $q(\tau) = u_y[x(\tau), y(\tau)]$ with respect to τ gives

$$\dot{p} = u_{xx}\dot{x} + u_{xy}\dot{y}, \qquad \dot{q} = u_{xy}\dot{x} + u_{yy}\dot{y}.$$

We have three linear equations in the second derivatives of $u(x, y)$,

$$au_{xx} + bu_{xy} + cu_{yy} = f(x, y, u, p, q), \tag{2.163a}$$

$$u_{xx}\dot{x} + u_{xy}\dot{y} = \dot{p}, \tag{2.163b}$$

$$u_{xy}\dot{x} + u_{yy}\dot{y} = \dot{q}. \tag{2.163c}$$

They have a unique solution for u_{xx}, u_{xy}, and u_{yy} in terms of τ when the determinant

$$\Delta = \det \begin{bmatrix} a & b & c \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{bmatrix} \neq 0$$

at every point on C'. When this is the case, differentiation of equation 2.163a with respect to x gives

$$au_{xxx} + a_x u_{xx} + bu_{xyx} + b_x u_{xy} + cu_{yyx} + c_x u_{yy} = f_x.$$

Furthermore, since u_{xx} and u_{xy} are now known as a functions of τ, we can write that

$$\dot{u}_{xx} = u_{xxx}\dot{x} + u_{xxy}\dot{y}, \qquad \dot{u}_{xy} = u_{xyx}\dot{x} + u_{xyy}\dot{y}.$$

These three equations constitute a set of linear equations in the third derivatives u_{xxx}, u_{xxy}, and u_{xyy},

$$au_{xxx} + bu_{xxy} + cu_{yyx} = f_x - a_x u_{xx} - b_x u_{xy} - c_x u_{yy}, \tag{2.164a}$$

$$\dot{x}u_{xxx} + \dot{y}u_{xxy} = \dot{u}_{xx}, \tag{2.164b}$$

$$\dot{x}u_{xxy} + \dot{y}u_{xyy} = \dot{u}_{xy}. \tag{2.164c}$$

Since the determinant of this system is once again Δ, solutions exist provided $\Delta \neq 0$. The third derivative u_{yyy} can then be found by differentiating equation 2.163a with respect to y. This process can be continued to obtain partial derivatives of all orders of $u(x, y)$ and hence the Taylor series of $u(x, y)$ in some neighbourhood of the initial curve C'.

When determinant Δ is identically equal to zero along the initial curve C', higher order derivatives of $u(x,y)$ cannot be determined along the curve. This occurs when

$$a\dot{y}^2 - b\dot{x}\dot{y} + c\dot{x}^2 = 0 \qquad \Longrightarrow \qquad a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0.$$

But this is the equation defining characteristic curves for the PDE (see Exercise 1 in Section 2.8). Hence, characteristic curves of PDE 2.161 are those curves along which second and higher order derivatives of the solution are unattainable from the PDE and the initial data. Other properties of characteristics will become clear as our discussions unfold.

Hyperbolic Equations and Their Characteristic Curves

To solve first-order PDEs we use characteristic curves. To reduce a second-order hyperbolic (or parabolic) PDE to canonical form we use characteristic curves. It is not coincidence that we use the term characteristic curves in both situations. We now demonstrate that characteristic curves for the wave equation can be obtained by reducing the wave equation to a pair of first-order PDEs. We begin by *factoring* the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

in the following way

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x}\right) = 0.$$

If we define a new variable u by $u = \dfrac{\partial y}{\partial t} - c\dfrac{\partial y}{\partial x}$, the second-order wave equation is replaced by a pair of first-order equations

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = u, \tag{2.165a}$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{2.165b}$$

Although this system is coupled, equation 2.165b can be solved independently of 2.165a. It is the unidirectional wave equation 1.18 of Section 1.3. When c is constant, its base C-curves are straight lines $x - ct =$ constant, one of the two families of characteristic curves in Section 2.8. Had we factored the wave equation in the form

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x}\right) = 0,$$

the family $x + ct =$ constant of characteristic curves would have arisen. We now demonstrate that d'Alembert's solution 2.119 of the wave equation on an infinite interval can be derived with the pair of first-order PDEs 2.165. At the same time, the second set of characteristic curves will arise naturally.

Hyperbolic Equations and Their Characteristic Curves on Infinite Intervals

The Cauchy problem for the wave equation on an infinite interval adds two initial conditions to the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.166a}$$

$$y(x,0) = f(x), \quad -\infty < x < \infty, \tag{2.166b}$$

$$y_t(x,0) = g(x), \quad -\infty < x < \infty. \tag{2.166c}$$

Initial conditions 2.166b,c add the following initial condition to PDE 2.165b,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \tag{2.167a}$$

$$u(x,0) = g(x) - cf'(x). \tag{2.167b}$$

Characteristic equations for this first-order DPE are

$$dt = \frac{dx}{c}, \quad du = 0.$$

A 2-parameter family of solutions is $x = ct + \alpha$, $u = \beta$. For a 1-parameter family of C-curves that contains the initial curve we set $\beta = \beta(\alpha)$, and

$$x = \alpha, \quad g(x) - cf'(x) = \beta(\alpha).$$

Thus, $\beta(\alpha) = g(\alpha) - cf'(\alpha)$, and the 1-parameter family of C-curves generating the solution to problem 2.167 is

$$x = ct + \alpha, \quad u = g(\alpha) - cf'(\alpha).$$

The explicit solution is

$$u(x,t) = g(x - ct) - cf'(x - ct).$$

We substitute this into PDE 2.165a, and attach initial condition 2.166b,

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = g(x - ct) - cf'(x - ct), \tag{2.168a}$$

$$y(x,0) = f(x). \tag{2.168b}$$

Characteristic equations for this PDE are

$$dt = \frac{dx}{-c} = \frac{dy}{g(x - ct) - cf'(x - ct)}.$$

The first two terms give $x = -ct + \gamma$, and when this is substituted into the first and last terms,

$$\frac{dy}{dt} = g(\gamma - 2ct) - cf'(\gamma - 2ct).$$

Integrations gives the 2-parameter family of C-curves

$$x = -ct + \gamma, \quad y = \int_0^t [g(\gamma - 2cv) - cf'(\gamma - 2cv)]\, dv + \beta.$$

For a 1-parameter family of C-curves that contains the initial curve we set $\beta = \beta(\gamma)$, and

$$x = \gamma, \quad f(x) = \beta(\gamma).$$

Thus, the 1-parameter of characteristic curves generating the solution surface is

$$x = \gamma - ct, \quad y = f(\gamma) + \int_0^t [g(\gamma - 2cv) - cf'(\gamma - 2cv)]\,dv.$$

The explicit solution is

$$y(x,t) = f(x+ct) + \int_0^t [g(x+ct-2cv) - cf'(x+ct-2cv)]\,dv.$$

If we set $u = x + ct - 2cv$ in the integral of $g(x + ct - 2cv)$, we obtain

$$\begin{aligned}
y(x,t) &= f(x+ct) + \int_{x+ct}^{x-ct} g(u)\left(-\frac{du}{2c}\right) - c\int_0^t f'(x+ct-2cv)\,dv \\
&= f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du - c\left\{-\frac{1}{2c}f(x+ct-2cv)\right\}_0^t \\
&= f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du + \frac{1}{2}[f(x-ct) - f(x+ct)] \\
&= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du.
\end{aligned}$$

This is d'Alembert's solution that we saw in Section 2.7.

This process of factoring the differential operator in the wave equation into first-order operators, thereby reducing the second-order wave equation to a pair of first-order equations, is also available for parabolic and elliptic equations, but the resulting system of first-order equations is not advantageous (see Exercises 11 and 12). Nor is it convenient if the wave equation has a damping term or a term proportional to displacement (see Exercises 9 and 10).

In Section 2.7, we used the transformation of independent variables $u = x + ct$ and $v = x - ct$ to obtain the general solution $y(x,t) = F(x+ct) + G(x-ct)$ of the one-dimensional wave equation 2.166. According to Section 2.8, we now know that this transformation reduced the wave equation to canonical form. Initial and boundary conditions in Section 2.7 then determined the functions F and G. The two, one-parameter families of characteristic curves for this problem are $x - ct = C_1$ and $x + ct = C_2$, so that specifying $y(x,t)$ and $y_t(x,t)$ on $t = 0$ is specifying the unknown function and its normal derivative along the curve $t = 0$, which is nowhere tangent to a characteristic curve (Figure 2.39a).

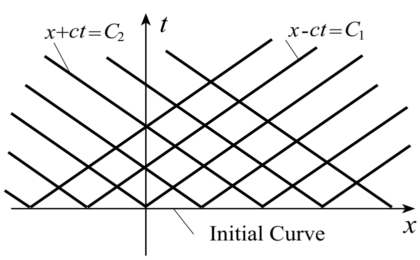


Figure 2.39a

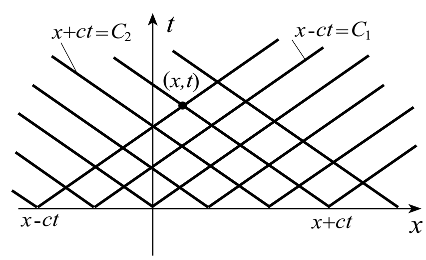


Figure 2.39b

A point x in the string at time t is represented by a point (x,t) in the first quadrant of Figure 2.39b. Characteristic curves through this point intersect the x-axis in

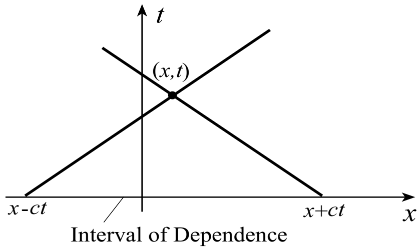


Figure 2.40

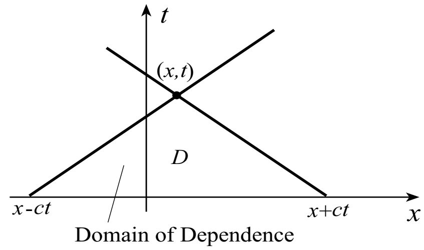


Figure 2.41

points with x-coordinates $x - ct$ and $x + ct$. To find the displacement of the string at this position and time using d'Alembert's solution 2.168, we evaluate $f(x)$ at these two intercepts and integrate $g(x)$ between the intercepts. We call this interval along the x-axis the **interval of dependence** of the solution at (x, t) (Figure 2.40). We shall see that for nonhomogeneous problems, the solution also depends on the triangular region D bounded by the x-axis and the characteristics through (x, t) (Figure 2.41). The triangle is called the **domain of dependence** of the solution at (x, t).

If we draw the characteristic curves through a point x on the positive x-axis (Figure 2.42), the region above the characteristics includes all points that would have x in their domain of dependences. We call this region the **range of influence** of the point x. In Figures 2.31 and 2.34 of Section 2.7, we demonstrated that discontinuities in the initial data or its derivatives are propagated along the string at velocities $\pm c$. In our present context, this means that discontinuities in the initial data or its derivatives at a point x will result in corresponding discontinuities across the characteristic curves through x.

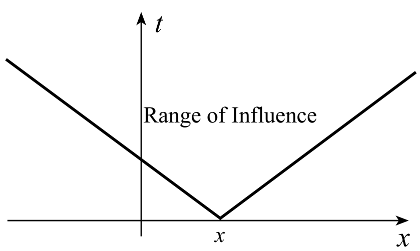


Figure 2.42

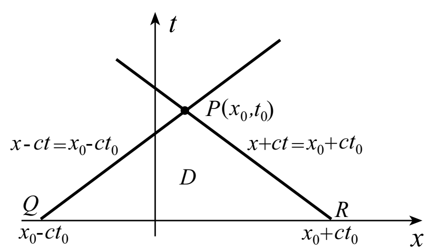


Figure 2.43

Consider now the nonhomogeneous problem corresponding to problem 2.166,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.169a}$$

$$y(x,0) = f(x), \quad -\infty < x < \infty, \tag{2.169b}$$

$$y_t(x,0) = g(x), \quad -\infty < x < \infty. \tag{2.169c}$$

The solution of this problem is the sum of d'Alembert's solution 2.168 to the homogeneous problem and a particular solution of nonhomogeneous PDE 2.169a that vanishes along with its first time-derivative at $t = 0$. We could simply quote the particular solution and verify its validity, but we prefer an alternative approach that derives the particular solution and simultaneously reproduces the d'Alembert solution. Let $P(x_0, t_0)$ be any point in the first quadrant of the xt-plane and let characteristics through P intersect the x-axis in points $Q(x_0 - ct_0, 0)$ and $R(x_0 + ct_0, 0)$

(Figure 2.43). We integrate both sides of PDE 2.169a over the triangular domain of dependence D,

$$\iint_D \left(\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2}\right) dA = \iint_D \frac{F(x,t)}{\rho} dA.$$

Using Green's Theorem, the double integral on the left can be replaced by a line integral around the boundary $\beta(D)$ of D,

$$\iint_D \left(\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2}\right) dA = \oint_{\beta(D)} -\frac{\partial y}{\partial t}dx - c^2\frac{\partial y}{\partial x}dt.$$

When we combine these two equations, and evaluate the line integral along the three lines forming $\beta(D)$, we obtain

$$\begin{aligned}\iint_D \frac{F(x,t)}{\rho} dA &= \int_Q^R -g(x)\,dx + \int_R^P -\frac{\partial y}{\partial t}(-c\,dt) - c^2\frac{\partial y}{\partial x}\left(-\frac{dx}{c}\right) + \int_P^Q -\frac{\partial y}{\partial t}(c\,dt) - c^2\frac{\partial y}{\partial x}\left(\frac{dx}{c}\right)\\
&= -\int_{x_0-ct_0}^{x_0+ct_0} g(x)\,dx + c\Big\{y(x,t)\Big\}_R^P - c\Big\{y(x,t)\Big\}_P^Q\\
&= -\int_{x_0-ct_0}^{x_0+ct_0} g(x)\,dx + 2cy(x_0,t_0) - cy(x_0+ct_0,0) - cy(x_0-ct_0,0)\\
&= -\int_{x_0-ct_0}^{x_0+ct_0} g(x)\,dx + 2cy(x_0,t_0) - cf(x_0+ct_0) - cf(x_0-ct_0).\end{aligned}$$

Since $P(x_0,t_0)$ represents any point, we drop the subscripts and solve for $y(x,t)$,

$$y(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta + \frac{1}{2c}\iint_D \frac{F(x,t)}{\rho} dA. \quad (2.170)$$

To the d'Alembert solution of the homogeneous problem is added the double integral of the nonhomogeneity over the domain of dependence at (x,t).

Characteristic curves are not needed in order to use d'Alembert's solution 2.168 for displacements $y(x,t)$ of an infinite string that is not subject to external forces (problem 2.166). They provide some geometric interpretations of the terms in the solution, but they are not required for evaluation of $y(x,t)$. On the other hand, when external forces act on the string, solution 2.170 of problem 2.169 does require characteristic curves; they define the domain of dependence D for the double integral.

Example 2.11 An infinitely long taut string has initial displacement $f(x)$ at time $t=0$, but no initial velocity. Find displacements of the string if gravity is taken into account.

Solution The initial value problem for displacements $y(x,t)$ is

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2} - g, \quad -\infty < x < \infty, \quad t > 0, \quad (g = 9.81),\\
y(x,0) &= f(x), \quad -\infty < x < \infty,\\
y_t(x,0) &= 0, \quad -\infty < x < \infty.\end{aligned}$$

According to equation 2.170, the solution of this problem is

$$y(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\iint_D -g\,dA,$$

where D is the triangle in Figure 2.41. Since the area of the triangle is ct^2, the solution is

$$y(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] - \frac{gt^2}{2}.$$

We have the left- and right-travelling waves due to the initial displacement and a free-fall term due to gravity.●

Example 2.12 An infinitely long taut string is at rest on the x-axis at time $t=0$. For $t>0$, each point of the string is subjected to the same sinusoidal force $\sin t$. Use equation 2.170 to find displacements of points in the string.

Solution Displacements $y(x,t)$ must satisfy the initial value problem

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{\sin t}{\rho}, \quad -\infty < x < \infty, \quad t>0,$$
$$y(x,0) = 0, \quad -\infty < x < \infty,$$
$$y_t(x,0) = 0, \quad -\infty < x < \infty.$$

According to equation 2.170, the solution of this problem is

$$y(x,t) = \frac{1}{2c}\iint_D \frac{\sin t}{\rho}\,dA,$$

where D is the triangle in Figure 2.41. To evaluate the integral, we must either introduce subscripts on x and t, or use alternative variables of integration. We choose the latter by replacing x and t by u and v (Figure 2.44). Integration now gives

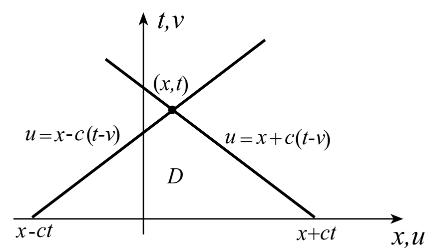


Figure 2.44

$$y(x,t) = \frac{1}{2c\rho}\int_0^t\int_{x-c(t-v)}^{x+c(t-v)} \sin v\,du\,dv = \frac{1}{\rho}\int_0^t (t-v)\sin v\,dv = \frac{1}{\rho}(t-\sin t).$$

As expected, the solution is independent of x (the forcing term has no x's), but perhaps unexpectedly, displacement is the sum of a linear term and a sinusoidal term.●

EXERCISES 2.9

1. Repeat Example 2.12 if the forcing function is e^{-t}.
2. Repeat Example 2.12 if the forcing function is $\sin x$ instead of $\sin t$. Use the solution to verify that points on the string that remain stationary for all time are points where the force vanishes.
3. Find displacements of points in an infinitely long taut string if, fictitiously, the initial displacement is $f(x)=5$, the initial velocity is $g(x)=x^2$, and the force on the string is $F(x,t)=e^x$.
4. Repeat Example 2.12 if the forcing function is $e^{-|x|}$.
5. Repeat Example 2.12 if the forcing function is $1/(x^2+1)$.

6. Repeat Example 2.11 if points in the string also have initial velocity $g(x)$.

7. Can we use formula 2.170 to calculate displacements in an infinite string if the only force acting on the string (besides tension) is a damping force proportional to velocity (see equation 2.46 in Section 2.3)?

8. Can we use formula 2.170 to calculate displacements in an infinite string if the only force acting on the string (besides tension) is a restoring force proportional to displacement (see equation 2.47 in Section 2.3)?

9. (a) Factor the operator in the wave equation containing a restoring force proportional to displacement,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - ky$$

to reduce it to a pair of first-order equations.

(b) Why is this system not as convenient as system 2.165 for the undamped wave equation?

10. (a) Show that the transformation $w = e^{\beta t/2} y$ of dependent variable replaces the damped wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \beta \frac{\partial y}{\partial t}$$

with

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} + \frac{\beta^2}{4} w.$$

(b) Now reduce the equation to the following pair of first-order PDEs

$$\frac{\partial w}{\partial t} - c\frac{\partial w}{\partial x} = u, \qquad \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\beta^2}{4} w.$$

(c) Why is this system not as convenient as system 2.165 for the undamped wave equation?

11. Use the complex factorization

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y}\right) = 0$$

for the two-dimensional Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

to show that it can be replaced by the system

$$\frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y} = u, \qquad \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = 0.$$

Although the second equation is independent of V, the presence of complex numbers, when u and V must be real, makes the system an unsatisfactory replacement.

12. (a) Set $v = \dfrac{\partial U}{\partial x}$ in the one-dimensional heat equation

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x},$$

to replace the second-order equation with a first-order system.

(b) Why is this system not as convenient as system 2.165 for the undamped wave equation?

§2.10 The Cauchy Problem on Semi-infinite Intervals

We now turn our attention to the wave equation on the semi-infinite interval $0 \leq x < \infty$. The homogeneous problem corresponding to problem 2.166 with a Dirichlet boundary condition at $x = 0$ is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \tag{2.171a}$$

$$y(0,t) = 0, \quad t > 0, \tag{2.171b}$$

$$y(x,0) = f(x), \quad 0 < x < \infty, \tag{2.171c}$$

$$y_t(x,0) = g(x), \quad 0 < x < \infty. \tag{2.171d}$$

A review of the analysis of problem 2.115 in Section 2.7 indicates that the d'Alembert solution of the wave equation and initial conditions is once again

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\, d\zeta, \tag{2.172a}$$

This cannot be the solution of problem 2.171 since we have not taken into account the boundary condition at $x = 0$, and surely the solution must depend on the boundary condition. The reason that the solution is not complete is that functions $f(x)$ and $g(x)$ are defined only for $0 \leq x < \infty$, and yet for 2.172a to represent $y(x,t)$ for all x and t, these functions must be defined for all real numbers. The boundary condition will show us how to extend definitions of $f(x)$ and $g(x)$ for negative values of x. Boundary condition 2.171b demands that

$$0 = \frac{1}{2}[f(ct) + f(-ct)] + \frac{1}{2c}\int_{-ct}^{ct} g(\zeta)\, d\zeta.$$

This will be satisfied for all $t > 0$ if we separately set

$$f(ct) + f(-ct) = 0, \qquad \int_{-ct}^{ct} g(\zeta)\, d\zeta = 0.$$

These imply that $f(x)$ and $g(x)$ are odd functions of x. In other words, if d'Alembert's solution 2.172a is to satisfy problem 2.171, $f(x)$ and $g(x)$ must be extended as odd functions of x. Just as we did in Section 2.7, we can give a graphical derivation of the position of the string and then a physical interpretation of what we see. Consider the case that $g(x) = 0$, in which case solution 2.172a reduces to

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

Graphically, we have the addition of one-half the graph of $f(x)$ shifted ct units to the left and one-half the graph of $f(x)$ shifted ct units to the right. Because $f(x)$ has been extended as an odd function, the right-shifting graph drags with it its odd extension. Suppose, for example, that the initial displacement of the string is that in Figure 2.31a. For $t < 7/(16c)$, the odd extension is to the left of the origin and therefore has no effect on the graphical determination of the position of the string. Displacements are exactly as shown in Figures 2.31a–f in Section 2.7. However, for $t > 7/(16c)$, the extension must be combined with $f(x+ct)/2$ to give the displacement of the string. We have shown results in Figures 2.45a–g.

Physically, we interpret the situation as follows. The position of the string is the superposition of two waves, both equal to one-half $f(x)$, one travelling to the right and the other to the left with speed c. At the boundary $x = 0$, the left-travelling wave is reflected with a reversal in sign to be combined with the original disturbance to give deflection in the string. From then on, there are two disturbances travelling to the right with speed c, one equal to $f(x)/2$ and the other equal to $-f(x)/2$. The second trails the first by unit distance.

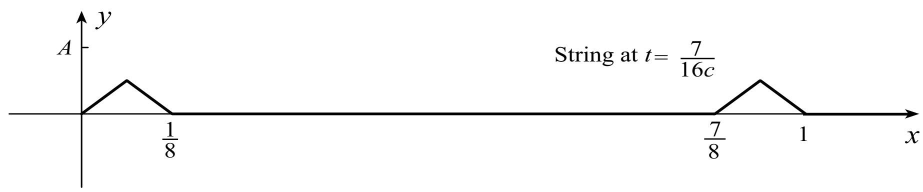


Figure 2.45a

Figure 2.45b

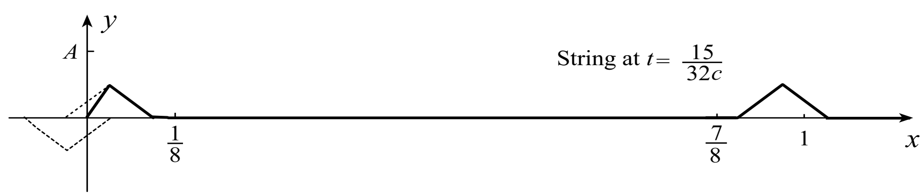


Figure 2.45c

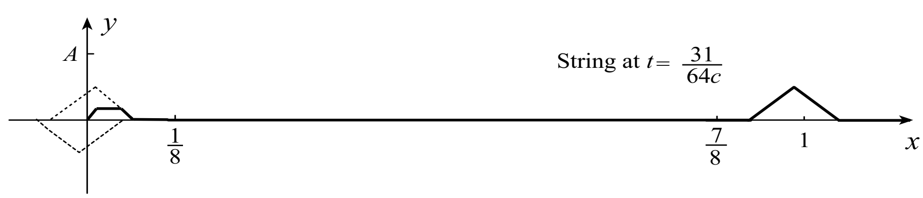


Figure 2.45d

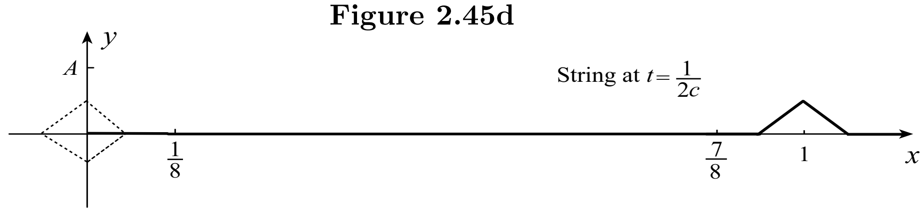


Figure 2.45e

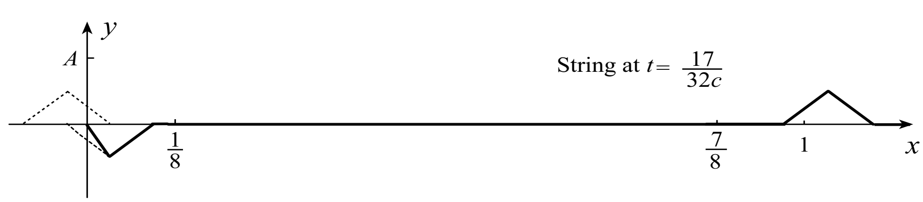


Figure 2.45f

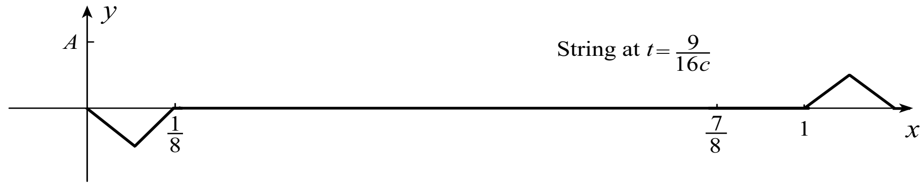


Figure 2.45g

As we did in Section 2.7, we could consider the case that $f(x) = 0$ and $g(x) \neq 0$, the string has zero initial displacement, but nonzero initial velocity. We shall leave consideration to the exercises. Displacement when both $f(x)$ and $g(x)$ are nonzero is an addition of the separate results.

Putting the travelling waves aside for the moment, let us determine the role of characteristic curves in problem 2.171. As long as $x > ct$ (see Figure 2.46a), the interval of dependence contains only points on the positive x-axis, and therefore solution 2.172a uses only given values of $f(x)$ and $g(x)$. When $x < ct$, however, (Figure 2.46b), the left characteristic curve intersects the x-axis at a negative value of x. We could certainly regard the interval between $x-ct$ and $x+ct$ as the interval of dependence, but an alternative approach leads to a formula for the solution of the nonhomogeneous problem corresponding to equation 2.171. Because $f(x)$ and $g(x)$ are extended as odd functions, we can write solution 2.172a in the form

$$y(x,t) = \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\zeta)\,d\zeta, \qquad (2.172b)$$

where $ct - x > 0$. We can find this point graphically by reflecting that part of the left characteristic to the left of the t-axis in the t-axis (Figure 2.46b). We now regard that part of the x-axis between $ct-x$ and $x+ct$ as the interval of dependence of the solution at (x,t), and the quadrilateral $PQRS$ as the domain of dependence of the solution at a point (x,t).

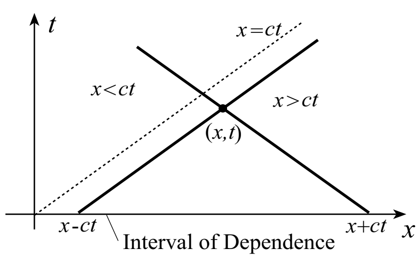


Figure 2.46a

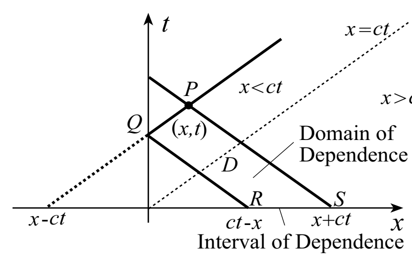


Figure 2.46b

Consider now the nonhomogeneous problem corresponding to problem 2.171,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \quad 0 < x < \infty, \quad t > 0, \qquad (2.173a)$$

$$y(0,t) = k(t), \quad t > 0, \qquad (2.173b)$$

$$y(x,0) = f(x), \quad 0 < x < \infty, \qquad (2.173c)$$

$$y_t(x,0) = g(x), \quad 0 < x < \infty. \qquad (2.173d)$$

For points below the characteristic curve $x = ct$ in Figure 2.46a, we can use solution 2.170 for the infinite problem (since the boundary has no effect for such points). For a point $P(x_0, t_0)$ above $x = ct$, we integrate the PDE over the domain of dependence D in Figure 2.47,

$$\iint_D \left(\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2}\right) dA = \iint_D \frac{F(x,t)}{\rho} dA.$$

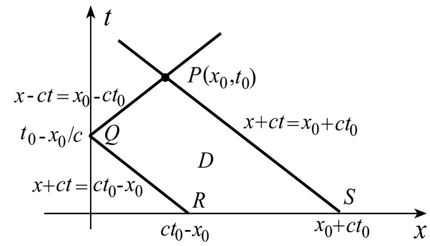


Figure 2.47

Using Green's Theorem, the double integral on the left can be replaced by a line integral around the boundary $\beta(D)$ of D,

$$\iint_D \left(\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2}\right) dA = \oint_{\beta(D)} -\frac{\partial y}{\partial t} dx - c^2 \frac{\partial y}{\partial x} dt.$$

When we combine these two equations, and evaluate the line integral along the four lines forming $\beta(D)$, we obtain

$$\begin{aligned}
\iint_D \frac{F(x,t)}{\rho} dA &= \int_R^S -g(x)\,dx + \int_S^P -\frac{\partial y}{\partial t}(-c\,dt) - c^2 \frac{\partial y}{\partial x}\left(-\frac{dx}{c}\right) + \int_P^Q -\frac{\partial y}{\partial t}(c\,dt) - c^2 \frac{\partial y}{\partial x}\left(\frac{dx}{c}\right) \\
&\quad + \int_Q^R -\frac{\partial y}{\partial t}(-c\,dt) - c^2 \frac{\partial y}{\partial x}\left(-\frac{dx}{c}\right) \\
&= -\int_{ct_0 - x_0}^{x_0 + ct_0} g(x)\,dx + c\Big\{y(x,t)\Big\}_S^P - c\Big\{y(x,t)\Big\}_P^Q + \Big\{y(x,t)\Big\}_Q^R \\
&= -\int_{ct_0 - x_0}^{x_0 + ct_0} g(x)\,dx + c[y(x_0,t_0) - y(x_0 + ct_0, 0) - y(0, t_0 - x_0/c) \\
&\qquad + y(x_0,t_0) + y(ct_0 - x_0, 0) - y(0, t_0 - x_0/c)] \\
&= -\int_{ct_0 - x_0}^{x_0 + ct_0} g(x)\,dx + c[2y(x_0,t_0) - 2k(t_0 - x_0/c) + f(ct_0 - x_0) - f(x_0 + ct_0)].
\end{aligned}$$

Since $P(x_0, t_0)$ represents any point, we drop the subscripts and solve for $y(x,t)$,

$$\begin{aligned}
y(x,t) = k(t - x/c) + \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\zeta)\,d\zeta \\
+ \frac{1}{2c}\iint_D \frac{F(x,t)}{\rho}\,dA.
\end{aligned} \qquad (2.174)$$

Thus, the solution of problem 2.173 is

$$y(x,t) = \begin{cases} \dfrac{1}{2}[f(x+ct) + f(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta + \dfrac{1}{2c}\iint_D \dfrac{F(x,t)}{\rho} dA, & x > ct \\ k(t - x/c) + \dfrac{1}{2}[f(x+ct) - f(ct-x)] + \dfrac{1}{2c}\displaystyle\int_{ct-x}^{x+ct} g(\zeta)\,d\zeta + \dfrac{1}{2c}\iint_D \dfrac{F(x,t)}{\rho}\,dA, & x < ct. \end{cases}$$

(2.175)

In this form, the solution uses only values of $f(x)$ and $g(x)$ from their original domain of definition $x > 0$ (and not their extensions as odd functions). The first line of the solution corresponds to points x that are sufficiently far to the right ($x > ct$) so that effects of the displacement at $x = 0$, which travel down the string at velocity c, have yet to arrive. The domain of dependence D in this case is the triangle in Figure 2.46a. When $x < ct$, effects of the boundary have reached x, and the second line of the solution must be used. The domain of dependence is the quadrilateral in Figure 2.46b.

EXERCISES 2.10

1. A semi-infinite string has its end at $x = 0$ fixed on the $x$axis. At time $t = 0$, it has displacement $f(x)$, but no velocity. Find displacements of the string if gravity is taken into account.

2. Can we use formula 2.175 to calculate displacements in a semi-infinite string if the only force acting on the string (besides tension) is a damping force proportional to velocity (see equation 2.46 in Section 2.3)?

3. Can we use formula 2.175 to calculate displacements in a semi-infinite string if the only force acting on the string (besides tension) is a restoring force proportional to displacement (see equation 2.47 in Section 2.3)?

4. Show that when the boundary condition in problem 2.171 is homogeneous and Neumann, equation 2.172a still represents the solution of the problem, but $f(x)$ and $g(x)$ must be extended as even functions.

5. Use a graphical technique to determine the position of a semi-infinite string with zero initial displacement and initial velocity

$$g(x) = \begin{cases} 0, & 0 \le x < 7/16, \\ k, & 7/16 \le x \le 9/16, \\ 0, & 9/16 < x \le 1, \end{cases}$$

where $k > 0$ is a constant at the times (a) $t = 1/(8c)$, (b) $t = 1/(4c)$, (c) $t = 3/(8c)$, (d) $t = 1/(2c)$, (e) $t = 5/(8c)$, (f) $t = 3/(4c)$. Assume that its end $x = 0$ is fixed on the x-axis.

§2.11 The Cauchy Problem on Finite Intervals

Our final consideration is the role of characteristic curves for the wave equation on a finite interval $0 \le x \le L$. The homogeneous initial boundary value problem for displacements of a finite string with fixed end points is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{2.176a}$$

$$y(0,t) = 0, \quad t > 0, \tag{2.176b}$$

$$y(L,t) = 0, \quad t > 0, \tag{2.176c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{2.176d}$$

$$y_t(x,0) = g(x), \quad 0 < x < L. \tag{2.176e}$$

For consistency, we assume that the initial displacement and velocity functions satisfy $f(0) = g(0) = f(L) = g(L) = 0$. Our discussions of the semi-infinite problem showed that the function that satisfies the PDE, the initial conditions, and the boundary condition at $x = 0$ is

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta, \tag{2.177}$$

where $f(x)$ and $g(x)$ are extended as odd functions from their original domains of definition for positive values of x. This still continues to be the case, but because $f(x)$ and $g(x)$ are only defined for $0 \le x \le L$ for the finite string, this extends their domains of definition only to $-L \le x \le L$. The boundary condition at $x = L$ will extend their domains to all values of x. The boundary condition demands that

$$0 = \frac{1}{2}[f(L+ct) + f(L-ct)] + \frac{1}{2c}\int_{L-ct}^{L+ct} g(\zeta)\,d\zeta.$$

This is satisfied if we choose

$$0 = f(L+ct) + f(L-ct), \qquad 0 = \int_{L-ct}^{L+ct} g(\zeta)\,d\zeta.$$

These imply that the odd extensions of $f(x)$ and $g(x)$ must also be $2L$-periodic. Solution 2.177 can now be used to calculate displacements in the finite string for any x in $0 < x < L$ and any time $t > 0$. It is the **d'Alembert's solution** of initial boundary value problem 2.176.

In Section 2.7, we gave a graphical derivation of the position of an infinite string and then a physical interpretation of what we saw as travelling waves. In Section 2.10, we gave a similar discussion for the semi-infinite string, but saw that the fixed end of the string reflected the left-travelling wave. We now show that for the finite string, multiple reflections of both waves occur at the ends of the string. Consider first the case that $g(x) = 0$, in which cases solution 2.172 reduces to

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]. \tag{2.178}$$

Graphically, we have the addition of one-half the graph of $f(x)$ shifted ct units to the left and one-half the graph of $f(x)$ shifted ct units to the right. Because $f(x)$ has been extended as an odd $2L$-periodic function, both waves drag odd, periodic extensions with them. Suppose, for example, that the initial displacement of the string is as shown in Figure 2.48.

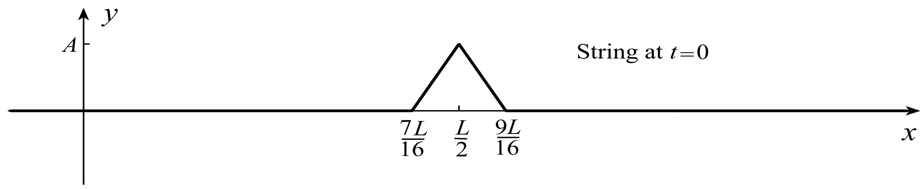


Figure 2.48

For $t < 7L/(8c)$, the odd, periodic extensions of $f(x)$ do not effect the graphical determination of the position of the string. Displacements are exactly as shown in Figures 2.31b–f in Section 2.7, but distances must be multiplied by L and times by L/c. For $t > 7L/(16c)$, however, the extensions become a part of the graphical solution for displacement of the string. We have shown results in Figures 2.49a–h.

Figure 2.49a

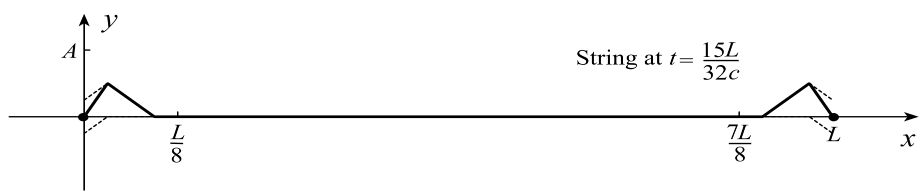


Figure 2.49b

Figure 2.49c

Figure 2.49d

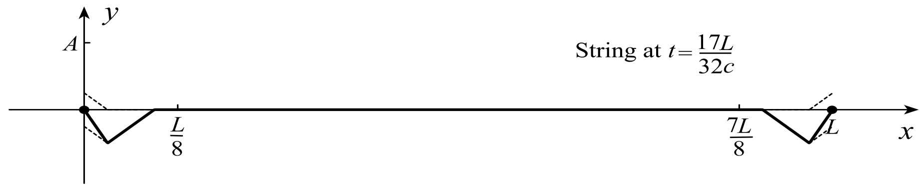


Figure 2.49e

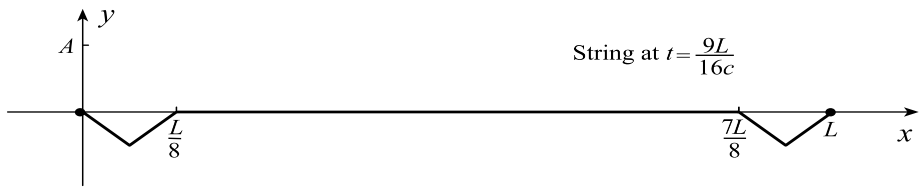


Figure 2.49f

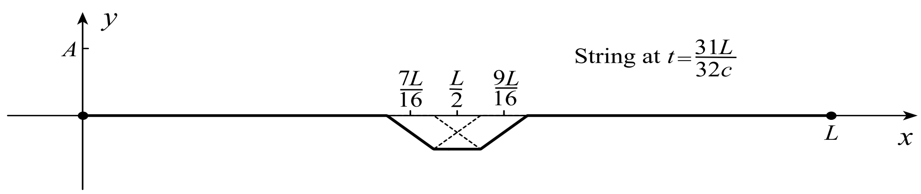


Figure 2.49g

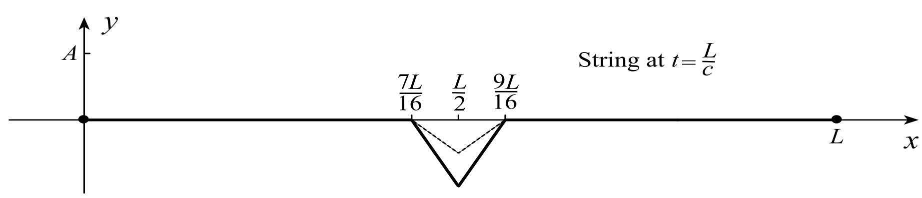


Figure 2.49h

These graphs suggest the following physical description for the motion of the string. Figures 2.31a–f indicate that the initial deflection $f(x)$ in the string divides into two parts, each equal to one-half of $f(x)$, one traveling to the left with velocity $-c$ and the other traveling to the right with velocity c. Figures 2.49a–h suggest that when these disturbances reach the fixed ends of the string at time $7L/(16c)$, they are reflected there with a reversal in sign. The reflected disturbance then combines with the original disturbance to yield the total deflection. Reflected disturbances then travel toward one another at speed c, eventually combining at time $t = L/c$ to give a disturbance identical to that in Figure 2.31a, but with a reversal in sign.

For times $t > L/c$, the disturbances separate again, travel to the ends of the string, are reflected there, and recombine at $t = 2L/c$ to yield the initial position in Figure 2.31a.

For times $t > 2L/c$, the two disturbances continue to travel back and forth along the string, interfering constructively near the centre of the string and destructively at the ends.

All of these things happed very quickly. For instance, if the tension in a 1-metre string with density $\rho = 2$ g/m is 50 N, then $2L/c = 0.0126$. Thus, the initial displacement separates into two parts, and these two disturbances travel twice the length of the string and recombine to give the initial displacement in 0.0126 s. In other words, all of this happens $1/0.0126 = 79$ times each second, too fast for the human eye, but not for modern cameras.

Example 2.13 Find the position of the string described by equation 2.178 at time $t = 1023L/(32c)$ when $f(x)$ is as shown in Figure 2.31a.

Solution In each time interval of length $2L/c$ after $t = 0$, the initial disturbance separates into two parts, each part travels to an end of the string and is reflected, then travels to the other end of the string and is reflected, and the parts then recombine to form $f(x)$ once again. Since $1023L/(32c) = 15(2L/c) + 63L/(32c)$, the position of the string at time $t = 1023L/(32c)$ is identical to that at $t = 63L/(32c)$. But this is 63/64 of the time for a complete cycle; that is, the two waves will be in positions shown in Figure 2.50a. These are combined to give the position of the string in Figure 2.50b.

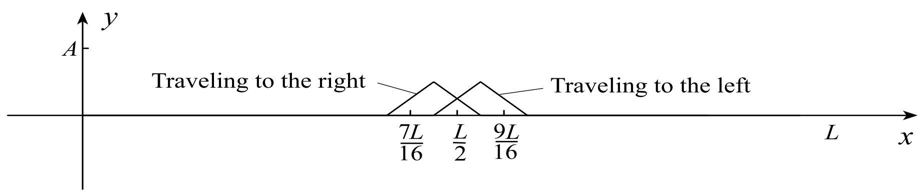


Figure 2.50a

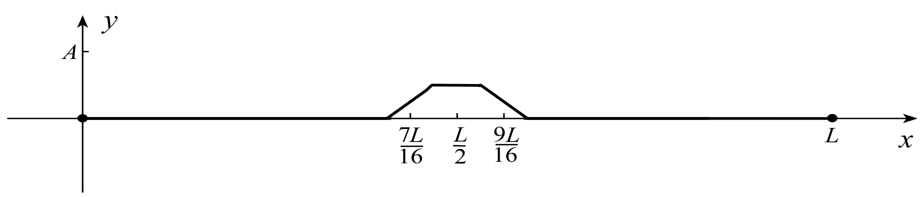


Figure 2.50b

An alternative procedure is to write solution 2.120 at $t = 1023L/(32c)$ in the form

$$\begin{aligned} y\left(x, \frac{1023L}{32c}\right) &= \frac{1}{2}\left[f\left(x + \frac{1023L}{32}\right) + f\left(x - \frac{1023L}{32}\right)\right] \\ &= \frac{1}{2}\left[f\left(x + 16(2L) - \frac{L}{32}\right) + f\left(x - 16(2L) + \frac{L}{32}\right)\right] \\ &= \frac{1}{2}\left[f\left(x - \frac{L}{32}\right) + f\left(x + \frac{L}{32}\right)\right], \end{aligned}$$

since $f(x)$ is $2L$-periodic. These functions are shown in Figure 2.50a and added in 2.50b.●

The above discussion and example have illustrated that the motion of a string with initial displacement $f(x)$ as shown in Figure 2.48 and zero initial velocity is easily described. For more complicated functions $f(x)$, such as in Figure 2.51, the principles are the same; the only difference is that reflections at the ends of the string begin immediately. Examples of this are given in Exercises 2 and 3.

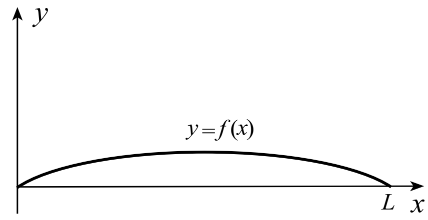


Figure 2.51

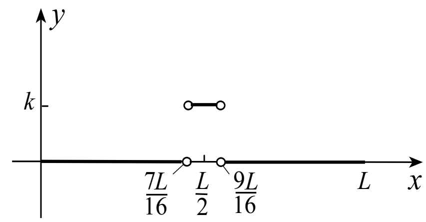


Figure 2.52

Consider now the situation in which the string is given a nonzero initial velocity $g(x)$, but no initial displacement, $f(x) = 0$. In this case, equation 2.119 yields

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta \tag{2.179a}$$

as the displacement of the string at position x and time t. Suppose, for example, that

$$g(x) = \begin{cases} 0, & 0 \le x < 7L/16, \\ k, & 7L/16 < x < 9L/16, \\ 0, & 9L/16 < x \le L \end{cases}$$

where $k > 0$ is a constant (Figure 2.52). (Think of only that part of the string $7L/16 < x < 9L/16$ being struck by a hammer.)

If we denote by $G(x)$ the antiderivative

$$G(x) = \frac{1}{2c}\int_0^x g(\zeta)\,d\zeta,$$

$y(x,t)$ can be expressed in the form

$$y(x,t) = G(x+ct) - G(x-ct), \tag{2.179b}$$

where, because $g(x)$ is extended as an odd, $2L$-periodic function (Figure 2.53a), the graph of $G(x)$ is shown in Figure 2.53b. The position of the string at any given time can now be obtained by the destructive combination of the left-traveling wave $G(x+ct)$ and the right traveling wave $G(x-ct)$. Results are shown for various times in Figures 2.54a–i.

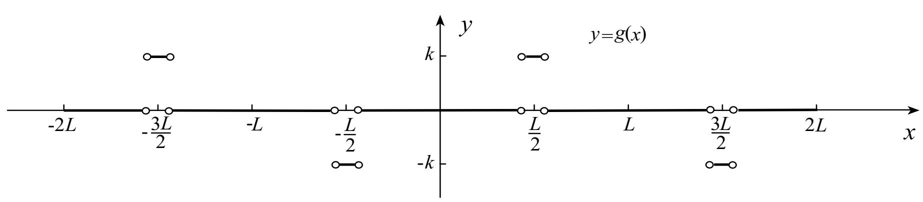


Figure 2.53a

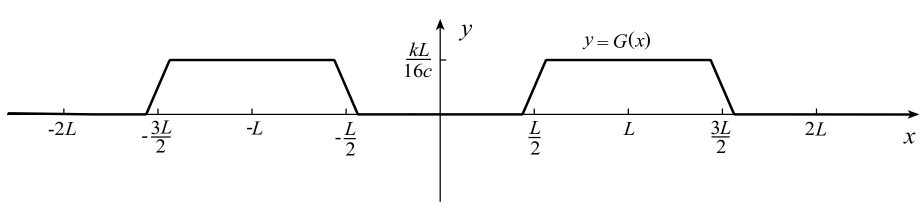


Figure 2.53b

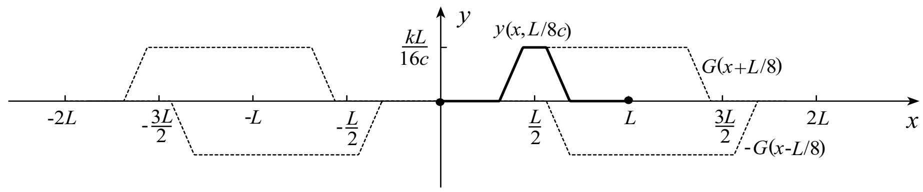


Figure 2.54a

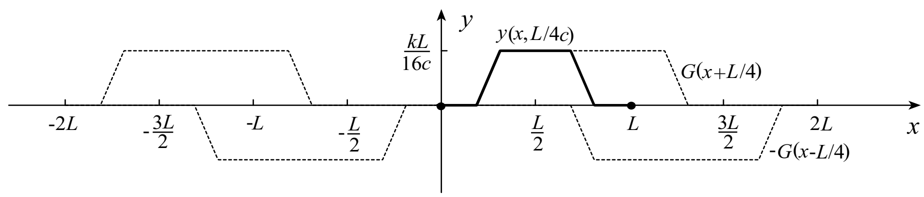


Figure 2.54b

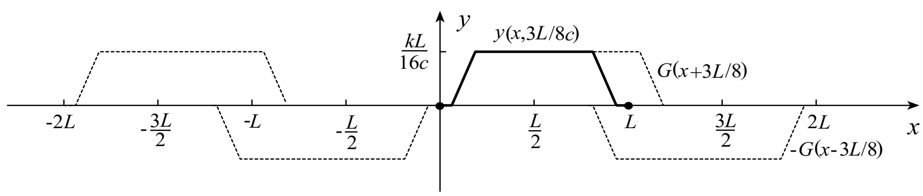


Figure 2.54c

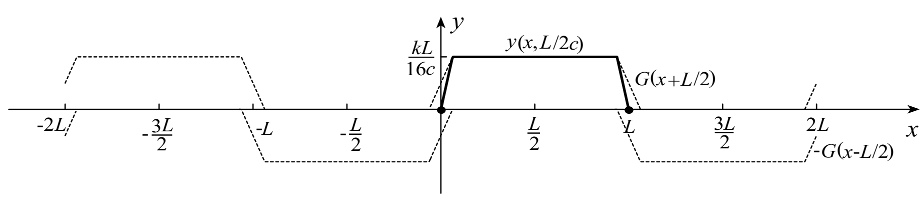


Figure 2.54d

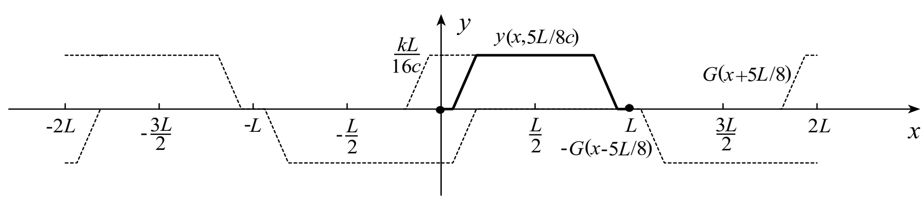


Figure 2.54e

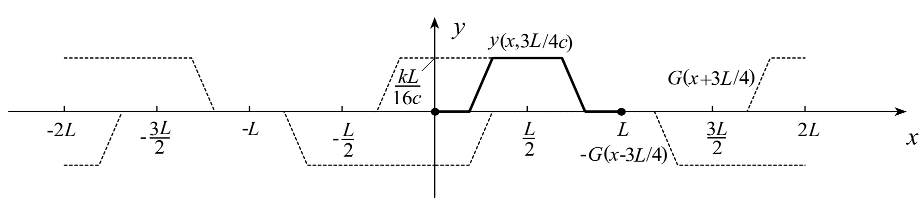


Figure 2.54f

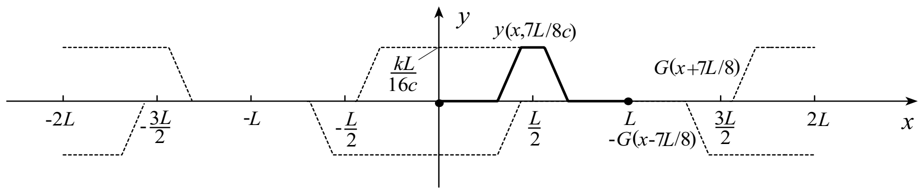


Figure 2.54g

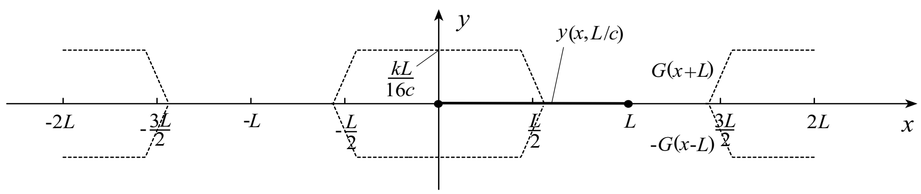


Figure 2.54h

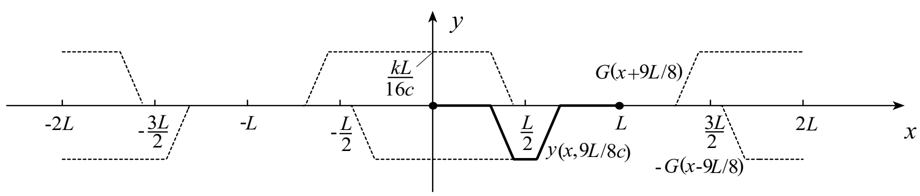


Figure 2.54i

When a string has both an initial displacement $f(x)$ and an initial velocity $g(x)$, graphical techniques may still be used to determine the solution of problem 2.176. We express $y(x,t)$ in the form $y(x,t) = u(x,t) + v(x,t)$, where $u(x,y)$ and $v(x,t)$ satisfy the problems

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} & \qquad \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial x^2} \\ u(0,t) &= 0 & v(0,t) &= 0 \\ u(L,t) &= 0 & v(L,t) &= 0 \\ u(x,0) &= f(x) & v(x,0) &= 0 \\ u_t(x,0) &= 0 & v_t(x,0) &= g(x). \end{aligned}$$

Consider now the role of characteristic curves in problem 2.176 for the finite string. The solution of the problem is

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta, \tag{2.180}$$

where $f(x)$ and $g(x)$ are extended as odd, $2L$-periodic functions. It was trivial to find the interval and domain of dependence for the infinite string. Because there is only one reflection at $x = 0$ for the semi-infinite string, determination of its interval and domain of dependence was also quite simple. This is not the case for the finite string with its multiple reflections at $x = 0$ and $x = L$. In other words, characteristic curves cannot be used to the same advantage here, especially for the corresponding nonhomogeneous problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \quad 0 < x < L, \quad t > 0, \tag{2.181a}$$

$$y(0,t) = k(t), \quad t > 0, \tag{2.181b}$$

$$y(L,t) = m(t), \quad t > 0, \tag{2.181c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{2.181d}$$

$$y_t(x,0) = g(x), \quad 0 < x < L. \tag{2.181e}$$

We make some investigations into the situation in Exercise 6–10.

EXERCISES 2.11

1. Determine the position of the string in Figure 2.48 when (a) $t = 3L/c$ and (b) $t = 49L/(8c)$.

2. Use the graphical techniques of this section to determine the displacements of a string with fixed ends on the x-axis, zero initial velocity, and initial displacement

$$f(x) = \begin{cases} x/8, & 0 \le x \le L/2 \\ (L-x)/8, & L/2 \le x \le L \end{cases}$$

at the times (a) $t = L/(8c)$ (b) $t = L/(4c)$ (c) $t = 3L/(8c)$ (d) $t = L/(2c)$ (e) $t = 5L/(8c)$ (f) $t = 3L/(4c)$ (g) $t = 7L/(8c)$ (h) $t = L/c$.

3. Repeat Exercise 2 with $f(x) = \sin(2\pi x/L)$, $0 \le x \le L$.

4. Use the graphical techniques of this section to determine the displacements of a string with fixed ends on the x-axis, zero initial displacement, and initial velocity

$$g(x) = \begin{cases} 0, & 0 \le x < L/4 \\ 1, & L/4 < x < 3L/4 \\ 0, & 3L/4 < x \le L. \end{cases}$$

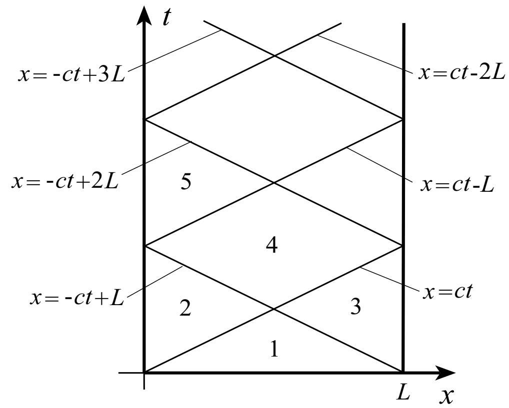
for the times in Exercise 2.

5. Repeat Exercise 4 with

$$g(x) = \begin{cases} 0, & 0 \le x < L/8 \\ 1, & L/8 < x < 3L/8 \\ 0, & 3L/8 < x < 5L/8 \\ 1, & 5L/8 < x < 7L/8 \\ 0, & 7L/8 < x \le L. \end{cases}$$

In Exercises 6–10 we use the figure to the right to discuss the role of characteristic curves for the wave equation on a finite interval.

6. Points in region 1 of the figure experience no reflections at the boundaries. What are the interval of dependence and domain of dependence for such a point? Find solutions of the homogeneous and the nonnhomogeneous wave equations with Dirichlet boundary conditions in this region?



7. Points in region 2 experience a reflection at the boundary $x = 0$. What are the interval of dependence and domain of dependence for such a point? Find solutions, involving only values of $f(x)$ and $g(x)$ in $0 \leq x \leq L$, of the homogeneous and the nonnhomogeneous wave equations with Dirichlet boundary conditions for points in this region?

8. Points in region 3 experience a reflection at the boundary $x = L$. Repeat Exercise 7 for points in this region.

9. Points in region 4 experience two reflections one at each boundary. Repeat Exercise 7 for points in this region.

10. Repeat Exercise 7 for points in region 5.

11. Show that when the boundary conditions in problem 2.176 are both homogeneous and Neumann, the solution is still 2.177, but $f(x)$ and $g(x)$ must be extended as $2L$-periodic, even functions.

CHAPTER 3 FOURIER SERIES

§3.1 Fourier Series

Power series play an integral part in real (and complex) analysis. Given a function $f(x)$ and a point $x = a$, it is investigated to what extent $f(x)$ can be expressed in the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n.$$

Perhaps one of the most important uses of such series (and one that we require in Chapter 8) is the solution of linear ODEs with variable coefficients. In this chapter we introduce a new type of series called a *Fourier series*; such series are indispensable to the study of PDEs. Fourier series are used in a theoretical way to determine properties of solutions of PDEs and in a practical way to find explicit representations of solutions. Some of the terminology associated with Fourier series is borrowed from ordinary vectors; in addition, many of the ideas in Fourier series have their origin in vector analysis. A quick review of pertinent ideas from vector analysis will therefore facilitate later comparisons and help to solidify underlying concepts in the new theory.

The Cartesian components of a vector $\mathbf{v}$ in space are three scalars v_x, v_y, and v_z such that $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$. Formulas for these components are

$$v_x = \mathbf{v} \cdot \hat{\mathbf{i}}, \qquad v_y = \mathbf{v} \cdot \hat{\mathbf{j}}, \qquad v_z = \mathbf{v} \cdot \hat{\mathbf{k}}. \tag{3.1}$$

These expressions are very simple, and the reason for this is that the basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are othonormal; that is, they are mutually orthogonal (or perpendicular) and have unit length. Given different basis vectors, say $\mathbf{e}_1 = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{e}_2 = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, and $\hat{\mathbf{e}}_3 = 3\hat{\mathbf{k}}$, which remain orthogonal, it is still possible to express $\mathbf{v}$ in terms of the $\mathbf{e}_j$,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3. \tag{3.2}$$

However, because the $\mathbf{e}_i$ do not have length 1, component formulas 3.1 must be replaced by somewhat more complicated expressions. Scalar products of representation 3.2 with $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ give

$$v_1 = \frac{\mathbf{v} \cdot \mathbf{e}_1}{|\mathbf{e}_1|^2}, \qquad v_2 = \frac{\mathbf{v} \cdot \mathbf{e}_2}{|\mathbf{e}_2|^2}, \qquad v_3 = \frac{\mathbf{v} \cdot \mathbf{e}_3}{|\mathbf{e}_3|^2}. \tag{3.3}$$

Were the $\mathbf{e}_i$ not orthogonal, but they must be linearly independent, expressions for components would be even more complicated, but we have no need for such generality here.

Thus, when an orthogonal basis is used for vectors, equations 3.3 yield components, and when the basis is orthonormal, the simpler expressions 3.1 prevail.

We now generalize these ideas to functions.

Definition 3.1 When two functions $f(x)$ and $g(x)$ are defined on the interval $a \le x \le b$, their **scalar product** with respect to a weight function $w(x)$ is defined as

$$\int_a^b w(x)f(x)g(x)\,dx. \tag{3.4}$$

This definition is much like the definition of the scalar product of two ordinary vectors, $\mathbf{u}\cdot\mathbf{v} = u_x v_x + u_y v_y + u_z v_z$, provided we think of a function as having an infinite number of components, its values at the points in the interval $a \le x \le b$. Corresponding components of $f(x)$ and $g(x)$ are then multiplied together and added in integral 3.4. The weight function in scalar products 3.1 and 3.3 is unity; definition 3.4 is more general; it permits a variable weight function $w(x)$ which is assumed to be nonnegative on the interval. Corresponding to the test for orthogonality of nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, namely $\mathbf{u}\cdot\mathbf{v} = 0$, we make the following definition for orthogonality of functions.

Definition 3.2 Two nonzero functions $f(x)$ and $g(x)$ are said to be **orthogonal** on the interval $a \le x \le b$ with respect to the weight function $w(x)$ if their scalar product vanishes:

$$\int_a^b w(x)f(x)g(x)\,dx = 0. \tag{3.5}$$

A sequence of nonzero functions $\{f_n(x)\} = f_1(x), f_2(x), \ldots$ is said to be orthogonal on $a \le x \le b$ with respect to $w(x)$ if every pair of functions is orthogonal:

$$\int_a^b w(x)f_n(x)f_m(x)\,dx = 0, \quad \text{when } n \neq m. \tag{3.6}$$

For example, since

$$\int_0^{2\pi} \sin nx \sin mx\,dx = \int_0^{2\pi} \frac{1}{2}[\cos{(n-m)x} - \cos{(n+m)x}]\,dx$$
$$= \frac{1}{2}\left\{\frac{\sin{(n-m)x}}{n-m} - \frac{\sin{(n+m)x}}{n+m}\right\}_0^{2\pi} = 0,$$

the sequence of functions $\{\sin nx\}$ is orthogonal on the interval $0 \le x \le 2\pi$ with respect to the weight function $w(x) = 1$. The sequence is also orthogonal with the same weight function on the interval $0 \le x \le \pi$.

By analogy with geometric vectors, where $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$, we regard the scalar product of a function $f(x)$ with itself as the square of its length.

Definition 3.3 The **length** of a function on the interval $a \le x \le b$ with respect to the weight function $w(x)$ is

$$\|f(x)\| = \sqrt{\int_a^b w(x)[f(x)]^2\,dx}. \tag{3.7}$$

Definition 3.4 A sequence of nonzero functions $\{f_n(x)\}$ is said to be **orthonormal** on $a \le x \le b$ with respect to the weight function $w(x)$ if

$$\int_a^b w(x)f_n(x)f_m(x)\,dx = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases} \tag{3.8}$$

This condition therefore requires the functions to be mutually orthogonal and of unit length.

Any orthogonal sequence can be made orthonormal simply by dividing each function by its length; that is, when $\{f_n(x)\}$ is orthogonal, then $\{f_n(x)/\|f_n(x)\|\}$ is orthonormal. For example, since

$$\int_0^{\pi} (\sin nx)^2\,dx = \frac{\pi}{2},$$

the sequence $\{\sqrt{2/\pi}\sin nx\}$ is orthonormal with respect to the weight function $w(x) = 1$ on $0 \le x \le \pi$.

With these preliminaries out of the way, we are now ready to consider Fourier series. In the theory of Fourier series, it is investigated to what extent a function $f(x)$ can be represented in an infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\Big), \tag{3.9}$$

where a_n and b_n are constants. The 2 in the first term of this series is included simply as a matter of convenience. (The formula for a_n for $n > 0$, will then include a_0 as well.)

Because $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ have period $2L/n$, it follows that any function $f(x)$ expressible in form 3.9 must necessarily be of period $2L$ (or of a period that evenly divides $2L$). That many $2L$-periodic functions can be expressed in this form is to a large extent attributable to the fact that the sine and cosine functions satisfy the following theorem.

Theorem 3.1 The set of functions $\{1, \cos(n\pi x/L), \sin(n\pi x/L)\}$ is orthogonal over the interval $0 \le x \le 2L$ with respect to the weight function $w(x) = 1$. Furthermore,

$$\int_0^{2L} 1^2\,dx = 2L; \qquad \int_0^{2L}\Big(\cos\frac{n\pi x}{L}\Big)^2 dx = \int_0^{2L}\Big(\sin\frac{n\pi x}{L}\Big)^2 dx = L. \tag{3.10}$$

(See Exercise 15 for a proof of this result.)

It follows that the functions

$$\frac{1}{\sqrt{2L}}, \qquad \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L}, \qquad \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}$$

are orthonormal with respect to the weight function $w(x) = 1$ on the interval $0 \le x \le 2L$.

Suppose we neglect, for the moment, questions of convergence and formally set

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\Big). \tag{3.11}$$

Just as v_x, v_y, and v_z are the components of $\mathbf{v}$ with respect to the basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, we regard the coefficients $a_0/2$, a_n, and b_n as components of $f(x)$ with respect to the basis functions 1, $\cos(n\pi x/L)$, and $\sin(n\pi x/L)$. If we integrate both side of equation 3.11 from $x = 0$ to $x = 2L$, and formally interchange the order of integration and summation on the right, we obtain

$$\int_0^{2L} f(x)\,dx = \frac{a_0}{2}(2L) \qquad \Longrightarrow \qquad a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx; \tag{3.12a}$$

that is, if representation 3.11 is to hold, the constant term $a_0/2$ must be the average value of $f(x)$ over the interval $0 \le x \le 2L$. When we multiply both sides of 3.11 by $\cos(k\pi x/L)$ and integrate from $x = 0$ to $x = 2L$, and once again interchange the order of integration and summation,

$$\begin{aligned}\int_0^{2L} f(x)\cos\frac{k\pi x}{L}dx &= \int_0^{2L} \frac{a_0}{2}\cos\frac{k\pi x}{L}dx \\ &\quad + \sum_{n=1}^{\infty}\left(\int_0^{2L} a_n \cos\frac{n\pi x}{L}\cos\frac{k\pi x}{L}dx + \int_0^{2L} b_n \sin\frac{n\pi x}{L}\cos\frac{k\pi x}{L}dx\right) \\ &= a_k(L) \quad \text{(by the orthogonality of Theorem 3.1).}\end{aligned}$$

Thus,

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx, \quad n > 0. \tag{3.12b}$$

Similarly,

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx, \quad n > 0. \tag{3.12c}$$

We have found, therefore, that if $f(x)$ can be represented in form 3.11, and if the series is suitably convergent, coefficients a_n and b_n can be calculated according to formulas 3.12. What we must answer is the converse question: If a_n and b_n are calculated according to 3.12, does series 3.11 converge to $f(x)$? Does it converge pointwise, uniformly, or in any other sense? When a_0, a_n, and b_n are calculated according to 3.12, the right side of 3.11 is called the **Fourier series** of $f(x)$. Numbers a_0, a_n, and b_n are called the **Fourier coefficients** of $f(x)$; they are, as we have already suggested, components of $f(x)$ with respect to the basis functions 1, $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$.

Theorem 3.2 which follows shortly, guarantees that series 3.11 essentially converges to $f(x)$ when $f(x)$ is piecewise continuous and has a piecewise continuous first derivative. A function $f(x)$ is **piecewise continuous** on an interval $a \le x \le b$ if the interval can be divided into a finite number of subintervals inside each of which $f(x)$ is continuous and has finite limits as x approaches either end point of the subinterval from the interior. A $2L$-periodic function is said to be piecewise continuous if it is piecewise continuous on the interval $0 \le x \le 2L$. Figure 3.1a illustrates a $2L$-periodic function that is piecewise continuous; its discontinuities at $x = c$ and $x = d$ are finite. Because the discontinuity at $x = c$ in Figure 3.1b is not finite, this $2L$-periodic function is not piecewise continuous.

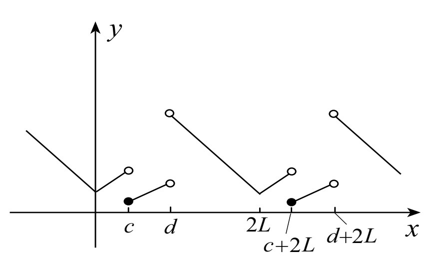


Figure 3.1a

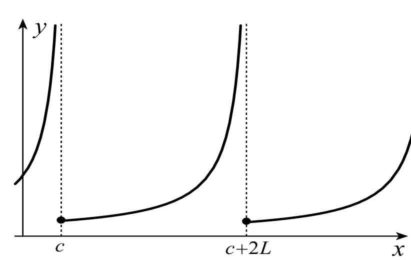


Figure 3.1b

A function $f(x)$ is said to be **piecewise smooth** on an interval $a \leq x \leq b$ if $f(x)$ and $f'(x)$ are both piecewise continuous therein. A $2L$-periodic function is piecewise smooth if it is piecewise smooth on $0 \leq x \leq 2L$. The periodic functions in Figure 3.2 are both continuous; that in Figure 3.2a is piecewise smooth, that in Figure 3.2b is not. The $2L$-periodic function in Figure 3.3 is piecewise smooth.

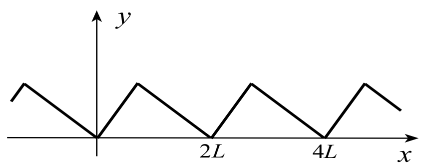


Figure 3.2a

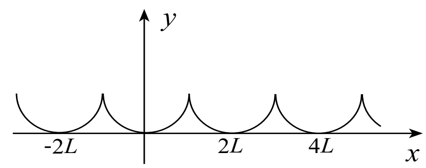


Figure 3.2b

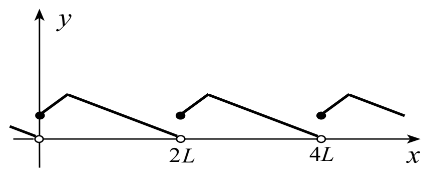


Figure 3.3

Theorem 3.2 The Fourier series of a periodic, piecewise continuous function $f(x)$ converges to $[f(x+) + f(x-)]/2$ at any point at which $f(x)$ has both a left- and right-derivative.

By $f(x+)$ we mean the right-hand limit of $f(x)$ at x, $\lim_{\epsilon \to 0^+} f(x+\epsilon)$. Similarly, $f(x-) = \lim_{\epsilon \to 0^+} f(x-\epsilon)$. The proof of this theorem is very lengthy; it requires verification of a number of preliminary results that, although interesting in their own right, detract from the flow of our discussion. We have therefore included the proof as Appendix A.

Since functions that are piecewise smooth must have right and left derivatives at all points, we may state the following corollary to Theorem 3.2.

Corollary When $f(x)$ is a periodic, piecewise smooth function, its Fourier series converges to $[f(x+) + f(x-)]/2$.

For such functions, we therefore write

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \qquad (3.13a)$$

where

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos \frac{n\pi x}{L} dx, \qquad b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx. \qquad (3.13b)$$

There is nothing sacrosanct about the limits $x = 0$ and $x = 2L$ on these integrals; all that is required is integration over one full period (of length $2L$). In other words, expressions 3.13b could be replaced by

$$a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos \frac{n\pi x}{L} dx, \qquad b_n = \frac{1}{L} \int_c^{c+2L} f(x) \sin \frac{n\pi x}{L} dx, \quad (3.13c)$$

where c is any number whatsoever.

If we make the agreement that at any point of discontinuity, $f(x)$ shall be defined (or redefined if necessary) as the average of its right- and left-hand limits, representation 3.13a may be replaced by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{3.14}$$

For example, the Fourier series of the function $f(x)$ in Figure 3.1a converges to the function in Figure 3.4; $f(x)$ must be defined as the average value of its right- and left-hand limits at $x = d + 2nL$, and redefined as the average of its right- and left-hand limits at $x = c + 2nL$.

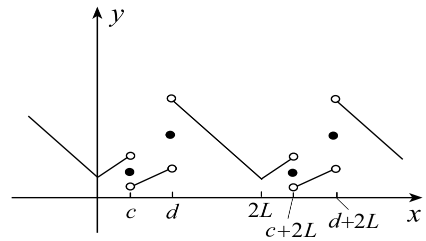


Figure 3.4

Example 3.1 Find the Fourier series of the function $f(x)$ that is equal to x for $0 < x < 2L$ and is $2L$-periodic.

Solution According to formulas 3.13b, the Fourier coefficients are

$$a_0 = \frac{1}{L}\int_0^{2L} x\,dx = \frac{1}{L}\left\{\frac{x^2}{2}\right\}_0^{2L} = 2L;$$

$$a_n = \frac{1}{L}\int_0^{2L} x \cos \frac{n\pi x}{L} dx = \frac{1}{L}\left\{\frac{Lx}{n\pi}\sin\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}\right\}_0^{2L} = 0, \quad n > 0;$$

$$b_n = \frac{1}{L}\int_0^{2L} x \sin \frac{n\pi x}{L} dx = \frac{1}{L}\left\{-\frac{Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right\}_0^{2L} = -\frac{2L}{n\pi}, \quad n > 0.$$

We may therefore write

$$f(x) = L + \sum_{n=1}^{\infty} -\frac{2L}{n\pi}\sin\frac{n\pi x}{L} = L\left(1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}\right),$$

provided we define $f(x)$ as L at its points of discontinuity $x = 2nL$. In other words, the Fourier series converges to the function in Figure 3.5.●

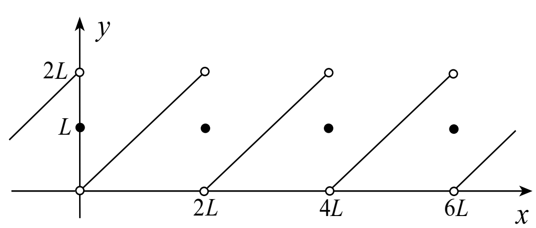


Figure 3.5

Example 3.2 Find the Fourier series of the function $f(x)$ that is equal to x^2 for $-L \le x \le L$ and is of period $2L$.

Solution In this example, it is more convenient to integrate over the interval $-L \le x \le L$. (You can see why if you examine the graph of the function in Figure

3.6.) In other words, we use formulas 3.13c with $c = -L$ to calculate the Fourier coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} x^2\,dx = \frac{1}{L}\left\{\frac{x^3}{3}\right\}_{-L}^{L} = \frac{2L^2}{3};$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x^2\cos\frac{n\pi x}{L}dx = \frac{1}{L}\left\{\left(\frac{Lx^2}{n\pi} - \frac{2L^3}{n^3\pi^3}\right)\sin\frac{n\pi x}{L} + \frac{2L^2x}{n^2\pi^2}\cos\frac{n\pi x}{L}\right\}_{-L}^{L}$$
$$= \frac{4L^2(-1)^n}{n^2\pi^2}, \quad n > 0;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} x^2\sin\frac{n\pi x}{L}dx = \frac{1}{L}\left\{\left(\frac{2L^3}{n^3\pi^3} - \frac{Lx^2}{n\pi}\right)\cos\frac{n\pi x}{L} + \frac{2L^2x}{n^2\pi^2}\sin\frac{n\pi x}{L}\right\}_{-L}^{L}$$
$$= 0, \quad n > 0.$$

Because $f(x)$ is continuous for all x (Figure 3.6), we may write

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{L} = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}.$$

This Fourier series can be used to find the sum of the series of constants $\sum_{n=1}^{\infty} 1/n^2$. When we set $x = L$, and note that $f(L) = L^2$,

$$L^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

This equation can be solved for

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4L^2}\left(L^2 - \frac{L^2}{3}\right) = \frac{\pi^2}{6}.$$

The sums of many series of constants can be obtained in this way. Unfortunately, given a series of constants to evaluate, say $\sum_{n=1}^{\infty} c_n$, it is difficult to determine the function $f(x)$ whose Fourier series would contain the series $\sum_{n=1}^{\infty} c_n$.●

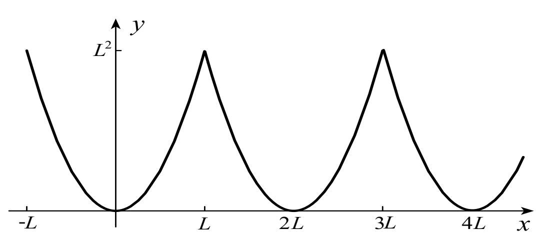


Figure 3.6

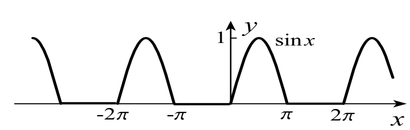


Figure 3.7

Example 3.3 Find the Fourier series for the 2π-periodic function $f(x)$ in Figure 3.7.

Solution With $L = \pi$ in formulas 3.13b,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}\Big\{-\cos x\Big\}_0^{\pi} = \frac{2}{\pi};$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\begin{cases} \left\{\dfrac{1}{2}\sin^2 x\right\}_0^{\pi}, & n = 1 \\ \left\{\dfrac{\cos(n-1)x}{2(n-1)} - \dfrac{\cos(n+1)x}{2(n+1)}\right\}_0^{\pi}, & n > 1 \end{cases}$$

$$= \begin{cases} 0, & n = 1 \\ -\dfrac{1+(-1)^n}{\pi(n^2-1)}, & n > 1; \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\sin nx\,dx$$

$$= \frac{1}{\pi}\begin{cases} \left\{\dfrac{x}{2} - \dfrac{\sin 2x}{4}\right\}_0^{\pi}, & n = 1 \\ \left\{\dfrac{\sin(n-1)x}{2(n-1)} - \dfrac{\sin(n+1)x}{2(n+1)}\right\}_0^{\pi} & n > 1 \end{cases}$$

$$= \begin{cases} 1/2, & n = 1 \\ 0, & n > 1. \end{cases}$$

Because $f(x)$ is continuous for all x, we may write

$$f(x) = \frac{1}{\pi} + \sum_{n=2}^{\infty} -\frac{1+(-1)^n}{\pi(n^2-1)}\cos nx + \frac{1}{2}\sin x.$$

Terms in the series vanish when n is odd. To display only the even terms, we replace n by $2n$:

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}.\bullet$$

By representation 3.14, we mean that the sequence of partial sums $\{S_n(x)\}$ of the series on the right converges to $f(x)$ for all x; that is, were we to plot the functions in the sequence

$$S_0(x) = \frac{a_0}{2}, \qquad S_1(x) = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{L} + b_1\sin\frac{\pi x}{L}\right),$$
$$S_2(x) = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{L} + b_1\sin\frac{\pi x}{L}\right) + \left(a_2\cos\frac{2\pi x}{L} + b_2\sin\frac{2\pi x}{L}\right),$$

and so forth, their graphs should resemble more and more closely that of $f(x)$. Figure 3.8 illustrates this fact with the partial sums $S_0(x)$, $S_1(x)$, $S_2(x)$, $S_3(x)$, $S_4(x)$, and $S_{10}(x)$ for the function $f(x)$ in Example 3.2. Graphs are plotted only for $-L \le x \le 3L$; they would be extended periodically outside this interval.

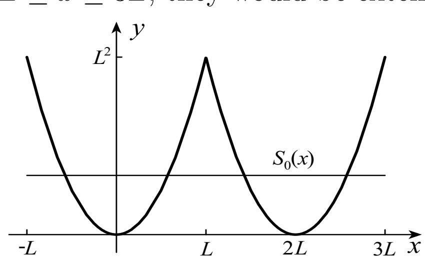


Figure 3.8a

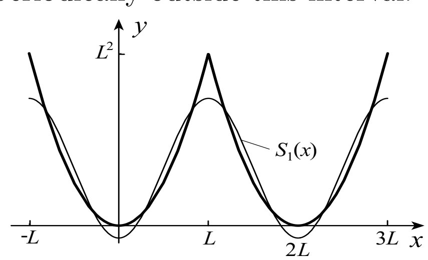


Figure 3.8b

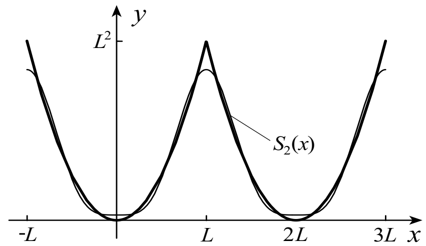


Figure 3.8c

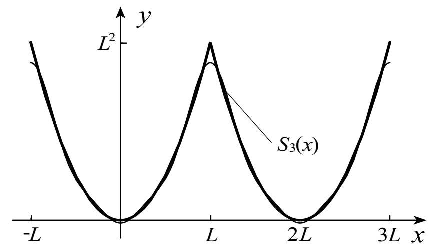


Figure 3.8d

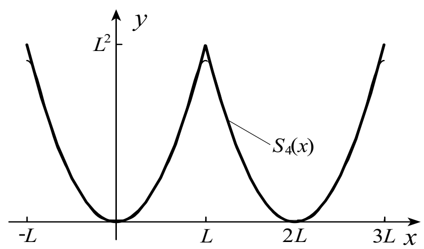


Figure 3.8e

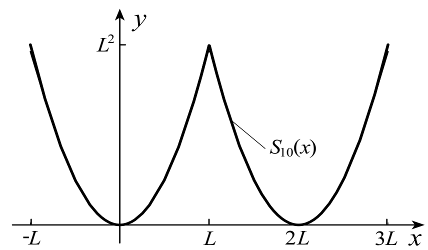


Figure 3.8f

Figure 3.9 illustrates the same partial sums for the function $f(x)$ in Example 3.1, but convergence in this case is much slower. This is easily explained by the fact that the Fourier coefficients in Example 3.2 have a factor n^2 in the denominator, whereas in Example 3.1 the factor is only n. Figure 3.9 also indicates a property of all Fourier series at points of discontinuity of the function $f(x)$. On either side of the discontinuity, the partial sums eventually overshoot $f(x)$, and this overshoot does not diminish in size as more and more terms of the Fourier series are included. This is known as the **Gibbs phenomenon**; it states that for large n, $S_n(x)$ overshoots the curve at a discontinuity by about 9% of the size of the jump in the function. Notice that when x is set equal to $2L$ in the series, all terms vanish except the first, resulting in the value L. Each partial sum in Figure 3.9 passes through the point $(2L, L)$.

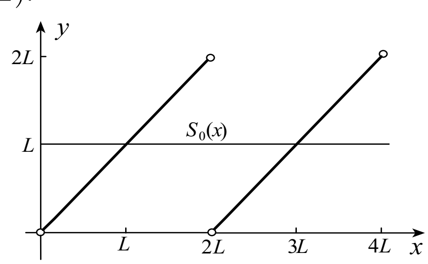


Figure 3.9a

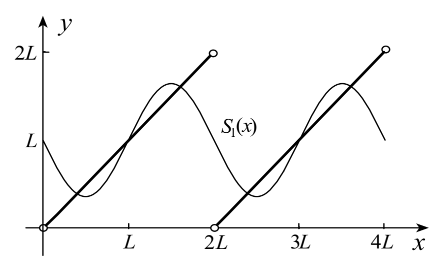


Figure 3.9b

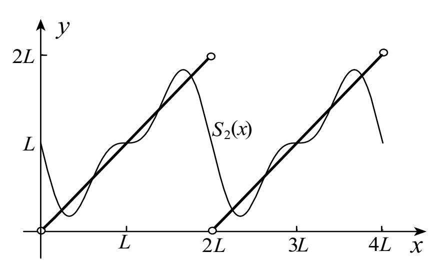


Figure 3.9c

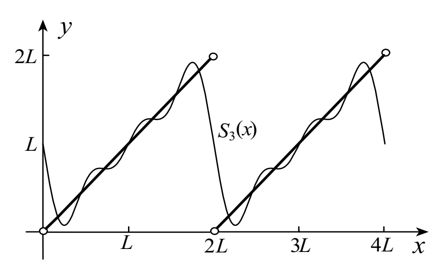


Figure 3.9d

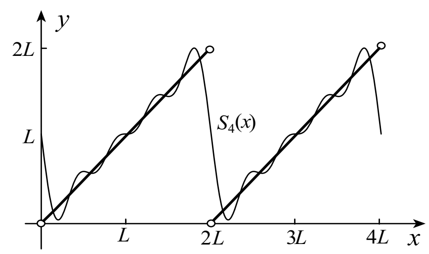


Figure 3.9e

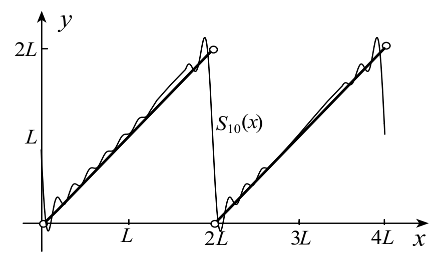


Figure 3.9f

Complex Form for Fourier Series

Integration formulas 3.13b for Fourier coefficients almost invariably involve integration by parts. These integrations can be combined by using what is called the complex form for a Fourier series. With the expressions $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$, we may express the Fourier series of a function $f(x)$ in the form

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{n\pi xi/L} + e^{-n\pi xi/L}}{2} + b_n \frac{e^{n\pi xi/L} - e^{-n\pi xi/L}}{2i}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right) e^{-n\pi xi/L} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right) e^{n\pi xi/L} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right) e^{-n\pi xi/L} + \sum_{n=-1}^{-\infty}\left(\frac{a_{-n} - ib_{-n}}{2}\right) e^{-n\pi x/L} \end{aligned}$$

or,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-n\pi xi/L}, \tag{3.15a}$$

where $c_0 = a_0/2$, $c_n = (a_n + ib_n)/2$ when $n > 0$, and $c_n = (a_{-n} - ib_{-n})/2$ when $n < 0$. It is straightforward to verify using formulas 3.13b that for all n,

$$c_n = \frac{1}{2L}\int_0^{2L} f(x)e^{n\pi xi/L}dx. \tag{3.15b}$$

This is called the **complex form** of Fourier series 3.13. Its compactness is evident, and only one integration is required to determine the complex coefficients c_n. In addition, Fourier coefficients a_n and b_n are easily extracted as real and imaginary parts of c_n.

Example 3.4 Use formula 3.15b to obtain the complex form for the Fourier series of the function in Example 3.1. From the result, derive the trigonometric form of the Fourier series.

Solution According to formula 3.15b,

$$c_n = \frac{1}{2L}\int_0^{2L} x\, e^{n\pi xi/L}dx = \frac{1}{2L}\left\{\frac{Lx}{n\pi i}e^{n\pi xi/L} + \frac{L^2}{n^2\pi^2}e^{n\pi xi/L}\right\}_0^{2L} = -\frac{Li}{n\pi},$$

provided of course that $n \neq 0$. For $n = 0$, we obtain

$$c_0 = \frac{1}{2L}\int_0^{2L} x\,dx = \frac{1}{2L}\left\{\frac{x^2}{2}\right\}_0^{2L} = L.$$

The complex form of the Fourier series for $f(x)$ is therefore

$$f(x) = L + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} -\frac{Li}{n\pi}e^{-n\pi xi/L} = L - \frac{Li}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}e^{-n\pi xi/L}.$$

To obtain the trigonometric form of the Fourier series, we can proceed in two ways. First, we can use the facts that $a_0 = 2c_0$, and $a_n + ib_n = 2c_n$. These give

$$a_0 = 2L, \qquad a_n + ib_n = -\frac{2Li}{n\pi},$$

and therefore $a_n = 0$ and $b_n = -2L/(n\pi)$. The trigonometric form of the Fourier series is

$$f(x) = L + \sum_{n=1}^{\infty} -\frac{2L}{n\pi}\sin\frac{n\pi x}{L}.$$

Alternatively, if we separate the complex series into summations over positive and negative values of n and change variables for the negative integers,

$$\begin{aligned} f(x) &= L - \frac{Li}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,e^{-n\pi xi/L} - \frac{Li}{\pi}\sum_{n=-\infty}^{-1}\frac{1}{n}\,e^{-n\pi xi/L} \\ &= L - \frac{Li}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,e^{-n\pi xi/L} - \frac{Li}{\pi}\sum_{n=1}^{\infty}\frac{1}{-n}\,e^{n\pi xi/L} \\ &= L + \frac{Li}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,(e^{n\pi xi/L} - e^{-n\pi xi/L}) = L + \frac{Li}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(2i\,\sin\frac{n\pi x}{L}\right) \\ &= L - \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\frac{n\pi x}{L}. \bullet \end{aligned}$$

EXERCISES 3.1

In Exercises 1–14 use formulas 3.13b or 3.13c to find the Fourier series for the function $f(x)$. Draw graphs of $f(x)$ and the function to which the series converges in Exercises 1–8, 13, and 14.

1. $f(x) = 3x + 2, \quad 0 < x < 4, \quad f(x+4) = f(x)$

2. $f(x) = 2x^2 - 1, \quad 0 \le x < 2L, \quad f(x+2L) = f(x)$

3. $f(x) = 2x^2 - 1, \quad -L \le x \le L, \quad f(x+2L) = f(x)$

4. $f(x) = 3x, \quad 0 < x \le 2L, \quad f(x+2L) = f(x)$

5. $f(x) = 3x, \quad -L < x \le L, \quad f(x+2L) = f(x)$

6. $f(x) = \begin{cases} 2(L-x), & 0 \le x \le L \\ x - L, & L < x < 2L \end{cases}, \quad f(x+2L) = f(x)$

7. $f(x) = \begin{cases} 2, & 0 < x < 1 \\ 1, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}, \quad f(x+3) = f(x)$

8. $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 2, & 2 \le x \le 4 \\ 6 - x, & 4 \le x \le 6 \end{cases}, \quad f(x+6) = f(x)$

9. $f(x) = 1 + \sin x - \cos 2x$

10. $f(x) = 2\cos x - 3\sin 10x + 4\cos 2x$

11. $f(x) = \cos^2 2x$

12. $f(x) = 3\cos 2x \sin 5x$

13. $f(x) = e^x, \quad 0 < x < 4, \quad f(x+4) = f(x)$

14. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ -2\sin x, & \pi \le x \le 2\pi \end{cases} \quad f(x+2\pi) = f(x)$

15. Verify that the functions in Theorem 3.1 are indeed orthogonal.

16. A student was once heard to say that the Fourier series of a periodic function is not unique. For example, in Example 3.1 the Fourier series of the function in Figure 3.5 was found. The student stated that this function also has period $4L$ and therefore has a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2L} + b_n \sin \frac{n\pi x}{2L} \right).$$

Is this series different from that found in Example 3.1?

In Exercises 17–26 use formula 3.15 to find the complex Fourier series for the given function. Use the result to derive the trigonometric Fourier series.

17. The function in Exercise 1

18. The function in Exercise 7

19. $f(x) = \begin{cases} 1, & 0 < x < L \\ -1, & L < x < 2L \end{cases}, \quad f(x+2L) = f(x)$

20. $f(x) = \begin{cases} x, & 0 < x < L \\ 2L - x, & L < x < 2L \end{cases}, \quad f(x+2L) = f(x)$

21. The function in Example 3.2

22. The function in Exercise 8

23. The function in Exercise 2

24. The function in Exercise 13

25. The function in Exercise 6

26. The function in Example 3.3

27. Is

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{n\pi x i/L} \quad \text{where} \quad d_n = \frac{1}{2L} \int_0^{2L} f(x) e^{-n\pi x i/L} dx$$

an alternative to equation 3.15 for the complex form of the Fourier series of a function $f(x)$? How is d_n related to a_n and b_n in this case?

28. A function $f(x)$ is said to be *odd-harmonic* if $f(x+L) = -f(x)$, wherever it is defined.
(a) Prove that such a function is $2L$-periodic.
(b) Illustrate an odd-harmonic function graphically.
(c) Show that the Fourier series for an odd-harmonic function takes the form

$$f(x) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{(2n-1)\pi x}{L} + b_n \sin \frac{(2n-1)\pi x}{L} \right],$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{L}dx \quad \text{and} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{L}dx.$$

§3.2 Fourier Sine and Cosine Series

When $f(x)$ is a $2L$-periodic, piecewise smooth function, it has Fourier series representation 3.13a with coefficients defined by 3.13b. If, in addition, $f(x)$ is an even function, it is a simple exercise to show that its Fourier coefficients satisfy

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad b_n = 0 \tag{3.16b}$$

(see, for instance, Example 3.2). Thus, the Fourier series of an even function has only cosine terms,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \tag{3.16a}$$

and is called a **Fourier cosine series**.

When $f(x)$ is an odd function, its Fourier coefficients are

$$a_n = 0, \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \tag{3.17b}$$

and therefore the Fourier series of an odd function has only sine terms,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{3.17a}$$

and is called a **Fourier sine series**.

Example 3.5 Find the Fourier series for the function $f(x)$ in Figure 3.10a.

Solution Because $f(x)$ is an odd function of period 2, its Fourier series must be a sine series of the form

$$\sum_{n=1}^{\infty} b_n \sin n\pi x.$$

Coefficients are

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^{1/2} 2x\sin n\pi x\,dx + 2\int_{1/2}^1 -2(x-1)\sin n\pi x\,dx \\ &= 4\left\{\frac{-x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right\}_0^{1/2} - 4\left\{\frac{-(x-1)}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right\}_{1/2}^{1} \\ &= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}. \end{aligned}$$

Because $f(x)$ is continuous for all x, the Fourier series of $f(x)$ converges to $f(x)$ for all x; that is,

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\sin n\pi x = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin(2n-1)\pi x.$$

We have shown the sum of the first two terms of the series in Figure 3.10b. Even with only two terms, we have a reasonable approximation of $f(x)$; in other words, convergence is very rapid, except at the points on the graph where the derivative $f'(x)$ does not exist.●

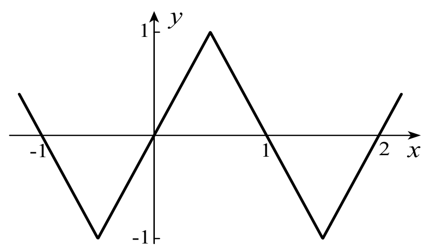


Figure 3.10a

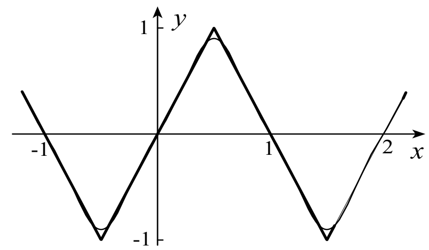


Figure 3.10b

Example 3.6 Find the Fourier series for the function $f(x)$ in Figure 3.11a.

Solution Because $f(x)$ is an even function of period 4, its Fourier series must be a cosine series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}.$$

Coefficients are

$$a_0 = \frac{2}{2}\int_0^2 (4 - x^2)\,dx = \left\{4x - \frac{x^3}{3}\right\}_0^2 = \frac{16}{3};$$

$$a_n = \frac{2}{2}\int_0^2 (4 - x^2)\cos\frac{n\pi x}{2}dx = \left\{\frac{2}{n\pi}(4 - x^2)\sin\frac{n\pi x}{2} - \frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} + \frac{16}{n^2\pi^3}\sin\frac{n\pi x}{2}\right\}_0^2$$
$$= \frac{16(-1)^{n+1}}{n^2\pi^2}.$$

Because $f(x)$ is continuous for all x, we may write

$$f(x) = \frac{8}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{2} = \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{2}.$$

Alternatively, we could have noted that this function is 4 minus the function in Example 3.2 when L is set equal to 2. Hence,

$$f(x) = 4 - \left[\frac{2^2}{3} + \frac{4(2)^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}\right] = \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{2}.$$

We have shown the sum of the first five terms of the series in Figure 3.11b. Even with only five terms, we have a reasonable approximation of $f(x)$, except at the points on the graph where the derivative $f'(x)$ does not exist.●

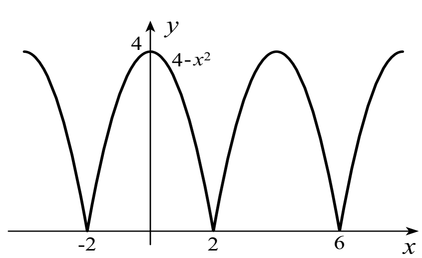


Figure 3.11a

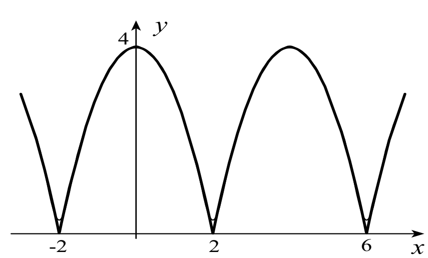


Figure 3.11b

Because we have treated Fourier sine and cosine series as special cases of the full Fourier series in Section 3.1, they have been approached from the following point of view: *Can an even (or odd)* $2L$*-periodic function* $f(x)$ *be expressed in a Fourier series of form* 3.16a (*or* 3.17a)?

When sine and cosine series are used to solve (initial) boundary value problems, they arise in a different way. Sine series arise from a need to answer the following question: *Suppose a function* $f(x)$ *is defined for* $0 < x < L$ *and is piecewise smooth for* $0 \le x \le L$. *Is it possible to represent* $f(x)$ *in a series of the form*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{3.18}$$

valid for $0 < x < L$?

Notice that $f(x)$ is not odd and it is not periodic; it is defined only between $x = 0$ and $x = L$. But by appropriately extending $f(x)$ outside the interval $0 < x < L$, we shall indeed be able to write it in form 3.18. First, we recognize that equation 3.18 is identical to 3.17a, the Fourier sine series of an odd function. We therefore extend the domain of definition of $f(x)$ to include $-L < x < 0$ by demanding that the extension be odd; that is, we define $f(x) = -f(-x)$ for $-L < x < 0$. For example, if $f(x)$ is as shown in Figure 3.12a, it is extended as shown in Figure 3.12b. Next, we know that series 3.17a represents a $2L$-periodic function. We therefore extend the domain of definition of $f(x)$ beyond $-L < x < L$ by making it $2L$-periodic (Figure 3.12c).

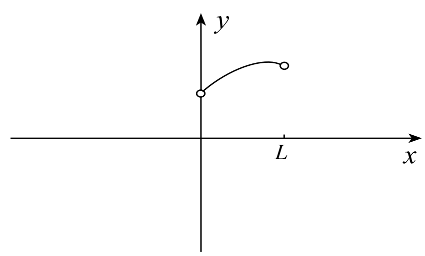


Figure 3.12a

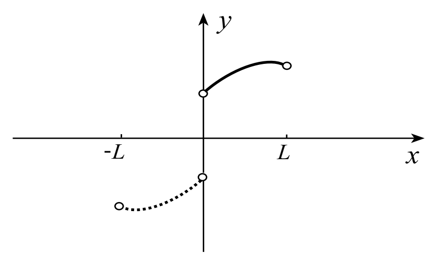


Figure 3.12b

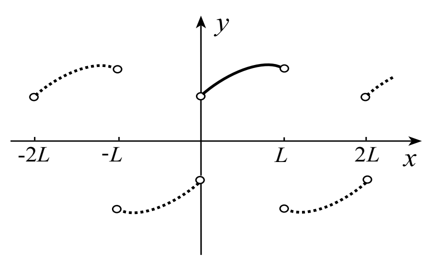


Figure 3.12c

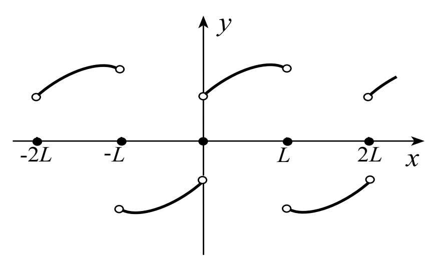


Figure 3.12d

We have now extended $f(x)$, which was originally defined only for $0 < x < L$, to an odd, $2L$-periodic function. Because $f(x)$ was piecewise smooth on $0 \le x \le L$, the extended function is piecewise smooth for all x. As a result, the extended function can be represented in Fourier sine series 3.17a, with coefficients defined by 3.17b, and this series converges to the average value of right- and left-hand limits at every point (Figure 3.12d). Since this extension does not affect its original values on $0 < x < L$, it follows that the Fourier sine series of the extension must represent

$f(x)$ on $0 < x < L$. Finally, we should note that the series converges to 0 at $x = 0$ and $x = L$.

In summary, when we are required to express a function $f(x)$, defined for $0 < x < L$, in form 3.18, we use the Fourier sine series of the odd, $2L$-periodic extension of $f(x)$.

In a similar way, if we are required to express a function $f(x)$, defined for $0 < x < L$, and piecewise smooth on $0 \leq x \leq L$, in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \tag{3.19}$$

we use the Fourier cosine series of the even, $2L$-periodic extension of $f(x)$. For the function $f(x)$ in Figure 3.12a, this extension is as shown in Figure 3.13a. The series converges to $\lim_{x\to 0^+} f(x)$ at $x = 0$, and to $\lim_{x\to L^-} f(x)$ at $x = L$; that is, to the continuous function in Figure 3.13b.

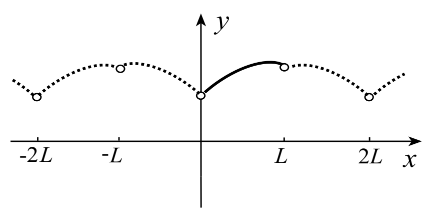


Figure 3.13a

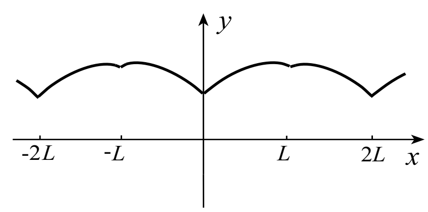


Figure 3.13b

Example 3.7 Find coefficients b_n so that

$$1 + 2x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}$$

for all x in the interval $0 < x < 3$.

Solution Constants b_n must be the coefficients in the Fourier sine series of the extension of $1 + 2x$ to an odd function of period 6 (Figure 3.14a). According to formulas 3.17b,

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^3 (1 + 2x) \sin \frac{n\pi x}{3} dx = \frac{2}{3} \left\{ \frac{-3}{n\pi}(1 + 2x) \cos \frac{n\pi x}{3} + \frac{18}{n^2\pi^2} \sin \frac{n\pi x}{3} \right\}_0^3 \\ &= \frac{2}{n\pi}[1 + 7(-1)^{n+1}]. \end{aligned}$$

Consequently,

$$1 + 2x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 + 7(-1)^{n+1}}{n} \sin \frac{n\pi x}{3}, \quad 0 < x < 3.$$

At $x = 0$ and $x = 3$, the series does not converge to $1 + 2x$; it converges to zero, the average value of right- and left-hand limits of the odd, periodic extension of $1 + 2x$. We have shown the twentieth partial sum of the series in Figure 3.14b in order to illustrate convergence of the series to $1 + 2x$. Convergence is slow at $x = 0$ and $x = 3$ because of the discontinuities of the extension at these points. The overshoot at $x = 3$ is larger than that at $x = 0$ because the magnitude of the discontinuity at

$x = 3$ is larger (as is predicted by the Gibb's phenomenon). Notice also that the partial sum crosses the x-axis at $x = 0$ and $x = 3$, as does every partial sum.●

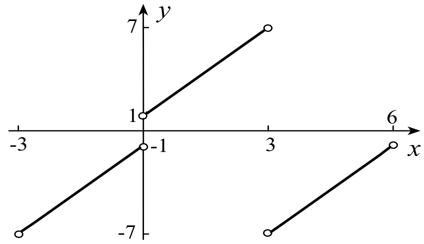


Figure 3.14a

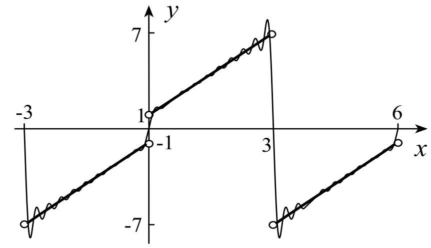


Figure 3.14b

Example 3.8 Find coefficients a_n so that

$$1 + 2x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3}$$

for all x in the interval $0 < x < 3$.

Solution Constants a_n must be the coefficients in the Fourier cosine series of the extension of $1 + 2x$ to an even function of period 6 (Figure 3.15a). According to formulas 3.16b,

$$a_0 = \frac{2}{3}\int_0^3 (1+2x)\,dx = \frac{2}{3}\left\{x + x^2\right\}_0^3 = 8;$$

$$a_n = \frac{2}{3}\int_0^3 (1+2x)\cos\frac{n\pi x}{3}dx = \frac{2}{3}\left\{\frac{3}{n\pi}(1+2x)\sin\frac{n\pi x}{3} + \frac{18}{n^2\pi^2}\cos\frac{n\pi x}{3}\right\}_0^3$$

$$= \frac{12}{n^2\pi^2}[(-1)^n - 1].$$

Consequently,

$$1 + 2x = 4 + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi x}{3}.$$

Terms in the series vanish when n is even. To display only the odd terms, we replace n by $2n - 1$ and sum from $n = 1$ to infinity:

$$1 + 2x = 4 - \frac{24}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{3}, \quad 0 < x < 3.$$

At $x = 0$ and $x = 3$, the series converges to 1 and 7, respectively (these being average values of right- and left-hand limits of the even, periodic extension), so that the series actually represents $1 + 2x$ for $0 \le x \le 3$. The third partial sum of the series is shown in Figure 3.15b. It is practically indistinguishable from $1+2x$ except at $x = 0$ and $x = 3$ where the extension has no derivative.●

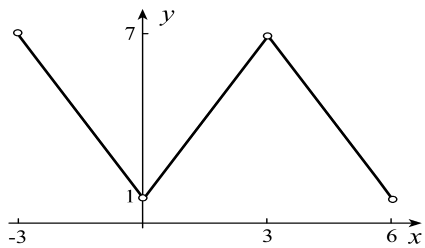


Figure 3.15a

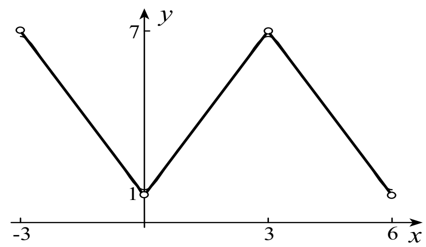


Figure 3.15b

EXERCISES 3.2

In Exercises 1–5 find the Fourier series for the function $f(x)$. Draw graphs of $f(x)$ and the function to which the series converges in Exercises 2–5.

1. $f(x) = 2\sin 4x - 3\sin x$

2. $f(x) = |x|, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x)$

3. $f(x) = \begin{cases} x, & -4 \le x \le 4, \\ 8 - x, & 4 \le x \le 12, \end{cases}, \quad f(x + 16) = f(x)$

4. $f(x) = 2x^2 - 1, \quad -L \le x \le L, \quad f(x + 2L) = f(x)$

5. $f(x) = \begin{cases} \cos x, & -\pi/2 \le x \le \pi/2 \\ 0, & \pi/2 < x < 3\pi/2 \end{cases} \quad f(x + 2\pi) = f(x)$

In Exercises 6–10 expand $f(x)$ in terms of the functions $\{\sin(n\pi x/L)\}$. Suppose the domain of $f(x)$ is extended to $0 \le x \le L$ by making it continuous from the right at $x = 0$ and continuous from the left at $x = L$. Determine whether the series converges to $f(0)$ and $f(L)$ in Exercises 6–9. Do this algebraically and also by using properties of Fourier series.

6. $f(x) = -x, \quad 0 < x < L$

7. $f(x) = \begin{cases} 1, & 0 < x < L/3 \\ 0, & L/3 < x < 2L/3 \\ -1, & 2L/3 < x < L \end{cases}$

8. $f(x) = \begin{cases} L/4, & 0 < x \le L/4 \\ L/2 - x, & L/4 < x \le L/2 \\ x - L/2, & L/2 < x < 3L/4 \\ L/4, & 3L/4 \le x < L \end{cases}$

9. $f(x) = Lx - x^2, \quad 0 < x < L$

10. $f(x) = \sin(\pi x/L)\cos(\pi x/L)$

In Exercises 11–15 expand $f(x)$ in terms of the functions $\{1, \cos(n\pi x/L)\}$. Suppose the domain of $f(x)$ is extended to $0 \le x \le L$ by making it continuous from the right at $x = 0$ and continuous from the left at $x = L$. Determine whether the series converges to $f(0)$ and $f(L)$ in Exercises 11–13. Do this algebraically and also by using properties of Fourier series.

11. $f(x) = -x, \quad 0 < x < L$

12. $f(x) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$

13. $f(x) = Lx - x^2, \quad 0 < x < L$ **14.** $f(x) = 1, \quad 0 < x < L$

15. $f(x) = \sin(\pi x/L)\cos(\pi x/L), \quad 0 < x < L$

16. Find the Fourier series for the function $f(x) = |\sin x|$ by using the fact that the function has period π. What series is obtained if period 2π is used?

17. Under what additional condition is it possible to express a function $f(x)$ that is piecewise smooth on $0 \le x \le L$ in the form

$$f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}?$$

18. In this exercise, we summarize results seen in Exercises 6–9 and 11–13. Illustrate with graphs that when a function $f(x)$, defined on the interval $0 \le x \le L$, is continuous (from the right) at $x = 0$,

(a) the Fourier cosine series of the even, $2L$-periodic extension of $f(x)$ always converges to $f(0)$ at $x = 0$;

(b) the Fourier sine series of the odd, $2L$-periodic extension of $f(x)$ converges to $f(0)$ at $x = 0$ if and only if $f(0) = 0$.

(c) Are similar statements to those in parts (a) and (b) correct at $x = L$?

19. (a) Find the Fourier series for the function

$$f(x) = \begin{cases} x, & 0 \le x \le L \\ 2L - x, & L < x \le 2L \end{cases}, \qquad f(x + 2L) = f(x).$$

Use this result to find Fourier series for the following functions:

(b) $f_1(x) = L - |x|, \quad -L \le x \le L, \quad f_1(x + 2L) = f_1(x)$

(c) $f_2(x) = 2L - |2L - x|, \quad 0 < x < 4L, \quad f_2(x + 4L) = f_2(x)$

(d) $f_3(x) = x, \quad -L < x < L, \quad f_3(L + x) = f_3(L - x), \quad f_3(x + 4L) = f_3(x)$

20. (a) A function $f(x)$ is said to be odd and odd-harmonic if it satisfies the conditions

$$f(-x) = -f(x), \qquad f(L + x) = f(L - x).$$

Show that such a function is $4L$-periodic.

(b) Illustrate an odd, odd-harmonic function graphically. Is it symmetric about the line $x = L$?

(c) Show that the Fourier series of an odd, odd-harmonic function takes the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2L} \quad \text{where} \quad b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} dx.$$

(d) Is an odd and odd-harmonic function odd-harmonic according to Exercise 28 in Section 3.1?

21. (a) A function $f(x)$ is said to be even and odd-harmonic if it satisfies the conditions

$$f(-x) = f(x), \qquad f(L + x) = -f(L - x).$$

Show that such a function is $4L$-periodic.

(b) Illustrate an even, odd-harmonic function graphically. Is it antisymmetric about the line $x = L$?

(c) Show that the Fourier series of an even, odd-harmonic function takes the form

$$f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{(2n-1)\pi x}{2L} \quad \text{where} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} dx.$$

(d) Is an even and odd-harmonic function odd-harmonic according to Exercise 28 in Section 3.1?

§3.3 Uniform Convergence and Convergence in the Mean

In Sections 3.1 and 3.2, we indicated that the Fourier series of a piecewise smooth function $f(x)$ converges on a point-by-point basis to $[f(x+) + f(x-)]/2$. In this section, we introduce two other types of convergence that are particularly important for Fourier series, but do so in a general setting rather than in the restricted environment of Fourier series. We discuss them in the context of Fourier series in Section 3.4.

Uniform Convergence

A series of functions $\sum_{n=1}^{\infty} u_n(x)$ converges to (or has sum) $S(x)$ if its sequence of partial sums $\{S_n(x)\}$ converges to $S(x)$. This is true if, given any $\epsilon > 0$, there exists an integer N such that $|S_n(x) - S(x)| < \epsilon$ whenever $n > N$. Usually N is a function of ϵ and x; in particular, the choice of N may vary from x to x. What this means is that convergence of $\{S_n(x)\}$ to $S(x)$ may be faster for some x's than others. If it is possible to find an N, independent of x, such that $|S_n(x) - S(x)| < \epsilon$ for all $n > N$ and all x in some interval I, then $\sum_{n=1}^{\infty} u_n(x)$ is said to converge **uniformly** to $S(x)$ in I. The word *uniform* is perhaps a misnomer. When N is independent of x, convergence is not necessarily uniformly fast for all x's; the rate of convergence may still, and does, vary from x to x. What we can say is that convergence does not become indefinitely slow for some x's in I. In practice, what often happens is that there is an x_0 in I at which convergence is slowest; for all other x's, convergence is more rapid than at this x_0. In this case, convergence is uniform. The most widely used test for uniform convergence of a series is the Weierstrass M-test.

Theorem 3.3 (**Weierstrass M-test**) If a convergent series of (positive) constants $\sum_{n=1}^{\infty} M_n$ can be found such that $|S_n(x)| \le M_n$ for each n and all x in I, then $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent in I.

An excellent example to illustrate these ideas is the geometric series $\sum_{n=0}^{\infty} x^n$. It is well known that this series converges to $1/(1-x)$ on the interval $-1 < x < 1$. In Figure 3.16 we show the five partial sums $S_1(x) = 1$, $S_2(x) = 1+x$, $S_3(x) = 1+x+x^2$, $S_4(x) = 1+x+x^2+x^3$, and $S_5(x) = 1+x+x^2+x^3+x^4$, as well as $S(x) = 1/(1-x)$. They indicate that convergence of the partial sums $S_n(x)$ to $S(x)$ is rapid for values of x close to zero, but as x approaches ± 1, convergence becomes much slower. We can demonstrate this algebraically by noting that

$$S(x) - S_n(x) = \frac{1}{1-x} - \frac{1-x^n}{1-x} = \frac{x^n}{1-x}.$$

This is the difference between the sum of the series and its n^{th} partial sum. As x approaches 1, the difference becomes very large; near $x = -1$, it oscillates back and forth between numbers close to $\pm 1/2$.

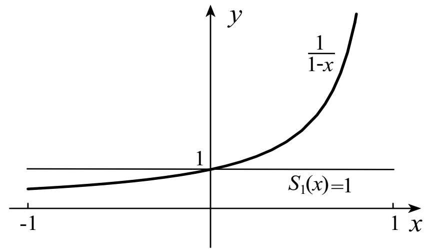


Figure 3.16a

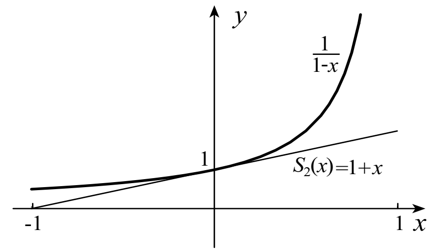


Figure 3.16b

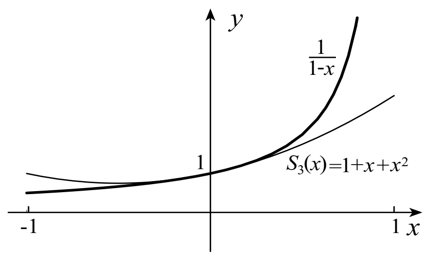


Figure 3.16c

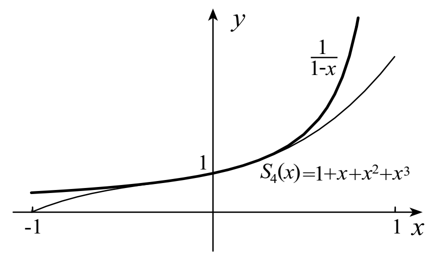


Figure 3.16d

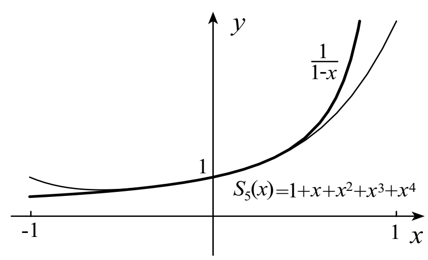


Figure 3.16e

When x is confined to the interval $|x| \leq a < 1$, we can state that $|x^n| \leq a^n$, and since $\sum_{n=0}^{\infty} a^n$ converges, it follows that the geometric series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $|x| \leq a < 1$. Convergence is slowest at $x = a$; at all other points in $|x| \leq a$, it converges more rapidly than it does at $x = a$. The series does not, however, converge uniformly on the interval $|x| < 1$; convergence becomes indefinitely slow as $x \to \pm 1$.

The Weierstrass M-test is easily generalized to series whose terms are functions of more than one variable. For example, $\sum_{n=1}^{\infty} u_n(x,y)$ is uniformly convergent for points (x,y) in a region R of the xy-plane if there exists a convergent series of constants $\sum_{n=1}^{\infty} M_n$ such that for each n and all (x,y) in R, $|u_n(x,y)| \leq M_n$.

Series of the following form arise in almost all phases of our work:

$$\sum_{n=1}^{\infty} X_n(x) Y_n(y),$$

that is, series in which each term is a function $X_n(x)$ of x multiplied by a function $Y_n(y)$ of y. We find Abel's test useful in establishing uniform convergence of such series.

Theorem 3.4 (**Abel's Test**) A series $\sum_{n=1}^{\infty} X_n(x)Y_n(y)$ converges uniformly in a region $\overline{R}$ of the xy-plane if:

(1) the series $\sum_{n=1}^{\infty} X_n(x)$ converges uniformly with respect to x for all x such that (x, y) is in $\overline{R}$;

(2) the functions $Y_n(y)$ are uniformly bounded* for all y such that (x, y) is in $\overline{R}$;

(3) for each y such that (x, y) is in $\overline{R}$, the sequence of constants $\{Y_n(y)\}$ is nonincreasing.

As further explanation of these conditions, suppose $\overline{R}$ is the *closed* region in Figure 3.17 consisting of the area R inside the curve plus the bounding curve $\beta(R)$. Condition 1 requires $\sum_{n=1}^{\infty} X_n(x)$ to be uniformly convergent for $a \leq x \leq b$. Conditions 2 and 3 must be satisfied for $c \leq y \leq d$. Of course, the roles of $X_n(x)$ and $Y_n(y)$ could be reversed.

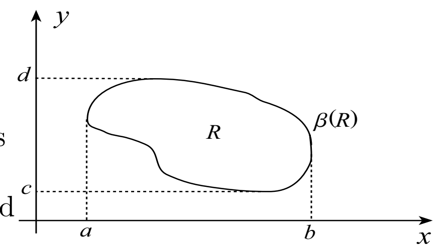


Figure 3.17

Example 3.9 Show that the series $\displaystyle\sum_{n=0}^{\infty} x^n y^n$ is uniformly convergent on the rectangle $|x| \leq a < 1$, $|y| \leq 1$.

Solution We have already seen that the series $\sum_{n=0}^{\infty} x^n$ is uniformly convergent for $|x| \leq a < 1$. The sequence of functions $\{y^n\}$ is uniformly bounded by 1 for $|y| \leq 1$, and for each fixed y in $|y| \leq 1$, the sequence is nonincreasing. Hence, by Abel's test, the series is uniformly convergent. Alternatively, since $|x^n y^n| \leq a^n$ and $\sum_{n=0}^{\infty} a^n$ converges, the given series converges uniformly by the Weierstrass M-test.●

It is a well-known fact that the sum of finitely many continuous functions is a continuous function. On the other hand, the sum of infinitely many continuous functions may not be a continuous function. Fourier series are prime examples; each term in a Fourier series is continuous, but the sum of the terms may well be discontinuous (see Example 3.1). When convergence is uniform, the following result indicates that this cannot happen.

Theorem 3.5 A uniformly convergent series of continuous functions converges to a continuous function.

This means that convergence of the Fourier series of a discontinuous function cannot be uniform over any interval that contains a point of discontinuity.

In many applications of series, it is necessary to integrate a series term-by-term. According to the following theorem, this is possible when the series converges uniformly.

Theorem 3.6 When a series $\displaystyle\sum_{n=1}^{\infty} u_n(x)$ of continuous functions converges uniformly to $S(x)$ on an interval $a \leq x \leq b$,

---

* A sequence of functions $\{Y_n(y)\}$ is said to be uniformly bounded on an interval I if there exists a constant M such that $|Y_n(y)| \leq M$ for all x in I and all n.

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx. \tag{3.20}$$

More important to our discussions of Fourier series and partial differential equations are sufficient conditions for term-by-term differentiation of a series. These are given in the next theorem.

Theorem 3.7 Suppose $\sum_{n=1}^{\infty} u_n(x) = S(x)$ for $a \leq x \leq b$. Then

$$S'(x) = \sum_{n=1}^{\infty} {u_n}'(x), \quad a \leq x \leq b, \tag{3.21}$$

provided each ${u_n}'(x)$ is continuous for $a \leq x \leq b$ and the series $\sum_{n=1}^{\infty} {u_n}'(x)$ is uniformly convergent on $a \leq x \leq b$.

Convergence in the Mean

Suppose that a series $\sum_{n=1}^{\infty} u_n(x)$ converges to $f(x)$ on some interval $a \leq x \leq b$. For any x in $a \leq x \leq b$, the difference $f(x) - \sum_{k=1}^{n} u_k(x)$, is the error in using the first n terms of the series to approximate $f(x)$ at that x. It might be positive; it might be negative; it might even be zero. One possibility to assign an overall error in the approximation of $f(x)$ by its n^{th} partial sum on $a \leq x \leq b$ would be to define it as

$$\max_{a \leq x \leq b} \left| f(x) - \sum_{k=1}^{n} u_k(x) \right|.$$

Overall error is error where the approximation is worst. Although this may be reasonable as an overall error, it turns out to be difficult to implement. Much more practical is what is called the *mean square error*, defined as follows.

Definition 3.5 Suppose $f(x) = \sum_{n=1}^{\infty} u_n(x)$ on the interval $a \leq x \leq b$. The **mean square error** in approximating $f(x)$ by its n^{th} partial sum, with respect to a weight function $p(x)$, is

$$E_n = \int_a^b \left[f(x) - \sum_{k=1}^{n} u_k(x) \right]^2 p(x)\,dx. \tag{3.22}$$

We will see the necessity for a weight function, which is always nonnegative, when we study Sturm-Liouville systems in Chapter 5. For ordinary Fourier series, the weight function is unity, and therefore we assume that $p(x) = 1$ for the remainder of this chapter. Mean square error regards the square of the difference between $f(x)$ and its partial sum as the error at each value of x, and adds, in integral form, errors at all points in the interval. We say the series **converges in the mean** to $f(x)$ if

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \int_a^b \left[f(x) - \sum_{k=1}^{n} u_k(x) \right]^2 p(x)\,dx = 0. \tag{3.23}$$

When a series converges in the mean, the mean square error becomes less and less as n gets larger and larger. The partial sums of the series approximate the sum better and better over the entire interval as more and more terms are included.

EXERCISES 3.3

In Exercises 1–5 show that the series is uniformly convergent on the given region R.

1. $\displaystyle\sum_{n=0}^{\infty} x^n y^n \qquad R\colon\ x^2+y^2 \le a^2,\ 0<a<1$

2. $\displaystyle\sum_{n=0}^{\infty} n^2 x^n y^n \qquad R\colon\ |x| \le a < 1,\ |y| \le 1$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n y^n \ln x}{2^n} \qquad 0 < x \le a < 1,\ |y| \le 2$

4. $\displaystyle\sum_{n=0}^{\infty} e^{-ny} \sin nx \qquad -\infty < x < \infty,\ y \ge a > 0$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\cosh n\pi y \sin n\pi x}{\sinh n\pi} \qquad 0 \le x \le 1,\ 0 \le y \le a < 1$

§3.4 Properties of Fourier Series

In Sections 3.1 and 3.2, we indicated that the Fourier series of a piecewise smooth function $f(x)$ converges on a point-by-point basis to $[f(x+)+f(x-)]/2$. In this section, we discuss conditions that guarantee that the Fourier series convergence uniformly to its sum, and also convergences in the mean to its sum. In addition, we take the opportunity to develop properties of Fourier series that are pertinent to our discussions on partial differential equations.

Theorem 3.7 in Section 3.3 states that a series can be differentiated term-by-term if the differentiated series converges uniformly. The following theorem gives conditions on a function $f(x)$, rather than its differentiated series, that determine when its Fourier series can be differentiated term-by-term.

Theorem 3.8 If $f(x)$ is a continuous function of period $2L$ with piecewise continuous derivatives $f'(x)$ and $f''(x)$, the Fourier series of $f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

can be differentiated term-by-term to yield

$$f'(x) = \frac{f'(x+)+f'(x-)}{2} = \frac{\pi}{L} \sum_{n=1}^{\infty} n \left(-a_n \sin \frac{n\pi x}{L} + b_n \cos \frac{n\pi x}{L} \right). \qquad (3.24)$$

Proof Because $f'(x)$ is piecewise smooth, its Fourier series converges to $[f'(x+)+f'(x-)]/2$ for each x,

$$\frac{f'(x+)+f'(x-)}{2} = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right),$$

where

$$A_0 = \frac{1}{L} \int_0^{2L} f'(x)\, dx = \frac{1}{L} \Big\{ f(x) \Big\}_0^{2L} = 0;$$

$$A_n = \frac{1}{L} \int_0^{2L} f'(x) \cos \frac{n\pi x}{L} dx = \frac{1}{L} \left\{ f(x) \cos \frac{n\pi x}{L} \right\}_0^{2L} + \frac{n\pi}{L^2} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx = \frac{n\pi}{L} b_n;$$

$$B_n = \frac{1}{L} \int_0^{2L} f'(x) \sin \frac{n\pi x}{L} dx = \frac{1}{L} \left\{ f(x) \sin \frac{n\pi x}{L} \right\}_0^{2L} - \frac{n\pi}{L^2} \int_0^{2L} f(x) \cos \frac{n\pi x}{L} dx = -\frac{n\pi}{L} a_n.$$

Consequently,

$$\frac{f'(x+)+f'(x-)}{2} = \frac{\pi}{L} \sum_{n=1}^{\infty} n \left(-a_n \sin \frac{n\pi x}{L} + b_n \cos \frac{n\pi x}{L} \right). \blacksquare$$

Example 3.10 The function in Figure 3.18a is the derivative of the function in Figure 3.7 (for Example 3.3). Find its Fourier series.

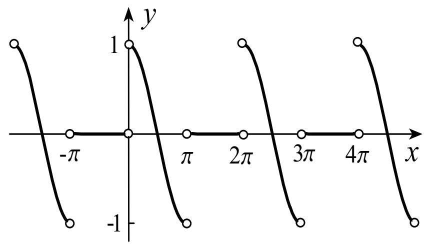


Figure 3.18a

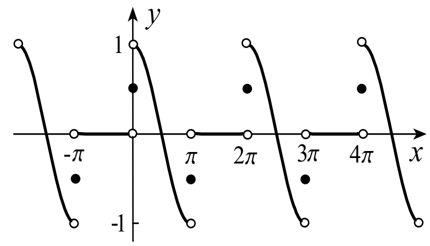


Figure 3.18b

Solution The function in Figure 3.7 is continuous and has Fourier series

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}.$$

Since $f'(x)$ and $f''(x)$ are piecewise continuous, we may differentiate this series term-by-term and write

$$f'(x) = \frac{1}{2}\cos x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2n\sin 2nx}{4n^2-1} = \frac{1}{2}\cos x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx,$$

provided we understand that $f'(x)$ is the function in Figure 3.18b; that is, provided we define $f'(n\pi) = (-1)^n/2$.●

According to the corollary to Theorem 3.2, the Fourier series of a periodic, piecewise smooth function $f(x)$ converges to $[f(x+) + f(x-)]/2$. Convergence is faster at some points, slower at others. In particular, our examples have shown that convergence is slow near points of discontinuity of the function. According to Theorem 3.5, a Fourier series cannot converge uniformly over an interval that contains a discontinuity because a uniformly convergent series of continuous functions always converges to a continuous function. Theorem 3.10 guarantees uniform convergence when $f(x)$ is continuous and has a piecewise continuous first derivative. In order to verify this, we require the following result.

Theorem 3.9 If $f(x)$ is a piecewise continuous funtion on $0 \le x \le 2L$, its Fourier coefficients must satisfy **Bessel's inequality**

$$\frac{{a_0}^2}{2} + \sum_{k=1}^{n}({a_k}^2 + {b_k}^2) \le \frac{1}{L}\int_0^{2L}[f(x)]^2\,dx. \tag{3.25}$$

Proof: The mean square error when a function $f(x)$ is approximated by the first $2n+1$ terms in its Fourier series is

$$\begin{aligned}E_n &= \int_0^{2L}\left[f(x) - \frac{a_0}{2} - \sum_{k=1}^{n}\left(a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right)\right]^2 dx\\ &= \int_0^{2L}[f(x)]^2\,dx - 2\int_0^{2L} f(x)\left[\frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right)\right]dx\\ &\quad + \int_0^{2L}\left\{\frac{a_0^2}{4} + \sum_{k=1}^{n}\left[a_k^2\cos^2\frac{k\pi x}{L} + b_k^2\sin^2\frac{k\pi x}{L}\right] + a_0\sum_{k=1}^{n}\left(a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right)\right.\end{aligned}$$

$$+ 2\sum_{i,j=1}^{n} a_i b_j \cos\frac{i\pi x}{L}\sin\frac{j\pi x}{L} + 2\sum_{i>j=1}^{n}\left(a_i a_j \cos\frac{i\pi x}{L}\cos\frac{j\pi x}{L} + b_i b_j \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}\right)\Bigg\}dx$$

$$= \int_0^{2L}[f(x)]^2\,dx - a_0(La_0) - 2\sum_{k=1}^{n}[a_k(La_k) + b_k(Lb_k)] + \frac{a_0^2}{4}(2L) + \sum_{k=1}^{n}[a_k^2(L) + b_k^2(L)]$$

$$= \int_0^{2L}[f(x)]^2\,dx - L\left[\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right].$$

Bessel's inequality is an immediate consequence of the fact that this quantity must be nonnegative. ∎

Since Bessel's inequality must be valid for any positive integer n, we can also state that

$$\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2) \le \frac{1}{L}\int_0^{2L}[f(x)]^2\,dx. \tag{3.26}$$

In Theorem 3.11 it is shown that, with more restrictive conditions on the function $f(x)$, inequality 3.26 may be replaced by an equality, the result being known as Parseval's theorem, and this leads to convergence in the mean of Fourier series. But first we discuss uniform convergence of Fourier series.

Theorem 3.10 If a $2L$-periodic function $f(x)$ is continuous and has a piecewise continuous first derivative, its Fourier series converges uniformly and absolutely to $f(x)$.

Proof The conditions on $f(x)$ and $f'(x)$ ensure pointwise convergence of the Fourier series of $f(x)$ to $f(x)$ for each x,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right).$$

Since each term in this series may be expressed in the form

$$a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L} = \sqrt{{a_n}^2 + {b_n}^2}\,\sin\left(\frac{n\pi x}{L} + \phi_n\right),$$

it follows that

$$\left|\frac{a_0}{2}\right| + \sum_{n=1}^{\infty}\left|a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right| \le \frac{|a_0|}{2} + \sum_{n=1}^{\infty}\sqrt{{a_n}^2 + {b_n}^2}. \tag{3.27}$$

Uniform and absolute convergence of the Fourier series of $f(x)$ will be established once the series $\sum_{n=1}^{\infty}\sqrt{{a_n}^2 + {b_n}^2}$ is shown to be convergent. If

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$$

is the Fourier series for $f'(x)$, its Fourier coefficients are related to those of $f(x)$ by the equations

$$A_n = \frac{n\pi}{L}b_n, \qquad B_n = \frac{-n\pi}{L}a_n$$

(see the proof of Theorem 3.8). Thus,

$$\sum_{n=1}^{m}\sqrt{{a_n}^2+{b_n}^2}=\sum_{n=1}^{m}\frac{L}{n\pi}\sqrt{{A_n}^2+{B_n}^2}=\frac{L}{\pi}\sum_{n=1}^{m}\frac{\sqrt{{A_n}^2+{B_n}^2}}{n}. \tag{3.28}$$

To proceed further, we require a result called **Schwarz's inequality**. It states that for arbitrary finite sequences $\{c_n\}$ and $\{d_n\}$ of nonnegative numbers,

$$\sum_{n=1}^{m}c_nd_n\le\left(\sum_{n=1}^{m}{c_n}^2\right)^{1/2}\left(\sum_{n=1}^{m}{d_n}^2\right)^{1/2}. \tag{3.29}$$

This is verified in Exercise 5. When it is applied to the series $\sum_{n=1}^{m}\sqrt{{A_n}^2+{B_n}^2}/n$ on the right side of equation 3.28, we obtain

$$\sum_{n=1}^{m}\sqrt{{a_n}^2+{b_n}^2}\le\frac{L}{\pi}\left(\sum_{n=1}^{m}\frac{1}{n^2}\right)^{1/2}\left[\sum_{n=1}^{m}({A_n}^2+{B_n}^2)\right]^{1/2}.$$

Since $\sum_{n=1}^{m}1/n^2<\sum_{n=1}^{\infty}1/n^2=\pi^2/6$ (see Example 3.2), it follows that

$$\sum_{n=1}^{m}\sqrt{{a_n}^2+{b_n}^2}\le\frac{L}{\sqrt{6}}\left[\sum_{n=1}^{m}({A_n}^2+{B_n}^2)\right]^{1/2}.$$

But Bessel's inequality 3.26 applied to the Fourier series for $f'(x)$ gives

$$\sum_{n=1}^{m}({A_n}^2+{B_n}^2)<\sum_{n=1}^{\infty}({A_n}^2+{B_n}^2)\le\frac{1}{L}\int_0^{2L}[f'(x)]^2\,dx-\frac{{A_0}^2}{2}.$$

Consequently,

$$\sum_{n=1}^{m}\sqrt{{a_n}^2+{b_n}^2}\le\frac{L}{\sqrt{6}}\left(\frac{1}{L}\int_0^{2L}[f'(x)]^2\,dx-\frac{{A_0}^2}{2}\right)^{1/2}.$$

Because this inequality is valid for any integer m whatsoever, it follows that the series $\sum_{n=1}^{\infty}\sqrt{{a_n}^2+{b_n}^2}$ converges. Inequality 3.27 then indicates that the Fourier series of $f(x)$ converges uniformly and absolutely. ∎

As mentioned prior to the theorem, continuity of $f(x)$ is indispensable for uniform convergence. A Fourier series cannot converge uniformly over an interval that contains a discontinuity because a uniformly convergent series of continuous functions always converges to a continuous function (Theorem 3.5). If $f(x)$ is defined only on the interval $0\le x\le 2L$, continuity of its periodic extension requires that $f(2L)=f(0)$.

When $f(x)$ satisfies the conditions of Theorem 3.10, Bessel's inequality 3.26 may be replaced by an equality. This result is contained in the next theorem.

Theorem 3.11 (**Parseval's Thoerem**) If $f(x)$ is a $2L$-periodic function that is continuous and has a piecewise continuous first derivative, its Fourier coefficients satisfy

$$\frac{{a_0}^2}{2}+\sum_{n=1}^{\infty}({a_n}^2+{b_n}^2)=\frac{1}{L}\int_0^{2L}[f(x)]^2\,dx. \tag{3.30}$$

Proof With the conditions cited on $f(x)$, the Fourier series of $f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

is uniformly convergent (Theorem 3.10). It may therefore be multiplied by $f(x)$ and integrated term-by-term between 0 and $2L$ to yield

$$\int_0^{2L} [f(x)]^2\, dx = \frac{a_0}{2} \int_0^{2L} f(x)\, dx + \sum_{n=1}^{\infty} \left[a_n \int_0^{2L} f(x) \cos \frac{n\pi x}{L} dx + b_n \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx \right]$$
$$= \frac{a_0}{2}(a_0 L) + \sum_{n=1}^{\infty} [a_n(La_n) + b_n(Lb_n)].$$

Thus,

$$\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} ({a_n}^2 + {b_n}^2) = \frac{1}{L} \int_0^{2L} [f(x)]^2\, dx. \blacksquare$$

This theorem can also be proved (albeit by different methods) when $f(x)$ is only piecewise smooth and $2L$-periodic. With Parseval's Theorem, it is now possible to verify that Fourier series converge in the mean. We state it for functions mentioned in the previous sentence.

Theorem 3.12 The Fourier series of a $2L$-periodic, piecewise smooth function $f(x)$ converges in the mean to $f(x)$.

Proof: According to Theorem 3.9, the mean square error when a $2L$-periodic, piecewise continuous function $f(x)$ is approximated by the first $2n+1$ terms of its Fourier series is

$$\int_0^{2L} \left[f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} \left(a_k \cos \frac{k\pi x}{L} + b_k \sin \frac{k\pi x}{L} \right) \right]^2 dx$$
$$= \int_0^{2L} [f(x)]^2\, dx - L \left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right].$$

If we take limits as $n \to \infty$, and invoke Parseval's Theorem (assuming now that $f(x)$ is piecewise smooth), we obtain

$$\lim_{n\to\infty} \int_0^{2L} \left[f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} \left(a_k \cos \frac{k\pi x}{L} + b_k \sin \frac{k\pi x}{L} \right) \right]^2 dx = 0.$$

This is definition 3.23 for convergence in the mean of the Fourier series of $f(x)$. $\blacksquare$

It is worth noting that Theorems 3.9, 3.11, and 3.12 involve $f(x)$ in integral form. When $f(x)$ has discontinuities, it does not matter therefore whether we use original values of $f(x)$ at these discontinuities or averages of right- and left-hand limits $[f(x+) + f(x-)]/2$. In particular, corresponding to Theorem 3.12, we could say that the Fourier series of a $2L$-periodic, piecewise smooth function $f(x)$ converges in the mean to $[f(x+) + f(x-)]/2$.

EXERCISES 3.4

1. Verify that Fourier coefficients for the function $f(x)$ in Example 3.1 satisfy equation 3.30. (You will need the fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.)

2. Use equation 3.30 and the Fourier coefficients in Example 3.2 to prove that $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$.

3. Use equation 3.30 and the Fourier coefficients in Example 3.8 to prove that $\sum_{n=1}^{\infty} 1/(2n-1)^4 = \pi^4/96$.

4. Use equation 3.30 and the Fourier coefficients in Example 3.7 to prove that $\sum_{n=1}^{\infty} (-1)^n/n^2 = \pi^2/12$. (You will need the fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.)

5. In this exercise we verify Schwarz's inequality 3.29.

(a) Show that inequality 3.29 becomes an equality when terms in the sequences $\{c_n\}$ and $\{d_n\}$ are proportional, that is, when $d_n = \lambda c_n$ for all n ($\lambda > 0$).

(b) Now suppose that the sequences $\{c_n\}$ and $\{d_n\}$ are not proportional. Consider the finite series

$$\sum_{n=1}^{m} (c_n x + d_n)^2 = x^2 \sum_{n=1}^{m} {c_n}^2 + 2x \sum_{n=1}^{m} c_n d_n + \sum_{n=1}^{m} {d_n}^2.$$

Establish that the quadratic expression on the right has no zeros, and use this to verify inequality 3.29.

6. (a) Prove that if a $2L$-periodic function is continuous with a piecewise continuous first derivative, its Fourier coefficients satisfy

$$\lim_{n\to\infty} n a_n = 0 = \lim_{n\to\infty} n b_n.$$

(Hint: See Theorem 3.10.)

(b) Does the result in part (a) hold if the function is only piecewise continuous?

7. Show that when $f(x)$ is a piecewise continuous function on $0 \le x \le L$,

$$\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} {a_n}^2 = \frac{2}{L}\int_0^L [f(x)]^2\,dx, \qquad \sum_{n=1}^{\infty} {b_n}^2 = \frac{2}{L}\int_0^L [f(x)]^2\,dx$$

when the a_n are calculated according to formula 3.16b and the b_n according to 3.17b.

8. Suppose that $f(x)$ is continuous on the interval $0 \le x \le L$ with piecewise continuous derivatives $f'(x)$ and $f''(x)$.

(a) Show that the Fourier sine series of the odd, $2L$-periodic extension $f_o(x)$ of $f(x)$ can be differentiated term-by-term to give a cosine series that converges to $[f_o'(x+) + f_0'(x-)]/2$ if $f(0) = f(L) = 0$. Does the differentiated series converge to $f'(0+)$ at $x = 0$ and $f'(L-)$ at $x = L$ when $f(0) = f(L) = 0$?

(b) Show that the Fourier cosine series of the even, $2L$-periodic extension $f_e(x)$ of $f(x)$ can always be differentiated term-by-term to give a sine series that converges to $[f_e'(x+) + f_e'(x-)]/2$. Does the differentiated series converge to $f'(0+)$ at $x = 0$ and $f'(L-)$ at $x = L$?

9. (a) Suppose a function $f(x)$ is continuous on the interval $0 \le x \le 2L$ and has a piecewise continuous first derivative. Does the Fourier series of the $2L$-periodic extension of $f(x)$ converge uniformly?

(b) Suppose a function $f(x)$ is continuous on the interval $0 \leq x \leq L$ and has a piecewise continuous first derivative. Does the Fourier sine series of the odd, $2L$-periodic extension of $f(x)$ converge uniformly?

(c) Is your conclusion in part (b) the same for the Fourier cosine series of the even, $2L$-periodic extension of $f(x)$?

10. Suppose you are to approximate a piecewise continuous function $f(x)$ of period $2L$ by a sum of the form

$$S_n(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n} \left(\alpha_k \cos \frac{k\pi x}{L} + \beta_k \sin \frac{k\pi x}{L} \right).$$

You are to choose coefficients α_0, α_k, and β_k so that the series is the best approximation to $f(x)$ in the mean square sense; that is, coefficients are to be chosen so that

$$\int_0^{2L} \left[f(x) - \frac{\alpha_0}{2} - \sum_{k=1}^{n} \left(\alpha_k \cos \frac{k\pi x}{L} + \beta_k \sin \frac{k\pi x}{L} \right) \right]^2 dx$$

is as small as possible. Show that the best choices for the coefficients are the Fourier coefficients defined by equations 3.12.

CHAPTER 4 SEPARATION OF VARIABLES

§4.1 Linearity and Superposition

Separation of variables is one of the most fundamental techniques for solving PDEs. It is a method that can by itself yield solutions to many initial boundary value problems; in addition, it is the basis for more advanced techniques that must be used on more complicated problems. Separation of variables is applied to linear PDEs. A PDE is said to **linear** if it is linear in the unknown function and all its derivatives (but not necessarily in the independent variables). For example, the most general linear second-order PDE for a function $u(x,y)$ of two independent variables is

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = F(x,y); \tag{4.1}$$

it is a linear combination of u and its partial derivatives, the coefficients being functions of only the independent variables x and y. Linear PDEs may be represented symbolically in the form

$$Lu = F, \tag{4.2}$$

where L is a *linear* differential operator. In particular, for PDE 4.1, $L = a\partial^2/\partial x^2 + b\partial^2/\partial x\partial y + c\partial^2/\partial y^2 + d\partial/\partial x + e\partial/\partial y + f$. Operator L is said to be **linear** because it satisfies the property that for any two functions $u(x,y)$ and $v(x,y)$, with continuous second partial derivatives, and any constants C_1 and C_2,

$$L(C_1 u + C_2 v) = C_1(Lu) + C_2(Lv). \tag{4.3}$$

When $F(x,y) \equiv 0$ in equation 4.1, the PDE is said to be **homogeneous**; otherwise, it is said to be **nonhomogeneous**.

The study of linear ordinary differential equations is based on the idea of *superposition* — when solutions to a linear, homogeneous ODE are added together, new solutions are obtained. These same principles are the basis for separation of variables in PDEs. We set them forth in the following two theorems.

Theorem 4.1 (**Superposition Principle 1**) If u_j $(j = 1, \ldots, n)$ are solutions of the same linear, homogeneous PDE, then so also is any linear combination of the u_j,

$$u = \sum_{j=1}^{n} c_j u_j, \qquad c_j = \text{constants}.$$

Furthermore, if each u_j satisfies the same linear, homogeneous boundary and/or initial conditions, then so also does u.

Proof: Suppose the u_j satisfy the homogeneous linear PDE $Lu = 0$. We can use property 4.3 to write

$$Lu = L\left(\sum_{j=i}^{n} c_j u_j\right) = \sum_{j=1}^{n} L(c_j u_j) = \sum_{j=1}^{n} c_j[L(u_j)] = \sum_{j=1}^{n} c_j(0) = 0.$$

A proof for homogeneous linear boundary conditions is similar when they are represented in the form $Bu = 0$. ∎

As an illustration, suppose that $y_1(x,t)$ and $y_2(x,t)$ are solutions of the one-dimensional wave equation $y_{tt} = (\tau/\rho)y_{xx}$ and the boundary conditions $y(0,t) = 0$ and $y(L,t) = 0$. Then $y(x,t) = c_1y_1 + c_2y_2$ must also satisfy the PDE and boundary conditions for any constants c_1 and c_2. Let us actually show this, even though Theorem 4.1 guarantees it. For the PDE,

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= \frac{\partial^2}{\partial t^2}(c_1y_1 + c_2y_2) = c_1\frac{\partial^2 y_1}{\partial t^2} + c_2\frac{\partial^2 y_2}{\partial t^2}\\ &= c_1\frac{\tau}{\rho}\frac{\partial^2 y_1}{\partial x^2} + c_2\frac{\tau}{\rho}\frac{\partial^2 y_2}{\partial x^2} = \frac{\tau}{\rho}\frac{\partial^2}{\partial x^2}(c_1y_1 + c_2y_2) = \frac{\tau}{\rho}\frac{\partial^2 y}{\partial x^2}.\end{aligned}$$

For the boundary conditions,

$$\begin{aligned}y(0,t) &= c_1y_1(0,t) + c_2y_2(0,t) = c_1(0) + c_2(0) = 0,\\ y(L,t) &= c_1y_1(L,t) + c_2y_2(L,t) = c_1(0) + c_2(0) = 0.\end{aligned}$$

Thus, $y(x,t)$ satisfies the same linear, homogeneous PDE and boundary conditions as y_1 and y_2.

In short, superposition principle 1 states that linear combinations of solutions to linear, homogeneous PDEs and linear, homogeneous subsidiary conditions are solutions of the same PDE and conditions. Superposition principle 2 addresses nonhomogeneous PDEs. It states that nonhomogeneous terms in a PDE may be handled individually, if it is desirable to do so.

Theorem 4.2 (**Superposition Principle 2**) If u_j ($j = 1, \ldots, n$) are, respectively, solutions of linear, nonhomogeneous PDEs $Lu = F_j$, then $u = \sum_{j=1}^n u_j$ is a solution of $Lu = \sum_{j=1}^n F_j$.

Proof: Verification requires only property 4.3,

$$Lu = L\left(\sum_{j=1}^n u_j\right) = \sum_{j=1}^n Lu_j = \sum_{j=1}^n F_j. \blacksquare$$

For example, if $U_1(x,y,t)$ and $U_2(x,y,t)$ satisfy the two-dimensional heat conduction equations

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{k}{\kappa}g_1(x,y,t), \qquad \frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{k}{\kappa}g_2(x,y,t),$$

respectively, then $U(x,y,t) = U_1(x,y,t) + U_2(x,y,t)$ satisfies

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{k}{\kappa}[g_1(x,y,t) + g_2(x,y,t)].$$

This principle can also be extended to incorporate nonhomogeneous boundary conditions. To illustrate, consider the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x,y), \quad 0 < x < L, \quad 0 < y < L',$$

$$\begin{aligned} V(0,y) &= g_1(y), \quad 0 < y < L', \\ V(L,y) &= g_2(y), \quad 0 < y < L', \\ V(x,0) &= h_1(x), \quad 0 < x < L, \\ V(x,L') &= h_2(x), \quad 0 < x < L, \end{aligned}$$

for potential in the rectangle of Figure 4.1. The solution is the sum of the functions $V_1(x,y)$, $V_2(x,y)$, and $V_3(x,y)$ satisfying the PDEs in Figure 4.2 together with indicated boundary conditions. The problem in Figure 4.2b could be further subdivided into two problems, each of which contained only one nonhomogeneous boundary condition (as could the problem in Figure 4.2c). In Section 4.2 we show that this is not necessary.

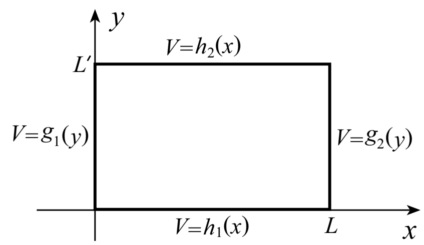
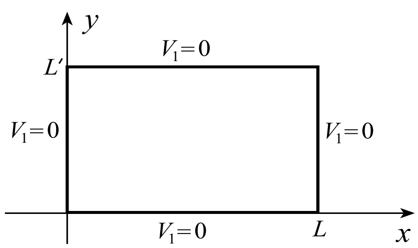
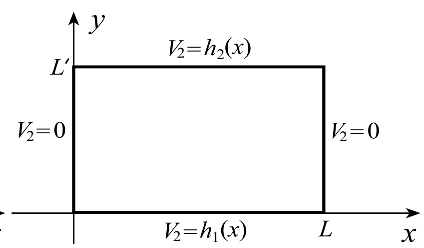


Figure 4.1



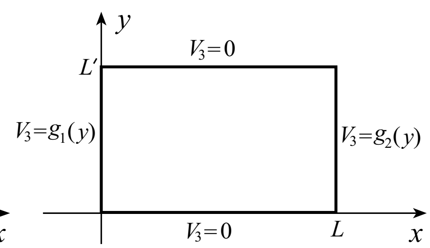
$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = F(x,y)$$

Figure 4.2a



$$\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} = 0$$

Figure 4.2b



$$\frac{\partial^2 V_3}{\partial x^2} + \frac{\partial^2 V_3}{\partial y^2} = 0$$

Figure 4.2c

EXERCISES 4.1

In Exercises 1–10 determine whether the PDE is linear. Which of the linear equations are homogeneous and which are nonhomogeneous?

1. $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2} + \dfrac{\partial y}{\partial t} + y$

2. $\dfrac{\partial^2 U}{\partial x^2} = 3\dfrac{\partial U}{\partial t} + U^2 + t^2 x$

3. $\dfrac{\partial^2 y}{\partial x^2}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial x}$

4. $\dfrac{\partial^2 y}{\partial x^2} + \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial y}{\partial t}\dfrac{\partial y}{\partial x}$

5. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = F(x,y,z)V$

6. $x^2\dfrac{\partial V}{\partial x} + x\dfrac{\partial^2 V}{\partial y^2} = xy$

7. $2\dfrac{\partial y}{\partial t} = xt\dfrac{\partial^2 y}{\partial x^2} + e^t\dfrac{\partial y}{\partial x} + t$

8. $\dfrac{\partial^2 U}{\partial t^2} + 2\dfrac{\partial^2 U}{\partial x \partial t} + \dfrac{\partial^2 U}{\partial x^2} = U\left(\dfrac{\partial U}{\partial x} + \dfrac{\partial U}{\partial t}\right)$

9. $\dfrac{\partial^2 U}{\partial x^2} - \dfrac{\partial^2 U}{\partial y^2} = 0$

10. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial V}{\partial x} - \dfrac{\partial V}{\partial y} = 3V$

11. Based on superposition principle 2, how would you subdivide the problem consisting of Poisson's equation $\nabla^2 V = F(x,y,z)$ inside the box $0 < x < L$, $0 < y < L'$, $0 < z < L''$, subject to the following boundary conditions?

$$\begin{aligned}
V(0,y,z) &= f_1(y,z), \quad 0<y<L', \quad 0<z<L'', \\
V(L,y,z) &= f_2(y,z), \quad 0<y<L', \quad 0<z<L'', \\
V(x,0,z) &= g_1(x,z), \quad 0<x<L, \quad 0<z<L'', \\
V(x,L',z) &= g_2(x,z), \quad 0<x<L, \quad 0<z<L'', \\
V(x,y,0) &= h_1(x,y), \quad 0<x<L, \quad 0<y<L', \\
V(x,y,L'') &= h_2(x,y), \quad 0<x<L, \quad 0<y<L'.
\end{aligned}$$

12. (a) Show that $u_1(x,y) = e^{x+y}$ and $u_2(x,y) = e^{x-y}$ are solutions of the PDE

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 2u^2.$$

(b) Is $u_1 + u_2$ a solution. Would you expect it to be?

§4.2 Separation of Variables

Before considering specific initial boundary value problems, we illustrate the basic idea of separation of variables on the PDE

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}. \tag{4.4}$$

Separation of variables determines functions $y(x,t)$ satisfying equation 4.4 that are functions $X(x)$ of x multiplied by functions $T(t)$ of t; that is, it determines solutions of the form

$$y(x,t) = X(x)T(t). \tag{4.5}$$

When this representation for $y(x,t)$ is substituted into the PDE,

$$\frac{d^2X}{dx^2}T(t) = X(x)\frac{dT}{dt},$$

and division by $X(x)T(t)$ gives

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = \frac{1}{T(t)}\frac{dT}{dt}. \tag{4.6}$$

The right side of this equation is a function of t only, and the left side is a function of x only. In other words, variables x and t have been *separated* from each other. Now, the only way this equation can hold for a range of value of x and t is for both sides to be equal to some constant, say α, which we take as real*; that is, we may write

$$\frac{1}{X}\frac{d^2X}{dx^2} = \alpha = \frac{1}{T}\frac{dT}{dt}. \tag{4.7}$$

We call this the **separation principle**.** This equation gives rise to two ordinary differential equations for $X(x)$ and $T(t)$,

$$\frac{d^2X}{dx^2} - \alpha X = 0, \qquad \text{and} \qquad \frac{dT}{dt} - \alpha T = 0. \tag{4.8}$$

Thus, by assuming that a function $y(x,t) = X(x)T(t)$ with variables separated satisfies equation 4.4, the PDE is replaced by the two ODEs 4.8. Boundary and/or initial conditions accompanying PDE 4.4 may give rise to subsidiary conditions to accompany ODEs 4.8. We shall see these in the examples to follow.

There is no reason to expect *a prior* that the solution to an initial boundary value problem should separate in form 4.5. In fact, separation of variables, by itself,

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* That α must be real for the problems of this chapter is proved in Exercise 46. That α must always be real is verified in Chapter 5.

** That the separation principle is valid can also be seen by differentiating 4.6 with respect to x. The result is

$$\frac{d}{dx}\left(\frac{1}{X}\frac{d^2X}{dx^2}\right) = 0,$$

and this implies that $(1/X)d^2X/dx^2$ must be equal to a numerical constant.

seldom yields the solution to an initial boundary value problem. However, separated functions can often be combined to yield the solution to an initial boundary value problem. We illustrate these ideas with the initial boundary value problem for transverse vibrations of a taut string in the following example.

Example 4.1 Solve the following initial boundary value problem for vibrations of a taut string with fixed ends at $x = 0$ and $x = L$ (Figure 4.3). The string has initial displacement (at time $t = 0$) of $f(x)$ and zero initial velocity.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{4.9a}$$

$$y(0,t) = 0, \quad t > 0, \tag{4.9b}$$

$$y(L,t) = 0, \quad t > 0, \tag{4.9c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{4.9d}$$

$$y_t(x,0) = 0, \quad 0 < x < L, \tag{4.9e}$$

Solve the problem for three initial displacement functions:

$$\text{(a)} \quad 3\sin\frac{\pi x}{L} \qquad \text{(b)} \quad 3\sin\frac{\pi x}{L} - \sin\frac{2\pi x}{L} \qquad \text{(c)} \quad x(L-x)$$

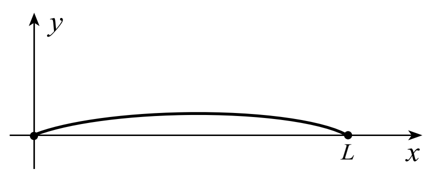


Figure 4.3

Solution We begin by searching for separated functions that satisfy the (linear) homogeneous PDE, the (linear) homogenous boundary conditions 4.9b,c, and the (linear) homogeneous initial condition 4.9e. We do not consider initial condition 4.9d; it is nonhomogeneous. As a general principle, then, separated functions are sought to satisy only linear and homogeneous PDEs, boundary conditions, and initial conditions.

When we substitute a separated function $y(x,t) = X(x)T(t)$ into PDE 4.9a,

$$XT'' = c^2 X''T \quad \Longrightarrow \quad \frac{X''}{X} = \frac{T''}{c^2 T},$$

where the $''$ on X'' indicates derivatives with respect to x, whereas on T'', it represents derivatives with respect to t. By the separation principle, we may set each side of this equation equal to a constant, say α, which is independent of both x and t. This results in two ODEs for $X(x)$ and $T(t)$,

$$X'' - \alpha X = 0, \qquad T'' - \alpha c^2 T = 0. \tag{4.10}$$

Homogeneous boundary condition 4.9b implies that

$$X(0)T(t) = 0, \quad t > 0.$$

Because $T(t) \not\equiv 0$ (why not?), it follows that $X(0) = 0$. Similarly, homogeneous boundary condition 4.9c and initial condition 4.9e require $X(L) = 0$ and $T'(0) = 0$. Thus, $X(x)$ and $T(t)$ must satisfy

$$X'' - \alpha X = 0, \quad 0 < x < L, \quad (4.11a) \qquad T'' - \alpha c^2 T = 0, \quad t > 0, \quad (4.12a)$$
$$X(0) = 0, \quad (4.11b) \qquad T'(0) = 0. \quad (4.12b)$$
$$X(L) = 0; \quad (4.11c)$$

Notice once again that we do not consider nonhomogeneous condition 4.9d at this time. For a separated function $y(x,t) = X(x)T(t)$, it would imply that $X(x)T(0) = f(x)$, but this would give no information about $X(x)$ and $T(t)$ separately. This is always the situation; nonhomogeneous boundary and/or initial conditions are never considered in conjunction with separation of the PDE.

Solutions of ODEs 4.11 and 4.12 depend on whether α is positive, negative, or zero. On purely physical grounds, a positive or zero value can be eliminated, for in these cases the time dependence of y is given by

$$T(t) = Ae^{c\sqrt{\alpha}t} + Be^{-c\sqrt{\alpha}t} \qquad \text{and} \qquad T(t) = At + B,$$

respectively, and these certainly do not yield oscillatory motions. Alternatively, for positive α, a general solution of ODE 4.11a is

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x},$$

and boundary conditions 4.11b,c imply that $A = B = 0$, and this in turn implies that $y(x,t) = 0$. Therefore, α cannot be positive. For $\alpha = 0$, we obtain $X(x) = Ax + B$, and the boundary conditions again imply that $A = B = 0$. Because α must therefore be negative, we set $\alpha = -\lambda^2$ $(\lambda > 0)$ and replace systems 4.11 and 4.12 with

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \quad (4.13a) \qquad T'' + c^2\lambda^2 T = 0, \quad t > 0, \quad (4.14a)$$
$$X(0) = 0, \quad (4.13b) \qquad T'(0) = 0. \quad (4.14b)$$
$$X(L) = 0; \quad (4.13c)$$

Boundary conditions 4.13b,c on the general solution $X(x) = A\cos\lambda x + B\sin\lambda x$ of 4.13a yield

$$0 = A, \qquad 0 = B\sin\lambda L.$$

Since we cannot set $B = 0$ (else $X(x) = 0$), we must therefore set $\sin\lambda L = 0$, and this implies that $\lambda L = n\pi$, where n is an integer. Thus,

$$X(x) = B\sin\frac{n\pi x}{L}.$$

Condition 4.14b on the general solution $T(t) = F\cos\dfrac{n\pi ct}{L} + G\sin\dfrac{n\pi ct}{L}$ of 4.14a yields

$$0 = \frac{n\pi c}{L}G \quad \Longrightarrow \quad G = 0.$$

We have now determined that the separated function

$$y(x,t) = X(x)T(t) = \left(B\sin\frac{n\pi x}{L}\right)\left(F\cos\frac{n\pi ct}{L}\right) = b\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \quad (4.15)$$

for an arbitrary constant b and any integer n is a solution of the one-dimensional wave equation 4.9a and conditions 4.9b,c,e. The initial displacement condition 4.9d requires b and n to satisfy

$$f(x) = b \sin \frac{n\pi x}{L}, \quad 0 < x < L. \tag{4.16}$$

We now consider the three cases for the initial displacement $f(x)$ following equation 4.9e, namely, $3\sin(\pi x/L)$, $3\sin(\pi x/L) - \sin(2\pi x/L)$, and $x(L-x)$. When $f(x) = 3\sin(\pi x/L)$, condition 4.16 becomes

$$3\sin\frac{\pi x}{L} = b\sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

Obviously, we should choose $b = 3$ and $n = 1$, in which case the solution of initial boundary value problem 4.9 is

$$y(x,t) = 3\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}.$$

This function is drawn for various value of t in Figure 4.4. The string oscillates back and forth between its initial position and the negative thereof, doing so once every $2L/c$ seconds.

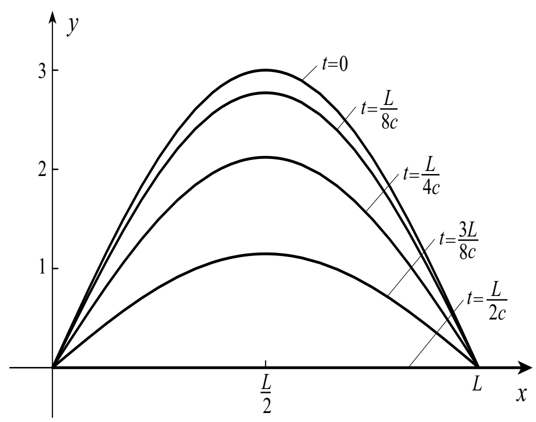


Figure 4.4

We have shown, then, that when the initial position of the string is $3\sin(\pi x/L)$, separation of variables leads to the solution of problem 4.9.

When $f(x) = 3\sin(\pi x/L) - \sin(2\pi x/L)$, condition 4.16 is

$$3\sin\frac{\pi x}{L} - \sin\frac{2\pi x}{L} = b\sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

It is not possible to choose values for b and n to satisfy this equation. In other words, the solution of 4.9 is not separable when $f(x) = 3\sin(\pi x/L) - \sin(2\pi x/L)$. Does this mean that we must abandon separation? Fortunately, the answer is no. Because PDE 4.9a, boundary conditions 4.9b,c, and initial condition 4.9e are all linear and homogeneous, superposition principle 1 states that linear combinations of solutions of 4.9a,b,c,e are also solutions. In particular, the function

$$y(x,t) = b\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + d\sin\frac{m\pi x}{L}\cos\frac{m\pi ct}{L}$$

satisfies 4.9a,b,c,e for arbitrary integers n and m and any constants b and d. If we apply initial condition 4.9d to this function, b, d, n and m must satisfy

$$3\sin\frac{\pi x}{L} - \sin\frac{2\pi x}{L} = b\sin\frac{n\pi x}{L} + d\sin\frac{m\pi x}{L}, \quad 0 < x < L.$$

Clearly, we should choose $b = 3$, $d = -1$, $n = 1$ and $m = 2$, in which case the solution of problem 4.9 is

$$y(x,t) = 3\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}.$$

This is not a separated solution; it is the sum of two separated functions. The motion of the string has two terms, called **modes**. The first term $3\sin(\pi x/L)\cos(\pi ct/L)$ is called the fundamental mode; it is shown in Figure 4.4. The second mode is $-\sin(2\pi x/L)\cos(2\pi ct/L)$; it is illustrated in Figure 4.5 for the same times. Oscillations of this mode occur twice as fast as those for the fundamental mode. The addition of these two modes gives the position of the string in Figure 4.6.

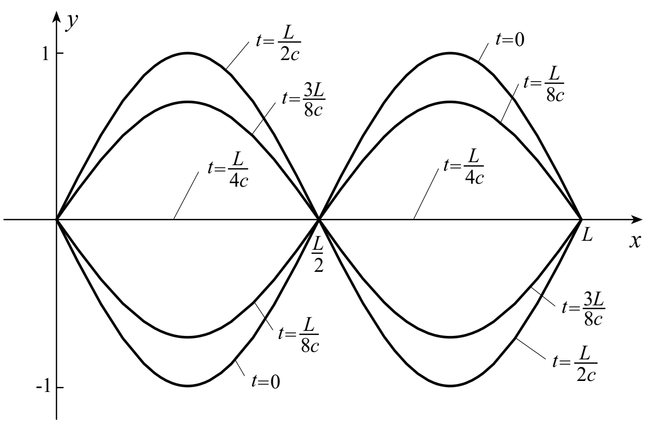


Figure 4.5

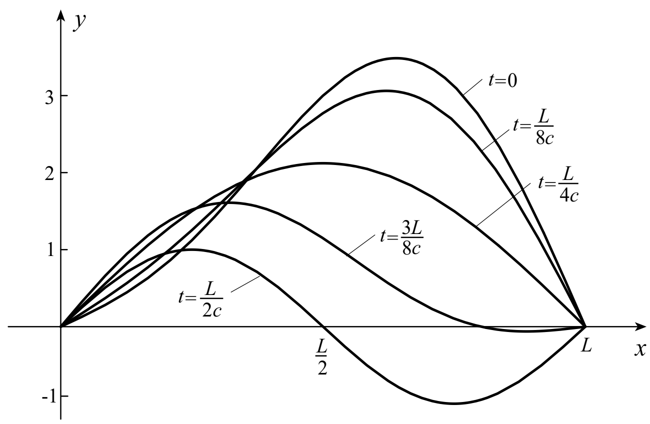


Figure 4.6

Finally, we consider the case in which the initial displacement in the string is parabolic, $f(x) = x(L-x)$. It is definitely not possible to satisfy condition 4.16,

$$x(L-x) = b\sin\frac{n\pi x}{L}, \quad 0 < x < L,$$

for any choice of b and n. Furthermore, no finite linear combination of terms of the form $b\sin(n\pi x/L)$ can satisfy this condition. Does this mean the ultimate demise of separation of variables? Again the answer is no. We superpose an infinity of separated functions in the form

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \tag{4.17}$$

where the constants b_n are arbitrary. No advantage is gained by including terms with negative values of n, for if we had a term in $-n$ (n positive), say

$$X_{-n}(x) = b_{-n}\sin\left(\frac{-n\pi x}{L}\right),$$

we could combine it with

$$X_n(x) = b_n \sin\frac{n\pi x}{L}$$

and write

$$X_n + X_{-n} = (b_n - b_{-n})\sin\frac{n\pi x}{L} = B_n \sin\frac{n\pi x}{L},$$

which is of the same form as $X_n(x)$.

Initial condition 4.9d requires the b_n in representation 4.17 to satisfy

$$x(L-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad 0 < x < L. \tag{4.18}$$

This equation is satisfied if the b_n are chosen as the coefficients in the Fourier sine series of the odd extension of $x(L-x)$ to a function of period $2L$. According to formula 3.17b,

$$b_n = \frac{2}{L}\int_0^L x(L-x)\sin\frac{n\pi x}{L}dx,$$

and integration by parts leads to

$$b_n = \frac{4L^2[1+(-1)^{n+1}]}{n^3\pi^3}$$

(see Exercise 9 in Section 3.2). Substitution of these into representation 4.17 gives displacements of the string when the initial position is $f(x) = x(L-x)$:

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} \frac{4L^2[1+(-1)^{n+1}]}{n^3\pi^3} \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \\ &= \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}\cos\frac{(2n-1)\pi ct}{L}. \end{aligned} \tag{4.19}$$

Each term in this series is called a mode of vibration of the string. The position of the string is the sum of an infinite number of modes, lower modes contributing more significantly than higher ones. We shall have more to say about them in Section 6.2.•

You would be wise in questioning whether the representation of $y(x,t)$ as an infinite series in equation 4.19 is really a solution of problem 4.9. Certainly it satisfies boundary conditions 4.9b,c, and, because $x(L-x)$ is continuously differentiable, our theory of Fourier series implies that initial condition 4.9d must also be satisfied. Conditions 4.9a,e present difficulties, however. First of all, because representation 4.17 is the superposition of an infinity of separated functions, and superposition principle 1 discusses only finite combinations, an infinite combination must be suspect. Second, because representation 4.19 is an infinite series, there is a question of its convergence. Does it, for instance, converge for $0 < x < L$ and $t > 0$, and do its derivatives satisfy wave equation 4.9a and initial condition 4.9e? Each of these questions must be answered, and we shall do so, but not at this time. In this chapter, we wish to illustrate the technique of separation of variables and some of its adaptations to more difficult problems. Verification that the resulting series are truly solutions of initial boundary value problems is discussed in Sections 6.6–6.8. To remind us that these series have not yet been verified as solutions of their respective problems, we call them **formal** solutions.

As a final consideration in this example, we show that the series solution can be expressed in closed form, d'Alembert's solution of Section 2.11. Using a trigonometric identity, we may write

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4L^2[1+(-1)^{n+1}]}{n^3\pi^3} \frac{1}{2}\left[\sin\frac{n\pi(x+ct)}{L} + \sin\frac{n\pi(x-ct)}{L}\right].$$

Because the above calculation showed that

$$f(x) = x(L-x) = \sum_{n=1}^{\infty} \frac{4L^2[1+(-1)^{n+1}]}{n^3\pi^3} \sin\frac{n\pi x}{L},$$

it follows that

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

This is d'Alembert's form of the solution. As noted earlier, $f(x)$ is now defined for all real values of x since it has been extended as an odd, $2L$-periodic function.

The one-dimensional wave equation 4.9a is a hyperbolic second-order equation (see Section 2.8). In the following two examples we show that separation of variables can be used on parabolic and elliptic equations as well.

Example 4.2 Solve the following initial boundary value problem for temperature in a homogeneous isotropic rod with insulated sides and no internal heat generation (Figure 4.7):

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{4.20a}$$

$$U_x(0,t) = 0, \quad t > 0, \tag{4.20b}$$

$$U_x(L,t) = 0, \quad t > 0, \tag{4.20c}$$

$$U(x,0) = x, \quad 0 < x < L. \tag{4.20d}$$

The ends of the rod are also insulated (conditions 4.20b,c), and its initial temperature increases linearly from $U = 0$ at $x = 0$ to $U = L$ at $x = L$.

Solution The assumption of a separated function $U(x,t) = X(x)T(t)$ satisfying PDE 4.20a leads to

$$XT' = kX''T \quad \Longrightarrow \quad \frac{X''}{X} = \frac{T'}{kT}.$$

The separation principle implies that both sides of the last equation must be equal to a constant, say α, in which case

$$X'' - \alpha X = 0, \qquad T' - \alpha kT = 0.$$

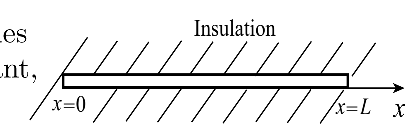


Figure 4.7

Homogeneous boundary conditions 4.20b,c imply that $X'(x) = 0 = X'(L)$, so that $X(x)$ and $T(t)$ must satisfy

$$X'' - \alpha X = 0, \quad 0 < x < L, \tag{4.21a}$$

$$X'(0) = 0, \tag{4.21b}$$

$$X'(L) = 0; \tag{4.21c}$$

$$T' - \alpha kT = 0, \quad t > 0. \tag{4.22}$$

For positive α, a general solution of ODE 4.21a is

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x},$$

and boundary conditions 4.21b,c require

$$0 = A - B, \qquad 0 = Ae^{\sqrt{\alpha}L} - Be^{-\sqrt{\alpha}L}.$$

From these, $A = B = 0$, and therefore α cannot be positive. For $\alpha = 0$, we obtain $X(x) = Ax+B$, and the boundary conditions imply that $A = 0$. Thus, when $\alpha = 0$, solutions of systems 4.21 and 4.22 are

$$X(x) = B = \text{constant} \quad \text{and} \quad T(t) = D = \text{constant}.$$

What we have shown, then, is that $U(x,t) = X(x)T(t) = \text{constant}$ satisfies PDE 4.20a and boundary conditions 4.20b,c.

When α is negative, we set $\alpha = -\lambda^2$ $(\lambda > 0)$, in which case systems 4.21 and 4.22 are replaced by

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \qquad (4.23a)$$
$$X'(0) = 0, \qquad (4.23b)$$
$$X'(L) = 0; \qquad (4.23c)$$

$$T' + k\lambda^2 T = 0, \quad t > 0. \qquad (4.24)$$

Boundary conditions 4.23b,c on the general solution $X(x) = A\cos\lambda x + B\sin\lambda x$ of ODE 4.23a require

$$0 = B, \qquad 0 = \lambda A \sin\lambda L.$$

Since we cannot set $A = 0$ (else $X(x) = 0$), we must therefore set $\sin\lambda L = 0$, and this implies that $\lambda L = n\pi$, n an integer. Thus,

$$X(x) = A\cos\frac{n\pi x}{L}.$$

A general solution of ODE 4.24 is

$$T(t) = De^{-n^2\pi^2 kt/L^2}.$$

Consequently, besides constant functions, we also have separated functions

$$X(x)T(t) = \left(A\cos\frac{n\pi x}{L}\right)\left(De^{-n^2\pi^2 kt/L^2}\right) = ae^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L},$$

which satisfy 4.20a,b,c for integers $n > 0$ and arbitrary a. Notice that when $n = 0$, this function reduces to the constant function corresponding to $\alpha = 0$. In other words, all separated functions satisfying 4.20a,b,c can be expressed in the form

$$ae^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L}, \quad n \geq 0.$$

(It is not necessary to include $n < 0$, since such separated functions are identical to those when $n > 0$.) Initial condition 4.20d would require a separated function to satisfy

$$x = a\cos\frac{n\pi x}{L}, \quad 0 < x < L,$$

an impossibility. But because the heat equation and boundary conditions are linear and homogeneous, we superpose separated functions and take

$$U(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L}$$

with arbitrary constants a_n. Initial condition 4.20d requires the a_n to satisfy

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 < x < L.$$

This equation is satisfied if the a_n are chosen as the coefficients in the Fourier cosine series of the even extension of the function $f(x) = x$ to a function of period $2L$. According to formula 3.16b,

$$a_n = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} dx,$$

and integration gives

$$a_0 = L, \qquad a_n = \frac{2L[(-1)^n - 1]}{n^2\pi^2}, \quad n > 0.$$

The formal solution of heat conduction problem 4.20 is therefore

$$U(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L[(-1)^n - 1]}{n^2\pi^2} e^{-n^2\pi^2 kt/L^2} \cos \frac{n\pi x}{L}$$
$$= \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2\pi^2 kt/L^2} \cos \frac{(2n-1)\pi x}{L}. \quad (4.25)$$

An interesting feature of this solution is its limit as time t becomes very large:

$$\lim_{t\to\infty} U(x,t) = \frac{L}{2}.$$

In other words, for large times, the temperature of the rod becomes constant throughout. But this is exactly what we should expect. Because the rod is totally insulated after $t = 0$, the original amount of heat in the rod will redistribute itself until a steady-state situation is achieved, the steady-state temperature being a constant value equal to the average of the initial temperature distribution. Since initially the temperature varies linearly from $U = 0$ at one end to $U = L$ at the other, its average value is $L/2$, precisely that predicted by the above limit.●

For a copper rod of length 1 m and diffusivity $k = 114 \times 10^{-6}$ m^2/s, representation 4.25 becomes

$$U(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-114\times 10^{-6}(2n-1)^2\pi^2 t} \cos(2n-1)\pi x,$$

This function is plotted in Figure 4.8 for various values of t to illustrate the transition from initial temperature $U(x,0) = x$ to final temperature $1/2$. These curves indicate that $U(x,t)$ is always an increasing function of x, and therefore heat always flows from right to left. Notice also that each curve is horizontal at $x = 0$ and $x = 1$. This reflects boundary conditions 4.20b,c.

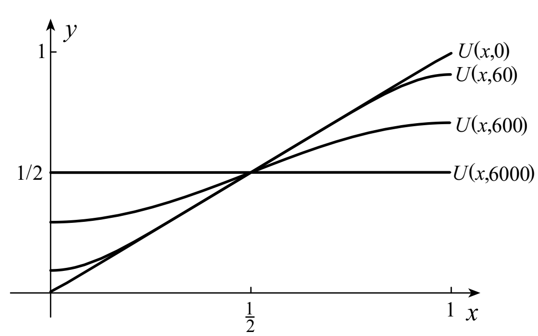


Figure 4.8

Example 4.3 Solve the following boundary value problem for potential in the rectangular plate of Figure 4.9 when the sides are maintained at the potentials shown:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L', \tag{4.26a}$$
$$V(0, y) = 0, \quad 0 < y < L', \tag{4.26b}$$
$$V(L, y) = 0, \quad 0 < y < L', \tag{4.26c}$$
$$V(x, L') = 0, \quad 0 < x < L, \tag{4.26d}$$
$$V(x, 0) = 1, \quad 0 < x < L. \tag{4.26e}$$

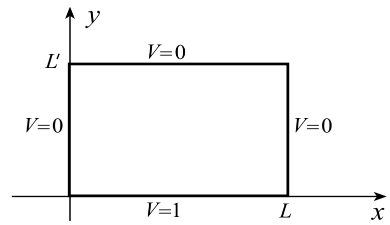


Figure 4.9

Solution When we assume that a function with variables separated, $V(x, y) = X(x)Y(y)$, satisfies PDE 4.26a,

$$X''Y + XY'' = 0 \quad \Longrightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}.$$

The separation principle requires X''/X and $-Y''/Y$ both to equal a constant α, so that

$$X'' - \alpha X = 0, \qquad Y'' + \alpha Y = 0.$$

Homogeneous boundary conditions 4.26b,c,d imply that $X(0) = X(L) = Y(L') = 0$, and therefore $X(x)$ and $Y(y)$ must satisfy

$$X'' - \alpha X = 0, \quad 0 < x < L, \tag{4.27a}$$
$$X(0) = 0, \tag{4.27b}$$
$$X(L) = 0; \tag{4.27c}$$

$$Y'' + \alpha Y = 0, \quad 0 < y < L', \tag{4.28a}$$
$$Y(L') = 0. \tag{4.28b}$$

System 4.27 is identical to 4.11; nontrivial solutions exist only when α is negative. If we set $\alpha = -\lambda^2$ $(\lambda > 0)$, then $\lambda = n\pi/L$, and the solution of 4.27 is

$$X(x) = B \sin \frac{n\pi x}{L}$$

for arbitrary B and n an integer. With $\alpha = -\lambda^2 = -n^2\pi^2/L^2$, a general solution of ODE 4.28a is

$$Y(y) = D \cosh \frac{n\pi y}{L} + E \sinh \frac{n\pi y}{L}.$$

We could also have expressed $Y(y)$ in terms of exponentials $e^{n\pi y/L}$ and $e^{-n\pi y/L}$, but hyperbolic functions turn out to be more convenient for problems on finite intervals. Exponentials are more suitable on infinite intervals as we shall see in Chapter 11. Condition 4.28b requires

$$0 = D \cosh \frac{n\pi L'}{L} + E \sinh \frac{n\pi L'}{L}.$$

We solve this for E in terms of D, in which case

$$\begin{aligned} Y(y) &= D \cosh \frac{n\pi y}{L} - D \frac{\cosh(n\pi L'/L)}{\sinh(n\pi L'/L)} \sinh \frac{n\pi y}{L} \\ &= \frac{D}{\sinh(n\pi L'/L)} \left(\sinh \frac{n\pi L'}{L} \cosh \frac{n\pi y}{L} - \cosh \frac{n\pi L'}{L} \sinh \frac{n\pi y}{L} \right) \end{aligned}$$

$$= F \sinh \frac{n\pi(L' - y)}{L}, \qquad \text{where } F = \frac{D}{\sinh(n\pi L'/L)}.$$

We have now determined that separated functions

$$X(x)Y(y) = b \sin \frac{n\pi x}{L} \sinh \frac{n\pi(L' - y)}{L} \qquad (b = BF)$$

for any constant b and any integer n are solutions of Laplace's equation 4.26a and boundary conditions 4.26b,c,d. Since these conditions and PDE are linear and homogeneous, we superpose separated functions and take

$$V(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(L' - y)}{L} \tag{4.29}$$

with arbitrary constants b_n. Boundary condition 4.26e requires the b_n to satisfy

$$1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi L'}{L} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L}, \quad 0 < x < L,$$

where $C_n = b_n \sinh(n\pi L'/L)$. But this equation is satisfied if the numbers C_n are chosen as the coefficients in the Fourier sine series of the odd extension of the function $f(x) = 1$ to a function of period $2L$. Hence

$$C_n = b_n \sinh \frac{n\pi L'}{L} = \frac{2}{L} \int_0^L (1) \sin \frac{n\pi x}{L} dx = \frac{2[1 + (-1)^{n+1}]}{n\pi}.$$

Formal solution 4.29 of potential problem 4.26 is therefore

$$V(x, y) = \sum_{n=1}^{\infty} \frac{2[1 + (-1)^{n+1}]}{n\pi \sinh \dfrac{n\pi L'}{L}} \sin \frac{n\pi x}{L} \sinh \frac{n\pi(L' - y)}{L}$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1) \sinh \dfrac{(2n-1)\pi L'}{L}} \sin \frac{(2n-1)\pi x}{L} \sinh \frac{(2n-1)\pi(L' - y)}{L}. \bullet \tag{4.30}$$

These three examples have illustrated the essentials of the method of separation of variables and Fourier series for boundary value and initial boundary value problems. In each, functions with variables separated are found to satisfy the linear, homogeneous PDE and the linear, homogeneous boundary and/or initial conditions. These separated functions invariably involve an arbitrary multiplicative constant and an integer parameter. To satisfy the one nonhomogeneous boundary or initial condition, these functions are superposed into an infinite series.

Our next example illustrates that separation of variables is not restricted to second-order PDEs.

Example 4.4 Transverse vibrations of a uniform beam with simply supported ends (Figure 4.10) are described by the initial boundary value problem

$$\frac{\partial^2 y}{\partial t^2} + c^2 \frac{\partial^4 y}{\partial x^4} = 0, \quad 0 < x < L, \quad t > 0, \tag{4.31a}$$

$$y(0, t) = y_{xx}(0, t) = 0, \quad t > 0, \tag{4.31b}$$

$$y(L, t) = y_{xx}(L, t) = 0, \quad t > 0, \tag{4.31c}$$

$$y(x, 0) = x \sin \frac{\pi x}{L}, \quad 0 < x < L, \tag{4.31d}$$

$$y_t(x, 0) = 0, \quad 0 < x < L, \tag{4.31e}$$

where $c^2 = EI/\rho$. The force of gravity on the beam has been assumed negligible relative to internal forces (see Section 2.5). Conditions 4.31d,e indicate an initial displacement $x \sin(\pi x/L)$ and zero initial velocity. Solve this problem.

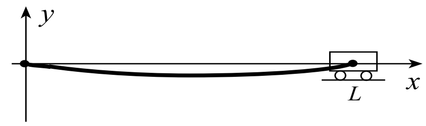


Figure 4.10

Solution Substitution of a function $y(x,t) = X(x)T(t)$ with variables separated into PDE 4.31a gives

$$XT'' + c^2 X''''T = 0 \quad \Longrightarrow \quad \frac{X''''}{X} = \frac{-T''}{c^2 T}.$$

The separation principle implies that

$$X'''' - \alpha X = 0 \quad \text{and} \quad T'' + \alpha c^2 T = 0$$

for some constant α. When $\alpha < 0$ and $\alpha = 0$, general solutions for $T(t)$ are

$$T(t) = A\cosh c\sqrt{-\alpha}t + B\sinh c\sqrt{-\alpha}t \quad \text{and} \quad T(t) = At + B,$$

respectively. Because the motion of the beam must be oscillatory, and neither of these functions displays this characteristic, we conclude that α must be positive. (The same conclusion can be obtained from the ODE $X'''' - \alpha X = 0$ in conjunction with boundary conditions 4.31b,c, but not so easily.) When we set $\alpha = \lambda^4$ $(\lambda > 0)$ and use separation on homogeneous boundary conditions 4.31b,c and initial condition 4.31e, $X(x)$ and $T(t)$ must satisfy the systems

$$X'''' - \lambda^4 X = 0, \quad 0 < x < L, \qquad (4.32a)$$
$$X(0) = X''(0) = 0, \qquad (4.32b)$$
$$X(L) = X''(L) = 0; \qquad (4.32c)$$

$$T'' + c^2\lambda^4 T = 0, \quad t > 0, \qquad (4.33a)$$
$$T'(0) = 0. \qquad (4.33b)$$

Boundary conditions 4.32b,c on the general solution

$$X(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x$$

of ODE 4.32a yield

$$\begin{aligned}
0 &= A + C,\\
0 &= A\cos\lambda L + B\sin\lambda L + C\cosh\lambda L + D\sinh\lambda L,\\
0 &= -\lambda^2 A + \lambda^2 C,\\
0 &= -\lambda^2 A\cos\lambda L - \lambda^2 B\sin\lambda L + \lambda^2 C\cosh\lambda L + \lambda^2 D\sinh\lambda L.
\end{aligned}$$

The first and third of these imply that $A = C = 0$, while the second and fourth require

$$B\sin\lambda L = 0, \qquad D\sinh\lambda L = 0.$$

Since $\lambda > 0$, we must set $D = 0$, in which case $B \neq 0$. It follows, then, that $\lambda L = n\pi$, where n is an integer, and

$$X(x) = B\sin\frac{n\pi x}{L}.$$

Condition 4.33b on the general solution

$$T(t) = E \cos \frac{n^2\pi^2 ct}{L^2} + F \sin \frac{n^2\pi^2 ct}{L^2}$$

of ODE 4.33 yields

$$0 = \frac{n^2\pi^2 c}{L^2} F,$$

from which $F = 0$. We have now determined that separated functions

$$X(x)T(t) = b \sin \frac{n\pi x}{L} \cos \frac{n^2\pi^2 ct}{L^2}$$

for an arbitrary constant b and any integer n are solutions of PDE 4.31a, its boundary conditions 4.31b,c, and initial condition 4.31e. Since the PDE and these conditions are linear and homogeneous, we superpose separated functions and take

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{n^2\pi^2 ct}{L^2}$$

with arbitrary constants b_n. Condition 4.31d requires the b_n to satisfy

$$x \sin \frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.$$

The b_n are therefore the coefficients in the Fourier sine series of the odd extension of $x \sin(\pi x/L)$ to a function of period $2L$. Hence,

$$b_n = \frac{2}{L} \int_0^L x \sin \frac{\pi x}{L} \sin \frac{n\pi x}{L} dx,$$

and integration leads to

$$b_1 = \frac{L}{2}, \qquad b_n = \frac{-4nL[1 + (-1)^n]}{(n^2 - 1)^2\pi^2}, \quad n > 1.$$

Transverse vibrations of the beam are therefore described formally by

$$\begin{aligned} y(x,t) &= \frac{L}{2} \sin \frac{\pi x}{L} \cos \frac{\pi^2 ct}{L^2} + \sum_{n=2}^{\infty} \frac{-4nL[1 + (-1)^n]}{(n^2 - 1)^2\pi^2} \sin \frac{n\pi x}{L} \cos \frac{n^2\pi^2 ct}{L^2} \\ &= \frac{L}{2} \sin \frac{\pi x}{L} \cos \frac{\pi^2 ct}{L^2} - \frac{16L}{\pi^2} \sum_{n=1}^{\infty} \frac{n}{(4n^2 - 1)^2} \sin \frac{2n\pi x}{L} \cos \frac{4n^2\pi^2 ct}{L^2}. \bullet \end{aligned} \tag{4.34}$$

You may have noticed that with the exception of Example 4.2, separation constant α was always negative, in which case, we set $\alpha = -\lambda^2$. This is not coincidence. In Chapter 5, we prove that α is indeed always negative except when both boundary conditions are Neumann, in which case $\alpha = 0$ is also acceptable. With this in mind, we suggest setting $\alpha = -\lambda^2$ immediately in any exercises giving rise to second order PDEs.

EXERCISES 4.2

Part A Heat Conduction

1. Determine $U(x,t)$ in Example 4.2 if the initial temperature is constant throughout.

2. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature $f(x)$, $0 \le x \le L$, at time $t = 0$. For time $t > 0$, its ends (at $x = 0$ and $x = L$) are held at temperature 0°C. Find a formula for the temperature $U(x, t)$ in the rod for $0 < x < L$ and $t > 0$.

3. (a) Find the temperature $U(x, t)$ of the rod in Exercise 2 when

$$f(x) = \begin{cases} x, & 0 \le x \le L/2 \\ L - x, & L/2 \le x \le L. \end{cases}$$

(b) The amount of heat per unit area per unit time flowing from left to right across the cross section of the rod at position x and time t is the x-component of the heat flux vector (this being the only component) $q(x, t) = -\kappa \partial U/\partial x$ (see Section 2.2). Find the heat flow rate for cross sections at positions $x = 0$, $x = L/2$, and $x = L$ by calculating

$$\lim_{x \to 0^+} q(x, t), \qquad q\left(\frac{L}{2}, t\right), \qquad \lim_{x \to L^-} q(x, t).$$

(c) Calculate limits of the heat flows in part (b) as $t \to 0^+$ and $t \to \infty$.

4. Repeat Exercise 3 if $f(x) = 10$, $0 \le x \le L$. In addition,
(d) Calculate

$$\lim_{x \to 0^+} U(x, 0) \qquad \text{and} \qquad \lim_{t \to 0^+} U(0, t).$$

(e) Draw what you feel $U(x, t)$ would look like as a function of x for various fixed values of t.

5. (a) Find the rate of flow of heat across the cross section at position $x = L/2$ for the rod in Example 4.2.
(b) What is the limit of your answer in part (a) as $t \to 0^+$?

6. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature $f(x)$, $0 \le x \le L$, at time $t = 0$. For time $t > 0$, its ends (at $x = 0$ and $x = L$) are insulated. Find a formula for the temperature $U(x, t)$ in the rod for $0 < x < L$ and $t > 0$. What is the limit of $U(x, t)$ as $t \to \infty$?

7. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature $L - x$, $0 \le x \le L$, at time $t = 0$. For time $t > 0$, its right end, $x = L$, is held at temperature zero and its left end, $x = 0$, is insulated. Use the result of Exercise 21 in Section 3.2 to find the temperature $U(x, t)$ in the rod for $0 < x < L$ and $t > 0$.

8. Repeat Exercise 7 with an unspecified initial temperature $f(x)$.

9. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature $L - x$, $0 \le x \le L$, at time $t = 0$. For time $t > 0$, its left end, $x = 0$, is held at temperature zero and its right end, $x = L$, is insulated. Use the result of Exercise 20 in Section 3.2 to find the temperature $U(x, t)$ in the rod for $0 < x < L$ and $t > 0$.

10. Repeat Exercise 9 with an unspecified initial temperature $f(x)$.

11. (a) A fuel rod of length L in neutron diffusion theory produces neutrons by fission. When the diffusion constant is k, and the ends of the rod are perfectly reflecting, the distribution $U(x, t)$ of neutrons in the rod must satisfy the initial boundary value problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \kappa U, \quad 0 < x < L, t > 0,$$
$$U_x(0, t) = U_x(L, t) = 0, \quad t > 0,$$
$$U(x, 0) = f(x), \quad 0 < x < L,$$

where $\kappa > 0$ is a constant, and $f(x)$ is the initial distribution of neutrons in the rod. Find $U(x,t)$ for $0 < x < L$ and $t > 0$, assuming that $\kappa \neq n^2\pi^2 k/L^2$ for any positive integer n.

(b) Simplify the solution when $f(x) =$ a constant.

(c) Simplify the solution when $f(x) = x$.

12. In Chapter 9 we deal with problems in polar, cylindrical, and spherical coordinates. Heat conduction problems in a sphere where temperature depends only on time and the radial coordinate r can be transformed into problems of the type that we have already considered when the temperature is specified on the bounding sphere. The initial boundary value problem for temperature in a sphere of radius a which has initial temperature $f(r)$ at time $t = 0$ and whose boundary is held at temperature 0°C for $t > 0$ is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < a, \quad t > 0,$$
$$U(a,t) = 0, \quad t > 0,$$
$$U(r,0) = f(r), \quad 0 < r < a.$$

(a) Show that with a change of dependent variable $V(r,t) = r\,U(r,t)$, the function $V(r,t)$ must satisfy the initial boundary value problem

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial r^2}, \quad 0 < r < a, \quad t > 0,$$
$$V(0,t) = 0, \quad t > 0,$$
$$V(a,t) = 0, \quad t > 0,$$
$$V(r,0) = rf(r), \quad 0 < r < a.$$

Note: You will need to use the fact that because $U(r,t)$ is to be continuous at $r = 0$, the function $V(r,t)$ must satisfy the condition $V(0,t) = 0$ for $t > 0$.

(b) Solve the problem in part (a) and hence find $U(r,t)$.

13. What happens if the transformation of Exercise 12 is applied to the problem when the surface of the sphere is insulated?

14. Two identical rods with the same thermal properties both have length $L/2$. One has constant temperature U_0 throughout, and the other has constant temperature U_1. At time $t = 0$ they are placed end-to-end along the positive x-axis, the one with temperature U_0 having its left end at $x = 0$. Their sides are insulated and heat flows freely through the contact face at $x = L/2$. If the ends at $x = 0$ and $x = L$ are held at temperature zero for $t > 0$, find the temperature in the combined rod for $t > 0$.

15. Show that when the temperature $f(x)$ of the rod in Exercise 2 is symmetric about $x = L/2$ (i.e., $f(L/2 - x) = f(L/2 + x)$), then no heat crosses the cross section at $x = L/2$. Is $U(x,t)$ symmetric about $x = L/2$ for all t?

16. A cylindrical, homogeneous, isotropic rod with insulated sides and length L is initially at temperature $f(x)$. For $t > 0$, its ends $x = 0$ and $x = L$ are held at temperature zero. Suppose that heat generation at each point of the rod is proportional to the temperature at the point; that is, suppose that $g(x,t)$ in the heat conduction equation 2.26 is relaced by aU, where a is a constant. Find the temperature in the rod as a function of x and t. For what values of a will the temperature of the rod remain finite for large t? Assume that $a \neq n^2\pi^2\kappa/L^2$ for any positive integer n.

17. Repeat Exercise 16 if both boundary conditions are homogeneous, Neumann.

Part B Vibrations

18. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial displacement

$$f(x) = \begin{cases} x/5, & 0 \le x \le L/2 \\ (L-x)/5, & L/2 \le x \le L \end{cases}$$

at time $t = 0$, but no initial velocity.
(a) Find a series representation for displacement of the string for $t > 0$ and $0 < x < L$.
(b) What is d'Alembert's solution for displacement of the string?

19. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial velocity $g(x) = x(L - x)$, $0 \le x \le L$ at time $t = 0$, but no initial displacement.
(a) Find a series representation for displacement of the string for $t > 0$ and $0 < x < L$.
(b) What is d'Alembert's solution for displacement of the string?

20. Suppose the string in Exercises 18 and 19 is given both the initial displacement $f(x)$ and the initial velocity $g(x)$ at time $t = 0$. Find its displacement for $t > 0$ and $0 < x < L$ in two ways:
(a) using superposition principle 2;
(b) using separation of variables.

21. (a) Find a series representation for displacement of the string in Exercise 20 if an external force (per unit x-length) $F = -ky$ $(k > 0)$ acts at each point in the string.
(b) Is there a d'Alembert solution?

22. Solve Exercise 21 if the external force $F = -ky$ is replaced by $F = -\beta \partial y/\partial t$. Assume that $\beta < 2\rho\pi c/L$.

23. A taut string is given initial displacement $f(x)$, $0 \le x \le L$ at time $t = 0$ and initial velocity $g(x)$, $0 \le x \le L$. The ends $x = 0$ and $x = L$ of the string are free to slide vertically without friction.
(a) Find a series repesentation for displacement of the string for $t > 0$ and $0 < x < L$.
(b) What is d'Alembert's solution for displacement of the string?

24. A taut string has its right end $x = L$ fixed on the x-axis. Its left end $x = 0$ is looped around a vertical support and slides without friction along the support. If its initial displacement at time $t = 0$ is $f(x)$ and it has no initial velocity, find displacements for $0 \le x \le L$ and $t > 0$. Hint: See Exercise 21 in Section 3.2. Is there a d'Alembert solution for displacements?

25. In Chapter 9 we deal with problems in polar, cylindrical, and spherical coordinates. Vibration problems in a sphere where displacement depends only on time and the radial coordinate r can be transformed into problems of the type that we have already considered when the displacement and velocity are specified on the bounding sphere. The initial boundary value problem for displacement in a sphere of radius a which has initial displacement $f(r)$ and initial velocity $g(r)$ at time $t = 0$ and whose bounding sphere is held fixed for $t > 0$ is

$$\begin{aligned} \frac{\partial^2 W}{\partial t^2} &= c^2\left(\frac{\partial^2 W}{\partial r^2} + \frac{2}{r}\frac{\partial W}{\partial r}\right), \quad 0 < r < a, \quad t > 0, \\ W(a,t) &= 0, \quad t > 0, \\ W(r,0) &= f(r), \quad 0 < r < a, \\ W_t(r,0) &= g(r), \quad 0 < r < a. \end{aligned}$$

(a) Show that with a change of dependent variable $V(r,t) = r\,W(r,t)$, the function $W(r,t)$ must satisfy the initial boundary value problem

$$\begin{aligned}
\frac{\partial V}{\partial t} &= c^2 \frac{\partial^2 V}{\partial r^2}, \quad 0 < r < a, \quad t > 0,\\
V(0,t) &= 0, \quad t > 0,\\
V(a,t) &= 0, \quad t > 0,\\
V(r,0) &= rf(r), \quad 0 < r < a,\\
V_t(r,0) &= rg(r).
\end{aligned}$$

Note: You will need to use the fact that because $W(r,t)$ is to be continuous at $r = 0$, the function $V(r,t)$ must satisfy the condition $V(0,t) = 0$ for $t > 0$.

(b) Solve the problem in part (a) and hence find $W(r,t)$.

26. What happens if the transformation of Exercise 25 is applied to the problem when the surface of the sphere satisfies a homogeneous Neumann condition?

27. A circular bar of natural length L is clamped at both ends and stretched until its length is L^*. At time $t = 0$ the left end of the bar is at position $x = 0$, and both clamps are removed. Subsequent horizontal vibrations occur along a frictionless surface.
(a) Find a series representation for displacements of cross sections of the bar.
(b) What is the d'Alembert solution for displacements of cross sections?
(c) Use the result in part (b) to find the velocity of the left end of the bar. Does it move smoothly?

28. A circular bar of natural length L is clamped at both ends and stretched until its length is L^*. At time $t = 0$ the left end of the bar is at position $x = 0$, and the clamp on the right end is removed. Subsequent horizontal vibrations occur along a frictionless surface.
(a) Find a series representation for displacements of cross sections of the bar. *Hint*: See Exercise 20 in Section 3.2.
(b) Is there a d'Alembert solution for displacements of cross sections?
(c) Use the result in part (b) to find the velocity of the right end of the bar. Does it move smoothly?

29. Solve the telegraphy equation of Exercise 27 in Section 2.3 for potential in a cable of length M when potential is zero at the ends of the cable, the initial potential in the cable is $f(x)$, and $V_t(x,0) = 0$.

Part C Potential, Steady-state Heat Conduction, Static Deflections of Membranes

30. A region A (in the xy-plane) is bounded by the lines $x = 0$, $y = 0$, $x = L$, and $y = L'$. If the edges $y = 0$, $y = L'$, and $x = L$ are held at potential zero, and $x = 0$ is at potential equal to 100, find the potential in A.

31. Solve exercise 30 if edges $x = 0$ and $y = 0$ are at potential 100, while $x = L$ and $y = L'$ are at zero potential. (Hint: See the extension of superposition principle 2 in Figure 4.2.)

32. Solve Exercise 30 if edges $x = 0$ and $x = L$ are at potential 100, while $y = 0$ and $y = L'$ are at zero potential.

33. Solve Exercise 30 if the condition $V(0,y) = 100$ along $x = 0$ is replaced by $\partial V(0,y)/\partial x = 100$, $0 < y < L'$.

34. Solve Exercise 30 if the boundary conditions are

$$-\frac{\partial V(0,y)}{\partial x} = -100, \quad 0 < y < L',$$
$$\frac{\partial V(L,y)}{\partial x} = 100, \quad 0 < y < L',$$
$$\frac{\partial V(x,0)}{\partial y} = \frac{\partial V(x,L')}{\partial y} = 0, \quad 0 < x < L.$$

Is the solution unique? What is the solution if $V(L/2, L'/2) = 0$?

35. Can Exercise 34 be solved if the condition along $x = L$ is $\partial V(L,y)/\partial x = -100$, $0 < y < L'$? Explain.

36. A thin rectangular plate occupies the region described by $0 \le x \le L$, $0 \le y \le L'$. Its top and bottom surfaces are insulated. If edges $x = 0$ and $x = L$ are held at temperature 0°C, while $y = 0$ and $y = L'$ have temperatures $x(L - x)$ and $-x(L - x)$, respectively, what is the steady-state temperature of the plate?

37. Solve Exercise 36 if edges $x = 0$, $x = L$, and $y = L'$ are held at temperature 0°C while heat is added along the edge $y = 0$ at a constant rate q W/m^2.

38. Solve Exercise 37 if heat is added to both edges $y = 0$ and $y = L'$ at rate q W/m^2 while edges $x = 0$ and $x = L$ are held at temperature 10°C.

39. A membrane is stretched tightly over the rectangle $0 \le x \le L$, $0 \le y \le L'$. Its edges are given deflections described by the following boundary conditions:

$$z(0,y) = kL(y - L')/L', \quad 0 < y < L',$$
$$z(L,y) = 0, \quad 0 < y < L',$$
$$z(x,0) = k(x - L), \quad 0 < x < L,$$
$$z(x,L') = 0, \quad 0 < x < L,$$

($k > 0$ a constant). Find static deflections of the membrane when all external forces are negligible compared with tensions in the membrane.

40. Find a formula for the solution of Laplace's equation inside the rectangle $0 \le x \le L$, $0 \le y \le L'$ of Figure 4.1 when (a) $g_1(y) = g_2(y) = h_2(x) = 0$ (b) $g_1(y) = g_2(y) = h_1(x) = 0$ (c) $g_1(y) = g_2(y) = 0$
(d) $h_1(x) = h_2(x) = 0$.

41. Solve Exercise 36 if edges $x = 0$ and $y = L'$ are insulated, $x = L$ is held at temperature 0°C, and $y = 0$ has temperature $(L - x)^2$, $0 \le x \le L$. (Hint: Use Exercise 21 in Section 3.2.)

42. Explain why Superposition Principle 2 cannot, in general, be used to solve the Neumann problem associated with Laplace's equation in a rectangle

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L',$$
$$-\frac{\partial V(0,y)}{\partial x} = f_1(y), \quad 0 < y < L',$$
$$\frac{\partial V(L,y)}{\partial x} = f_2(y), \quad 0 < y < L',$$

$$-\frac{\partial V(x,0)}{\partial y} = f_3(x), \quad 0 < x < L,$$
$$\frac{\partial V(x,L')}{\partial y} = f_4(x), \quad 0 < x < L.$$

43. Find potential in the semi-infinite strip $0 < x < L$, $y > 0$ if potential is zero along the vertical sides and $x(L-x)$ along the horizontal side.

Part D General Results

44. Prove that a second-order, linear, homogeneous PDE in two independent variables with constant coefficients is always separable. (A more general result is proved in the next exercise.)

$*$**45.** (a) Show that the homogeneous PDE

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = 0$$

is separable if $a(x,y) = a(x)$, $b(x,y) =$ constant, $c(x,y) = c(y)$, $d(x,y) = d(x)$, $e(x,y) = e(y)$, and $f(x,y) = f_1(x) + f_2(y)$.

(b) Are the conditions in part (a) necessary for separation?

46. Verify that we cannot have a complex separation constant α for problems 4.11 and 4.21.

§4.3 Nonhomogeneities and Variation of Constants

In Section 4.2 we stressed the fact that separation of variables is carried out on linear, homogeneous PDEs and linear, homogeneous boundary and/or initial conditions. Separated functions are then superposed in order to satisfy nonhomogeneous conditions. When nonhomogeneities are present in the PDE, or in the boundary conditions of time-dependent problems, separation by itself fails. To illustrate, we reconsider vibration problem 4.9 for displacement of a taut string with fixed end points, but take gravity into account:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0, \quad (g = 9.81), \tag{4.35a}$$

$$y(0,t) = 0, \quad t > 0, \tag{4.35b}$$

$$y(L,t) = 0, \quad t > 0, \tag{4.35c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{4.35d}$$

$$y_t(x,0) = 0, \quad 0 < x < L. \tag{4.35e}$$

Only the partial differential equation is affected; it becomes nonhomogeneous. The boundary conditions remain homogeneous. Substitution of a separated function $y(x,t) = X(x)T(t)$ into PDE 4.35a gives

$$XT'' = c^2 X''T - g.$$

Our usual procedure of dividing by $X(x)T(t)$ does not lead to a separated equation; in fact, this equation cannot be separated. Likewise, were boundary condition 4.35b not homogeneous, say $y(0,t) = f(t)$, in which case the left end of the string would be forced to undergo specific motion, substitution of $y(x,t) = X(x)T(t)$ would not lead to information about $X(x)$ and $T(t)$ separately.

In this section we illustrate two methods for handling nonhomogeneities. The first method uses steady-state solutions for heat conduction problems and static deflections for vibration problems. It applies, however, only to time-independent nonhomogeneities. The second method is called *variation of constants*; it applies to time-dependent as well as time-independent nonhomogeneities.

Time-Independent Nonhomogeneities

Partial differential equation 4.35a has a time-independent nonhomogeneity (it is also independent of x, but that is immaterial). To solve this problem, we define a new dependent variable $z(x,t)$ according to

$$y(x,t) = z(x,t) + \psi(x), \tag{4.36}$$

where $\psi(x)$ is the solution of the corresponding static-deflection problem

$$0 = c^2 \frac{d^2\psi}{dx^2} - g, \quad 0 < x < L, \tag{4.37a}$$

$$\psi(0) = 0, \quad \psi(L) = 0. \tag{4.37b}$$

Differential equation 4.37a implies that

$$\psi(x) = \frac{gx^2}{2c^2} + Ax + B,$$

and boundary conditions 4.37b require

$$0 = B, \qquad 0 = \frac{gL^2}{2c^2} + AL + B.$$

From these we obtain the position of the string were it to hang motionless under gravity:

$$\psi(x) = \frac{gx}{2c^2}(x - L). \tag{4.38}$$

We expect that the string will vibrate about this position and that $z(x, t)$ represents displacements from this position. A PDE satisfied by $z(x, t)$ can be found by substituting representation 4.36 into PDE 4.35a:

$$\frac{\partial^2}{\partial t^2}[z(x,t) + \psi(x)] = c^2 \frac{\partial^2}{\partial x^2}[z(x,t) + \psi(x)] - g.$$

This equation simplifies to the following homogeneous PDE when we note that $\psi(x)$ is only a function of x that satisfies ODE 4.37a:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \tag{4.39a}$$

Boundary conditions for $z(x, t)$ are obtained by setting $x = 0$ and $x = L$ in representation 4.36 and using boundary conditions 4.35b,c for $y(x, t)$:

$$z(0, t) = y(0, t) - \psi(0) = 0, \quad t > 0, \tag{4.39b}$$

$$z(L, t) = y(L, t) - \psi(L) = 0, \quad t > 0. \tag{4.39c}$$

Finally, by setting $t = 0$ in 4.36 and its partial derivative with respect to t, and using initial conditions 4.35d,e for $y(x, t)$, we obtain initial conditions for $z(x, t)$:

$$z(x, 0) = y(x, 0) - \psi(x) = f(x) + \frac{gx}{2c^2}(L - x), \quad 0 < x < L, \tag{4.39d}$$

$$z_t(x, 0) = y_t(x, 0) = 0, \quad 0 < x < L. \tag{4.39e}$$

We have therefore replaced problem 4.35, which has a nonhomogeneous PDE, with problem 4.39, which has a homogeneous PDE. We have complicated one of the initial conditions, but this is a small price to pay. If a function with variables separated is to satisfy PDE 4.39a, boundary conditions 4.39b,c, and initial condition 4.39e, it must be of the form

$$b \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L},$$

for arbitrary b and n an integer (see problem 4.9). Because PDE 4.39a and conditions 4.39b,c,e are linear and homogeneous, we superpose these functions and take

$$z(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}. \tag{4.40}$$

Initial condition 4.39d requires the constants b_n to satisfy

$$f(x) + \frac{gx}{2c^2}(L - x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.$$

Consequently, the b_n are coefficients in the Fourier sine series of the odd extension of $f(x) + gx(L-x)/(2c^2)$ to a function of period $2L$; that is,

$$b_n = \frac{2}{L}\int_0^L \left[f(x) + \frac{gx}{2c^2}(L-x)\right] \sin\frac{n\pi x}{L} dx. \tag{4.41}$$

The formal solution of vibration problem 4.35 is therefore

$$y(x,t) = \frac{gx}{2c^2}(x-L) + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \tag{4.42}$$

where the b_n are given by the integral in equation 4.41.

This technique of spliting off static deflections can be applied to any nonhomogeneity that is only a function of position, be it in the PDE or in a boundary condition. We illustrate nonhomogeneities in boundary conditions in the next example. Before doing so, however, we express solution 4.42 of problem 4.35 in d'Alembert's form, by first writing

$$y(x,t) = \frac{gx}{2c^2}(x-L) + \sum_{n=1}^{\infty}\frac{b_n}{2}[\sin\frac{n\pi(x+ct)}{L} + \sin\frac{n\pi(x-ct)}{L}].$$

Since

$$f(x) - \psi(x) = f(x) - \frac{gx}{2c^2}(x-L) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

it follows that

$$\begin{aligned} y(x,t) &= \psi(x) + \frac{1}{2}[f(x+ct) - \psi(x+ct) + f(x-ct) - \psi(x-ct)] \\ &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2}\left[2\psi(x) - \psi(x+xt) - \psi(x-ct)\right]. \end{aligned}$$

This function is defined for all $0 < x < L$ and $t > 0$ since $f(x)$ and $\psi(x)$ are both extended as odd, $2L$-periodic functions.

Example 4.5 Solve the initial boundary value problem for temperature in a homogeneous, isotropic rod with insulated sides when the ends of the rod are held at constant temperatures

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{4.43a}$$

$$U(0,t) = U_0, \quad t > 0, \tag{4.43b}$$

$$U(L,t) = U_L, \quad t > 0, \tag{4.43c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{4.43d}$$

Solution We define a new dependent variable $V(x,t)$ by

$$U(x,t) = V(x,t) + \psi(x), \tag{4.44}$$

where $\psi(x)$ is the solution of the associated steady-state problem

$$0 = k\frac{d^2\psi}{dx^2}, \quad 0 < x < L, \tag{4.45a}$$

$$\psi(0) = U_0, \tag{4.45b}$$

$$\psi(L) = U_L. \tag{4.45c}$$

Differential equation 4.45a implies that $\psi(x) = Ax + B$, and boundary conditions 4.45b,c require

$$U_0 = B, \qquad U_L = AL + B.$$

From these, we obtain the steady-state solution

$$\psi(x) = U_0 + \frac{(U_L - U_0)x}{L} \tag{4.46}$$

(the temperature in the rod after a very long time). With this choice for $\psi(x)$, the PDE for $V(x,t)$ can be found by substituting representation 4.44 into 4.43a:

$$\frac{\partial}{\partial t}[V(x,t) + \psi(x)] = k\frac{\partial^2}{\partial x^2}[V(x,t) + \psi(x)].$$

Because $\psi(x)$ is only a function of x that has a vanishing second derivative, this equation simplifies to

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \tag{4.47a}$$

Boundary conditions for $V(x,t)$ are obtained from representation 4.44 and boundary conditions 4.43b,c for $U(x,t)$:

$$V(0,t) = U(0,t) - \psi(0) = U_0 - U_0 = 0, \quad t > 0, \tag{4.47b}$$

$$V(L,t) = U(L,t) - \psi(L) = U_L - U_L = 0, \quad t > 0. \tag{4.47c}$$

Finally, $V(x,t)$ must satisfy the initial condition

$$V(x,0) = U(x,0) - \psi(x) = f(x) - U_0 - \frac{(U_L - U_0)x}{L}, \quad 0 < x < L. \tag{4.47d}$$

Separation of variables $V(x,t) = X(x)T(t)$ on 4.47a,b,c leads to the ordinary differential equations

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \tag{4.48a}$$

$$X(0) = X(L) = 0; \tag{4.48b}$$

$$T' + k\lambda^2 T = 0, \quad t > 0. \tag{4.49}$$

These give separated functions

$$be^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}$$

for arbitrary b and n an integer. To satisfy the initial condition, we superpose separated functions and take

$$V(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}. \tag{4.50}$$

Initial condition 4.47d requires the constants b_n to satisfy

$$f(x) - U_0 - \frac{(U_L - U_0)x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.$$

Consequently, the b_n are the coefficients in the Fourier sine series of the odd extension of $f(x) - U_0 - (U_L - U_0)x/L$ to a function of period $2L$:

$$b_n = \frac{2}{L} \int_0^L \left[f(x) - U_0 - \frac{(U_L - U_0)x}{L} \right] \sin \frac{n\pi x}{L} dx. \tag{4.51}$$

The formal solution of problem 4.43 is therefore

$$U(x,t) = V(x,t) + U_0 + \frac{(U_L - U_0)x}{L}, \tag{4.52}$$

where $V(x,t)$ is given by the series in 4.50 and b_n by the integral in 4.51. Function $V(x,t)$ represents the transient part of the temperature function, which, because of the exponential factor $e^{-n^2\pi^2kt/L^2}$, decreases with time. Temperature approaches the steady-state solution.•

It is interesting and informative to analyze solution 4.52 further for two specific initial temperature distributions $f(x)$. First, suppose that the initial temperature of the rod is 0°C throughout; that is, $f(x) \equiv 0$. In this case, equations 4.50–4.52 yield, for the temperature in the rod,

$$U(x,t) = U_0 + \frac{(U_L - U_0)x}{L} + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2kt/L^2} \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L \left[-U_0 - (U_L - U_0)\frac{x}{L} \right] \sin \frac{n\pi x}{L} dx = \frac{-2}{n\pi}[U_0 + (-1)^{n+1}U_L].$$

This function is plotted for various fixed values of t in Figure 4.11 (using a diffusivity of $k = 12.4 \times 10^{-6}$ m^2/s). What is important to notice is the smooth transition from initial temperature 0°C to final steady-state temperature at every point in the rod except for its ends $x = 0$ and $x = L$. Here the transition is instantaneous, as is dictated by problem 4.43 when $f(x)$ is chosen to vanish identically. Physically, this is an impossibility, but the mathematics required to describe a very quick but smooth change in temperature from 0°C at $x = 0$ and $x = L$ to U_0 and U_L would complicate the problem enormously. In practice, then, we are willing to live with the anomaly of the solution at time $t = 0$ for $x = 0$ and $x = L$ in order to avoid these added complications. This anomaly is manifested in heat transfer across the ends of the rod at time $t = 0$. According to equation 2.19 in Section 2.2, the amount of heat flowing left to right through any cross section of the rod is

$$\begin{aligned} q(x,t) = -\kappa\frac{\partial U}{\partial x} &= -\kappa \left(\frac{U_L - U_0}{L} + \frac{\pi}{L} \sum_{n=1}^{\infty} n b_n e^{-n^2\pi^2kt/L^2} \cos \frac{n\pi x}{L} \right) \\ &= \frac{\kappa}{L} \left\{ U_0 - U_L + 2\sum_{n=1}^{\infty} [U_0 + (-1)^{n+1}U_L] e^{-n^2\pi^2kt/L^2} \cos \frac{n\pi x}{L} \right\}. \end{aligned}$$

The series in this expression diverges (to infinity) when $x = 0$ at $t = 0$. In other words, the instantaneous temperature change at time $t = 0$ from 0°C to U_0°C is predicated on an infinite heat flux at that time. A similar situation occurs at the end $x = L$.

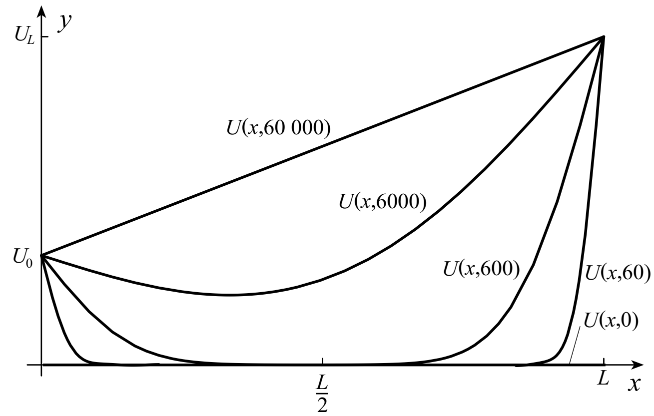


Figure 4.11

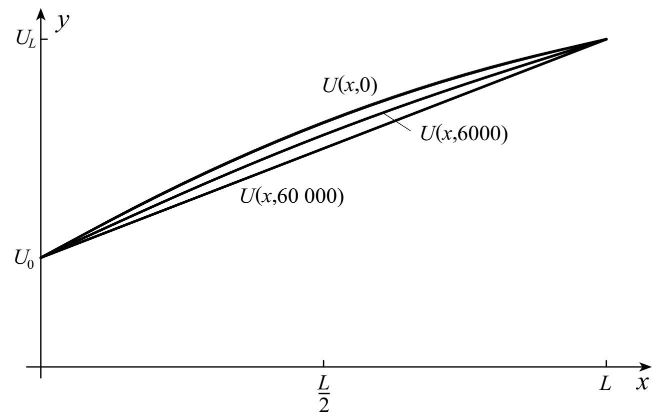


Figure 4.12

The second initial temperature function we consider is $f(x) = U_0(1 - x^2/L^2) + U_L x/L$, a distribution that does not give rise to abrupt temperature changes at time $t = 0$ since $f(0) = U_0$ and $f(L) = U_L$. In this case, coefficients b_n in 4.51 are $b_n = 4U_0[1 + (-1)^{n+1}]/(n^3\pi^3)$, and

$$U(x,t) = U_0 + \frac{(U_L - U_0)x}{L} + \frac{8U_0}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}.$$

As shown in Figure 4.12, the transition from initial to steady-state temperature is smooth for all $0 \le x \le L$. Supporting this is the heat flux vector

$$q(x,t) = \frac{\kappa}{L}\left[U_0 - U_L - \frac{8U_0}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt/L^2}\cos\frac{(2n-1)\pi x}{L}\right].$$

The series herein converges uniformly for $0 \le x \le L$ and $t \ge 0$. If we take limits as $x \to 0^+$ and $t \to 0^+$, we find the initial heat flux across the end $x = 0$,

$$\begin{aligned} q(0+,0+) &= \frac{\kappa}{L}\left[U_0 - U_L - \frac{8U_0}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\right] \\ &= \frac{\kappa}{L}\left[U_0 - U_L - \frac{8U_0}{\pi^2}\left(\frac{\pi^2}{8}\right)\right] = -\frac{\kappa U_L}{L} \end{aligned}$$

(since $\sum_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$). Perhaps unexpectedly, we find that the direction of heat flow across $x = 0$ at time $t = 0$ is completely dictated by the sign of U_L. When $U_L < 0$, heat flows into the rod, and when $U_L > 0$, heat flows out. This is most easily seen by calculating the derivative of the initial temperature distribution in the rod at $x = 0$, $f'(0) = U_L/L$. If $U_L < 0$, points in the rod near $x = 0$ have temperature less than those in the end $x = 0$, and heat flows into the rod; if $U_L > 0$, points near $x = 0$ are at a higher temperature than those at $x = 0$, and heat flows out of the rod.

Time-Dependent Nonhomogeneities

When the nonhomogeneity in a PDE is time dependent, it is necessary to adopt a different approach. The technique used is called *variation of constants*. Because variation of constants resembles the method of variation of parameters for ODEs, we digress for a quick review of variation of parameters. Consider the ODE

$$y'' + y = f(x), \tag{4.53a}$$

where $f(x)$ is as yet an unspecified function. The general solution of the associated homogeneous equation $y'' + y = 0$ is $y(x) = A\cos x + B\sin x$, to which must be added a particular solution of 4.53a. When $f(x)$ is a polynomial, a sine, a cosine, an exponential, or a combination of these, various techniques (such as undetermined coefficients, or operators) yield this particular solution. Variation of parameters also gives a particular solution in these cases, but it realizes its true potential when $f(x)$ is not one of these, or when a general solution is required for arbitrary $f(x)$. The method assumes that a general (or particular) solution of 4.53a can be found in the form $A\cos x + B\sin x$, but where A and B are functions of x; that is, it assumes that a general solution of the nonhomogeneous equation is $y(x) = A(x)\cos x + B(x)\sin x$. To obtain $A(x)$ and $B(x)$, this function is substituted into the differential equation. Because this imposes only one condition on two functions $A(x)$ and $B(x)$, the opportunity is taken to impose a second condition, and this condition is always taken as $A'(x)\cos x + B'(x)\sin x = 0$. The result is the following system of linear equations in $A'(x)$ and $B'(x)$:

$$A'(x)\cos x + B'(x)\sin x = 0, \tag{4.54a}$$
$$-A'(x)\sin x + B'(x)\cos x = f(x). \tag{4.54b}$$

These can be solved for

$$A'(x) = -f(x)\sin x, \qquad B'(x) = f(x)\cos x,$$

from which

$$A(x) = -\int f(x)\sin x\,dx + C_1, \qquad B(x) = \int f(x)\cos x\,dx + C_2,$$

where C_1 and C_2 are constants of integration. The general solution of ODE 4.53a is therefore

$$y(x) = \left[C_1 - \int f(x)\sin x\,dx\right]\cos x + \left[C_2 + \int f(x)\cos x\,dx\right]\sin x. \tag{4.55}$$

(If C_1 and C_2 are omitted, this is a particular solution of the differential equation.) A simplified form results if we express the antiderivatives as definite integrals:

$$\begin{aligned} y(x) &= C_1\cos x + C_2\sin x - \cos x\int_0^x f(t)\sin t\,dt + \sin x\int_0^x f(t)\cos t\,dt \\ &= C_1\cos x + C_2\sin x + \int_0^x f(t)\sin(x-t)\,dt. \end{aligned} \tag{4.56}$$

In this form, any initial conditions

$$y(0) = y_0, \qquad y'(0) = y_0', \tag{4.53b}$$

that might accompany ODE 4.53a are easily incorporated. They require that

$$y_0 = C_1, \qquad y_0' = C_2,$$

and therefore the final solution of differential equation 4.53a subject to initial conditions 4.53b is

$$y(x) = y_0 \cos x + y_0' \sin x + \int_0^x f(t) \sin(x-t)\,dt. \tag{4.57}$$

We now develop the analogous method for solving initial boundary value problems that have time-dependent nonhomogeneities in their PDEs. The one-dimensional vibration problem for displacement of a taut string with a time-dependent forcing function $F(x,t) = e^{-t}$ is a convenient vehicle:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{e^{-t}}{\rho}, \quad 0 < x < L, \quad t > 0, \tag{4.58a}$$

$$y(0,t) = 0, \quad t > 0, \tag{4.58b}$$

$$y(L,t) = 0, \quad t > 0, \tag{4.58c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{4.58d}$$

$$y_t(x,0) = 0, \quad 0 < x < L. \tag{4.58e}$$

We have taken a forcing function that does not depend on x to simplify calculations, but the technique works when the forcing function is a function of x as well as t. If the forcing term were absent, the PDE would be homogeneous, and according to our solution to problem 4.9, separation of variables on 4.58a,b,c,e would lead to a superposed solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L},$$

where the C_n are arbitrary constants (see equation 4.17). To incorporate a nonzero forcing term, we use a method called **variation of constants**. This method is much like variation of parameters for ODEs; we attempt to find a solution in this form, but where $C_n = C_n(t)$ are functions of t,

$$y(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}. \tag{4.59}$$

Because the $C_n(t)$ are unknown functions at this point, it is more convenient, and no less general, to group $C_n(t)$ and $\cos(n\pi ct/L)$ together as the unknown function, say $d_n(t) = C_n(t)\cos(n\pi ct/L)$. In other words, we replace the above solution with

$$y(x,t) = \sum_{n=1}^{\infty} d_n(t) \sin\frac{n\pi x}{L}. \tag{4.60}$$

For any choice of $d_n(t)$ whatsoever, representation 4.60 satisfies boundary conditions 4.58b,c. To satisfy initial condition 4.58e, we must have

$$\sum_{n=1}^{\infty} d_n'(0) \sin\frac{n\pi x}{L} = 0, \quad 0 < x < L.$$

This requires the unknown functions $d_n(t)$ to have vanishing first derivatives at $t = 0$, $d_n'(0) = 0$.

To determine whether a function of form 4.60 can satisfy PDE 4.58a, we substitute 4.60 into 4.58a and formally differentiate term-by-term:

$$\sum_{n=1}^{\infty} d_n''(t) \sin \frac{n\pi x}{L} = c^2 \sum_{n=1}^{\infty} -\frac{n^2\pi^2}{L^2} d_n(t) \sin \frac{n\pi x}{L} + \frac{e^{-t}}{\rho}. \tag{4.61}$$

In its present form, this equation is intractable, but the fact that two of the terms are series in $\sin(n\pi x/L)$ suggests that the function e^{-t}/ρ be expressed in this way also; that is, we should write

$$\frac{e^{-t}}{\rho} = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{L}. \tag{4.62a}$$

We have seen equations of this form before; they are Fourier sine series representations for the function on the left. However, should not the function on the left be a function of x, not t? Indeed it should, but e^{-t}/ρ is trivially a function of x, and in addition it is a function of t. In other words, it is not that e^{-t}/ρ is a function of the wrong variable; it is a function of both x and t, and we wish to express this function of x and t as a Fourier sine series in x for any given t. Clearly, this can happen only if coefficients are functions of t; that is, we really want to express e^{-t}/ρ in the form

$$\frac{e^{-t}}{\rho} = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi x}{L}. \tag{4.62b}$$

For each fixed t, 4.62 is the Fourier sine series of the odd, $2L$-periodic extension of the constant function e^{-t}/ρ (of x). According to equation 3.17b then,

$$F_n(t) = \frac{2}{L} \int_0^L \frac{1}{\rho} e^{-t} \sin \frac{n\pi x}{L} dx = \frac{2e^{-t}[1 + (-1)^{n+1}]}{n\pi\rho},$$

and therefore

$$\frac{e^{-t}}{\rho} = \frac{2e^{-t}}{\rho\pi} \sum_{n=1}^{\infty} \frac{[1 + (-1)^{n+1}]}{n} \sin \frac{n\pi x}{L}. \tag{4.63}$$

If representation 4.63 is now substituted into equation 4.61, the result is

$$\sum_{n=1}^{\infty} d_n''(t) \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} -\frac{n^2\pi^2c^2}{L^2} d_n(t) \sin \frac{n\pi x}{L} + \frac{2e^{-t}}{\rho\pi} \sum_{n=1}^{\infty} \frac{[1 + (-1)^{n+1}]}{n} \sin \frac{n\pi x}{L}$$

or,

$$\sum_{n=1}^{\infty} \left[d_n''(t) + \frac{n^2\pi^2c^2}{L^2} d_n(t) - \frac{2e^{-t}[1 + (-1)^{n+1}]}{\rho n\pi} \right] \sin \frac{n\pi x}{L} = 0.$$

But for each fixed t, the series on the left side of this equation is the Fourier sine series of the function on the right, the function that is identically equal to zero. It follows that all coefficients must be zero; that is,

$$d_n''(t) + \frac{n^2\pi^2c^2}{L^2} d_n(t) - \frac{2e^{-t}[1 + (-1)^{n+1}]}{\rho n\pi} = 0.$$

In other words, each unknown function $d_n(t)$ must satisfy the differential equation

$$\frac{d^2d_n}{dt^2} + \frac{n^2\pi^2c^2}{L^2}d_n = \frac{2e^{-t}[1+(-1)^{n+1}]}{\rho n\pi}.$$

A general solution of this equation is

$$d_n(t) = b_n \cos\frac{n\pi ct}{L} + a_n \sin\frac{n\pi ct}{L} + \frac{2L^2[1+(-1)^{n+1}]e^{-t}}{n\pi\rho(L^2+n^2\pi^2c^2)},$$

where a_n and b_n are constants. The condition $d_n'(0) = 0$ implies that

$$a_n = \frac{2L^3[1+(-1)^{n+1}]}{n^2\pi^2\rho c(L^2+n^2\pi^2c^2)},$$

and therefore

$$d_n(t) = b_n \cos\frac{n\pi ct}{L} + \frac{2L^2[1+(-1)^{n+1}]}{n^2\pi^2\rho c(L^2+n^2\pi^2c^2)}\left(n\pi ce^{-t} + L\sin\frac{n\pi ct}{L}\right).$$

Substitution of this expression into representation 4.60 gives

$$y(x,t) = \sum_{n=1}^{\infty}\left[b_n \cos\frac{n\pi ct}{L} + \frac{2L^2[1+(-1)^{n+1}]}{n^2\pi^2\rho c(L^2+n^2\pi^2c^2)}\left(n\pi ce^{-t} + L\sin\frac{n\pi ct}{L}\right)\right]\sin\frac{n\pi x}{L}. \quad (4.64)$$

Initial condition 4.58d requires

$$f(x) = \sum_{n=1}^{\infty}\left[b_n + \frac{2L^2[1+(-1)^{n+1}]}{n\pi\rho(L^2+n^2\pi^2c^2)}\right]\sin\frac{n\pi x}{L}, \quad 0 < x < L,$$

from which

$$b_n + \frac{2L^2[1+(-1)^{n+1}]}{n\pi\rho(L^2+n^2\pi^2c^2)} = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \quad (4.65)$$

The formal solution of problem 4.58 is now complete; it is 4.64 with the b_n defined by 4.65.

Perhaps a summary of the variation of constants technique would be valuable at this juncture. When a PDE has a nonhomogeneity, the method proceeds as follows:

1. Find separated functions satisfying the homogeneous boundary conditions (and homogeneous initial conditions) and the corresponding *homogeneous* PDE. Suppose we denote the functions of x by $X_n(x)$. ($X_n(x) = \sin(n\pi x/L)$ in our previous problem.)

2. Represent the unknown function in a series of the form

$$\sum_{n=1}^{\infty} d_n(t)X_n(x)$$

with unknown coefficients $d_n(t)$.

3. Substitute the series of step 2 into the PDE, at the same time expanding the nonhomogeneity in terms of the functions $X_n(x)$.

4. Obtain and solve ordinary differential equations for the $d_n(t)$.

5. Use initial conditions on the PDE to determine any constants of integration in step 4.

When time-dependent nonhomogeneities are present in boundary conditions, they are transformed into nonhomogeneities in the PDE. They can then be handled by variation of constants. This is illustrated in the following example.

Example 4.6 Solve the following initial boundary value problem for temperature in a homogeneous, isotropic rod with insulated sides:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{4.66a}$$

$$U(0,t) = \phi_0(t), \quad t > 0, \tag{4.66b}$$

$$U(L,t) = \phi_L(t), \quad t > 0, \tag{4.66c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{4.66d}$$

The rod is free of internal heat generation, and its ends are kept at prescribed temperatures.

Solution We define a new dependent variable $V(x,t)$ according to

$$U(x,t) = V(x,t) + \psi(x,t), \tag{4.67}$$

where $\psi(x,t)$ is to be chosen so that $V(x,t)$ will satisfy homogeneous boundary conditions. Boundary conditions 4.66b,c require

$$V(0,t) = \phi_0(t) - \psi(0,t), \qquad V(L,t) = \phi_L(t) - \psi(L,t).$$

Consequently, $V(x,t)$ will satisfy homogeneous boundary conditions

$$V(0,t) = 0, \quad t > 0, \tag{4.68a}$$

$$V(L,t) = 0, \quad t > 0, \tag{4.68b}$$

if $\psi(x,t)$ is chosen so that

$$\psi(0,t) = \phi_0(t), \qquad \psi(L,t) = \phi_L(t).$$

These are accommodated if $\psi(x,t)$ is chosen as the following linear function in x

$$\psi(x,t) = \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)]. \tag{4.69}$$

This is not the only choice for $\psi(x,t)$, but it is perhaps the simplest. With this choice

$$U(x,t) = V(x,t) + \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)]. \tag{4.70}$$

The PDE for $V(x,t)$ can be obtained by substituting representation 4.70 into PDE 4.66a,

$$\frac{\partial}{\partial t}\left[V(x,t) + \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)]\right] = k\frac{\partial^2}{\partial x^2}\left[V(x,t) + \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)]\right],$$

or,

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + G(x,t), \tag{4.68c}$$

where

$$G(x,t) = -\phi_0'(t) - \frac{x}{L}[\phi_L'(t) - \phi_0'(t)]. \tag{4.68d}$$

Initial condition 4.66d yields the initial condition for $V(x,t)$,

$$V(x,0) = f(x) - \phi_0(0) - \frac{x}{L}[\phi_L(0) - \phi_0(0)], \quad 0 < x < L. \tag{4.68e}$$

Our problem now is to solve PDE 4.68c,d subject to homogeneous boundary conditions 4.68a,b and initial condition 4.68e; that is, $V(x,t)$ must satisfy

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + G(x,t), \quad 0 < x < L, \quad t > 0, \tag{4.71a}$$

$$V(0,t) = 0, \quad t > 0, \tag{4.71b}$$

$$V(L,t) = 0, \quad t > 0, \tag{4.71c}$$

$$V(x,0) = f(x) - \phi_0(0) - \frac{x}{L}[\phi_L(0) - \phi_0(0)], \quad 0 < x < L, \tag{4.71d}$$

where

$$G(x,t) = -\phi_0'(t) - \frac{x}{L}[\phi_L'(t) - \phi_0'(t). \tag{4.71e}$$

What we have done is transform the nonhomogeneities in boundary conditions 4.66b,c into PDE 4.71a. But this presents no difficulty; variation of constants handles nonhomogeneous PDEs. Were $G(x,t)$ not present, separation of variables would lead to a solution of the form

$$V(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}.$$

We therefore assume a solution for nonhomogeneous problem 4.71 in the form

$$V(x,t) = \sum_{n=1}^{\infty} d_n(t) \sin\frac{n\pi x}{L}, \tag{4.72}$$

where the exponential has been absorbed into the unknown function $d_n(t)$. This function satisfies boundary conditions 4.71b,c and will satisfy PDE 4.71a if

$$\sum_{n=1}^{\infty} d_n'(t) \sin\frac{n\pi x}{L} = k\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{L^2} d_n(t) \sin\frac{n\pi x}{L} + G(x,t). \tag{4.73}$$

To simplify this equation, we extend $G(x,t)$ as an odd, $2L$-periodic function and expand it in a Fourier sine series

$$G(x,t) = \sum_{n=1}^{\infty} G_n(t) \sin\frac{n\pi x}{L}, \qquad \text{where} \qquad G_n(t) = \frac{2}{L}\int_0^L G(x,t)\sin\frac{n\pi x}{L}dx. \tag{4.74}$$

Substitution of this series into equation 4.73 gives

$$\sum_{n=1}^{\infty} \left[d_n'(t) + \frac{n^2\pi^2 k}{L^2} d_n(t) - G_n(t)\right] \sin\frac{n\pi x}{L} = 0.$$

But for each fixed t, the series on the left of this equation is the Fourier sine series of the function on the right, the function that is identically zero. It follows that all coefficients must vanish; that is,

$$d_n'(t) + \frac{n^2\pi^2 k}{L^2} d_n(t) = G_n(t).$$

A general solution of this linear, first-order ODE is

$$d_n(t) = b_n e^{-n^2\pi^2 kt/L^2} + \int_0^t G_n(u) e^{n^2\pi^2 k(u-t)/L^2} du,$$

where b_n is a constant. Substitution of this into 4.72 gives

$$V(x,t) = \sum_{n=1}^{\infty} \left[b_n e^{-n^2\pi^2 kt/L^2} + \int_0^t G_n(u) e^{n^2\pi^2 k(u-t)/L^2} du \right] \sin \frac{n\pi x}{L}. \quad (4.75)$$

To satisfy initial condition 4.71d, we must have

$$f(x) - \phi_0(0) - \frac{x}{L}[\phi_L(0) - \phi_0(0)] = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L,$$

and this implies that

$$b_n = \frac{2}{L} \int_0^L \left[f(x) - \phi_0(0) - \frac{x}{L}[\phi_L(0) - \phi_0(0)] \right] \sin \frac{n\pi x}{L} dx. \quad (4.76)$$

The formal solution of problem 4.66 is therefore

$$U(x,t) = V(x,t) + \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)],$$

with $V(x,t)$ given by 4.75, 4.76, and 4.74.•

Let us summarize the techniques for handling nonhomogeneities.

1. When nonhomogeneous boundary conditions are associated with Laplace's equation, all that is needed is superposition principle 2. The problem is divided into two or more subproblems, each of which can be solved by separation of variables, and the solutions of these subproblems are then added together. (For example, when $F(x,y) \equiv 0$ in the problem of Figure 4.1, $V(x,y)$ is the sum of $V_1(x,y)$ and $V_2(x,y)$.) Nonhomogeneities that turn Laplace's equation into Poisson's equation require variation of constants (see Exercise 28).

2. When time-independent nonhomogeneities occur in initial boundary value problems (be they in the boundary conditions or in the PDE), it is advantageous to split off steady-state or static solutions. The remaining part of the solution then satisfies a homogeneous PDE and homogeneous boundary conditions.

3. When nonhomogeneities in boundary conditions of initial boundary value problems are time dependent, they can be transformed into time-dependent nonhomogeneities in the PDE. (See, for example, transformation 4.70 in Example 4.6.) Variation of constants then takes care of time-dependent nonhomogeneities in PDEs.

Because time-independent nonhomogeneities (in technique 2) are trivially functions of time, it is natural to ask whether technique 2 is necessary now that we have technique 3. To answer this question, we use technique 3 on problem 4.35. Separation of variables on 4.35a,b,c,e in the absence of the nonhomogeneity leads to a superposition of separated functions in the form

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$

Variation of constants suggests a solution of problem 4.35 (with g now present) in the form

$$y(x,t)=\sum_{n=1}^{\infty} d_n(t)\sin\frac{n\pi x}{L}.$$

When this solution is pursued, the result obtained is

$$y(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi ct}{L}-\frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\left(1-\cos\frac{n\pi ct}{L}\right)\right]\sin\frac{n\pi x}{L}, \quad (4.77\text{a})$$

where

$$a_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \quad (4.77\text{b})$$

This does not appear to be the same as solution 4.42 of problem 4.35, namely,

$$y(x,t)=\frac{gx(x-L)}{2c^2}+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \quad (4.42)$$

where

$$b_n=\frac{2}{L}\int_0^L\left[f(x)+\frac{gx(L-x)}{2c^2}\right]\sin\frac{n\pi x}{L}dx. \quad (4.41)$$

They do, however, represent the same function as we now show. Integration by parts gives

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx+\frac{2}{L}\int_0^L\frac{gx(L-x)}{2c^2}\sin\frac{n\pi x}{L}dx=a_n+\frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}$$

and therefore 4.42 may be expressed as

$$y(x,t)=\frac{gx(x-L)}{2c^2}+\sum_{n=1}^{\infty}\left[a_n+\frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\right]\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}.$$

If we divide the summation in 4.77 into two parts, this function may be written in the form

$$y(x,t)=\sum_{n=1}^{\infty}\frac{-2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\sin\frac{n\pi x}{L}+\sum_{n=1}^{\infty}\left[a_n+\frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\right]\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}.$$

These expressions are indeed identical, since the first series in the latter equation is the Fourier sine series of the odd, $2L$-periodic extension of $gx(x-L)/(2c^2)$,

$$\frac{gx(x-L)}{2c^2}=\sum_{n=1}^{\infty}\frac{-2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\sin\frac{n\pi x}{L}, \quad 0\le x\le L.$$

Although this example illustrates that variation of constants can also be used to solve problems when nonhomogeneities are time independent, we would not suggest abandoning technique 2. There is a definite advantage to solution 4.42 over 4.77. Contained in 4.42 is a closed-form part, $gx(x-L)/(2c^2)$. This is also a part of 4.77a, but it is in the form of a Fourier series. This is the advantage of technique

2; it always splits off, in closed form, a steady-state or static part of the solution. Technique 3 does not; it delivers steady-state or static parts in series form. Given only the Fourier series for steady-state or static solutions, it could be very difficult to recognize their closed forms.

It is not just for cosmetic reasons that it is preferable to split off closed form solutions. The series part of the solution converges much more rapidly when the closed form part has been removed. This is effectively illustrated with the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{kG}{\kappa}, \quad 0 < x < L, \quad t > 0, \tag{4.78a}$$

$$U(0,t) = 0, \quad t > 0, \tag{4.78b}$$

$$U(L,t) = 0, \quad t > 0, \tag{4.78c}$$

$$U(x,0) = 0, \quad 0 < x < L, \tag{4.78d}$$

for temperature in a rod with zero initial and end temperatures and constant heat generation represented by G. When the steady-state temperature $\psi(x) = Gx(L-x)/(2\kappa)$ is split off from $U(x,t)$, the solution is found to be

$$U(x,t) = \frac{Gx(L-x)}{2\kappa} + \frac{2GL^2}{\pi^3\kappa}\sum_{n=1}^{\infty}\frac{1}{n^3}[(-1)^n - 1]e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}. \tag{4.79}$$

On the other hand, when variation of constants is used, the solution takes the form

$$U(x,t) = \frac{2GL^2}{\pi^3\kappa}\sum_{n=1}^{\infty}\frac{1}{n^3}[1 + (-1)^{n+1}][1 - e^{-n^2\pi^2kt/L^2}]\sin\frac{n\pi x}{L}. \tag{4.80}$$

Both series contain the factor $1/n^3$. The series in solution 4.79 also contains the exponential $e^{-n^2\pi^2kt/L^2}$ which enhances convergence for large n and/or t. Series solution 4.80, on the other hand, contains the factor $1 - e^{-n^2\pi^2kt/L^2}$ which approaches unity for large n and/or t. Convergence is much more rapid when the closed form solution is removed from the series.

In Sections 4.2 and 4.3 we have shown how the method of separation of variables leads to the use of Fourier series in the solution of various initial boundary value problems. We have considered problems with one and more than one nonhomogeneous condition, many second-order equations, and one fourth-order equation. All equations contained two independent variables in order that the method not be obscured by overly complicated calculations. Certainly, however, the method can, and will be used for problems in several independent variables.

We do not yet know whether we have solved any of the initial boundary value problems in these sections; we have found only what we call *formal* solutions. They are formal because of the questionable validity of superposing an infinity of separated functions. Each formal solution must therefore be verified as a valid solution to its initial boundary value problem. We do this in Sections 6.6–6.8 when we take up detailed analyses of convergence properties of formal solutions.

In problems 4.9, 4.26, 4.35, 4.43, and 4.66, separation of variables led to the system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = 0 = X(L),$$

and in problem 4.20 to the system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X'(0) = 0 = X'(L).$$

Each of these problems is a special case of a general mathematical system called a *Sturm-Liouville system*. It consists of an ordinary differential equation

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) - q(x)]y = 0 \tag{4.81a}$$

on some interval $a < x < b$, together with two boundary conditions

$$-l_1 y'(a) + h_1 y(a) = 0, \tag{4.81b}$$
$$l_2 y'(b) + h_2 y(b) = 0, \tag{4.81c}$$

where λ is a parameter and h_1, h_2, l_1, and l_2 are constants.

In Chapter 5 we discuss Sturm-Liouville systems in a general context and obtain properties of solutions of such systems. These systems lead to *generalized Fourier series* containing not only trigonometric functions but many other types of functions, such as Bessel functions and Legendre polynomials.

Finally, it is obvious that the steps in the solutions of boundary value and initial boundary value problems in Sections 4.2 and 4.3, and even the wording of the steps, are almost identical. Surely, then, we should be able to devise a method that would eliminate the tedious repetition of these steps in every problem. Indeed, *finite Fourier transforms* associated with Sturm-Liouville systems can be used for this purpose. They are discussed in Chapter 7.

EXERCISES 4.3

Part A Heat Conduction

1. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature 20°C throughout $(0 \le x \le L)$ at time $t = 0$. For $t > 0$, a constant electric current I is passed along the length of the rod, creating heat generation $g(x,t) = I^2/(A^2\sigma)$, where σ is the electrical conductivity of the rod and A is its cross-sectional area (see Exercise 42 in Section 2.2). If the ends of the rod are held at temperature 0°C for $t > 0$, find the temperature in the rod for $t > 0$ and $0 < x < L$.

2. Repeat Exercise 1 if the ends of the rod are held at temperature 100°C for $t > 0$.

3. Repeat Exercise 1 if the ends $x = 0$ and $x = L$ of the rod are held at constant temperatures U_0 and U_L, respectively, for $t > 0$.

4. Repeat Exercise 1 if the electric current is a function of time $I = e^{-\alpha t}$. Assume that $\alpha \ne n^2\pi^2 k/(2L^2)$ for any positive integer n.

5. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature 100°C throughout ($0 \le x \le L$) at time $t = 0$. For $t > 0$, its left end ($x = 0$) is held at temperature zero and its right end has temperature $100e^{-t}$. Find the temperature in the rod for $t > 0$ and $0 < x < L$. Assume first that $k \neq L^2/(n^2\pi^2)$ for any integer n, and secondly that $k = L^2/(m^2\pi^2)$ for some positive integer m.

6. Repeat Exercise 1 if the ends of the rod are insulated for $t > 0$.

7. Repeat Exercise 1 if the ends of the rod are insulated and $I = e^{-\alpha t}$.

8. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature zero throughout ($0 \le x \le L$). For $t > 0$, its ends $x = 0$ and $x = L$ continue to be held at temperature zero, and heat generation at each point of the rod is described by $g(x,t) = e^{-\alpha t}\sin(m\pi x/L)$, where $\alpha > 0$, and m is a positive integer. Find the temperature in the rod as a function of x and t. Consider the cases that (a) $\alpha \neq m^2\pi^2 k/L^2$, and (b) $\alpha = m^2\pi^2 k/L^2$.

9. The general one-dimensional heat conduction problem for a homogeneous, isotropic rod with insulated sides is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \quad 0 < x < L, \quad t > 0,$$
$$-l_1\frac{\partial U}{\partial x} + h_1 U = f_1(t), \quad x = 0, \quad t > 0,$$
$$l_2\frac{\partial U}{\partial x} + h_2 U = f_2(t), \quad x = L, \quad t > 0,$$
$$U(x,0) = f(x), \quad 0 < x < L.$$

Show that when the nonhomogeneities $g(x,t)$ $f_1(t)$, and $f_2(t)$ are independent of time, the change of dependent variable $U(x,t) = V(x,t) + \psi(x)$, where $\psi(x)$ is the solution of the corresponding steady-state problem, leads to an initial boundary value problem in $V(x,t)$ that has a homogeneous PDE and homogeneous boundary conditions.

10. Explain how to solve Exercise 1 if the current is turned on for only 100 seconds beginning at time $t = 0$. Do not solve the problem; just explain the steps that you would take to solve it.

11. Suppose that heat generation in the thin wire of Exercise 41 in Section 2.2 is caused by an electric current I. When the temperature of the material surrounding the wire is a constant 0°C and σ is the electrical conductivity of the material in the wire, temperature at points in the wire must satisfy the PDE

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} - hU + \frac{kI^2}{\kappa\sigma A^2}, \quad 0 < x < L, \quad t > 0$$

(see Exercise 42 in Section 2.2).

(a) Assuming that the ends of the wire are held at temperature 0°C and the initial temperature in the wire at time $t = 0$ is also 0°C, show that when $U(x,t)$ is split into steady-state and transient parts, $U(x,t) = V(x,t) + \psi(x)$,

$$\psi(x) = \frac{kI^2}{\kappa h\sigma A^2}\left[1 - \frac{\sinh\sqrt{h/k}\,x + \sinh\sqrt{h/k}(L-x)}{\sinh\sqrt{h/k}\,L}\right].$$

(b) Find $V(x,t)$ and hence $U(x,t)$.

12. Use variation of constants to solve Example 4.5.

13. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature zero throughout. For time $t > 0$, there is located at cross section $x = a$ $(0 < a < L)$ a plane heat source of constant strength g. If the ends $x = 0$ and $x = L$ of the rod are kept at zero temperature, the initial boundary value problem for temperature in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial t^2} + \frac{kg}{\kappa}\delta(x-a), \quad 0 < x < L, \quad t > 0,$$
$$U(0,t) = 0, \quad t > 0,$$
$$U(L,t) = 0, \quad t > 0,$$
$$U(x,0) = 0, \quad 0 < x < L,$$

where $\delta(x-a)$ is the Dirac delta function. Solve this problem for $U(x,t)$.

14. Replace the delta function in Exercise 13 with $[h(x-a)-h(x-b)]/(b-a)$, where $h(x-a)$ is the Heaviside unit step function, so that the heat source is distributed over the interval $a \le x \le b$. Solve the problem and then take the limit as b approaches a to arrive at the same solution as with the delta function.

15. Suppose that constant heat generation occurs at each point in the sphere of Exercise 12 in Section 4.2. Show that temperature in the sphere can be obtained using the transformation $V = rU$ and the steady-state solution of the resulting problem in $V(r,t)$.

16. Repeat Exercise 8 if $g(x,t) = e^{-\alpha t}$, $(\alpha > 0)$, and the initial temperature in the rod is 10°C throughout. Consider the cases that (a) $\alpha \ne n^2\pi^2 k/L^2$ for any positive integer n, and (b) $\alpha = m^2\pi^2 k/L^2$ for some positive integer m.

Part B Vibrations

17. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial displacement at time $t = 0$ of $f(x)$, $0 \le x \le L$, and an initial velocity $g(x)$, $0 \le x \le L$.
(a) If an external force per unit length of constant magnitude acts vertically downward at every point on the string, find a series representation for displacements in the string for $t > 0$ and $0 < x < L$.
(b) Find the d'Alembert solution for displacements of the string.

18. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial displacement at time $t = 0$ of $f(x)$, $0 \le x \le L$, but no initial velocity.
(a) If a concentrated mass m is fastened to the string at point $x = a$, $0 < a < L$, find a series representation for displacements in the string for $t > 0$ and $0 < x < L$. Use the Dirac delta function to represent the concentrated mass.
(b) Find the d'Alembert solution for displacements of the string.

19. A taut string has an end at $x = 0$ fixed on the x-axis, but the end at $x = L$ is removed a small amount y_L away from the x-axis and kept at this position.
(a) If the string has initial position $f(x)$ and velocity $g(x)$ (at time $t = 0$), find a series representation for displacements for $t > 0$ and $0 < x < L$.
(b) Find the d'Alembert solution for displacements of the string.

20. A horizontal cyindrical bar is originally at rest and unstrained along the x-axis between $x = 0$ and $x = L$. For time $t > 0$, the left end is fixed and the right end is subjected to a constant elongating force per unit area F parallel to the bar. Displacements $y(x,t)$ of cross sections then satisfy the initial boundary value problem

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\ y(0,t) &= 0, \quad t > 0,\\ \frac{\partial y(L,t)}{\partial x} &= \frac{F}{E}, \quad t > 0,\\ y(x,0) &= y_t(x,0) = 0, \quad 0 < x < L.\end{aligned}$$

(a) Can this problem be solved by separation of variables $[y(x,t) = X(x)T(t)]$ and superposition? It has only one nonhomogeneous condition.
(b) Replace this initial boundary value problem by one in $z(x,t)$ in which $y(x,t) = z(x,t) + \psi(x)$ and $\psi(x)$ is the solution of the associated static deflection problem.
(c) If separation of variables and superposition are used on the problem for $z(x,t)$, what form does the series take? Finish the problem using the result of Exercise 20 in Section 3.2.
(d) Find a closed form for the solution.

21. A taut string, with ends $x = 0$ and $x = L$ fixed on the x-axis, is at rest along the x-axis at time $t = 0$. A pulsating force per unit length $F_0 \sin\omega t$, F_0 a constant, acts at every point on the string for $t > 0$.
(a) Derive the following series representation for displacements of the string in the case that $\omega \neq n\pi c/L$, for any odd, positive integer n,

$$y(x,t) = \frac{4F_0L^2}{\rho\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)[(2n-1)^2\pi^2c^2 - \omega^2L^2]}\left[\sin\omega t - \frac{\omega L}{(2n-1)\pi c}\sin\frac{(2n-1)\pi ct}{L}\right]\sin\frac{(2n-1)\pi x}{L}.$$

(b) Show that the terms involving $\sin\omega t$ can be expressed in closed form by finding the Fourier sine series of the odd, $2L$-periodic extension of the function

$$\psi(x) = \sec\frac{\omega L}{2c}\sin\frac{\omega x}{2c}\sin\frac{\omega(L-x)}{2c}, \quad 0 \le x \le L.$$

The resonant case when $\omega = n\pi c/L$ for some odd integer will be discusssed in Exercise 26 of Section 7.2.

22. Repeat part (a) of Exercise 21 if the pulsating force acts only on that part of the string between $x = a$ and $x = b$ $(a < b)$.

23. Repeat part (a) of Exercise 21 if the pulsating force acts only on the point $x = x_0$ of the string.

24. A beam of uniform cross section and length L has its ends simply supported at $x = 0$ and $x = L$. The beam has constant density ρ kg/m and is subjected to an additional uniform load of k kg/m. If the beam is released from rest at a horizontal position at time $t = 0$, find subsequent displacements.

25. Repeat Exercise 24 if the beam is at rest at time $t = 0$ with displacement $f(x)$, $0 \le x \le L$.

26. Repeat Exercise 9 for the general one-dimensional vibration problem

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \quad 0 < x < L, \quad t > 0,\\ -l_1\frac{\partial y}{\partial x} + h_1 y &= f_1(t), \quad x = 0, \quad t > 0,\\ l_2\frac{\partial y}{\partial x} + h_2 y &= f_2(t), \quad x = L, \quad t > 0,\end{aligned}$$

$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = g(x), \quad 0 < x < L.$$

Part C Potential, Steady-State Heat Conduction, Static Deflections of Membranes

27. Find a formula for the solution of Laplace's equation inside the rectangle $0 \le x \le L$, $0 \le y \le L'$ when the boundary conditions are as indicated in Figure 4.1.

28. Nonhomogeneities in Laplace's equation $\nabla^2 V = 0$ convert it into Poisson's equation. For example, suppose a charge distribution with density $\sigma(x,y)$ coulombs per cubic metre occupies the volume R in space bounded by the planes $x = 0$, $y = 0$, $x = L$, and $y = L'$.
(a) If the bounding planes are maintained at zero potential, what is the boundary value problem for potential in R?
(b) Use variation of constants to solve the boundary value problem in part (a) when σ is constant. Find two series, one in terms of $\sin(n\pi x/L)$ and the other in terms of $\sin(n\pi y/L')$. Is either series preferred?
(c) Solve the problem in part (a) when σ is constant by setting $V(x,y) = U(x,y) + \psi(x)$, where $\psi(x)$ satisfies

$$\frac{d^2\psi}{dx^2} = \frac{-\sigma}{\epsilon_0}, \quad 0 < x < L,$$
$$\psi(0) = \psi(L) = 0.$$

Is this the same solution as in (b)?
(d) If $\sigma = \sigma(x)$ is a function of x only, which type of expansion in part (b) is preferred? Find the potential in this case.
(e) Find the potential when $\sigma = xy$.

29. Solve Exercise 41 in Section 4.2 if heat is generated at a constant rate at every point in the plate.

30. Solve Exercise 42 in Section 4.2 by using variation of constants with functions in terms of x. Assume for simplicity of calculations, (but not out of necessity for the procedure), that $f_1'(0) = f_1'(L') = f_2'(0) = f_2'(L') = f_3(0) = f_3(L) = f_4(0) = f_4(L) = 0$.

31. Solve the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -k_1, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$U(0,y) = 0, \quad 0 < y < 1,$$
$$U_x(1,y) = k_2, \quad 0 < y < 1,$$
$$U_y(x,0) = 0, \quad 0 < x < L,$$
$$U(x,1) = 0, \quad 0 < x < L,$$

where k_1 and k_2 are constants.

32. Solve the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = k, \quad -L < x < L, \quad -L < y < L,$$
$$U(-L,y) = U(L,y) = 0, \quad -L < y < L,$$
$$U(x,-L) = U(x,L) = 0, \quad -L < x < L,$$

where k is a constant.

CHAPTER 5 STURM-LIOUVILLE SYSTEMS

§5.1 Eigenvalues and Eigenfunctions

When separation of variables was carried out on linear (initial) boundary value problems in Chapter 4, we repeatedly encountered boundary value problems for the ODE $X'' + \lambda^2 X = 0$ for a function $X(x)$. When the initial boundary problem had two Dirichlet boundary conditions, we were led to the system

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad 0 < x < L, \tag{5.1a}$$

$$X(0) = 0, \tag{5.1b}$$

$$X(L) = 0; \tag{5.1c}$$

and when the problem had two Neumann boundary conditions, we were led to

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad 0 < x < L, \tag{5.2a}$$

$$X'(0) = 0, \tag{5.2b}$$

$$X'(L) = 0. \tag{5.2c}$$

These are examples of what are called *Sturm-Liouville systems*. In this chapter we undertake a general study of such systems. The results obtained are then applied to Sturm-Liouville systems that arise from (initial) boundary value problems that have combinations of Dirichlet, Neumann, and Robin boundary conditions, and also from initial boundary value problems in polar, cylindrical, and spherical coordinates.

Nontrivial solutions of systems 5.1 and 5.2 do not exist for arbitrary λ. On the contrary, only for specific values of λ, namely $\lambda = n\pi/L$, do nontrivial solutions exist, and to each value there corresponds a solution (unique to a multiplicative constant).

In general, a **Sturm-Liouville system** consists of a linear, second-order, homogeneous differential equation, together with two linear, homogeneous boundary conditions for an unknown function $y(x)$:

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) - q(x)]y = 0, \quad a < x < b, \tag{5.3a}$$

$$-l_1 y'(a) + h_1 y(a) = 0, \tag{5.3b}$$

$$l_2 y'(b) + h_2 y(b) = 0. \tag{5.3c}$$

Constants h_1, h_2, l_1, and l_2 are real and independent of parameter λ. When functions p, q, r, and r' are real and continuous for $a \le x \le b$, and $p > 0$ and $r > 0$ for $a \le x \le b$, the Sturm-Liouville system is said to be **regular**. The negative signs in 5.3a,b are chosen simply as a matter of convenience for applications.

The differential equation associated with a Sturm-Liouville system may not always be given in form 5.3a. When $R(x) > 0$ for $a \le x \le b$, any linear second order differential equation

$$R(x)\frac{d^2y}{dx^2} + S(x)\frac{dy}{dx} + [\lambda P(x) - Q(x)]y = 0$$

can be expressed in form 5.3a. Since $R(x) > 0$, we may divide this equation by $R(x)$ to express it in the form

$$\frac{d^2y}{dx^2} + T(x)\frac{dy}{dx} + [\lambda U(x) - V(x)]y = 0.$$

If each term is multiplied by the function $e^{\int T(x)\,dx}$,

$$e^{\int T(x)\,dx}\frac{d^2y}{dx^2} + e^{\int T(x)\,dx}T(x)\frac{dy}{dx} + \left[\lambda e^{\int P(x)\,dx}U(x) - e^{\int T(x)\,dx}V(x)\right]y = 0,$$

and this can be written in form 5.3a,

$$\frac{d}{dx}\left[e^{\int T(x)\,dx}\frac{dy}{dx}\right] + \left[\lambda e^{\int T(x)\,dx}U(x) - e^{\int T(x)\,dx}V(x)\right]y = 0.$$

No matter what the value of λ, the trivial function $y(x) \equiv 0$ always satisfies 5.3, but for certain values of λ, called **eigenvalues**, the system has nontrivial solutions. We shall see that there is always a countable (but infinite) number of such eigenvalues, which we denote by λ_n $(n = 1, 2, \ldots)$. A solution of 5.3 corresponding to an eigenvalue λ_n is called an **eigenfunction** and is denoted by $y_n(x)$. Eigenfunctions are to satisfy the usual conditions for solutions of second-order differential equations, namely that y_n and dy_n/dx be continuous for $a \le x \le b$. The second derivative d^2y_n/dx^2 will also be continuous for regular Sturm-Liouville systems since 5.3a implies that

$$\frac{d^2y_n}{dx^2} = \frac{1}{r(x)}\left\{[-\lambda p(x) + q(x)]y_n(x) - r'(x)y_n'(x)]\right\}.$$

When $\lambda = 0$ is an eigenvalue of a Sturm-Liouville system, it is customary to denote it by $\lambda_0 = 0$. Such is the case for system 5.2.

Eigenfunctions $\sin(n\pi x/L)$ of system 5.1 form the basis for Fourier sine series, and in Chapter 3 we saw that they were orthogonal on the interval $0 \le x \le L$. Eigenfunctions $\cos(n\pi x/L)$ of system 5.2 are also orthogonal on this interval. This is not coincidence; the following theorem verifies orthogonality for eigenfunctions of every Sturm-Liouville system. (See equation 3.6 in Section 3.1 for the definition of orthogonality of a sequence of functions.)

Theorem 5.1 Eigenvalues of a regular Sturm-Liouville system are real, and eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $p(x)$,

$$\int_a^b p(x)y_n(x)y_m(x)\,dx = 0. \tag{5.4}$$

Proof If $[\lambda_n, y_n(x)]$ and $[\lambda_m, y_m(x)]$ are eigenpairs of Sturm-Liouville system 5.3, where $\lambda_n \ne \lambda_m$, then

$$(ry_n')' = -(\lambda_n p - q)y_n, \qquad (ry_m')' = -(\lambda_m p - q)y_m.$$

Multiplication of the first by y_m and the second by y_n, and subtraction of the two equations, eliminates q:

$$y_m(ry_n')' - y_n(ry_m')' = -\lambda_n p y_n y_m + \lambda_m p y_m y_n$$

or,

$$(\lambda_n - \lambda_m)py_ny_m = (ry'_m)'y_n - (ry'_n)'y_m.$$

We now integrate both sides of this equation over the interval $a \le x \le b$ and use integration by parts on the right,

$$\begin{aligned}(\lambda_n - \lambda_m)\int_a^b py_ny_m\,dx &= \int_a^b [(ry'_m)'y_n - (ry'_n)'y_m]\,dx \\ &= \Big\{(ry'_m)y_n - (ry'_n)y_m\Big\}_a^b - \int_a^b [(ry'_m)y'_n - (ry'_n)y'_m]\,dx \\ &= r(b)[y'_m(b)y_n(b) - y'_n(b)y_m(b)] - r(a)[y'_m(a)y_n(a) - y'_n(a)y_m(a)] \\ &= r(b)\begin{vmatrix} y_n(b) & y_m(b) \\ {y_n}'(b) & {y_m}'(b)\end{vmatrix} - r(a)\begin{vmatrix} y_n(a) & y_m(a) \\ {y_n}'(a) & {y_m}'(a)\end{vmatrix}.\end{aligned}$$

Since $y_n(x)$ and $y_m(x)$ both satisfy boundary condition 5.3b,

$$-l_1{y_n}'(a) + h_1y_n(a) = 0,$$
$$-l_1{y_m}'(a) + h_1y_m(a) = 0.$$

Because at least one of h_1 and l_1 is not zero, these equations (regarded as homogeneous, linear equations in l_1 and h_1) must have nontrivial solutions. Consequently, the determinant of their coefficients must vanish:

$$\begin{vmatrix} {y_n}'(a) & y_n(a) \\ {y_m}'(a) & y_m(a)\end{vmatrix} = 0.$$

A similar discussion with boundary condition 5.3c indicates that

$$\begin{vmatrix} {y_n}'(b) & y_n(b) \\ {y_m}'(b) & y_m(b)\end{vmatrix} = 0.$$

It follows now that

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n(x)y_m(x)\,dx = 0,$$

and because $\lambda_n \ne \lambda_m$, orthogonality condition 5.4 has been established.

To prove that eigenvalues are real, we assume to the contrary that $\lambda = \alpha + \beta i$ ($\beta \ne 0$) is a complex eigenvalue with eigenfunction $y(x)$. This eigenfunction could be complex, but if it is, it is a complex-valued function of the real variable x. If we divide $y(x)$ into real and imaginary parts, $y(x) = u(x) + v(x)i$, the complex conjugate of dy/dx is

$$\overline{\frac{dy}{dx}} = \overline{\frac{d}{dx}(u+vi)} = \overline{\frac{du}{dx} + \frac{dv}{dx}i} = \frac{du}{dx} - \frac{dv}{dx}i = \frac{d}{dx}(u - vi) = \frac{d\overline{y}}{dx}.$$

With this result it is straightforward to take complex conjugates of equation 5.3. Because the functions $r(x)$, $p(x)$, and $q(x)$ are real, as are the constants h_1, h_2, l_1, and l_2, we find that $\overline{\lambda}$ and $\overline{y(x)}$ must satisfy

$$(r\overline{y}')' + (\overline{\lambda}p - q)\overline{y} = 0,$$

$$-l_1\overline{y(a)}' + h_1\overline{y(a)} = 0, \quad l_2\overline{y(b)}' + h_2\overline{y(b)} = 0.$$

These imply that $\overline{y(x)}$ is an eigenfunction of Sturm-Liouville system 5.3 corresponding to the eigenvalue $\overline{\lambda}$. Since $\lambda \neq \overline{\lambda}$, the eigenfunctions $y(x)$ and $\overline{y(x)}$ must therefore be orthogonal; that is,

$$\int_a^b p(x)\overline{y(x)}y(x)\,dx = 0.$$

But this is impossible because $p(x) > 0$ for $a < x < b$, and $\overline{y(x)}y(x) = |y(x)|^2 \geq 0$. Consequently, λ canot be complex. ∎

It is evident from the above proof that the theorem is also valid under the circumstances in the following corollary.

Corollary The results of Theorem 5.1 are valid when:

1. $r(a) = 0$ (boundary condition 5.3b then being unnecessary);
2. $r(b) = 0$ (boundary condition 5.3c then being unnecessary);
3. $r(a) = r(b)$ if boundary conditions 5.3b,c are replaced by the periodic conditions

$$y(a) = y(b), \qquad y'(a) = y'(b). \tag{5.5}$$

A Sturm-Liouville system is said to be **singular** if either or both of its boundary conditions is absent; it is said to be **periodic** if $r(a) = r(b)$ and boundary conditions 5.3b,c are replaced by periodic boundary conditions 5.5. Theorem 5.1 and its corollary state that eigenfunctions of regular and periodic Sturm-Liouville systems are always orthogonal. They are also orthogonal for singular systems when boundary conditions 5.3b or 5.3c or both are absent, provided either $r(a) = 0$ or $r(b) = 0$, or both, respectively. We consider only regular and periodic Sturm-Liouville systems in this chapter; singular systems are discussed in Chapter 8.

Example 5.1 Find eigenvalues and eigenfunctions of the Sturm-Liouville system

$$\frac{d^2X}{dx^2} + \lambda X = 0, \quad 0 < x < L,$$
$$X(0) = 0 = X'(L).$$

Solution When $\lambda < 0$, a general solution of the differential equation is

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

The boundary conditions require

$$0 = X(0) = A + B, \qquad 0 = X'(L) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}L} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}L},$$

the only solution of which is $A = B = 0$.

When $\lambda = 0$, $X(x) = Ax + B$, and the boundary conditions once again imply that $A = B = 0$.

Thus, eigenvalues of the Sturm-Liouville system must be positive, and when $\lambda > 0$, the boundary conditions require constants A and B in the general solution $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ of the differential equation to satisfy

$$0 = X(0) = A, \quad 0 = X'(L) = -A\sqrt{\lambda}\sin\sqrt{\lambda}L + B\sqrt{\lambda}\cos\sqrt{\lambda}L.$$

With A vanishing, the second condition reduces to $B\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$. Since neither B nor λ can vanish, $\cos\sqrt{\lambda}L$ must be zero. Hence, $\sqrt{\lambda}L$ must be equal to $-\pi/2$

plus an integer multiple of π; that is, permissible values of λ are λ_n where $\sqrt{\lambda_n}L = n\pi - \pi/2$, and n is an integer. Corresponding eigenfunctions are

$$X_n(x) = B\sin\sqrt{\lambda_n}x = B\sin\frac{(2n-1)\pi x}{2L}.$$

But the set of functions for $n \leq 0$ is identical to that for $n > 0$. In other words, eigenvalues of the Sturm-Liouville system are $\lambda_n = (2n-1)^2\pi^2/(4L^2)$, $n \geq 1$, with corresponding eigenfunctions $X_n(x) = B\sin[(2n-1)\pi x/(2L)]$.•

Example 5.2 Discuss the periodic Sturm-Liouville system

$$y'' + \lambda y = 0, \quad -L < x < L, \tag{5.6a}$$

$$y(-L) = y(L), \tag{5.6b}$$

$$y'(-L) = y'(L). \tag{5.6c}$$

Solution If $\lambda > 0$, a general solution of the differential equation is

$$y(\lambda, x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

The boundary conditions require

$$A\cos\sqrt{\lambda}L - B\sin\sqrt{\lambda}L = A\cos\sqrt{\lambda}L + B\sin\sqrt{\lambda}L,$$

$$\sqrt{\lambda}A\sin\sqrt{\lambda}L + \sqrt{\lambda}B\cos\sqrt{\lambda}L = -\sqrt{\lambda}A\sin\sqrt{\lambda}L + \sqrt{\lambda}B\cos\sqrt{\lambda}L.$$

These equations require $\sin\sqrt{\lambda}L = 0$, and this implies that $\sqrt{\lambda}L = n\pi$. In other words, eigenvalues of the Sturm-Liouville system are $\lambda_n = n^2\pi^2/L^2$, where n is an integer that we take as positive. Corresponding to these eigenvalues are the eigenfunctions

$$y_n(x) = A\cos\frac{n\pi x}{L} + B\sin\frac{n\pi x}{L}.$$

When $\lambda = 0$, $y(x) = A + Bx$, and the boundary conditions require $B = 0$. Thus, corresponding to the eigenvalue $\lambda_0 = 0$, we have the eigenfunction $y_0(x) = A$. The only solution when $\lambda < 0$ is the trivial solution.

Theorem 5.1 guarantees that for nonnegative integers m and n ($n \neq m$), the eigenfunctions

$$y_n(x) = A\cos\frac{n\pi x}{L} + B\sin\frac{n\pi x}{L} \quad \text{and} \quad y_m(x) = C\cos\frac{m\pi x}{L} + D\sin\frac{m\pi x}{L}$$

are orthogonal over the interval $-L \leq x \leq L$. It is true, however, that all functions in the set

$$\left\{1, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\right\}$$

are orthogonal. These are precisely the "eigenfunctions" found in the Fourier series expansion of a function of period $2L$. We shall return to this point in Section 5.2.•

Because differential equation 5.3a and boundary conditions 5.3b,c are homogeneous, if $[\lambda_n, y_n(x)]$ is an eigenpair for a Sturm-Liouville system, then so also is $[\lambda_n, cy_n(x)]$ for any constant $c \neq 0$. In other words, eigenfunctions are not unique; if $y_n(x)$ is an eigenfunction corresponding to an eigenvalue λ_n, then any constant times $y_n(x)$ is also an eigenfunction corresponding to the same λ_n. This fact is reflected in Example 5.1, where eigenfunctions were determined only to multiplicative constants. In this example, there is, except for the multiplicative constant, only one eigenfunction, $\sin[(2n-1)\pi x/(2L)]$, corresponding to each eigenvalue. This is not the case in Example 5.2. Corresponding to each positive eigenvalue, there are two linearly independent eigenfunctions, $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$. The difference is that in Example 5.1 the Sturm-Liouville system is regular, but in Example 5.2 it is periodic. It can be shown (see Exercise 14) that a regular Sturm-Liouville system cannot have two linearly independent eigenfunctions corresponding to the same eigenvalue. Such an eigenvalue is said to have multiplicity one. In Example 5.2, $\lambda = 0$ has multiplicity one, and all other eigenvalues have multiplicity two.

In regular Sturm-Liouville systems, it is customary to single out one of the eigenfunctions $y_n(x)$ corresponding to an eigenvalue as special and refer all other eigenfunctions to it. The one that is chosen is an eigenfunction with *length* unity; that is, an eigenfunction $y_n(x)$ satisfying

$$\|y_n(x)\| = \sqrt{\int_a^b p(x)[y_n(x)]^2\,dx} = 1.$$

Such an eigenfunction is said to be **normalized**. Normalized eigenfunctions can always be found by dividing nonnormalized eigenfunctions by their lengths. Consider, for example, Sturm-Liouville system 5.1. Since $\sin(n\pi x/L)$ is an eigenfunction of this system corresponding to the eigenvalue $\lambda_n^2 = n^2\pi^2/L^2$, so also is $c\sin(n\pi x/L)$ for any constant $c \neq 0$. A normalized eigenfunction corresponding to this eigenvalue is

$$\frac{\sin(n\pi x/L)}{\|\sin(n\pi x/L)\|}, \quad \text{where} \quad \left\|\sin\frac{n\pi x}{L}\right\|^2 = \int_0^L \sin^2\frac{n\pi x}{L}dx = \frac{L}{2}.$$

Thus, with each eigenvalue $\lambda_n^2 = n^2\pi^2/L^2$ of the Sturm-Liouville system, we associate the normalized eigenfunction

$$X_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}.$$

All other eigenfunctions for λ_n^2 are then $cX_n(x)$.

Similarly, normalized eigenfunctions for Sturm-Liouville system 5.2 are

$$X_0(x) = \frac{1}{\sqrt{L}} \quad \text{corresponding to } \lambda_0^2 = 0$$

and

$$X_n(x) = \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L} \quad \text{corresponding to } \lambda_n^2 = n^2\pi^2/L^2,\ n > 0.$$

In general, if $y_n(x)$ is an eigenfunction of Sturm-Liouville system 5.3, we replace it by the normalized eigenfunction

$$\frac{1}{N}y_n(x) \quad \text{where} \quad N^2 = \|y_n(x)\|^2 = \int_a^b p(x)[y_n(x)]^2\,dx. \tag{5.7}$$

The complete set of normalized eigenfunctions, one for each eigenvalue, then constitutes a set of orthonormal eigenfunctions for the Sturm-Liouville system. Unless otherwise stated, we shall always regard $y_n(x)$ as normalized eigenfunctions of a

Sturm-Liouville system. Notice that any number of the $y_n(x)$ could be replaced by $-y_n(x)$, and the new set would also be orthonormal. In other words, orthonormal eigenfunctions are determined only to a factor of ± 1.

Example 5.3 Find eigenvalues and normalized eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \lambda y = 0, \quad 0 < x < 1,$$
$$y(0) = 0 = y(1).$$

Solution Roots of the auxiliary equation $m^2 + m + \lambda = 0$ associated with the differential equation are $m = (-1 \pm \sqrt{1-4\lambda})/2$. When $\lambda < 1/4$, these roots are real; denote them by $\omega_1 = (-1+\sqrt{1-4\lambda})/2$ and $\omega_2 = (-1-\sqrt{1-4\lambda})/2$. A general solution of the differential equation in this case is $y(x) = Ae^{\omega_1 x} + Be^{\omega_2 x}$, and the boundary conditions require

$$0 = A + B, \qquad 0 = Ae^{\omega_1} + Be^{\omega_2}.$$

The only solution of these equations is $A = B = 0$, leading to the trivial solution.

When $\lambda = 1/4$, the auxiliary equation has equal roots, and $y(x) = (A + Bx)e^{-x/2}$. Once again, the boundary conditions require $A = B = 0$.

Consequently, λ must be greater than $1/4$, in which case we set $m = -1/2 \pm \omega i$, where $\omega = \sqrt{4\lambda - 1}/2$. The boundary conditions require constants A and B in the general solution $y(x) = e^{-x/2}(A\cos\omega x + B\sin\omega x)$ to satisfy

$$0 = A, \qquad 0 = e^{-1/2}(A\cos\omega + B\sin\omega).$$

With vanishing A, the second condition requires $\sin\omega = 0$; that is, $\omega = n\pi$, where n is an integer. In other words, eigenvalues of the Sturm-Liouville system are given by

$$\frac{\sqrt{4\lambda_n - 1}}{2} = n\pi \qquad \Longrightarrow \qquad \lambda_n = \frac{1}{4} + n^2\pi^2.$$

Except for a multiplicative constant, corresponding eigenfunctions are $e^{-x/2}\sin n\pi x$. Clearly, we need only take $n > 0$. To normalize these functions, we express the differential equation in standard Sturm-Liouville form 5.3a. This can be done by multiplying by e^x (see the discussion following equation 5.3),

$$0 = e^x\frac{d^2y}{dx^2} + e^x\frac{dy}{dx} + \lambda e^x y = \frac{d}{dx}\left(e^x\frac{dy}{dx}\right) + \lambda e^x y.$$

With the weight function now identified as $p(x) = e^x$, we calculate lengths of the eigenfunctions,

$$\|e^{-x/2}\sin n\pi x\|^2 = \int_0^1 e^x(e^{-x/2}\sin n\pi x)^2\,dx = \int_0^1 \sin^2 n\pi x\,dx = \frac{1}{2}.$$

Normalized eigenfunctions are therefore $y_n(x) = \sqrt{2}e^{-x/2}\sin n\pi x$.•

EXERCISES 5.1

In Exercises 1–9 find eigenvalues and orthonormal eigenfunctions for the Sturm-Liouville system.

1. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 3, \quad y(0) = 0 = y(3)$

2. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 4, \quad y'(0) = 0 = y'(4)$

3. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 9, \quad y(0) = 0 = y'(9)$

4. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 1, \quad y'(0) = 0 = y(1)$

5. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < L, \quad y'(0) = 0 = y(L)$

6. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 1 < x < 10, \quad y(1) = 0 = y(10)$
(Do this directly and also by making the change of independent variable $z = x - 1$.)

7. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + \lambda y = 0, \quad 0 < x < 1, \quad y(0) = 0 = y(1)$

8. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \lambda y = 0, \quad 1 < x < 5, \quad y'(1) = 0 = y'(5)$
(Hint: Use the change of variable $z = x - 1$.)

9. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + \lambda y = 0, \quad 1 < x < e, \quad y(1) = 0 = y(e)$

10. Find eigenvalues and eigenfunctions of the periodic Sturm-Liouville system

$$\begin{aligned} y'' + \lambda y = 0&, \quad 0 < x < 2L, \\ y(0) = y(2L)&, \\ y'(0) = y'(2L)&. \end{aligned}$$

11. Consider the Sturm-Liouville system

$$\begin{aligned} \frac{d^2y}{dx^2} + 4\lambda y = 0&, \quad 0 < x < L, \\ y(0) = 0 = y(L)&. \end{aligned}$$

We could regard this system as one with eigenvalues λ and weight function $p(x) = 4$, or, alternatively, as one with eigenvalues 4λ and weight function $p(x) = 1$. Is there a difference as far as normalized eigenfunctions are concerned?

12. Heat equation 2.25 is valid when thermal constants ρ, s, and κ are not constant. In the one-dimensional case, it reads

$$\rho s\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(\kappa\frac{\partial U}{\partial x}\right)$$

when there is no heat generation. Show that when ρ, s, and κ are functions of x, but not t, separation of variables leads to a Sturm-Liouville differential equation.

13. The one-dimensional wave equation 2.43 is valid when ρ and τ are not constant. When no external forces act on the string (or bar), it reads

$$\frac{\partial}{\partial t}\left(\rho \frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial x}\left(\tau \frac{\partial y}{\partial x}\right).$$

Show that when ρ and τ are functions of x, but not t, separation of variables leads to a Sturm-Liouville differential equation.

14. In this exercise we prove that a regular Sturm-Liouville system cannot have two linearly independent eigenfunctions corresponding to the same eigenvalue; that is, all eigenvalues have multiplicity one.

(a) Suppose that $y(x)$ and $z(x)$ are eigenfunctions of system 5.3 corresponding to the same eigenvalue λ. Show that $w(x) = y'(a)z(x) - z'(a)y(x)$ satisfies 5.3a and that $w(a) = w'(a) = 0$. This implies that $w(x) \equiv 0$ (and therefore that $y(x)$ and $z(x)$ are linearly dependent) unless $y'(a) = z'(a) = 0$.

(b) If $y'(a) = z'(a) = 0$, then $h_1 = 0$. Define $w(x) = y(a)z(x) - z(a)y(x)$ to show once again that $w(x) \equiv 0$.

15. Use the result of Exercise 14 to show that up to a multiplicative constant, eigenfunctions of regular Sturm-Liouville systems are real.

16. In Exercises 6 and 8 we suggested the change of variable $z = x - 1$ in order to find eigenfunctions of the Sturm-Liouville system. Does it make any difference whether normalization is carried out in the z-variable or in the x-variable?

§5.2 Generalized Fourier Series

In Chapters 3 and 4 we learned how to express functions $f(x)$, which are piecewise smooth on the interval $0 \le x \le L$, in the form of Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad \text{where} \quad b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx. \tag{5.8}$$

We regard the Fourier coefficients b_n as the components of the function $f(x)$ with respect to the basis functions $\{\sin(n\pi x/L)\}$. In Section 5.1 we discovered that the $\sin(n\pi x/L)$ are eigenfunctions of Sturm-Liouville system 5.1, and it has become our practice to replace eigenfunctions with normalized eigenfunctions, namely $\sqrt{2/L}\sin(n\pi x/L)$. Representation 5.8 can easily be replaced by an equivalent expression in terms of these normalized eigenfunctions,

$$f(x) = \sum_{n=1}^{\infty} c_n \left(\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}\right) \quad \text{where} \quad c_n = \int_0^L f(x) \left(\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}\right) dx. \tag{5.9}$$

Constants c_n are the components of $f(x)$ with respect to the orthonormal basis $\{\sqrt{2/L}\sin(n\pi x/L)\}$. Equation 5.8 should be compared with equation 3.3 in Section 3.1, together with the fact that the length of $\sin(n\pi x/L)$ is $\sqrt{L/2}$. Equation 5.9 is analogous to equation 3.1.

The same function $f(x)$ can be represented by a Fourier cosine series in terms of normalized eigenfunctions of system 5.2,

$$f(x) = \frac{c_0}{\sqrt{L}} + \sum_{n=1}^{\infty} c_n \left(\sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}\right) \tag{5.10a}$$

where

$$c_0 = \int_0^L f(x) \left(\frac{1}{\sqrt{L}}\right) dx \quad \text{and} \quad c_n = \int_0^L f(x) \left(\sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}\right) dx, \quad n > 0. \tag{5.10b}$$

A natural question to ask now is the following: Given a function $f(x)$, defined on the interval $a \le x \le b$, and given a Sturm-Liouville system on the same interval, is it always possible to express $f(x)$ in terms of the orthonormal eigenfunctions of the Sturm-Liouville system? It is still not clear that every Sturm-Liouville system has an infinity of eigenfunctions, but, as we shall see, this is indeed the case. We wish then to investigate the possibility of finding coefficients c_n such that on $a \le x \le b$,

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \tag{5.11}$$

where $y_n(x)$ are the orthonormal eigenfunctions of Sturm-Liouville system 5.3. If we formally multiply equation 5.11 by $p(x)y_m(x)$, and integrate term-by-term between $x = a$ and $x = b$,

$$\int_a^b p(x)f(x)y_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b p(x)y_n(x)y_m(x)\,dx.$$

Because of the orthogonality of eigenfunctions, only the m^{th} term in the series does not vanish, and therefore

$$\int_a^b p(x)f(x)y_m(x)\,dx = c_m. \tag{5.12}$$

This has been strictly a formal procedure. It has illustrated that *if* $f(x)$ can be represented in form 5.11, and *if* the series is suitably convergent, coefficients c_n can be calculated according to formula 5.12. What we must answer is the converse question: If coefficients c_n are calculated according to 5.12, where $y_n(x)$ are orthonormal eigenfunctions of a Sturm-Liouville system, does series 5.11 converge to $f(x)$? This question is answered in the following theorem.

Theorem 5.2 Let p, q, r, r', and $(pr)''$ be real and continuous functions of x for $a \le x \le b$, and let $p > 0$ and $r > 0$ for $a \le x \le b$. Let l_1, l_2, h_1, and h_2 be real constants independent of λ. Then Sturm-Liouville system 5.3 has a countable infinity of simple eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ (all real), not more than a finite number of which are negative, and $\lim_{n\to\infty} \lambda_n = \infty$. Corresponding orthonormal eigenfunctions $y_n(x)$ are such that $y_n(x)$ and $y_n{}'(x)$ are continuous and $|y_n(x)|$ and $|\lambda_n{}^{-1/2} y_n{}'(x)|$ are uniformly bounded with respect to x and n. If $f(x)$ is piecewise smooth on $a \le x \le b$, then for any x in $a < x < b$,

$$\frac{f(x+)+f(x-)}{2} = \sum_{n=1}^{\infty} c_n y_n(x), \qquad \text{where} \qquad c_n = \int_a^b p(x)f(x)y_n(x)\,dx. \tag{5.13}$$

Series 5.13 is called the **generalized Fourier series** for $f(x)$ with respect to the eigenfunctions $y_n(x)$, and the c_n are the **generalized Fourier coefficients**. They are the components of $f(x)$ with respect to the orthonormal basis of eigenfunctions $\{y_n(x)\}$. Notice the similarity between this theorem and Theorem 3.2 in Section 3.1 for Fourier series. Both guarantee pointwise convergence of Fourier series for a piecewise smooth function to the value of the function at a point of continuity of the function, and to the average value of right- and left-hand limits at a point of discontinuity. Because the eigenfunctions in Theorem 3.2 of Section 3.1 are periodic, convergence is also assured at the end points of the interval $0 \le x \le 2L$. This is not the case in Theorem 5.2 above. Eigenfunctions are not generally periodic, and convergence at $x = a$ and $x = b$ is not guaranteed. It should be clear, however, that when $l_1 = 0$ (in which case $y_n(a) = 0$) convergence of the series in 5.13 at $x = a$ can be expected only if $f(a) = 0$ also. A similar statement can be made at $x = b$.

Because series 5.13 is a representation of the function $f(x)$ in terms of normalized eigenfunctions of a regular Sturm-Liouville system, it is also called an **eigenfunction expansion** of $f(x)$. We use both terms, namely, generalized Fourier series and eigenfunction expansion, freely and interchangeably.

We say that the normalized eigenfunctions of Sturm-Liouville system 5.3 form a **complete set** for the space of piecewise smooth functions on the interval $a \le x \le b$; this means that every piecewise smooth function can be expressed in a convergent series of the eigenfunctions.

When a regular Sturm-Liouville system satisfies the conditions of this theorem as well as the conditions that $q(x) \ge 0$ for $a \le x \le b$, and $l_1h_1 \ge 0$, $l_2h_2 \ge 0$, it is said to be a **proper Sturm-Liouville system**. For such a system we shall take l_1, l_2, h_1, and h_2 all nonnegative, in which case we can prove the following corollary.

Corollary All eigenvalues of a proper Sturm-Liouville system are nonnegative. Furthermore, zero is an eigenvalue of a proper Sturm-Liouville system only when $q(x) \equiv 0$ and $h_1 = h_2 = 0$.

Proof Let λ and $y(x)$ be an eigenpair of a regular Sturm-Liouville system. Multiplication of differential equation 5.3a by $y(x)$ and integration from $x = a$ to $x = b$ gives

$$\begin{aligned}\lambda \int_a^b p(x) y^2(x)\, dx &= \int_a^b q(x) y^2(x)\, dx - \int_a^b y(x)[r(x)y'(x)]'\, dx \\ &= \int_a^b q(x) y^2(x)\, dx - \Big\{ r(x) y(x) y'(x) \Big\}_a^b + \int_a^b r(x)[y'(x)]^2\, dx.\end{aligned}$$

When we solve boundary conditions 5.3b,c for $y'(b)$ and $y'(a)$ and substitute into the second term on the right, we obtain

$$\begin{aligned}\lambda \int_a^b p(x) y^2(x)\, dx = \int_a^b q(x) y^2(x)\, dx &+ \int_a^b r(x)[y'(x)]^2\, dx \\ &+ \frac{h_2}{l_2} r(b) y^2(b) + \frac{h_1}{l_1} r(a) y^2(a).\end{aligned}$$

When the Sturm-Liouville system is proper, every term on the right is nonnegative, as is the integral on the left, and therefore $\lambda \geq 0$. (If either $l_1 = 0$ or $l_2 = 0$, the corresponding terms in the above equation are absent and the result is the same.)

Furthermore, if $\lambda = 0$ is an eigenvalue, then each of the four terms on the right side of the above equation must vanish separately. The first requires that $q(x) \equiv 0$ and the second that $y'(x) = 0$. But the fact that $y(x)$ is constant implies that the last two terms can vanish only if $h_1 = h_2 = 0$. ∎

Since eigenvalues of a proper Sturm-Liouville system must be nonnegative, we may replace λ by λ^2 in differential equation 5.3a whenever it is convenient to do so,

$$\frac{d}{dx}\left[r(x) \frac{dy}{dx} \right] + [\lambda^2 p(x) - q(x)] y = 0, \quad a < x < b.$$

This often has the advantage of eliminating unnecessary square roots in calculations.

Example 5.4 Expand the function $f(x) = L - x$ in terms of normalized eigenfunctions of the Sturm-Liouville system of Example 5.1.

Solution According to Example 5.1, eigenfunctions of the Sturm-Liouville system are $\sin \dfrac{(2n-1)\pi x}{2L}$. Because

$$\left\| \sin \frac{(2n-1)\pi x}{2L} \right\|^2 = \int_0^L \left[\sin \frac{(2n-1)\pi x}{2L} \right]^2 dx = \frac{L}{2},$$

normalized eigenfunctions are $X_n(x) = \sqrt{\dfrac{2}{L}} \sin \dfrac{(2n-1)\pi x}{2L}$. In terms of these eigenfunctions, the generalized Fourier series for $f(x) = L - x$ is

$$L - x = \sum_{n=1}^{\infty} c_n X_n(x),$$

where

$$c_n = \int_0^L (L-x)\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}dx = \frac{2\sqrt{2}L^{3/2}}{\pi^2}\left[\frac{\pi}{2n-1} + \frac{2(-1)^n}{(2n-1)^2}\right].$$

Thus,

$$L - x = \frac{2\sqrt{2}L^{3/2}}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\pi}{2n-1} + \frac{2(-1)^n}{(2n-1)^2}\right]\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}.$$

Theorem 5.2 guarantees convergence of the series to $L - x$ for $0 < x < L$. It obviously does not converge to $L - x$ at $x = 0$, but it does converge to $L - x$ at $x = L$. This follows from the facts that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Figure 5.1 shows a few partial sums of this series to illustrate convergence of the series to $L - x$. It is very slow because of the term $\pi/(2n-1)$.•

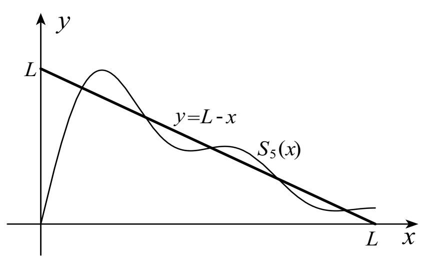


Figure 5.1a

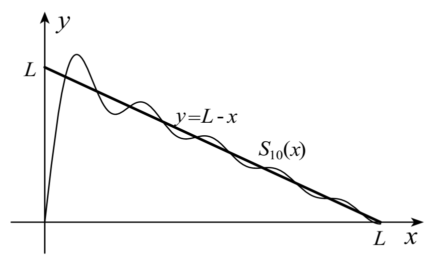


Figure 5.1b

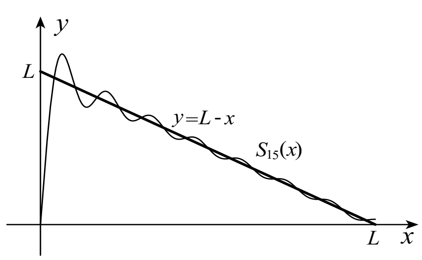


Figure 5.1c

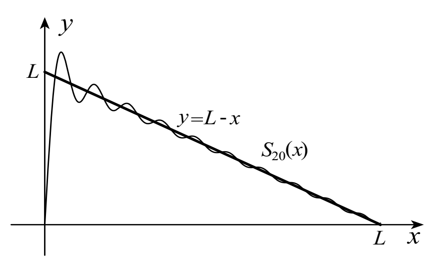


Figure 5.1d

In the examples of Chapter 4, when separation of variables was applied to (initial) boundary value problems, all boundary conditions in a given problem were either of Dirichlet type or Neumann type. These led to Fourier sine and cosine series, series that we now know are eigenfunction expansions in terms of eigenfunctions of Sturm-Liouville systems 5.1 and 5.2. We did not consider problems with Robin conditions, but in some of the exercises, we mixed Dirichlet and Neumann conditions. We were able to do so because of the results in Exercises 20 and 21 of Section 3.2. With our results on Sturm-Liouville systems in this section, we will be well prepared to tackle any combination of Dirichlet, Neumann, and Robin boundary conditions.

A proper Sturm-Liouville system that arises repeatedly in our discussions is

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad 0 < x < L, \tag{5.14a}$$

$$-l_1 X'(0) + h_1 X(0) = 0, \tag{5.14b}$$

$$l_2 X'(L) + h_2 X(L) = 0. \tag{5.14c}$$

where l_1, l_2, h_1, and h_2 are non-negative constants. (Systems 5.1 and 5.2 are special cases of 5.14 when $l_1 = l_2 = 0$ and $h_1 = h_2 = 0$, respectively. Examples 5.1 and 5.4 contain the special case of $l_1 = h_2 = 0$ and $l_2 = h_1 = 1$.) We consider here the most general case, in which $h_1 h_2 l_1 l_2 \neq 0$; special cases in which one or two of h_1, h_2, l_1, and l_2 vanish are tabulated later. In the general case when $h_1 h_2 l_1 l_2 \neq 0$, we could divide boundary condition 5.14b by either l_1 or h_1. This would lead to a boundary condition with only one arbitrary constant (h_1/l_1 or l_1/h_1). Likewise, we could divide boundary condition 5.14c by l_2 or h_2 and express the boundary condition in terms of the ratio h_2/l_2 or l_2/h_2. However, when this is done, it is not quite so transparent how to specialize the results in the cases in which one or two of h_1, h_2, l_1, and l_2 vanish. For this reason, we prefer to leave 5.14b,c in their present forms.

We are justified in representing the eigenvalues of system 5.14 by λ^2 rather than λ, because all eigenvalues of a proper Sturm-Liouville system are nonnegative. A general solution of differential equation 5.14a is

$$X(x) = A\cos\lambda x + B\sin\lambda x, \tag{5.15}$$

and when we impose boundary conditions 5.14b,c,

$$-l_1\lambda B + h_1 A = 0, \tag{5.16a}$$

$$l_2(-A\lambda\sin\lambda L + B\lambda\cos\lambda L) + h_2(A\cos\lambda L + B\sin\lambda L) = 0. \tag{5.16b}$$

We solve equation 5.16a for $B = h_1 A/(l_1\lambda)$ and substitute into 5.16b. After rearrangement, we obtain

$$\tan\lambda L = \frac{\lambda\left(\dfrac{h_1}{l_1} + \dfrac{h_2}{l_2}\right)}{\lambda^2 - \dfrac{h_1 h_2}{l_1 l_2}}, \tag{5.17}$$

the equation that must be satisfied by λ. We denote by λ_n ($n = 1, 2, \ldots$) eigenvalues of this transcendental equation, although, in fact, λ_n^2 are the eigenvalues of the Sturm-Liouville system. Corresponding to these eigenvalues are orthonormal eigenfunctions

$$X_n(x) = \frac{1}{N}\left(\cos\lambda_n x + \frac{h_1}{\lambda_n l_1}\sin\lambda_n x\right), \tag{5.18a}$$

where

$$N^2 = \int_0^L \left(\cos\lambda_n x + \frac{h_1}{\lambda_n l_1}\sin\lambda_n x\right)^2 dx. \tag{5.18b}$$

In Exercise 1, integration is shown to lead to

$$2N^2 = \left[1 + \left(\frac{h_1}{\lambda_n l_1}\right)^2\right]\left[L + \frac{h_2/l_2}{\lambda_n^2 + (h_2/l_2)^2}\right] + \frac{h_1/l_1}{\lambda_n^2}. \tag{5.18c}$$

Of the nine possible combinations of boundary conditions at $x = 0$ and $x = L$, we have considered only one, the most general in which none of h_1, h_2, l_1, and l_2 vanishes. Results for the remaining eight cases can be obtained from equations 5.17 and 5.18, or by similar analyses; they are tabulated in Table 5.1.

Each eigenvalue equation in Table 5.1 is unchanged if λ is replaced by $-\lambda$, so that for every positive solution λ of the equation, $-\lambda$ is also a solution. Since NX_n is invariant (up to a sign change) by the substitution of $-\lambda_n$ for λ_n, it is necessary only to consider nonnegative solutions of the eigenvalue equations. This agrees with the fact that eigenvalues of the Sturm-Liouville system are λ_n^2 and that there cannot be two linearly independent eigenfunctions corresponding to the same eigenvalue. Table 5.1 gives the eigenvalues explicitly in only four of the nine cases. Eigenvalues in the remaining five cases are illustrated geometrically below.

If $h_1h_2l_1l_2 \neq 0$, eigenvalues are illustrated graphically in Figure 5.2 as points of intersection of the curves

$$y = \tan\lambda L, \qquad y = \frac{\lambda(h_1/l_1 + h_2/l_2)}{\lambda^2 - h_1h_2/(l_1l_2)}.$$

It might appear that $\lambda = 0$ is an eigenvalue in this case. However, the corollary to Theorem 5.2 indicates that zero is an eigenvalue only when $h_1 = h_2 = 0$. This can also be verified using conditions 5.16, which led to the eigenvalue equation (see Exercise 3).

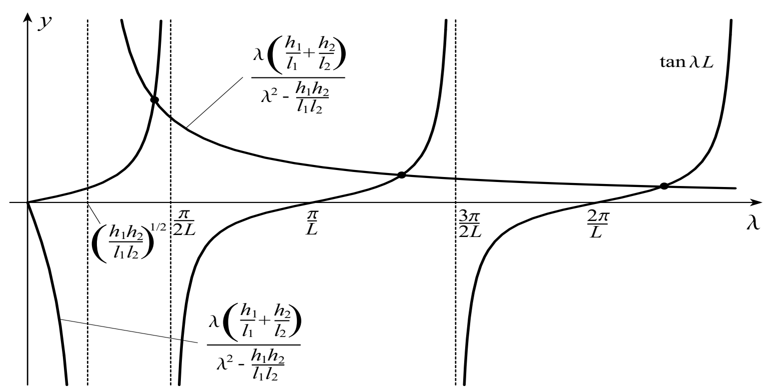


Figure 5.2

Condition at $x = 0$	Condition at $x = L$	Eigenvalue Equation	NX_n	$2N^2$
$h_1 l_1 \neq 0$	$h_2 l_2 \neq 0$	$\tan \lambda L = \dfrac{\lambda\left(\dfrac{h_1}{l_1} + \dfrac{h_2}{l_2}\right)}{\lambda^2 - \dfrac{h_1 h_2}{l_1 l_2}}$	$\cos \lambda_n x + \dfrac{h_1}{\lambda_n l_1} \sin \lambda_n x$	$\dfrac{h_1/l_1}{\lambda_n^2} + \left[1 + \left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right] \times \left[L + \dfrac{h_2/l_2}{\lambda_n^2 + (h_2/l_2)^2}\right]$
$h_1 l_1 \neq 0$	$h_2 = 0$ $(l_2 = 1)$	$\tan \lambda L = \dfrac{h_1}{\lambda l_1}$	$\dfrac{\cos \lambda_n (L - x)}{\cos \lambda_n L}$	$L\left[1 + \left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right] + \dfrac{h_1/l_1}{\lambda_n^2}$
$h_1 l_1 \neq 0$	$l_2 = 0$ $(h_2 = 1)$	$\cot \lambda L = -\dfrac{h_1}{\lambda l_1}$	$\dfrac{\sin \lambda_n (L - x)}{\sin \lambda_n L}$	$L\left[1 + \left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right] + \dfrac{h_1/l_1}{\lambda_n^2}$
$h_1 = 0$ $(l_1 = 1)$	$h_2 l_2 \neq 0$	$\tan \lambda L = \dfrac{h_2}{\lambda l_2}$	$\cos \lambda_n x$	$L + \dfrac{h_2/l_2}{\lambda_n^2 + (h_2/l_2)^2}$
$h_1 = 0$ $(l_1 = 1)$	$h_2 = 0$ $(l_2 = 1)$	$\sin \lambda L = 0$ $\lambda_n = \dfrac{n\pi}{L},\ n = 0, 1, 2, \ldots$	$\cos \lambda_n x$	$L\ (n \neq 0)$ $2L\ (n = 0)$
$h_1 = 0$ $(l_1 = 1)$	$l_2 = 0$ $(h_2 = 1)$	$\cos \lambda L = 0$ $\lambda_n = \dfrac{(2n-1)\pi}{2L},\ n = 1, 2, \ldots$	$\cos \lambda_n x$	L
$l_1 = 0$ $(h_1 = 1)$	$h_2 l_2 \neq 0$	$\cot \lambda L = -\dfrac{h_2}{\lambda l_2}$	$\sin \lambda_n x$	$L + \dfrac{h_2/l_2}{\lambda_n^2 + (h_2/l_2)^2}$
$l_1 = 0$ $(h_1 = 1)$	$h_2 = 0$ $(l_2 = 1)$	$\cos \lambda L = 0$ $\lambda_n = \dfrac{(2n-1)\pi}{2L},\ n = 1, 2, \ldots$	$\sin \lambda_n x$	L
$l_1 = 0$ $(h_1 = 1)$	$l_2 = 0$ $(h_2 = 1)$	$\sin \lambda L = 0$ $\lambda_n = \dfrac{n\pi}{L},\ n = 1, 2, \ldots$	$\sin \lambda_n x$	L

Table 5.1

If $h_1 l_1 \neq 0$ and $h_2 = 0$ (in which case we set $l_2 = 1$), eigenvalues are illustrated graphically in Figure 5.3 as points of intersection of the curves

$$y = \tan \lambda L, \qquad y = h_1/(\lambda l_1).$$

A similar situation arises when $h_2 l_2 \neq 0$ and $h_1 = 0$.

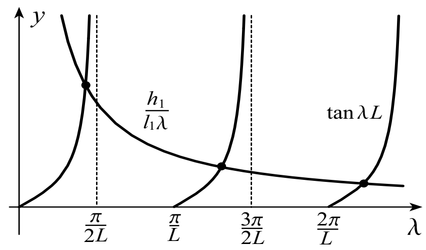


Figure 5.3

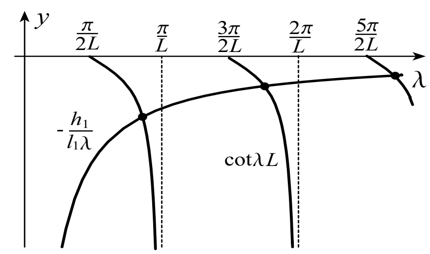


Figure 5.4

If $h_1 l_1 \neq 0$ and $l_2 = 0$ (in which case we set $h_2 = 1$), eigenvalues are illustrated graphically in Figure 5.4 as points of intersection of the curves

$$y = \cot \lambda L, \qquad y = -\frac{h_1}{\lambda l_1}.$$

A similar situation arises when $h_2 l_2 \neq 0$ and $l_1 = 0$.

Theorem 5.2 states that when a function $f(x)$ is piecewise smooth on the interval $0 \le x \le L$, we may write for $0 < x < L$

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x) \qquad \text{where} \qquad c_n = \int_0^L f(x) X_n(x)\, dx. \tag{5.19}$$

Example 5.5 Expand the function $f(x) = 2x - 1$, $0 \le x \le 4$ in terms of orthonormal eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < 4,$$
$$X'(0) = 0 = X(4).$$

Solution When we set $L = 4$ in line 6 of Table 5.1, normalized eigenfunctions of the Sturm-Liouville system are

$$X_n(x) = \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8}, \quad n = 1, 2, \ldots$$

For $0 < x < 4$, we may write that $\quad 2x - 1 = \sum_{n=1}^{\infty} c_n X_n(x), \quad$ where

$$\begin{aligned} c_n &= \int_0^4 (2x-1) X_n(x)\, dx \\ &= \frac{1}{\sqrt{2}} \left\{ \frac{8(2x-1)}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{8} + \frac{128}{(2n-1)^2\pi^2} \cos \frac{(2n-1)\pi x}{8} \right\}_0^4 \\ &= \frac{-8[16 + 7(-1)^n(2n-1)\pi]}{\sqrt{2}(2n-1)^2\pi^2}. \end{aligned}$$

Thus,

$$\begin{aligned} 2x - 1 &= \sum_{n=1}^{\infty} \frac{-8[16 + 7(-1)^n(2n-1)\pi]}{\sqrt{2}(2n-1)^2\pi^2} \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8} \\ &= -\frac{4\sqrt{2}}{\pi^2} \sum_{n=1}^{\infty} \frac{16 + 7(-1)^n(2n-1)\pi}{(2n-1)^2} \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8}, \quad 0 < x < 4. \end{aligned}$$

Figure 5.5 shows a few partial sums of the series to illustrate convergence of the series to $2x-1$. Slowness of convergence is the result of the term $7\pi(-1)^n/(2n-1)$.●

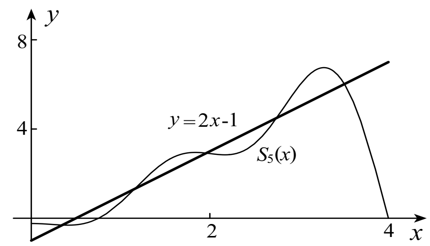


Figure 5.5a

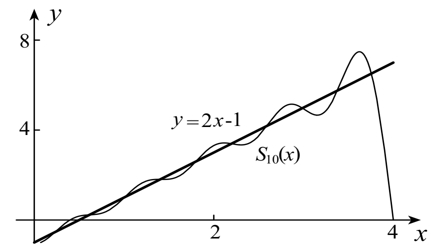


Figure 5.5b

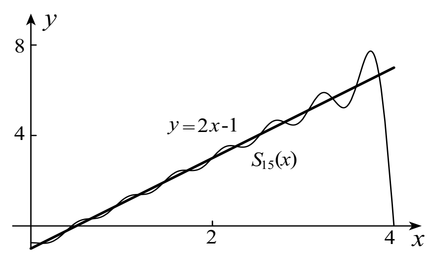


Figure 5.5c

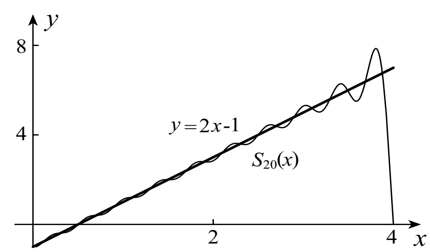


Figure 5.5d

Figure 5.5

Periodic Sturm-Liouville systems do not come under the purview of Theorem 5.2. In particular, this theorem does not guarantee expansions in terms of normalized eigenfunctions of periodic Sturm-Liouville systems. For instance, eigenvalues for the periodic Sturm-Liouville system of Example 5.2 are $\lambda_n = n^2\pi^2/L^2$ $(n = 0, 1, 2, \ldots)$, with corresponding eigenfunctions

$$\lambda_0 \leftrightarrow 1, \qquad \lambda_n \leftrightarrow \sin\frac{n\pi x}{L},\ \cos\frac{n\pi x}{L} \quad (n > 0).$$

Normalized eigenfunctions are

$$\lambda_0 \leftrightarrow \frac{1}{\sqrt{2L}}, \qquad \lambda_n \leftrightarrow \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L},\ \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L} \quad (n > 0).$$

Theorem 5.2 does not ensure the expansion of a function $f(x)$ in terms of these eigenfunctions, but our theory of ordinary Fourier series does. These are precisely the basis functions for ordinary Fourier series, except for normalizing factors, so we may write

$$f(x) = \frac{a_0}{\sqrt{2L}} + \sum_{n=1}^{\infty}\left(a_n\frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L} + b_n\frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}\right), \tag{5.20a}$$

where

$$a_0 = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{2L}}\right)dx, \qquad a_n = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L}\right)dx, \tag{5.20b}$$

$$b_n = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}\right)dx. \tag{5.20c}$$

As a final consideration in this section, we show that the Sturm-Liouville systems in Table 5.1 arise when separation of variables is applied to (initial) boundary value problems involving the second-order PDE

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \tag{5.21}$$

where p, q, and s are constants, t is time, and the Laplacian is expressed in Cartesian coordinates. We consider this PDE because it includes as special cases many of those in Chapter 2. In particular,

1. if $V = V(\mathbf{r}, t)$, $p = s = 0$, and $q = k^{-1}$, then 5.21 is the one-, two-, or three-dimensional heat conduction equation;
2. if $V = V(\mathbf{r}, t)$, $p = \rho/\tau$ (or ρ/E), then 5.21 is the one-, two-, or three-dimensional wave equation;
3. if $V = V(\mathbf{r})$, $p = q = s = 0$, then 5.21 is the one-, two-, or three-dimensional Laplace equation.

Thus, the results obtained here are valid for heat conduction, vibration, and potential problems, problems that we discuss in detail in Chapter 6.

When PDE 5.21 is to be solved in some finite region, boundary conditions and possibly initial conditions are associated with the PDE. If this region is a rectangular parallelopiped (box) in space, Cartesian coordinates can be chosen to specify the region in the form $0 \le x \le L$, $0 \le y \le L'$, $0 \le z \le L''$ (Figure 5.6). Boundary conditions must then be specified on the six faces. Suppose, for example, that the following homogeneous Dirichlet, Neumann, and Robin conditions accompany equation 5.21:

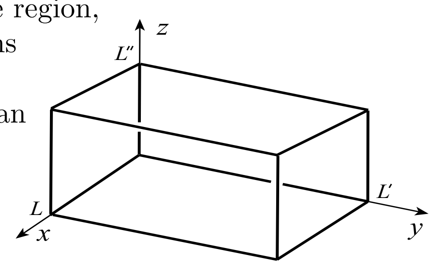


Figure 5.6

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \quad 0 < x < L, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{5.22a}$$

$$V = 0, \quad x = 0, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{5.22b}$$

$$\frac{\partial V}{\partial x} = 0, \quad x = L, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{5.22c}$$

$$-l_3\frac{\partial V}{\partial y} + h_3 V = 0, \quad y = 0, \quad 0 < x < L, \quad 0 < z < L'', \quad t > 0, \tag{5.22d}$$

$$V = 0, \quad y = L', \quad 0 < x < L, \quad 0 < z < L'', \quad t > 0, \tag{5.22e}$$

$$\frac{\partial V}{\partial z} = 0, \quad z = 0, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{5.22f}$$

$$l_6\frac{\partial V}{\partial z} + h_6 V = 0, \quad z = L'', \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{5.22g}$$

Initial conditions, if applicable. (5.22h)

If we assume that a function $V(x, y, z, t) = X(x)Y(y)Z(z)T(t)$ with variables separated satisfies PDE 5.22a,

$$X''YZT + XY''ZT + XYZ''T = pXYZT'' + qXYZT' + sXYZT.$$

Division by $XYZT$ gives

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{pT'' + qT' + sT}{T},$$

or,

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T}.$$

The separation principle (see Section 4.1) implies that each side of this equation must be equal to a constant, say α:

$$-\frac{X''}{X} = \alpha = \frac{Y''}{Y} + \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T}. \tag{5.23}$$

Thus, $X(x)$ must satisfy the ODE $X'' + \alpha X = 0$, $0 < x < L$. When the separated function is substituted into boundary conditions 5.22b,c,

$$X(0)Y(y)Z(z)T(t) = 0, \qquad X'(L)Y(y)Z(z)T(t) = 0.$$

From these, $X(0) = 0 = X'(L)$, and hence $X(x)$ must satisfy

$$X'' + \alpha X = 0, \quad 0 < x < L, \tag{5.24a}$$

$$X(0) = 0 = X'(L). \tag{5.24b}$$

This is proper Sturm-Liouville system 5.14 with $l_1 = h_2 = 0$ and $h_1 = l_2 = 1$. When we set $\alpha = \lambda^2$, eigenvalues λ_n^2 and orthonormal eigenfunctions $X_n(x)$ are then given in line 8 of Table 5.1

$$\lambda_n^2 = \frac{(2n-1)^2\pi^2}{4L^2}, \qquad X_n(x) = \sqrt{\frac{2}{L}} \sin\frac{(2n-1)\pi x}{2L}.$$

Further separation of equation 5.23 gives

$$-\frac{Y''}{Y} = \beta = \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T} - \lambda_n^2, \tag{5.25}$$

where β is a constant. Boundary conditions 5.22d,e imply that $Y(y)$ must satisfy

$$Y'' + \beta Y = 0, \quad 0 < y < L', \tag{5.26a}$$

$$-l_3 Y'(0) + h_3 Y(0) = 0, \tag{5.26b}$$

$$Y(L') = 0. \tag{5.26c}$$

This is Sturm-Liouville system 5.14 with y's replacing x's, with h_3, l_3, and L' replacing h_1, l_1, and L, and with $l_2 = 0$ and $h_2 = 1$. When we set $\beta = \mu^2$, the eigenvalue equation and orthonormal eigenfunctions are found in line 3 of Table 5.1,

$$\cot\mu L' = -\frac{h_3}{\mu l_3}, \quad NY_m(y) = \frac{1}{\sin\mu_m L'}\sin\mu_m(L' - y), \quad 2N^2 = L'\left[1 + \left(\frac{h_3}{\mu_m l_3}\right)^2\right] + \frac{h_3/l_3}{\mu_m^2}.$$

Continued separation of equation 5.25 yields

$$-\frac{Z''}{Z} = \gamma = -\frac{pT'' + qT' + sT}{T} - \lambda_n^2 - \mu_m^2, \tag{5.27}$$

where γ is a constant. When this is combined with boundary conditions 5.22f,g, $Z(z)$ must satisfy the Sturm-Liouville system

$$Z'' + \gamma Z = 0, \quad 0 < z < L'', \tag{5.28a}$$
$$Z'(0) = 0, \tag{5.28b}$$
$$l_6 Z'(L'') + h_6 Z(L'') = 0. \tag{5.28c}$$

With changes in notation, this is the Sturm-Liouville system in line 4 of Table 5.1. Eigenvalues $\gamma = \nu^2$ are defined by

$$\tan \nu L'' = \frac{h_6}{\nu l_6},$$

with orthonormal eigenfunctions

$$\frac{1}{N} \cos \nu_j z \qquad \text{where} \qquad 2N^2 = L'' + \frac{h_6/l_6}{\nu_j^2 + (h_6/l_6)^2}.$$

The time-dependent part $T(t)$ of $V(x, y, z, t)$ is obtained from the ODE

$$pT'' + qT' + sT = -(\lambda_n^2 + \mu_m^2 + \nu_j^2)T. \tag{5.29}$$

In summary, separation of variables on (initial) boundary value problem 5.22 has led to the Sturm-Liouville systems in lines 3, 4, and 8 of Table 5.1. Other choices for boundary conditions lead to the remaining five Sturm-Liouville systems in Table 5.1 (see Exercises 31–33).

EXERCISES 5.2

1. Obtain expression 5.18c for $2N^2$ by direct integration of 5.18b. Hint: Show that

$$\sin \lambda_n L = \frac{(-1)^{n+1} \lambda_n \left(\dfrac{h_1}{l_1} + \dfrac{h_2}{l_2} \right)}{\left[\left(\lambda_n^2 + \dfrac{{h_1}^2}{{l_1}^2} \right) \left(\lambda_n^2 + \dfrac{{h_2}^2}{{l_2}^2} \right) \right]^{1/2}}, \qquad \cos \lambda_n L = \frac{(-1)^{n+1} \left(\lambda_n^2 - \dfrac{h_1 h_2}{l_1 l_2} \right)}{\left[\left(\lambda_n^2 + \dfrac{{h_1}^2}{{l_1}^2} \right) \left(\lambda_n^2 + \dfrac{{h_2}^2}{{l_2}^2} \right) \right]^{1/2}}.$$

2. For each Sturm-Liouville system in Table 5.1, find expressions for $\sin \lambda_n L$ and $\cos \lambda_n L$ that involve only h_1, h_2, l_1, l_2, and/or λ_n. These should be tabulated and attached to Table 5.1 for future reference.

3. Use equations 5.16 to verify that $\lambda = 0$ is an eigenvalue of Sturm-Liouville system 5.14 only when $h_1 = h_2 = 0$.

In Exercises 4–9 express the function $f(x) = x$, $0 \le x \le L$, in terms of orthonormal eigenfunctions of the Sturm-Liouville system. In the first four exercises, discuss convergence of the expansion at $x = 0$ and $x = L$.

4. $X'' + \lambda^2 X = 0, \quad X(0) = X(L) = 0$

5. $X'' + \lambda^2 X = 0, \quad X'(0) = X'(L) = 0$

6. $X'' + \lambda^2 X = 0, \quad X(0) = X'(L) = 0$

7. $X'' + \lambda^2 X = 0, \quad X'(0) = X(L) = 0$

8. $X'' + \lambda^2 X = 0, \quad X'(0) = 0, \quad l_2 X'(L) + h_2 X(L) = 0$

9. $X'' + \lambda^2 X = 0, \quad X(0) = 0, \quad l_2 X'(L) + h_2 X(L) = 0$

10. Express the function $f(x) = x^2$, $0 \le x \le L$, in terms of orthonormal eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = 0 = X'(L).$$

Does the expansion converge to $f(x)$ at $x = 0$ and $x = L$?

In Exercises 11–13 find eigenvalues and orthonormal eigenfunctions of the proper Sturm-Liouville sytem.

11. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L, \quad y'(0) = 0 = y'(L)$

12. $\dfrac{d^2y}{dx^2} + \beta\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L, \quad y(0) = 0 = y(L) \quad (\beta \neq 0$ a given constant$)$

13. $\dfrac{d^2y}{dx^2} + \beta\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L, \quad y'(0) = 0 = y'(L) \quad (\beta \neq 0$ a given constant$)$

14. (a) Find eigenvalues and (nonnormalized) eigenfunctions for the proper Sturm-Liouville system

$$y'' + \lambda^2 y = 0, \quad -L < x < L,$$
$$y'(-L) = 0 = y'(L).$$

(b) Show that eigenfunctions in part (a) can be expressed in the compact form $\cos \dfrac{n\pi(x+L)}{2L}$, $n = 0, 1, 2, \ldots$.

(c) Normalize the eigenfunctions.

15. Find normalized eigenfunctions for the Sturm-Liouville system

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda^2 y = 0, \quad 1 < x < L,$$
$$y(1) = 0 = y(L).$$

Hint: Since the differential equation is of Cauchy-Euler type, set $y = x^m$.

16. Find normalized eigenfunctions of the Sturm-Liouville system in Exercise 15 if the boundary conditions are (a) $y'(1) = 0 = y(L)$ and (b) $y'(1) = 0 = y'(L)$.

17. On the basis of Exercises 15 and 16, we might be led to believe that eigenvalues and eigenfunctions of Sturm-Liouville systems associated with the differential equation in Exercise 15 on the interval $1 < x < L$, could be obtained by replacing x and L with $\ln x$ and $\ln L$ in Table 5.1. Show that this is not always the case by finding normalized eigenfunctions of the Sturm-Liouville system in Exercise 15 when boundary conditions are $y(1) = 0 = ly'(L) + hy(L)$.

18. Find nonnormalized eigenfunctions of the Sturm-Liouville system in Exercise 15 if the boundary conditions are $-l_1 y'(1) + h_1 y(1) = 0$ and $l_2 y'(L) + h_2 y(L) = 0$ with $h_1 h_2 l_1 l_2 \neq 0$.

19. Find normalized eigenfunctions of the Sturm-Liouville system of Exercise 15 when the interval is $a \le x \le b$ and boundary conditions are (a) $y(a) = 0 = y(b)$, (b) $y'(a) = 0 = y(b)$, and (c) $y'(a) = 0 = y'(b)$.

20. Repeat Exercise 19 with the boundary conditions $y(a) = 0 = ly'(b) + hy(b)$.

In Exercises 21–23 find six-figure approximations for the four smallest eigenvalues of the Sturm-Liouville system.

21. $X'' + \lambda^2 X = 0, \quad 0 < x < 1, \quad -X'(0) + 2000X(0) = 0, \quad X'(1) = 0$

22. $X'' + \lambda^2 X = 0, \quad 0 < x < 1, \quad X(0) = 0, \quad 3X'(1) + 2000X(1) = 0$

23. $X'' + \lambda^2 X = 0, \quad 0 < x < 1, \quad -X'(0) + 2X(0) = 0, \quad 2X'(1) + X(1) = 0$

24. (a) Expand the function

$$f(x) = \begin{cases} 1, & 0 < x < L/2 \\ -1, & L/2 < x < L \end{cases}$$

in terms of the normalized eigenfunctions of Sturm-Liouville system 5.2.
(b) What does the series converge to at $x = L/2$? Is this to be expected?
(c) What does the series converge to at $x = 0$ and $x = L$? Are these to be expected?

25. Repeat Exercise 24 with the eigenfunctions of Sturm-Liouville system 5.1.

26. In Exercise 11 of Section 5.1, we suggested two ways of interpreting the 4 in the differential equation. Does it make a difference as far as generalized Fourier series are concerned?

27. The initial boundary value problem for transverse vibrations $y(x,t)$ of a beam simply supported at one end ($x = L$) and horizontally built in at the other end ($x = 0$) when gravity is negligible compared with internal forces is

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} + c^2 \frac{\partial^4 y}{\partial x^4} &= 0, \quad 0 < x < L, \quad t > 0, \\ y(0,t) = y_x(0,t) &= 0, \quad t > 0, \\ y(L,t) = y_{xx}(L,t) &= 0, \quad t > 0, \\ y(x,0) &= f(x), \quad 0 < x < L, \\ y_t(x,0) &= g(x), \quad 0 < x < L. \end{aligned}$$

(a) Show that when $y(x,t)$ is set equal to $X(x)T(t)$, eigenfunctions obtained are

$$X_n(x) = \frac{1}{\cos \lambda_n L} \sin \lambda_n (L - x) - \frac{1}{\cosh \lambda_n L} \sinh \lambda_n (L - x),$$

where eigenvalues λ_n must satisfy

$$\tan \lambda L = \tanh \lambda L.$$

(b) Prove that these eigenfunctions are orthogonal on the interval $0 \le x \le L$ with respect to the weight function $p(x) = 1$. (Hint: Use the differential equation defining $X_n(x)$ and a construction like that in Theorem 5.1.)

28. Does the Sturm-Liouville system in line 6 of Table 5.1 give rise to the expansion in Exercise 21 of Section 3.2 for even and odd-harmonic functions?

29. Does the Sturm-Liouville system in line 8 of Table 5.1 give rise to the expansion in Exercise 20 of Section 3.2 for odd and odd-harmonic functions?

30. Show that the Sturm-Liouville system

$$\begin{aligned} \frac{d^2 X}{dx^2} + \lambda X &= 0, \quad 0 < x < L, \\ X'(0) &= 0, \\ l_2 X'(L) - h_2 X(L) &= 0, \quad (l_2 > 0, \ h_2 > 0) \end{aligned}$$

has exactly one negative eigenvalue. What is the corresponding eigenfunction?

In Exercises 31–33 determine all Sturm-Liouville systems that result when separation of variables is used to solve the problem. Do not solve the problem; simply find the Sturm-Liouville systems. Find eigenvalues (or eigenvalue equations) and orthonormal eigenfunctions for each Sturm-Liouville system. Give a physical interpretation of each problem.

31.

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= k^{-1}\frac{\partial U}{\partial t}, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \\
U(0, y, t) &= 0, \quad 0 < y < L', \quad t > 0, \\
\frac{\partial U(L, y, t)}{\partial x} + 200U(L, y, t) &= 0, \quad 0 < y < L', \quad t > 0, \\
\frac{\partial U(x, 0, t)}{\partial y} &= 0, \quad 0 < x < L, \quad t > 0, \\
\frac{\partial U(x, L', t)}{\partial y} &= 0, \quad 0 < x < L, \quad t > 0, \\
U(x, y, 0) &= f(x, y), \quad 0 < x < L, \quad 0 < y < L'.
\end{aligned}$$

32.

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2} - \beta\frac{\partial y}{\partial t}, \quad 0 < x < L, \quad t > 0, \\
-\tau\frac{\partial y(0, t)}{\partial x} + ky(0, t) &= 0, \quad t > 0, \\
y(L, t) &= 0, \quad t > 0, \\
y(x, 0) &= f(x), \quad 0 < x < L, \\
y_t(x, 0) &= 0, \quad 0 < x < L.
\end{aligned}$$

33.

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= 0, \quad 0 < x < L, \quad 0 < y < L', \quad 0 < z < L'', \\
V(0, y, z) &= 0, \quad 0 < y < L', \quad 0 < z < L'', \\
\frac{\partial V(L, y, z)}{\partial x} &= 0, \quad 0 < y < L', \quad 0 < z < L'', \\
V(x, 0, z) &= 0, \quad 0 < x < L, \quad 0 < z < L'', \\
V(x, L', z) &= 0, \quad 0 < x < L, \quad 0 < z < L'', \\
V(x, y, 0) &= f(x, y), \quad 0 < x < L, \quad 0 < y < L', \\
V(x, y, L'') &= 0, \quad 0 < x < L, \quad 0 < y < L'.
\end{aligned}$$

34. A fourth-order Sturm-Liouville system consists of a fourth-order, homogeneous differential equation of the following form, together with four linear, homogeneous boundary conditions for a function $y(\lambda, x)$:

$$\frac{d^2}{dx^2}\left[r(x)\frac{d^2y}{dx^2}\right] + [\lambda p(x) - q(x)]y = 0, \quad a < x < b,$$

$$l_1(ry'')' + h_1 y = 0, \quad x = a,$$

$$l_2(ry'') + h_2 y' = 0, \quad x = a,$$

$$l_3(ry'')' + h_3 y = 0, \quad x = b,$$

$$l_4(ry'') + h_4 y' = 0, \quad x = b,$$

where $p(x)$, $q(x)$, and $r''(x)$ are continuous on $a \le x \le b$, and $p > 0$ and $r > 0$ for $a \le x \le b$. Assuming that the system has eigenfunctions, show that eigenfunctions corresponding to distinct eigenvalues are orthogonal on $a \le x \le b$ with respect to the weight function $p(x)$.

35. Show that when separation of variables is applied to the homogeneous beam equation 2.95 and boundary conditions corresponding to simple supports, ends built-in horizontally, and/or cantilevered ends, the Sturm-Liouville system of Exercise 34 results.

§5.3 Further Properties of Sturm-Liouville Systems

In Section 3.4, we discussed uniform convergence and convergence in the mean for ordinary Fourier series. We discuss similar results for generalized Fourier series in this section. Let $y_n(x)$ be orthonormal eigenfunctions of Sturm-Liouville system 5.3 and $f(x)$ be a piecewise continuous function on $a \le x \le b$. Consider approximating $f(x)$ with a linear combination of the first n eigenfunctions

$$S_n(x) = \sum_{k=1}^{n} \alpha_k y_k(x). \tag{5.30}$$

so that the mean square error between $f(x)$ and $S_n(x)$ be as small as possible. Coefficients α_k should then be chosen so that

$$E_n = \int_a^b p(x) \left[f(x) - \sum_{k=1}^{n} \alpha_k y_k(x) \right]^2 dx \tag{5.31}$$

is minimized. The weight function $p(x)$ is that in Definition 3.5 of Section 3.4, but it also the weight function of Sturm-Liouville system 5.3. We could treat this is a multivariable extrema problem to minimize E_n as a function of the n coefficients, setting partial derivatives of E_n with respect to α_k equal to zero. We prefer the following derivation. When we expand expression 5.31 and use orthogonality of the eigenfunctions and definitions of generalized Fourier coefficients c_k for $f(x)$, we obtain

$$\begin{aligned}
E_n &= \int_a^b p(x) \left[f(x) - \sum_{k=1}^{n} \alpha_k y_k(x) \right]^2 dx \\
&= \int_a^b p(x)[f(x)]^2\,dx - 2\sum_{k=1}^{n} \int_a^b \alpha_k p(x) f(x) y_k(x)\,dx + \sum_{k=1}^{n} \int_a^b \alpha_k^2 p(x) y_k^2(x)\,dx \\
&\quad + 2 \sum_{i>j=1}^{n} \int_a^b \alpha_j \alpha_k p(x) y_j(x) y_k(x)\,dx \\
&= \int_a^b p(x)[f(x)]^2\,dx - 2\sum_{k=1}^{n} \alpha_k c_k + \sum_{k=1}^{n} \alpha_k^2 \\
&= \int_a^b p(x)[f(x)]^2\,dx + \sum_{k=1}^{n} (\alpha_k - c_k)^2 - \sum_{k=1}^{n} c_k^2.
\end{aligned} \tag{5.32}$$

This expression shows that E_n, as a function of the α_k is minimized for $\alpha_k = c_k$; that is, the best approximation 5.30 of a function $f(x)$ by orthonormal eigenfunctions is when the coefficients α_k are chosen as the generalized Fourier coefficients of $f(x)$. In other words, the partial sums of the generalized Fourier series of a function approximate the function in the mean square sense better than any other linear combination of the eigenfunctions. When we set $\alpha_k = c_k$ in equation 5.32, we obtain an expression for the mean square error when the n^{th} partial sum of a generalized Fourier series is used to approximate its sum,

$$\int_a^b p(x) \left[f(x) - \sum_{k=1}^{n} c_k y_k(x) \right]^2 dx = \int_a^b p(x)[f(x)]^2\,dx - \sum_{k=1}^{n} c_k^2. \tag{5.33}$$

Eventually, we use this expression to show that the generalized Fourier series of a function $f(x)$ converges in the mean to $f(x)$. Some preliminary results are required. The first is an immediate consequence of expression 5.33.

Theorem 5.3 If $f(x)$ is a piecewise continuous function on the interval $a \le x \le b$ of Sturm-Liouville system 5.3, its generalized Fourier coefficients must satisfy the inequality

$$\sum_{k=1}^{n} c_k^2 \le \int_a^b p(x)[f(x)]^2\,dx. \tag{5.34}$$

Proof: Since the left side of equation 5.33 is nonnegative,

$$\int_a^b p(x)[f(x)]^2\,dx - \sum_{k=1}^{n} c_k^2 \ge 0,$$

and this gives inequality 5.34. By letting n become infinite, we can also state that

$$\sum_{n=1}^{\infty} c_n^2 \le \int_a^b p(x)[f(x)]^2\,dx. \blacksquare \tag{5.35}$$

This is known as **Bessel's inequality**. In Theorem 5.5 it is shown that, with more restrictive conditions on the function $f(x)$, inequality 5.35 may be replaced by an equality, the result being known as Parseval's theorem, and this leads to convergence in the mean of generalized Fourier series. Because our proof of Parseval's theorem requires uniform convergence of the generalized Fourier series of a function, we digress to discuss this type of convergence. The counterpart of Theorem 3.10 in Section 3.4 is contained in the following theorem (which we state without proof).

Theorem 5.4 Suppose that $f(x)$ is continuous and $f'(x)$ is piecewise continuous on $a \le x \le b$. If $f(x)$ satisfies the boundary conditions of a proper Sturm-Liouville system on $a \le x \le b$, then the generalized Fourier series 5.13 for $f(x)$ converges absolutely and uniformly to $f(x)$ for $a \le x \le b$.

We illustrate this result in the following example.

Example 5.6 (a) Without finding the generalized Fourier series of the function $f(x) = x(L - x)$ in terms of normalized eigenfunctions of Sturm-Liouville system 5.1, show that the series converges uniformly for $0 \le x \le L$. (b) Now find the series and use the Weierstrass M-test (see Section 3.4) to verify uniform convergence.

Solution (a) Because $f(x)$ is infinitely differentiable, and satisfies the boundary conditions of proper Sturm-Liouville system 5.1, namely that $y(0) = y(L) = 0$, Theorem 5.4 guarantees uniform convergence of the generalized Fourier series of $f(x)$.
(b) A short calculation shows that the generalized Fourier series of $f(x)$ is

$$\begin{aligned} x(L-x) &= \frac{2\sqrt{2}L^{5/2}}{\pi^3}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^3}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \\ &= \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}, \quad 0 \le x \le L. \end{aligned}$$

Since

$$\sum_{n=1}^{\infty}\left|\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}\right| \le \sum_{n=1}^{\infty}\frac{1}{(2n-1)^3},$$

and the latter series converges, the generalized Fourier series converges uniformly by the Weierstrass M-test.•

Exercise 9 in Section 3.4 contains special cases of Theorem 5.4.

When $f(x)$ satisfies the conditions of Theorem 5.4, Bessel's inequality 5.35 may be replaced by an equality. This result is contained in the next theorem.

Theorem 5.5 (**Parseval's Theorem**) Suppose that $f(x)$ is continuous and $f'(x)$ is piecewise continuous on $a \le x \le b$. If $f(x)$ satisfies the boundary conditions of a proper Sturm-Liouville system on $a \le x \le b$, its Fourier coefficients satisfy

$$\sum_{n=1}^{\infty} c_n^2 = \int_a^b p(x)[f(x)]^2\,dx. \tag{5.36}$$

Proof With the conditions cited on $f(x)$, the generalized Fourier series of $f(x)$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x),$$

where $y_n(x)$ are the normalized eigenfunctions of the Sturm-Liouville system, is uniformly convergent (Theorem 5.4). It may therefore be multiplied by $p(x)f(x)$ and integrated term-by-term between $x = a$ and $x = b$ to yield

$$\int_a^b p(x)[f(x)]^2\,dx = \sum_{n=1}^{\infty} c_n \int_a^b p(x)f(x)y_n(x)\,dx = \sum_{n=1}^{\infty} c_n^2. \blacksquare$$

It is now possible to verify that generalized Fourier series converge in the mean.

Theorem 5.6 Suppose that $f(x)$ is continuous and $f'(x)$ is piecewise continuous on $a \le x \le b$. If $f(x)$ satisfies the boundary conditions of a proper Sturm-Liouville system on $a \le x \le b$, its generalized Fourier series converges in the mean to $f(x)$.

Proof: Expression 5.33 gives the mean square error when a function $f(x)$ is approximated by the n^{th} partial sum of its generalized Fourier series,

$$\int_a^b p(x)\left[f(x) - \sum_{k=1}^{n} c_k y_k(x)\right]^2 dx = \int_a^b p(x)[f(x)]^2\,dx - \sum_{k=1}^{n} c_k^2.$$

If we take limits as $n \to \infty$, and invoke Parseval's Theorem, we obtain

$$\lim_{n\to\infty}\int_a^b p(x)\left[f(x) - \sum_{k=1}^{n} c_k y_k(x)\right]^2 dx = 0.$$

This is condition 3.23 for convergence in the mean of the generalized Fourier series. ▮

Expansions of functions as generalized Fourier series are very different from power series expansions. A function can be represented in a Taylor series on an interval only if the function and all of its derivatives are continuous throughout the interval, and even these conditions may not be sufficient to guarantee convergence of the series to the function. Eigenfunction expansions, however, are valid even though a function and its first derivative may each possess a finite number of finite discontinuities.

On the other hand, whereas Taylor series expansions may be differentiated term-by-term inside the interval of convergence of the series, such may not be the case for generalized Fourier series. The following result is analogous to Theorem 3.8 in Section 3.1.

Theorem 5.7 Suppose that $f(x)$ is continuous and $f'(x)$ and $f''(x)$ are piecewise continuous on $a \le x \le b$. If $f(x)$ satisfies the boundary conditions of a proper Sturm-Liouville system, then for any x in $a < x < b$, series 5.13 may be differentiated term-by-term with the resulting series converging to $[f'(x+) + f'(x-)]/2$.

We now prove the Sturm comparison theorem, a result that has implications when we study singular Sturm-Liouville systems in Chapter 8.

Theorem 5.8 (**Sturm Comparison Theorem**) Let $r(x)$ be a function that is positive on the interval $a < x < b$ and has a continuous first derivative for $a \le x \le b$. Suppose that $s_1(x)$ and $s_2(x)$ are continuous functions for $a < x < b$ such that $s_2(x) > s_1(x)$ thereon. If $y_1(x)$ and $y_2(x)$ satisfy

$$\frac{d}{dx}\left[r(x)\frac{dy_1}{dx}\right] + s_1(x)y_1 = 0, \qquad \frac{d}{dx}\left[r(x)\frac{dy_2}{dx}\right] + s_2(x)y_2 = 0, \tag{5.37}$$

there is at least one zero of $y_2(x)$ between every consecutive pair of zeros of $y_1(x)$ in $a < x < b$.

Proof Let α and β be any two consecutive zeros of $y_1(x)$ in $a < x < b$, and suppose that $y_2(x)$ has no zero between α and β. We assume, without loss in generality, that $y_1(x) > 0$ and $y_2(x) > 0$ on $\alpha < x < \beta$. (If this were not true, we would work with $-y_1(x)$ and $-y_2(x)$.) When equations 5.37 are multiplied by y_2 and y_1, respectively, and the results are subtracted,

$$0 = y_1\left[\frac{d}{dx}\left(r\frac{dy_2}{dx}\right) + s_2y_2\right] - y_2\left[\frac{d}{dx}\left(r\frac{dy_1}{dx}\right) + s_1y_1\right].$$

Integration of this equation from α to β gives

$$\begin{aligned}\int_\alpha^\beta (s_2 - s_1)y_1y_2\,dx &= \int_\alpha^\beta [(ry_1')'y_2 - (ry_2')'y_1]\,dx = \int_\alpha^\beta (ry_1'y_2 - ry_2'y_1)'\,dx \\ &= \Big\{(ry_1'y_2 - ry_2'y_1)\Big\}_\alpha^\beta \\ &= r(\beta)[y_2(\beta)y_1'(\beta) - y_1(\beta)y_2'(\beta)] - r(\alpha)[y_2(\alpha)y_1'(\alpha) - y_1(\alpha)y_2'(\alpha)] \\ &= r(\beta)y_2(\beta)y_1'(\beta) - r(\alpha)y_2(\alpha)y_1'(\alpha),\end{aligned}$$

since $y_1(\alpha) = y_1(\beta) = 0$. Because $y_1(x) > 0$ for $\alpha < x < \beta$, it follows that $y_1'(\alpha) \ge 0$ and $y_1'(\beta) \le 0$. Furthermore, because $r(\alpha)$, $r(\beta)$, $y_2(\alpha)$, and $y_2(\beta)$ are all positive, we must have

$$r(\beta)y_2(\beta)y_1'(\beta) - r(\alpha)y_2(\alpha)y_1'(\alpha) \le 0.$$

But this contradicts the fact that

$$\int_{\alpha}^{\beta} [s_2(x) - s_1(x)]y_2(x)y_1(x)\,dx > 0,$$

since $s_2 > s_1$ on $\alpha \le x \le \beta$. Consequently, $y_2(x)$ must have a zero between α and β. ∎

To see the implication of this theorem in Sturm-Liouville theory, we set $s_1(x) = \lambda_1 p(x) - q(x)$ and $s_2(x) = \lambda_2 p(x) - q(x)$, where $\lambda_2 > \lambda_1$ are eigenvalues of system 5.3. It then follows that between every pair of zeros of the eigenfunction $y_1(x)$ corresponding to λ_1, there is at least one zero of the eigenfunction $y_2(x)$ associated with λ_2. Figure 5.7 illustrates the situation for eigenfunctions $X_3(x)$ and $X_4(x)$ of Sturm-Liouville system 5.1.

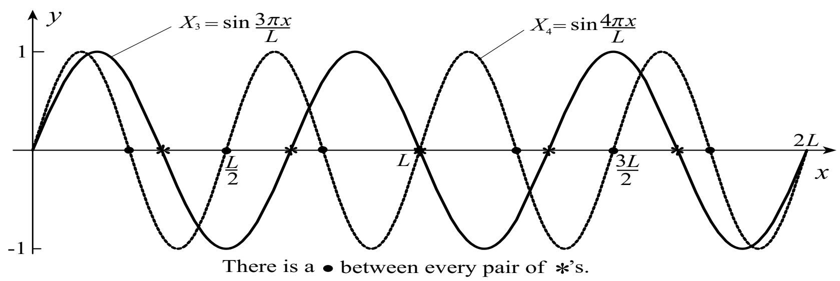


Figure 5.7

EXERCISES 5.3

1. Theorem 5.7 indicates that generalized Fourier series from proper Sturm-Liouville systems may be differentiated term-by-term when $f(x)$ is continuous, $f'(x)$ and $f''(x)$ are piecewise continuous, and $f(x)$ satisfies the boundary conditions of the system. We illustrate with two examples.

(a) Find the generalized Fourier series for

$$f(x) = \begin{cases} x, & 0 \le x \le L/2 \\ L - x, & L/2 \le x \le L \end{cases}$$

in terms of the normalized eigenfunctions of Sturm-Liouville system 5.1. Show graphically that $f(x)$ is continuous, and $f'(x)$ and $f''(x)$ are piecewise continuous on $0 \le x \le L$. Since $f(0) = f(L) = 0$, Theorem 5.7 guarantees that term-by-term differentiation of the eigenfunction expansion for $f(x)$ yields a series that converges to $[f'(x+) + f'(x-)]/2$ for $0 < x < L$. Verify that this is indeed true, but do so without using Theorem 5.7.

(b) Find the generalized Fourier series for $g(x) = 1$, $0 \le x \le L$, in terms of the eigenfunctions of Sturm-Liouville 5.1. Show that term-by-term differentiation of this series gives a series that converges only for $x = L/2$. Which of the conditions in Theorem 5.7 are violated by $g(x)$?

2. (a) Expand the function $f(x) = x(L - x)$ in terms of the eigenfunctions of Sturm-Liouville system 5.1.

(b) Use Parseval's theorem 5.5 to prove that $\sum_{n=1}^{\infty} 1/(2n-1)^6 = \pi^6/960$.

3. (a) Show that an eigenvalue λ_n of a regular Sturm-Liouville system can be expressed in terms of its corresponding normalized eigenfunction $y_n(x)$ according to

$$\lambda_n = \int_a^b \{r(x)[y_n{}'(x)]^2 + q(x)[y_n(x)]^2\}\,dx - \Big\{r(x)y_n(x)y_n{}'(x)\Big\}_a^b.$$

This is often called the **Rayleigh quotient**. The word *quotient* is used because it is often stated with nonnormalized eigenfunctions in which case there is a denominator to the expression.

(b) What form does the Rayleigh quotient take when boundary conditions are Dirichlet or Neumann?

CHAPTER 6 SOLUTION OF HOMOGENEOUS PROBLEMS BY SEPARATION OF VARIABLES

§6.1 Introduction

In Chapter 2 we developed boundary value and initial boundary value problems to describe physical phenomena such as heat conduction, vibrations, and electrostatic potentials. In Chapter 3 we introduced Fourier series, which we then used in Chapter 4, in conjunction with separation of variables, to solve very simple problems. These straightforward examples led to consideration of Sturm-Liouville systems in Chapter 5. We are now ready to apply these results in more complex homogeneous problems. In Chapter 7 we introduce finite Fourier transforms to solve nonhomogeneous problems. They are a more effective technique for handling nonhomogeneities than variation of constants of Section 4.3, especially for higher dimensional problems.

A great variety of homogeneous problems could be considered — heat conduction, vibration, or potential; one-, two-, or three-dimensional; time dependent or steady-state. Because we cannot hope to consider all of these problems, we select a few straightforward examples to illustrate the technique; this puts us in a position to consider quite general PDEs, such as

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \tag{6.1}$$

where p, q and s are constants. We pointed out in Section 5.2 that this PDE contains many of the PDEs in Chapter 2 (see equation 5.22 in Chapter 5). It follows that initial boundary value problems associated with PDE 6.1 contain as special cases many of the (initial) boundary value problems of Chapter 2. In fact, when we solve PDE 6.1 subject to Robin boundary conditions, we obtain general formulas that may be specialized to give solutions to many problems. We begin in Sections 6.2 and 6.3 with problems in two independent variables. In Section 6.4 we generalize to problems in higher dimensions.

§6.2 Homogeneous Initial Boundary Value Problems in Two Variables

We begin this section by using separation of variables to solve two initial boundary value problems, one in heat conduction and the other in vibrations. What we learn from these examples will prepare us for separation of variables in more general problems. The heat conduction problem is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{6.2a}$$

$$U_x(0,t) = 0, \quad t > 0, \tag{6.2b}$$

$$\kappa\frac{\partial U(L,t)}{\partial x} + \mu U(L,t) = 0, \quad t > 0, \tag{6.2c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{6.2d}$$

Physically described is a rod of uniform cross section and insulated sides that at time $t = 0$ has temperature $f(x)$ (Figure 6.1). For time $t > 0$, the end $x = 0$ is also insulated, and heat is exchanged at the other end with an environment at temperature 0°C. The problem is said to be homogeneous because PDE 6.2a and boundary conditions 6.2b,c are homogeneous.

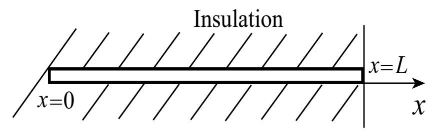


Figure 6.1

If we assume that a function $U(x,t)$, separated in the form $U(x,t) = X(x)T(t)$, satisfies PDE 6.2a, then

$$XT' = kX''T \qquad \Longrightarrow \qquad \frac{X''}{X} = \frac{T'}{kT} = \alpha = \text{constant}.$$

When this is combined with boundary conditions 6.2b,c, $X(x)$ must satisfy the system

$$X'' - \alpha X = 0, \quad 0 < x < L, \tag{6.3a}$$
$$X'(0) = 0, \tag{6.3b}$$
$$\kappa X'(L) + \mu X(L) = 0, \tag{6.3c}$$

and $T(t)$ must satisfy the ODE

$$T' - \alpha k T = 0, \quad t > 0. \tag{6.4}$$

System 6.3 is a special case of proper Sturm-Liouville system 5.14 in Section 5.2. Since eigenvalues $(-\alpha)$ must be positive, we set $-\alpha = \lambda^2$, in which case line 4 in Table 5.1 defines eigenvalues as solutions of the equation

$$\tan \lambda L = \frac{\mu}{\kappa\lambda}$$

and orthonormal eigenfunctions as

$$X_n(x) = \frac{1}{N}\cos\lambda_n x, \qquad \text{where} \qquad 2N^2 = L + \frac{\mu/\kappa}{\lambda_n^2 + (\mu/\kappa)^2}.$$

For these eigenvalues, a general solution of ODE 6.4 is $T(t) = ce^{-k\lambda_n^2 t}$, where c is an arbitrary constant. It follows that separated functions $ce^{-k\lambda_n^2 t}X_n(x)$ for any constant c and any eigenvalue λ_n satisfy PDE 6.2a and boundary conditions 6.2b,c. To satisfy initial condition 6.2d, we superpose separated functions (the PDE and boundary conditions being linear and homogeneous) and take

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(x), \tag{6.5}$$

where the c_n are constants. Condition 6.2d now requires that

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x), \quad 0 < x < L. \tag{6.6}$$

But this equation states that the c_n are the Fourier coefficients in the generalized Fourier series of $f(x)$ in terms of the $X_n(x)$. According to equation 5.19, they are given by

$$c_n = \int_0^L f(x)X_n(x)\,dx = \frac{1}{N}\int_0^L f(x)\cos\lambda_n x\,dx. \tag{6.7a}$$

The final formal solution of problem 6.2 is therefore

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t}\frac{1}{N}\cos\lambda_n x. \tag{6.7b}$$

To see how the boundary conditions affect temperature in the rod, we consider a specific initial temperature distribution. Suppose, for example, that the rod is 1 m long and that $f(x) = 100(1-x)$. Furthermore, suppose that the conductivity κ and diffusivity k of the material in the rod are 48 W/mK and 12×10^{-6} m^2/s and that the heat transfer coefficient at $x = L$ is $\mu = 96$ $\text{W/m}^2\text{K}$. With these physical attributes, eigenvalues are defined by $\tan\lambda = 2/\lambda$, and normalizing factors are $2N^2 = 1 + 2/(\lambda_n^2 + 4)$. Coefficients c_n are given by

$$c_n = \frac{1}{N}\int_0^1 100(1-x)\cos\lambda_n x\,dx = \frac{100}{N\lambda_n^2}(1-\cos\lambda_n).$$

Thus,

$$\begin{aligned} U(x,t) &= \sum_{n=1}^{\infty}\frac{100}{N\lambda_n^2}(1-\cos\lambda_n)e^{-12\times10^{-6}\lambda_n^2 t}\frac{1}{N}\cos\lambda_n x \\ &= \sum_{n=1}^{\infty}\frac{200(\lambda_n^2+4)(1-\cos\lambda_n)}{\lambda_n^2(\lambda_n^2+6)}e^{-12\times10^{-6}\lambda_n^2 t}\cos\lambda_n x. \end{aligned}$$

When this series is approximated by its first four terms, plots for various values of t are as shown in Figure 6.2. (The four smallest positive solutions of $\tan\lambda = 2/\lambda$ are 1.076874, 3.643597, 6.578334, and 9.629560.)

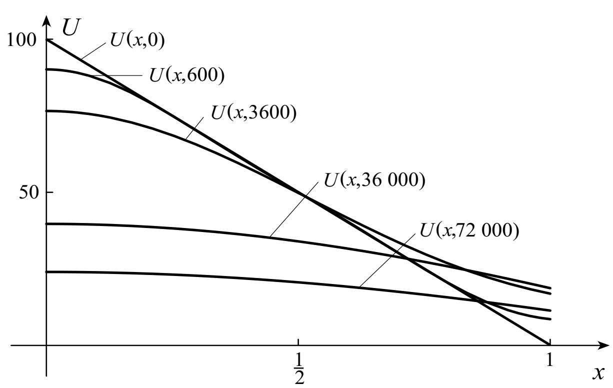


Figure 6.2

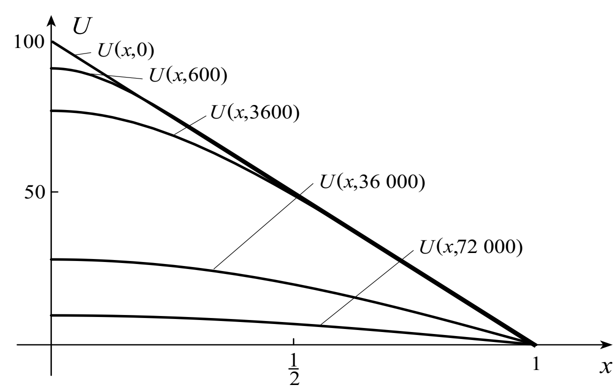


Figure 6.3

In Figure 6.3 we show temperature in the rod for the same times when boundary condition 6.2c is replaced by $U(L,t) = 0$. What this means is that the heat transfer coefficient μ in 6.2c has become very large and there is essentially no resistance to heat transfer across the boundary $x = L$. The solution in this case is

$$U(x,t) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-12\times10^{-6}(2n-1)^2\pi^2 t/4}\cos\frac{(2n-1)\pi x}{2}.$$

Ultimately, the solution approaches the situation in which temperature in the rod is identically zero, as it does in problem 6.2, but it does so more quickly.

Our second illustrative example is concerned with displacements of the taut string in Figure 6.4. The end at $x = L$ is fixed on the x-axis, while the end at $x = 0$ is looped around a vertical support and can move thereon without friction. If the position of the string is initially parabolic, $x(L-x)$, and it is motionless, subsequent displacements are described by the homogeneous initial boundary value problem

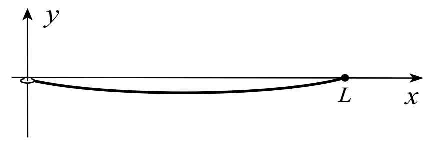


Figure 6.4

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{6.8a}$$
$$y_x(0,t) = 0, \quad t > 0, \tag{6.8b}$$
$$y(L,t) = 0, \quad t > 0, \tag{6.8c}$$
$$y(x,0) = x(L-x), \quad 0 < x < L, \tag{6.8d}$$
$$y_t(x,0) = 0, \quad 0 < x < L. \tag{6.8e}$$

If we assume a function $y(x,t)$, separated in the form $y(x,t) = X(x)T(t)$, satisfies PDE 6.8a then

$$XT'' = c^2 X'' T \qquad \Longrightarrow \qquad \frac{X''}{X} = \frac{T''}{c^2 T} = \alpha = \text{constant}.$$

When this is combined with boundary conditions 6.8b,c, and initial condition 6.8e, $X(x)$ must satisfy the system

$$X'' - \alpha X = 0, \quad 0 < x < L,$$
$$X'(0) = 0,$$
$$X(L) = 0,$$

and $T(t)$ must satisfy

$$T'' - \alpha c^2 T = 0, \quad t > 0,$$
$$T'(0) = 0.$$

The system in $X(x)$ is a special case of Sturm-Liouville system 5.14 in Section 5.2. Since eigenvlaues $(-\alpha)$ must be positive, we set $-\alpha = \lambda^2$, in which case line 6 in Table 5.1 gives eigenvalues of the Sturm-Liouville system, ${\lambda_n}^2 = (2n-1)^2\pi^2/(4L^2)$, where $n \geq 1$, with orthonormal eigenfunctions $X_n(x) = \sqrt{2/L}\cos\lambda_n x$. For these eigenvalues, the solution of system in $T(t)$ is $T(t) = A\cos c\lambda_n t$, where A is an arbitrary constant. We have shown, therefore, that separated functions $A\cos c\lambda_n t X_n(x)$ for any constant A and any eigenvalue λ_n satisfy PDE 6.8a, boundary condition 6.8b,c, and initial condition 6.8e. To satisfy initial condition 6.8d, we superpose separated functions and take

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos c\lambda_n t X_n(x), \tag{6.9}$$

where the A_n are constants. Condition 6.8d now requires

$$x(L-x) = \sum_{n=1}^{\infty} A_n X_n(x), \quad 0 < x < L. \tag{6.10}$$

Consequently, the A_n are coefficients in the generalized Fourier series of $x(L-x)$; that is,

$$A_n = \int_0^L x(L-x)X_n(x)\,dx = \int_0^L x(L-x)\sqrt{\frac{2}{L}}\cos\frac{(2n-1)\pi x}{2L}dx$$
$$= \frac{16\sqrt{2}L^{5/2}(-1)^{n+1}}{(2n-1)^3\pi^3} - \frac{4\sqrt{2}L^{5/2}}{(2n-1)^2\pi^2}.$$

When these are substituted into representation 6.9, the formal solution is

$$y(x,t) = \frac{-8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(2n-1)\pi + 4(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}. \tag{6.11}$$

Each term in this solution is called a **normal mode of vibration** of the string. The first term, let us denote it by

$$H_1(x,t) = \frac{-8L^2(\pi-4)}{\pi^3}\cos\frac{\pi ct}{2L}\cos\frac{\pi x}{2L} = 0.22L^2\cos\frac{\pi ct}{2L}\cos\frac{\pi x}{2L},$$

is called the **fundamental mode** or **first harmonic**. As a separated function, $H_1(x,t)$ satisfies 6.8a,b,c,e; at time $t=0$, it reduces to $0.22L^2\cos[\pi x/(2L)]$. In other words, $H_1(x,t)$ describes displacements of a string identical to that in problem 6.8, except that the initial displacement is $0.22L^2\cos[\pi x/(2L)]$ instead of $x(L-x)$. Positions of this string for various values of t are shown in Figure 6.5. The string vibrates back and forth between the enveloping curves $\pm 0.22L^2\cos[\pi x/(2L)]$, always maintaining the shape of a cosine.

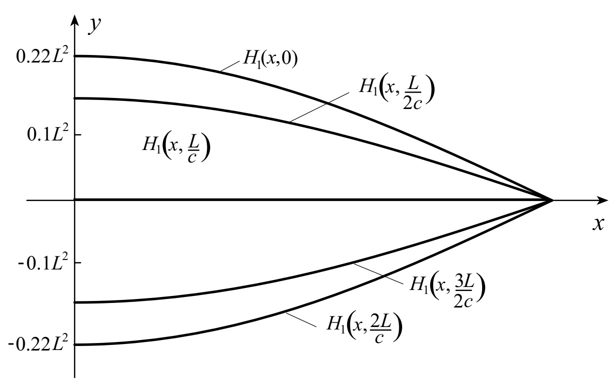


Figure 6.5

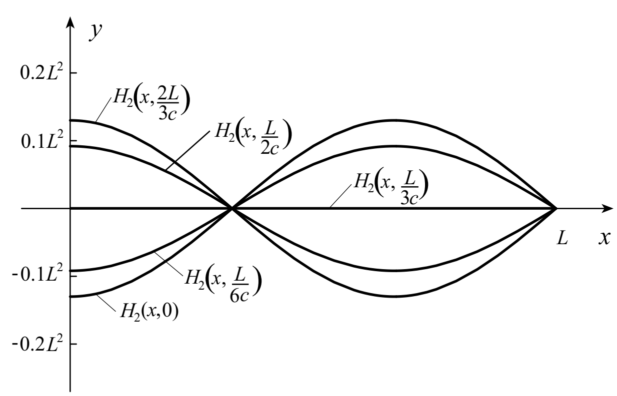


Figure 6.6

The second harmonic is the second term

$$H_2(x,t) = -0.13L^2\cos\frac{3\pi ct}{2L}\cos\frac{3\pi x}{2L};$$

it represents displacements of the same string were the initial displacement described by $-0.13L^2\cos[3\pi x/(2L)]$. Positions of this string for various values of t are shown in Figure 6.6. The point at $x = L/3$ in the string remains motionless; it is called a **node** of $H_2(x, t)$.

The third harmonic

$$H_3(x,t) = -0.024L^2 \cos\frac{5\pi ct}{2L}\cos\frac{5\pi x}{2L},$$

is shown in Figure 6.7. It has two nodes, one at $x = L/5$ and the other at $x = 3L/5$.

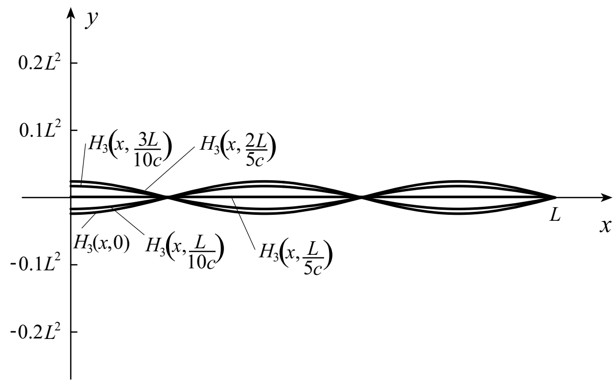


Figure 6.7

Solution 6.11 of 6.8 is the sum of all its harmonics. Because A_n decreases rapidly with increasing n, lower harmonics are more significant than higher ones.

We are now in a position to consider the general homogeneous initial boundary value problem

$$\frac{\partial^2 V}{\partial x^2} = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \quad 0 < x < L, \quad t > 0, \tag{6.12a}$$

$$-l_1\frac{\partial V}{\partial x} + h_1 V = 0, \quad x = 0, \quad t > 0, \tag{6.12b}$$

$$l_2\frac{\partial V}{\partial x} + h_2 V = 0, \quad x = L, \quad t > 0, \tag{6.12c}$$

$$V(x, 0) = f(x), \quad 0 < x < L, \tag{6.12d}$$

$$V_t(x, 0) = g(x), \quad 0 < x < L. \tag{6.12e}$$

It is said to be homogeneous because the PDE and boundary conditions are homogeneous. This problem includes as special cases the following problems from Chapter 2:

1. If $V(x,t) = U(x,t)$, $p = s = 0$, and $q = k^{-1}$, then 6.12 is the one-dimensional heat conduction problem with no internal heat generation but with heat transfer at ends $x = 0$ and $x = L$ into or from media at temperature zero. In this case, initial condition 6.12e would be absent.
2. If $V(x,t) = y(x,t)$, $p = \rho/\tau$ (or ρ/E), $q = \beta/\tau$, and $s = k/\tau$, then 6.12 is the one-dimensional vibration problem with a damping force proportional to velocity and a restoring force proportional to displacement.

When a function separated in the form $V(x,t) = X(x)T(t)$ is substituted into the PDE

$$X''T = pXT'' + qXT' + sXT \quad \Longrightarrow \quad \frac{X''}{X} = \frac{pT'' + qT' + sT}{T}.$$

We have customarily set each side of this equation equal to a constant which we have called α. Subsequent analysis always showed that α must be negative so that we set $\alpha = -\lambda^2$. In future problems, we will insert $-\lambda^2$ directly. When we do this for the above equation, and introduce boundary conditions 6.12b,c, we are led to a Sturm-Liouville system in $X(x)$,

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \tag{6.13a}$$

$$-l_1 X' + h_1 X = 0, \quad x = 0, \tag{6.13b}$$

$$l_2 X' + h_2 X = 0, \quad x = L, \tag{6.13c}$$

and an ODE in $T(t)$,

$$pT'' + qT' + (s + \lambda^2)T = 0, \quad t > 0. \tag{6.14}$$

System 6.13 is Sturm-Liouville system 5.14 in Chapter 5. Eigenvalues and orthonormal eigenfunctions are listed in Table 5.1.

When $p = 0$, ODE 6.14 has general solution

$$T(t) = ce^{-(s+\lambda_n^2)t/q}, \tag{6.15}$$

where c is a constant. We have shown, therefore, that separated functions

$$V(x,t) = X(x)T(t) = ce^{-(s+\lambda_n^2)t/q}X_n(x),$$

for any constant c, and any eigenvalue λ_n are solutions of PDE 6.12a and boundary conditions 6.12b,c. There is but one initial condition when $p = 0$, namely 6.12d, and to satisfy it, we superpose separated functions (the PDE and boundary conditions being linear and homogeneous) and take

$$V(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) e^{-(s+\lambda_n^2)t/q}, \tag{6.16}$$

where the c_n are constants. Initial condition 6.12d now implies that the c_n must satisfy

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x), \quad 0 < x < L. \tag{6.17}$$

The constants c_n are therefore Fourier coefficents in the generalised Fourier series of $f(x)$,

$$c_n = \int_0^L f(x) X_n(x)\, dx. \tag{6.18}$$

The formal solution of problem 6.12 for $p = 0$ is therefore 6.16 with the c_n defined by 6.18.

When $p \neq 0$, ODE 6.14 has general solution

$$T(t) = c\phi_1(t) + d\phi_2(t), \tag{6.19}$$

where $\phi_1(t)$ and $\phi_2(t)$ are independent solutions of 6.14 and c and d are arbitrary constants. In this case, separated functions

$$V(x,t) = X(x)T(t) = X_n(x)[c\phi_1(t) + d\phi_2(t)],$$

for any constants c and d and any eigenvalue λ_n, are solutions of PDE 6.12a and boundary conditions 6.12b,c. To satisfy the initial conditions, we superpose separated functions and take

$$V(x,t) = \sum_{n=1}^{\infty} X_n(x)[c_n\phi_1(t) + d_n\phi_2(t)], \tag{6.20}$$

where c_n and d_n are constants. Initial conditions 6.12d,e now imply that the c_n and d_n must satisfy

$$f(x) = \sum_{n=1}^{\infty} X_n(x)[c_n\phi_1(0) + d_n\phi_2(0)], \quad 0 < x < L, \tag{6.21a}$$

$$g(x) = \sum_{n=1}^{\infty} X_n(x)[c_n{\phi_1}'(0) + d_n{\phi_2}'(0)], \quad 0 < x < L. \tag{6.21b}$$

If we multiply the first by ${\phi_2}'(0)$, multiply the second by $\phi_2(0)$, and subtract,

$${\phi_2}'(0)f(x) - \phi_2(0)g(x) = \sum_{n=1}^{\infty} c_n[\phi_1(0){\phi_2}'(0) - {\phi_1}'(0)\phi_2(0)]X_n(x). \tag{6.22}$$

This equation implies that $c_n[\phi_1(0){\phi_2}'(0) - {\phi_1}'(0)\phi_2(0)]$ must be the Fourier coefficients in the generalized Fourier series of ${\phi_2}'(0)f(x) - \phi_2(0)g(x)$ in terms of $X_n(x)$ and are therefore defined by equation 5.19,

$$c_n[\phi_1(0){\phi_2}'(0) - {\phi_1}'(0)\phi_2(0)] = \int_0^L [{\phi_2}'(0)f(x) - \phi_2(0)g(x)]X_n(x)\,dx. \tag{6.23}$$

(This equation can also be obtained by multiplying 6.22 by $X_m(x)$ and integrating with respect to x from $x = 0$ to $x = L$.) Thus,

$$c_n = \frac{1}{\phi_1(0){\phi_2}'(0) - {\phi_1}'(0)\phi_2(0)} \int_0^L [{\phi_2}'(0)f(x) - \phi_2(0)g(x)]X_n(x)\,dx. \tag{6.24}$$

Similarly, it can be shown that

$$d_n = \frac{1}{{\phi_1}'(0)\phi_2(0) - \phi_1(0){\phi_2}'(0)} \int_0^L [{\phi_1}'(0)f(x) - \phi_1(0)g(x)]X_n(x)\,dx. \tag{6.25}$$

The formal solution of problem 6.12 for $p \neq 0$ is 6.20, where c_n and d_n are defined by 6.24 and 6.25.

We have demonstrated that separation of variables can be used to solve initial boundary value problems of form 6.12 and therefore, as special cases, problems 1. and 2. following 6.12. In fact, 6.16 and 6.20 represent formulas for solutions of many of these problems. For example, to solve heat conduction problem 4.20 in Section 4.2, we could set $p = s = h_1 = h_2 = 0$, $l_1 = l_2 = 1$, and $q = 1/k$ in 6.12, delete initial condition 6.12e, and set $f(x) = x$. According to 6.16, the solution is

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(x) \qquad \text{where} \qquad c_n = \int_0^L f(x)X_n(x)\,dx.$$

Eigenpairs are found in line 5 of Table 5.1,

$$\lambda_0 = 0 \leftrightarrow X_0(x) = \frac{1}{\sqrt{L}}, \qquad \lambda_n = \frac{n\pi}{L} \leftrightarrow X_n(x) = \frac{2}{L}\cos\frac{n\pi x}{L}.$$

With these,

$$c_0 = \int_0^L x\frac{1}{\sqrt{L}}dx = \frac{L^{3/2}}{2}, \qquad c_n = \int_0^L x\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}dx = \frac{\sqrt{2}L^{3/2}[(-1)^n - 1]}{n^2\pi^2},$$

and therefore

$$\begin{aligned} U(x,t) &= \frac{L^{3/2}}{2}\left(\frac{1}{\sqrt{L}}\right) + \sum_{n=1}^{\infty}\frac{\sqrt{2}L^{3/2}[(-1)^n - 1]}{n^2\pi^2}e^{-n^2\pi^2kt/L^2}\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L} \\ &= \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt/L^2}\cos\frac{(2n-1)\pi x}{L}. \end{aligned}$$

This is solution 4.25 of problem 4.20 in Section 4.2.

We are not in the habit of recommending the use of results such as 6.16 and 6.20 as formulas. Formulas are fine for those who have mastered fundamentals and are now looking for shortcuts in solving large classes of problems. We prefer to regard our analysis of problem 6.12 as an illustration of the fact that any problem of this form can be solved by separation of variables. The procedure leading from problem 6.12 to either solution 6.16 or 6.20 should be used as a guideline for solving other problems — separate variables, obtain the appropriate Sturm-Liouville system, solve the system (perhaps by quoting Table 5.1), solve the ODE in $T(t)$, superpose separated functions, and apply the nonhomogeneous initial condition(s).

EXERCISES 6.2

Part A Heat Conduction

1. (a) A homogeneous, isotropic rod with insulated sides has temperature $f(x) = L - x$, $0 \le x \le L$, at time $t = 0$. If, for time $t > 0$, the end $x = 0$ is insulated and the end $x = L$ is held at temperature 0°C, find the temperature in the rod.

(b) Find an expression (in series form) for the amount of heat leaving the end $x = L$ of the rod as a function of time.

(c) Plot a graph of the function in part (b) if $\kappa = 48$ W/mK, $k = 12 \times 10^{-6}$ m^2/s, and $L = 1$ m.

2. What is the solution to Exercise 1(a) for an arbitrary initial temperature $f(x)$?

3. A homogeneous, isotropic rod with insulated sides has temperature $f(x)$, $0 \le x \le L$, at time $t = 0$. For time $t > 0$, heat is transferred at end $x = 0$ according to Newton's law of cooling to a medium at temperature 0°C. If the end $x = L$ is held at temperature 0°C, find the temperature in the rod.

4. Let $U(x,t)$ denote temperature in the thin-wire problem (see Exercise 41 of Section 2.2) of a thin wire of length L lying along the x-axis. When the temperature of the surrounding medium is zero and there is no heat generation, $U(x,t)$ must satisfy the PDE

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} - hU, \quad 0 < x < L, \quad t > 0,$$

where $h > 0$ is a constant.

(a) If the ends of the wire are insulated and the initial temperature distribution is denoted by $f(x)$, find and solve the initial boundary value problem for $U(x,t)$.

(b) Compare the solution in part (a) with that obtained when the lateral sides are also insulated.

5. Exercise 4 suggests the following result. The general homogeneous thin-wire problem (see Exercise 41 of Section 2.2) is

$$\begin{aligned}\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} - hU, \quad 0 < x < L, \quad t > 0,\\ -l_1\frac{\partial U}{\partial x} + h_1 U &= 0, \quad x = 0, \quad t > 0,\\ l_2\frac{\partial U}{\partial x} + h_2 U &= 0, \quad x = L, \quad t > 0,\\ U(x,0) &= f(x), \quad 0 < x < L.\end{aligned}$$

(Homogeneity requires an environmental temperature identically zero. Nonzero environmental temperatures and other nonhomogeneities are considered in the exercises in Section 7.2.) Show that the solution of this problem is always e^{-ht} times that of the corresponding problem when no heat transfer takes place over the surface of the wire.

Part B Vibrations

6. (a) A taut string is given an initial displacement (at time $t = 0$) of $f(x)$, $0 \le x \le L$, and initial velocity $g(x)$, $0 \le x \le L$. If the ends $x = 0$ and $x = L$ of the string are fixed on the x-axis, find displacements of points in the string thereafter.

(b) As functions of time, what are the amplitudes of the first, second, and third harmonics? Sketch graphs of these harmonics for various fixed values of t. Are frequencies of higher harmonics integer multiples of the frequency of the fundamental mode?

(c) What are the nodes for the first three harmonics?

7. A taut string is given an initial displacement (at time $t = 0$) of $f(x)$, $0 \le x \le L$, and initial velocity $g(x)$, $0 \le x \le L$. The end $x = 0$ is fixed on the x-axis, while the end $x = L$ is looped around a vertical support and can move thereon without friction.

(a) Find a series representation for displacements in the string for $0 < x \le L$ and $t > 0$.

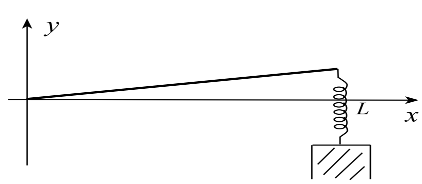
(b) Find the d'Alembert form for displacements of the string.

8. Repeat Exercise 6(a) if an external force (per unit x-length) $F = -ky$ ($k > 0$) acts at each point in the string.

(b) Compare the normal modes of vibration with those in Exercise 6.

9. Repeat Exercise 6(a) if an external force (per unit x-length) $F = -\beta\partial y/\partial t$ ($0 < \beta < 2\pi\rho c/L$) acts at every point on the string.

10. A taut string is given displacement bx, b a constant, $0 \le x \le L$, and zero initial velocity. The end $x = 0$ is fixed on the x-axis, and the right end moves vertically but is restrained by a spring (constant k) that is unstretched on the x-axis (figure below).



(a) Show that subsequent displacements of points on the string can be expressed in the form

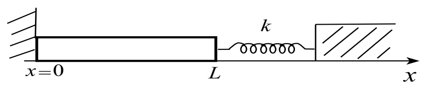
$$y(x,t) = \frac{2b(kL+\tau)}{\tau} \sum_{n=1}^{\infty} \frac{(\tau^2\lambda_n^2 + k^2)\sin \lambda_n L}{\lambda_n^2[L(\tau^2{\lambda_n}^2 + k^2) + k\tau]} \cos c\lambda_n t \sin \lambda_n x,$$

where λ_n are the positive solutions of the equation $\cot \lambda L = -k/(\tau\lambda)$, τ is the constant tension in the string, and $c^2 = \tau/\rho$, where ρ is the constant density of the string.

(b) Reduce the expression in part (a) to

$$y(x,t) = 2b(kL+\tau) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{k^2+\tau^2\lambda_n^2}}{\lambda_n[L(k^2+\tau^2\lambda_n^2)+k\tau]} \cos c\lambda_n t \sin \lambda_n x.$$

11. A bar of uniform cross section and length L lies along the x-axis. Its left end is fixed at $x = 0$, and its right end is attached to a spring with constant k that is unstretched when the bar is unstrained (figure below). If, at time $t = 0$, the bar is pulled to the right so that cross sections are displaced according to $f(x) = x/100$, then released from rest at this position, find subsequent displacements of cross sections.



Equations 6.12a–e describe displacements of a taut string when $p \neq 0$. Separation leads to a solution in form 6.20 with coefficients c_n and d_n given by 6.24 and 6.25. The normal modes of this solution are

$$H_n(x,t) = X_n(x)[c_n\phi_1(t) + d_n\phi_2(t)],$$

where $X_n(x)$ are the eigenfunction in Table 5.1. Nodes of $H_n(x,t)$ are points that remain motionless for all t. They are the zeros of $X_n(x)$. In Exercises 12–17 we show that the number of nodes of the n^{th} mode is $n-1$ (except when both ends of the string are looped around vertical supports and move freely without friction).

12. Show that when both ends are fixed on the x-axis, the distance between successive nodes is L/n, and hence there are $n-1$ equally spaced nodes between $x = 0$ and $x = L$.

13. Show that when the end $x = 0$ is fixed on the x-axis and the end $x = L$ is looped around a vertical support and moves without friction thereon (a free end), there are $n-1$ nodes between $x = 0$ and $x = L$. A similar result holds when the left end is free and the right end is fixed.

14. Verify that when both ends are free, the n^{th} mode has n nodes.

15. (a) Verify that when the end $x = 0$ is fixed on the x-axis and the end $x = L$ satisfies a homogeneous Robin condition, nodes of the n^{th} mode occur for $x_m = m\pi/\lambda_n$, $m > 0$ an integer.

(b) Use Figure 5.4 to establish that eigenvalues λ_n satisfy

$$\frac{(n-1)\pi}{L} < \lambda_n < \frac{n\pi}{L}.$$

Use this to verify the existence of $n-1$ nodes. A similar result holds when the right end is fixed and the left end satisfies a homogeneous Robin condition.

16. (a) Verify that when end $x = 0$ is free and end $x = L$ satisfies a homogeneous Robin condition, nodes of the n^{th} mode occur for $x_m = (2m-1)\pi/(2\lambda_n)$, $m > 0$ an integer.

(b) Establish the inequality

$$\frac{(n-1)\pi}{L} < \lambda_n < \frac{(2n-1)\pi}{2L}$$

for this case, and use this to verify that there are $n-1$ nodes. A similar result holds when the right end is free and the left end satisfies a homogeneous Robin condition.

17. The final case is when both ends of the string satisfy homogeneous Robin conditions, in which case $X_n(x)$ is given in line 1 of Table 5.1.

(a) Show that zeros of $X_n(x)$ occur for $x_m = m\pi/\lambda_n - \phi_n$, where m is an integer, and $\phi_n = (1/\lambda_n)\text{Tan}^{-1}(\lambda_n l_1/h_1)$.

(b) Establish the inequality in Exercise 16(b) and the fact that the difference between successive nodes is π/λ_n.

(c) Use the results in part (b) to verify that there are $n-1$ nodes between $x = 0$ and $x = L$.

§6.3 Homogeneous Boundary Value Problems in Two Variables

The Helmholtz equation on a rectangle $0 \le x \le L$, $0 \le y \le L'$ takes the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + k^2 V = 0, \quad 0 < x < L, \quad 0 < y < L', \tag{6.26a}$$

where k is some given constant. When $k = 0$, we obtain the important special case of Laplace's equation. A boundary value problem accompanying either of these equations is said to be homogeneous if the boundary conditions on a pair of parallel sides are homogeneous. For example, the following conditions on $x = 0$ and $x = L$ yield a homogeneous problem:

$$V(0, y) = 0, \quad 0 < y < L', \tag{6.26b}$$
$$\frac{\partial V(L, y)}{\partial x} = 0, \quad 0 < y < L', \tag{6.26c}$$
$$V(x, 0) = f(x), \quad 0 < x < L, \tag{6.26d}$$
$$V(x, L') = g(x), \quad 0 < x < L. \tag{6.26e}$$

No real difficulty is encountered in the solution of problem 6.26 if 6.26b,c are not homogeneous, if say

$$V(0, y) = h(y), \quad 0 < y < L', \tag{6.26f}$$
$$\frac{\partial V(L, y)}{\partial x} = k(y), \quad 0 < y < L'. \tag{6.26g}$$

We simply use superposition to write $V(x, y) = V_1(x, y) + V_2(x, y)$, where V_1 and V_2 both satisfy PDE 6.26a and the following boundary conditions:

$$\begin{aligned} V_1(0, y) &= 0, \quad 0 < y < L', & V_2(0, y) &= h(y), \quad 0 < y < L', \\ \frac{\partial V_1(L, y)}{\partial x} &= 0, \quad 0 < y < L', & \frac{\partial V_2(L, y)}{\partial x} &= k(y), \quad 0 < y < L', \\ V_1(x, 0) &= f(x), \quad 0 < x < L, & V_2(x, 0) &= 0, \quad 0 < x < L, \\ V_1(x, L') &= g(x), \quad 0 < x < L; & V_2(x, L') &= 0, \quad 0 < x < L. \end{aligned}$$

In other words, the nonhomogeneous boundary value problem 6.26a,d,e,f,g can be divided into two homogeneous problems. It follows then, that separation of variables as illustrated here on problem 6.26a–e is typical for all boundary value problems on rectangles (provided the PDE is homogeneous).

Substitution of a separated function $V(x, y) = X(x)Y(y)$ into 6.26a,b,c leads to a Sturm-Liouville system in $X(x)$,

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = 0 = X'(L),$$

and an ODE in $Y(y)$,

$$Y'' - (\lambda^2 - k^2)Y = 0, \quad 0 < y < L'.$$

Eigenpairs of the Sturm-Liouville system are $\lambda_n^2 = (2n-1)^2\pi^2/(4L^2)$ with eigenfunctions $X_n(x) = \sqrt{2/L}\sin \lambda_n x$. If we assume that $\lambda_n^2 > k^2$, for all n, corresponding solutions for $Y(y)$ are $Y(y) = A\cosh\sqrt{\lambda_n^2 - k^2}y + B\sinh\sqrt{\lambda_n^2 - k^2}y$. We superpose separated functions and take

$$V(x,y) = \sum_{n=1}^{\infty}[A_n\cosh\sqrt{\lambda_n^2 - k^2}y + B_n\sinh\sqrt{\lambda_n^2 - k^2}y]X_n(x). \quad (6.27a)$$

Boundary conditions 6.26d,e require that for $0 < x < L$,

$$f(x) = \sum_{n=1}^{\infty}A_nX_n(x), \quad g(x) = \sum_{n=1}^{\infty}[A_n\cosh\sqrt{\lambda_n^2 - k^2}L' + B_n\sinh\sqrt{\lambda_n^2 - k^2}L']X_n(x).$$

These imply that

$$A_n = \int_0^L f(x)X_n(x)\,dx \quad (6.27b)$$

and

$$A_n\cosh\sqrt{\lambda_n^2 - k^2}L' + B_n\sinh\sqrt{\lambda_n^2 - k^2}L' = \int_0^L g(x)X_n(x)\,dx \quad (6.27c)$$

or,

$$B_n = \frac{1}{\sinh\sqrt{\lambda_n^2 - k^2}L'}\left[\int_0^L g(x)X_n(x)\,dx - A_n\cosh\sqrt{\lambda_n^2 - k^2}L'\right]. \quad (6.27d)$$

The formal solution of problem 6.26a–e is therefore

$$V(x,y) = \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}[A_n\cosh\sqrt{\lambda_n^2 - k^2}y + B_n\sinh\sqrt{\lambda_n^2 - k^2}y]\sin\lambda_n x, \quad (6.28)$$

where A_n and B_n are calculated according to formulas 6.27b,d.

As a specific example, suppose $k = 0$, so that PDE 6.26a becomes Laplace's equation, and suppose that $f(x) = 0$ and $g(x) = x$. One possible interpretation of problem 6.26 would be that for steady-state temperature in a rectangle in which sides $x = 0$ and $y = 0$ are held at temperature 0°C, side $x = L$ is insulated, and $y = L'$ has temperature x. The solution of this problem is

$$V(x,y) = \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}B_n\sinh\frac{(2n-1)\pi y}{2L}\sin\frac{(2n-1)\pi x}{2L},$$

where

$$B_n = \frac{1}{\sinh\lambda_n L'}\int_0^L x\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}dx = \frac{4\sqrt{2}L^{3/2}(-1)^{n+1}}{(2n-1)^2\pi^2\sinh[(2n-1)\pi L'/(2L)]}.$$

Thus,

$$\begin{aligned} V(x,y) &= \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\frac{4\sqrt{2}L^{3/2}(-1)^{n+1}}{(2n-1)^2\pi^2\sinh(2n-1)\pi L'/(2L)]}\sinh\frac{(2n-1)\pi y}{2L}\sin\frac{(2n-1)\pi x}{2L} \\ &= \frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2\sinh[(2n-1)\pi L'/(2L)]}\sinh\frac{(2n-1)\pi y}{2L}\sin\frac{(2n-1)\pi x}{2L}. \end{aligned}$$

We now consider Laplace's equation in a circle of radius a with a Dirichlet boundary condition (Figure 6.8),

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0, \quad 0 < r < a, \quad -\pi < \theta \le \pi, \tag{6.29a}$$

$$V(a,\theta) = f(\theta), \quad -\pi < \theta \le \pi. \tag{6.29b}$$

The solution of this problem describes a number of physical phenomena. It represents (axially symmetric) electrostatic potential in a source-free cylinder $r \le a$, with potential prescribed on the surface of the cylinder $r = a$ as $f(\theta)$. Also described is steady-state temperature in a thin circular plate, insulated top and bottom, with circumferential temperature $f(\theta)$. Finally, $V(r,\theta)$ represents static deflections of a circular membrane subjected to no external forces but with edge deflections $f(\theta)$.

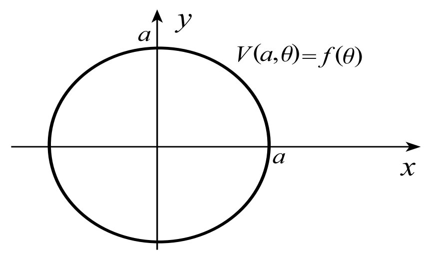


Figure 6.8

When we substitute a separated function $V(r,\theta) = R(r)H(\theta)$ into the PDE and multiply by $r^2/V(r,\theta)$, separation results,

$$-\frac{r^2 R''}{R} - \frac{rR'}{R} = \frac{H''}{H} = -\lambda^2 = \text{constant}.$$

Thus, $R(r)$ and $H(\theta)$ must satisfy the ODEs

$$r^2 R'' + rR' - \lambda^2 R = 0, \qquad H'' + \lambda^2 H = 0.$$

Now, $V(r,\theta)$ must be 2π-periodic in θ, as must its first derivative with respect to θ; that is,

$$V(r,\theta+2\pi) = V(r,\theta),$$
$$\frac{\partial V(r,\theta+2\pi)}{\partial\theta} = \frac{\partial V(r,\theta)}{\partial\theta}.$$

These imply that $H(\theta)$ and $H'(\theta)$ must also be periodic. It follows that $H(\theta)$ must satisfy the periodic Sturm-Liouville system

$$H'' + \lambda^2 H = 0, \quad -\pi < \theta < \pi,$$
$$H(-\pi) = H(\pi),$$
$$H'(-\pi) = H'(\pi).$$

According to Example 5.2 in Section 5.1 and equations 5.20 in Section 5.2, eigenvalues of this system are $\lambda_n^2 = n^2$ $(n \ge 0)$, with a single eigenfunction $1/\sqrt{2\pi}$ corresponding to $\lambda_0 = 0$, and a pair of eigenfunctions $(1/\sqrt{\pi})\cos n\theta$ and $(1/\sqrt{\pi})\sin n\theta$ corresponding to $\lambda_n^2 = n^2$ $(n > 0)$.

The differential equation in $R(r)$ is a Cauchy-Euler equation, which can be solved (in the case when $n > 0$) by setting $R(r) = r^m$, m an unknown constant. This results in the general solution

$$R(r) = \begin{cases} A + B \ln r, & n = 0 \\ Ar^n + Br^{-n}, & n \geq 1. \end{cases} \tag{6.30}$$

For these solutions to remain bounded near $r = 0$, we must set $B = 0$. Separated functions have now been determined to be $A/\sqrt{2\pi}$ corresponding to $\lambda_0 = 0$, and $(Ar^n/\sqrt{\pi}) \cos n\theta$ and $(Ar^n/\sqrt{\pi}) \sin n\theta$ corresponding to $\lambda_n = n$ $(n > 0)$. To satisfy boundary condition 6.29b, we superpose separated functions and take

$$V(r, \theta) = \frac{A_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} r^n \left(A_n \frac{\cos n\theta}{\sqrt{\pi}} + B_n \frac{\sin n\theta}{\sqrt{\pi}} \right). \tag{6.31a}$$

The boundary condition requires

$$f(\theta) = \frac{A_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} a^n \left(A_n \frac{\cos n\theta}{\sqrt{\pi}} + B_n \frac{\sin n\theta}{\sqrt{\pi}} \right), \quad -\pi < \theta \leq \pi,$$

from which

$$A_0 = \int_{-\pi}^{\pi} f(\theta) \frac{1}{\sqrt{2\pi}} d\theta, \ A_n = \frac{1}{a^n} \int_{-\pi}^{\pi} f(\theta) \frac{\cos n\theta}{\sqrt{\pi}} d\theta, \ B_n = \frac{1}{a^n} \int_{-\pi}^{\pi} f(\theta) \frac{\sin n\theta}{\sqrt{\pi}} d\theta. \tag{6.31b}$$

(see equations 5.20 in Section 5.2 with $L = \pi$ and x replaced by θ.) The formal solution of problem 6.29 is now complete; it is series 6.31a with coefficients defined by 6.31b. An integral expression for the solution can be obtained by substituting coefficients A_n and B_n into 6.31a. In order to keep variable θ distinct from the variable of integration in 6.31b, we replace θ by u in 6.31b,

$$\begin{aligned} V(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\, du + \sum_{n=1}^{\infty} \frac{1}{\pi} \left(\frac{r}{a} \right)^n \left[\cos n\theta \int_{-\pi}^{\pi} f(u) \cos nu\, du + \sin n\theta \int_{-\pi}^{\pi} f(u) \sin nu\, du \right] \\ &= \frac{1}{\pi} \left[\frac{1}{2} \int_{-\pi}^{\pi} f(u)\, du + \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \int_{-\pi}^{\pi} f(u) \cos n(\theta - u)\, du \right]. \end{aligned} \tag{6.32}$$

If we interchange the order of integration and summation,

$$V(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \cos n(\theta - u) \right] f(u)\, du.$$

The series can be summed in closed form by noting that $\cos n(\theta - u)$ is the real part of a complex exponential, $\cos n(\theta - u) = \mathrm{Re}[e^{in(\theta - u)}]$,

$$\sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \cos n(\theta - u) = \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \mathrm{Re}[e^{in(\theta - u)}] = \mathrm{Re} \left[\sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta - u)} \right)^n \right].$$

Since the right side is a geometric series with common ratio $(r/a)e^{i(\theta - u)}$, converging when $r < a$, we may write

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \cos n(\theta - u) &= \mathrm{Re} \left[\frac{(r/a)e^{i(\theta - u)}}{1 - (r/a)e^{i(\theta - u)}} \right] = \mathrm{Re} \left[\frac{r[\cos(\theta - u) + i \sin(\theta - u)]}{a - r[\cos(\theta - u) + i \sin(\theta - u)]} \right] \\ &= \frac{ar \cos(\theta - u) - r^2}{a^2 + r^2 - 2ar \cos(\theta - u)}. \end{aligned} \tag{6.33}$$

Consequently,

$$V(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \frac{ar\cos(\theta-u) - r^2}{a^2+r^2-2ar\cos(\theta-u)}\right]f(u)\,du$$
$$= \frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)}{a^2+r^2-2ar\cos(\theta-u)}du. \qquad (6.34)$$

This result is called **Poisson's integral formula for a circle**. It expresses the solution to Laplace's equation inside the circle $r \le a$ in terms of its values on the circle. Immediate consequences of the formula are the following two results.

Theorem 6.1 When $V(r,\theta)$ is the solution to Dirichlet's problem for Laplace's equation in a circle $r \le a$, the value $V(0,\theta)$ at the centre of the circle is the average of its values on $r = a$.

Proof According to formula 6.34, the value of $V(r,\theta)$ at $r = 0$ is

$$V(0,\theta) = \frac{a^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)}{a^2}du = \frac{1}{2\pi a}\int_{-\pi}^{\pi} f(\theta)\,a\,d\theta,$$

the average value of $f(\theta)$ on $r = a$. ∎

Corollary When $V(r,\theta)$ is the solution to Dirichlet's problem for Laplace's equation in a circle $r \le a$, the average value of $V(r,\theta)$ around every circle centred at $r = 0$ is $V(0,\theta)$.

EXERCISES 6.3

1. (a) Solve Exercise 30 from Section 4.2.
(b) Find an approximate value for the potential at the centre of the plate if the plate is square.

2. The temperature in a circular plate with insulated faces is in a steady-state situation. The temperature at one point along the edge of the plate is zero and then increases linearly with respect to angle around the circle to temperature 100°C at the opposite end of the diameter of the plate. Temperature then decreases linearly back to zero around the other half of the edge. What is the temperature at the centre of the plate?

3. Temperature in a square plate is in a steady-state situation. Three of the edges are at temperature zero and temperature along the fourth edge is a constant value U_0. Without solving the boundary value problem for the steady-state temperature in the plate, find the temperature at the centre of the plate?

4. (a) Find the steady-state temperature $U(x,y)$ inside a plate $0 \le x, y \le L$ if the sides $x = 0$, $y = 0$, and $x = L$ are all insulated and the boundary condition on $y = L$ is $\partial U(x,L)/\partial y = f(x)$. Can $f(x)$ be specified arbitrarily?
(b) What is the solution when $f(x) = (L-2x)/2$ and the temperature at the centre of the plate is 50°C?
(c) What is the solution when $f(x) = x(L-x)$?

5. (a) Find the steady-state temperature $U(x,y)$ inside a rectangular plate $0 \le x \le L$, $0 \le y \le L'$ if the sides $y = 0$ and $y = L'$ are insulated, the temperature along $x = L$ is prescribed by the function $f_2(y)$, $0 < y < L'$, and the boundary condition along $x = 0$ is $\partial U(0,y)/\partial x = f_1(y)$, $0 < y < L'$.
(b) Simplify the solution in part (a) when $f_1(y)$ and $f_2(y)$ are constants.

6. A membrane is stretched tightly over the rectangle $0 \le x \le L$, $0 \le y \le L'$. Its edges are given deflections that are described by the following boundary conditions:

$$\begin{aligned} z(0,y) &= f_1(y), \quad 0 < y < L', \\ z(L,y) &= f_2(y), \quad 0 < y < L', \\ z(x,0) &= g_1(x), \quad 0 < x < L, \\ z(x,L') &= g_2(x), \quad 0 < x < L. \end{aligned}$$

Find static deflections of the membrane when all external forces are negligible compared with tensions in the membrane.

7. Solve Laplace's equation in the rectangle $0 \le x \le L$, $0 \le y \le L'$ subject to the following boundary conditions:

$$\begin{aligned} V(0,y) &= f_1(y), \quad 0 < y < L', \\ V_x(L,y) &= f_2(y), \quad 0 < y < L', \\ V_y(x,0) &= 0, \quad 0 < x < L, \\ V(x,L') &= g(x), \quad 0 < x < L. \end{aligned}$$

8. (a) Solve Laplace's equation in a semicircle $r \le a$, $0 \le \theta \le \pi$ when the unknown function is zero on the diameter and $f(\theta)$ on the semicircle.
(b) Simplify the solution when $f(\theta) = 1$. Evaluate this solution along the y-axis.

9. (a) Along the circle $r = a$, a solution $V(r,\theta)$ of Laplace's equation must take on the value 1 for $0 < \theta < \pi$ and 0 for $-\pi < \theta < 0$. Show that the series solution for $V(r,\theta)$ is

$$V(r,\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(r/a)^{2n-1}}{2n-1} \sin(2n-1)\theta.$$

A closed-form solution of this problem is found in Exercise 29.
(b) What is the value of $V(r,\theta)$ along the x-axis?

10. Find the steady-state temperature inside the quarter-circle $r \le a$, $0 \le \theta \le \pi/2$ if its straight edges are insulated and the temperature along the curved edge is $\sin\theta$.

11. (a) Solve boundary value problem 6.29 when boundary condition 6.29b is of Neumann type:

$$\frac{\partial V(a,\theta)}{\partial r} = f(\theta), \quad -\pi < \theta \le \pi.$$

(b) Show that the solution can be expressed in the form

$$V(r,\theta) = C - \frac{a}{2\pi}\int_{-\pi}^{\pi} f(u) \ln[a^2 + r^2 - 2ar\cos(\theta - u)]\,du,$$

where C is an arbitrary constant. This result is called **Dini's integral**.

12. Solve boundary value problem 6.29 when boundary condition 6.29b is of Robin type:

$$l\frac{\partial V(a,\theta)}{\partial r} + hV(a,\theta) = f(\theta), \quad -\pi < \theta \le \pi.$$

13. (a) Show that the negative of Poisson's integral formula 6.34 is the solution to Laplace's equation exterior to the circle $r = a$ if $V(r, \theta)$ is required to be bounded at infinity [i.e., $V(r, \theta)$ must be bounded for large r].
(b) Show that if $V(r, \theta)$ is the solution to the interior problem, then $V(a^2/r, \theta)$ is the solution to the exterior problem.
(c) Is the solution to the exterior problem different if $V(r, \theta)$ must vanish at infinity?

14. (a) Show that if $V(r, \theta)$ is required to be bounded at infinity, then Dini's integral of Exercise 11 is also the solution of Laplace's equation exterior to the circle $r = a$ when the boundary condition is Neumann, $-\partial V(a, \theta)/\partial r = f(\theta)$, $-\pi < \theta \le \pi$.
(b) Is the solution different if $V(r, \theta)$ must vanish at infinity?

15. (a) Solve Laplace's equation exterior to the circle $r = a$ when the solution is required to be bounded at infinity and satisfy a Robin boundary condition at $r = a$,

$$-l\frac{\partial V(a,\theta)}{\partial r} + hV(a,\theta) = f(\theta), \quad -\pi < \theta \le \pi.$$

(b) Is the solution different if $V(r, \theta)$ must vanish at infinity?

16. Solve Laplace's equation inside a circular annulus $a < r < R$ with Dirichlet boundary conditions

$$V(a,\theta) = f_1(\theta), \qquad V(R,\theta) = f_2(\theta), \qquad -\pi < \theta \le \pi.$$

17. Solve Exercise 16 when the boundary conditions are Neumann:

$$-\frac{\partial V(a,\theta)}{\partial r} = f_1(\theta), \qquad \frac{\partial V(R,\theta)}{\partial r} = f_2(\theta), \qquad -\pi < \theta \le \pi.$$

18. Solve Exercise 16 when the boundary conditions are Robin:

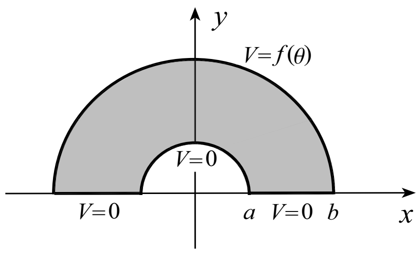
$$-l_1\frac{\partial V(a,\theta)}{\partial r} + h_1V(a,\theta) = f_1(\theta), \quad l_2\frac{\partial V(R,\theta)}{\partial r} + h_2V(R,\theta) = f_2(\theta), \qquad -\pi < \theta \le \pi.$$

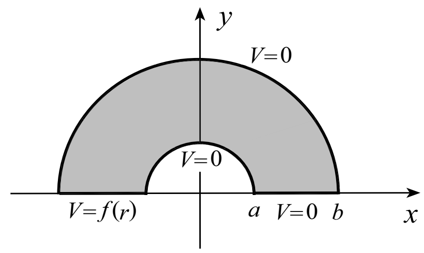
19. (a) A plate with insulated faces is in the shape of a wedge bounded by the edges $\theta = 0$, $\theta = \alpha$, and $r = a$. If its straight edges are also insulated and edge $r = a$ is held at temperature $f(\theta)$, find the steady-state temperature in the plate.
(b) Simplify the solution in part (a) when $f(\theta) = 1$ and when $f(\theta) = \theta$.

20. Repeat Exercise 19 if the curved side is insulated and the straight sides have prescribed temperatures.

21. (a) A circular membrane of radius a is in a static position with radial lines $\theta = 0$ and $\theta = \alpha$ clamped on the xy-plane. If the displacement of the edge $r = a$ is $f(\theta)$ for $0 < \theta < \alpha$, find the displacement in the sector $0 < \theta < \alpha$.
(b) Take the limit of your answer in part (a) as $\alpha \to 2\pi$. What does this function represent physically?
(c) What is the answer in part (b) if $f(\theta) = \sin(\theta/2)$?

22. (a) Solve Laplace's equation in the region shown in the left figure below with the given boundary conditions.
(b) Simplify the solution when $f(\theta) = V_0$, a constant.

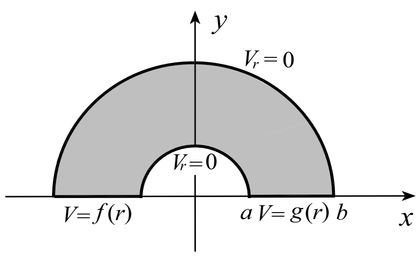


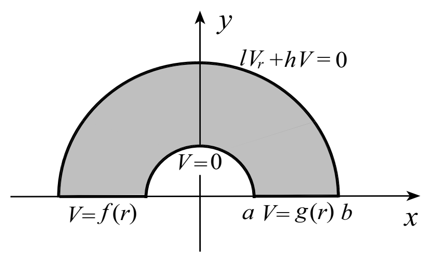


23. (a) Solve Laplace's equation in the region shown in the right figure above with the given boundary conditions. Hint: See Exercise 19 in Section 5.2 for eigenfunctions.

(b) Simplify the solution when $f(r) = V_0$, a constant.

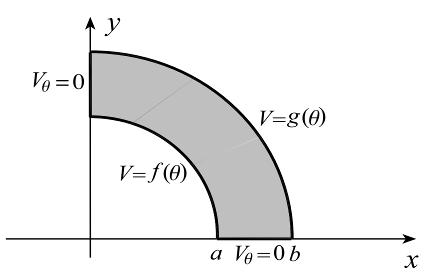
24. Solve Laplace's equation in the region shown in the left figure below with the given boundary conditions. Hint: See Exercise 19 in Section 5.2 for eigenfunctions.

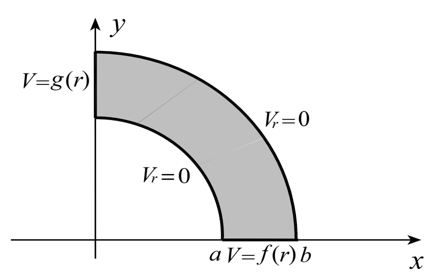




25. Solve Laplace's equation in the region shown in the right figure above with the given boundary conditions. Hint: See Exercise 20 in Section 5.2 for eigenfunctions.

26. Solve Laplace's equation in the region shown in the left figure below with the given boundary conditions.





27. Solve Laplace's equation in the region shown in the right figure above with the given boundary conditions. Hint: See Exercise 19 in Section 5.2 for eigenfunctions.

When $f(\theta)$ in the boundary condition for Dirichlet problem 6.29 is piecewise constant, Poisson's integral formula for a circle can be evaluated analytically. We illustrate this in Exercises 28–30.

28. Show that when $a > r > 0$,

$$\int \frac{1}{a^2 + r^2 - 2ar\cos(\theta - u)}\,du = \frac{-2}{a^2 - r^2}\,\mathrm{Tan}^{-1}\left[\frac{a+r}{a-r}\tan\left(\frac{\theta - u}{2}\right)\right] + C,$$

provided $u \neq \theta \pm \pi$.

29. (a) When

$$f(\theta) = \begin{cases} 0, & -\pi < \theta < 0 \\ 1, & 0 < \theta < \pi \end{cases},$$

use the result of Exercise 28 to obtain the following solution for problem 6.29:

$$V(r,\theta) = \begin{cases} 1 + \dfrac{1}{\pi}\mathrm{Tan}^{-1}\left[\dfrac{a+r}{a-r}\tan\left(\dfrac{\theta}{2}\right)\right] + \dfrac{1}{\pi}\mathrm{Tan}^{-1}\left[\dfrac{a+r}{a-r}\cot\left(\dfrac{\theta}{2}\right)\right], & -\pi < \theta < 0 \\ \dfrac{1}{\pi}\mathrm{Tan}^{-1}\left[\dfrac{a+r}{a-r}\tan\left(\dfrac{\theta}{2}\right)\right] + \dfrac{1}{\pi}\mathrm{Tan}^{-1}\left[\dfrac{a+r}{a-r}\cot\left(\dfrac{\theta}{2}\right)\right], & 0 < \theta < \pi. \end{cases}$$

(b) For $\theta = 0$ and $\theta = \pi$, the solution in part (a) must be regarded in the sense of limits as $\theta \to 0^+$ and $\theta \to \pi^-$. What are $V(r,0)$ and $V(r,\pi)$?

(c) Use trigonometry to combine the description for $V(r,\theta)$ in part (a) into the single expression

$$V(r,\theta) = \frac{1}{2} + \frac{1}{\pi}\mathrm{Tan}^{-1}\left(\frac{2ar\sin\theta}{a^2 - r^2}\right).$$

(d) Solve the expression in part (c) for r in terms of V and θ, and use the result to plot equipotential curves for $V = 1/8, 1/4, 3/8, 5/8, 3/4$, and $7/8$.

30. Use the result of Exercise 29 to solve problem 6.29 when

$$f(\theta) = \begin{cases} V_2, & -\pi < \theta < 0 \\ V_1, & 0 < \theta < \pi \end{cases}.$$

31. Find expressions similar to those in Exercise 29(a) when the boundary condition is

$$f(\theta) = \begin{cases} 0, & -\pi < \theta < 0 \\ 1, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta \le \pi \end{cases}.$$

§6.4 Homogeneous Problems in Three and Four Variables (Cartesian Coordinates Only)

In this section we extend the technique of separation of variables to homogeneous problems in two and three space variables, but confine our discussions to rectangles $0 \le x \le L$, $0 \le y \le L'$ in the xy-plane and boxes $0 \le x \le L$, $0 \le y \le L'$, $0 \le z \le L''$ in space. In other words, boundaries of the region under consideration must be coordinates curves $x =$ constant and $y =$ constant in the xy-plane and coordinate surfaces $x =$ constant, $y =$ constant, and $z =$ constant in space. This is an inherent restriction on the method of separation of variables for any problem whatsoever, be it initial boundary value or boundary value; be it two- or three- dimensional; be it in Cartesian, polar, cylindrical, or spherical coordinates. Separation of variables requires a region bounded by coordinate curves or surfaces; then and only then will separation of variables lead to Sturm-Liouville systems in space variables.

First consider the homogeneous initial boundary value problem

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{6.35a}$$

$$z(0, y, t) = 0, \quad 0 < y < L', \quad t > 0, \tag{6.35b}$$

$$z(L, y, t) = 0, \quad 0 < y < L', \quad t > 0, \tag{6.35c}$$

$$z(x, 0, t) = 0, \quad 0 < x < L, \quad t > 0, \tag{6.35d}$$

$$z(x, L', t) = 0, \quad 0 < x < L, \quad t > 0, \tag{6.35e}$$

$$z(x, y, 0) = f(x, y), \quad 0 < x < L, \quad 0 < y < L', \tag{6.35f}$$

$$z_t(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L'. \tag{6.35g}$$

Physically described are the vertical oscillations of a rectangular membrane that is released from rest at time $t = 0$ with displacement described by $f(x, y)$. Its edges are fixed on the xy-plane for all time, and no external forces act on the membrane.

If a function separated in the form $z(x, y, t) = X(x)Y(y)T(t)$ is substituted into the PDE, the x-dependence can be separated from the y- and t-dependence,

$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T''}{c^2 T} = -\lambda^2 = \text{constant independent of } x,\ y, \text{ and } t.$$

When this is combined with boundary conditions 6.35b,c, a Sturm-Liouville system is obtained,

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \tag{6.36a}$$

$$X(0) = 0 = X(L). \tag{6.36b}$$

Corresponding to eigenvalues $\lambda_n^2 = n^2\pi^2/L^2$ are normalized eigenfunctions $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$.

We continue to separate the equation in $Y(y)$ and $T(t)$,

$$\frac{Y''}{Y} = \frac{T''}{c^2 T} + \lambda_n^2 = -\mu^2 = \text{constant independent of } y \text{ and } t.$$

Combine this with boundary conditions 6.35d,e, and the result is

$$Y'' + \mu^2 Y = 0, \quad 0 < y < L', \tag{6.37a}$$

$$Y(0) = 0 = Y(L'). \tag{6.37b}$$

Eigenvalues of this proper Sturm-Liouville system are $\mu_m^2 = m^2\pi^2/L'^2$, with orthonormal eigenfunctions $Y_m(y) = \sqrt{2/L'}\sin(m\pi y/L')$.

The ordinary differential equation

$$T'' + c^2(\lambda_n^2 + \mu_m^2)T = 0$$

has general solution $A\cos c\sqrt{\lambda_n^2+\mu_m^2}\,t + B\sin c\sqrt{\lambda_n^2+\mu_m^2}\,t$. But initial condition 6.35g requires $B = 0$, and therefore $T(t) = A\cos c\sqrt{\lambda_n^2+\mu_m^2}\,t$. We have determined that separated functions

$$X_n(x)Y_m(y)T(t) = A\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\cos c\sqrt{\lambda_n^2+\mu_m^2}\,t,$$

for any positive integers n and m and any constant A, satisfy PDE 6.35a, boundary conditions 6.35b,c,d,e, and initial condition 6.35g. Since these conditions are all linear and homogeneous, we superpose separated functions in an attempt to satisfy the initial displacement condition,

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\cos c\sqrt{\lambda_n^2+\mu_m^2}\,t, \qquad (6.38a)$$

where A_{mn} are constants. Condition 6.35f requires

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}, \quad 0 < x < L, \quad 0 < y < L'.$$

If we multiply this equation by $\sqrt{2/L}\sin(k\pi x/L)$, integrate with respect to x from $x = 0$ to $x = L$, and interchange orders of summation and integration on the right, orthogonality of the eigenfunctions $\sqrt{2/L}\sin(n\pi x/L)$ leads to

$$\int_0^L f(x,y)\sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}dx = \sum_{m=1}^{\infty} A_{mk}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}, \quad 0 < y < L'.$$

Multiplication by $\sqrt{2/L'}\sin(j\pi y/L')$ and integration with respect to y from $y = 0$ to $y = L'$ gives, similarly,

$$\int_0^{L'}\left[\int_0^L f(x,y)\sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}dx\right]\sqrt{\frac{2}{L'}}\sin\frac{j\pi y}{L'}dy = A_{jk}.$$

Thus, coefficients A_{mn} in solution 6.38a are given by

$$A_{mn} = \int_0^{L'}\int_0^L f(x,y)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}dx\,dy, \qquad (6.38b)$$

and the formal solution is complete.

As a special case, suppose $f(x,y) = xy(L-x)(L'-y)$ so that cross sections of the initial displacement parallel to the xz- and yz-planes are parabolic. Integration by parts yields

$$A_{mn} = \frac{8(LL')^{5/2}[1+(-1)^{n+1}][1+(-1)^{m+1}]}{n^3m^3\pi^6},$$

and hence,

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{8(LL')^{5/2}[1+(-1)^{n+1}][1+(-1)^{m+1}]}{n^3m^3\pi^6}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'} *$$
$$\cos c\sqrt{\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{L'^2}}t.$$

Since terms are nonzero only when both m and n are odd integers, we may write

$$z(x,y,t) = \frac{64(LL')^2}{\pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3(2m-1)^3}\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2m-1)\pi y}{L'} *$$
$$\cos\pi c\sqrt{\frac{(2n-1)^2}{L^2}+\frac{(2m-1)^2}{L'^2}}t. \tag{6.39}$$

Terms in this series are called the **normal modes of vibration for the membrane** (similar to the normal modes of a vibrating string in Section 6.2). The first term corresponding to $n=1$ and $m=1$,

$$H_{1,1}(x,y,t) = \frac{64(LL')^2}{\pi^6}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L'}\cos\pi c\sqrt{\frac{1}{L^2}+\frac{1}{L'^2}}t,$$

is called the **fundamental mode of vibration**. It represents displacements of a membrane identical to that in problem 6.35, except that the initial displacement is described by $[64(LL')^2/\pi^6]\sin(\pi x/L)\sin(\pi y/L')$. For such an initial displacement, the membrane oscillates back and forth between the enveloping surfaces $\pm[64(LL')^2/\pi^6]\sin(\pi x/L)\sin(\pi y/L')$; the shape of the membrane is always the same, the cosine factor describes the time dependence of the oscillations.

The $n=1$ and $m=2$ term in series 6.39 is

$$H_{2,1}(x,y,t) = \frac{64(LL')^2}{27\pi^6}\sin\frac{\pi x}{L}\sin\frac{3\pi y}{L'}\cos\pi c\sqrt{\frac{1}{L^2}+\frac{9}{L'^2}}t.$$

It represents vibrations of the same membrane but with an initial displacement given by $[64(LL')^2/(27\pi^6)]\sin(\pi x/L)\sin(3\pi y/L')$. The membrane oscillates back and forth between this surface and its negative. The lines $y=L'/3$ and $y=2L'/3$, which always remain motionless, are called **nodal curves** for this mode of vibration.

The mode

$$H_{1,2}(x,y,t) = \frac{64(LL')^2}{27\pi^6}\sin\frac{3\pi x}{L}\sin\frac{\pi y}{L'}\cos\pi c\sqrt{\frac{9}{L^2}+\frac{1}{L'^2}}t$$

is similar with nodal curves $x=L/3$ and $x=2L//3$.

Solution 6.39 is the sum of an infinity of modes of vibration, the modes of lower orders contributing more significantly than higher-order ones.

We now consider a three-dimensional boundary value problem,

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0, \quad 0<x<L, \quad 0<y<L', \quad 0<z<L'', \tag{6.40a}$$
$$\frac{\partial U(0,y,z)}{\partial x}=0, \quad 0<y<L', \quad 0<z<L'', \tag{6.40b}$$
$$U(L,y,z)=0, \quad 0<y<L', \quad 0<z<L'', \tag{6.40c}$$

$$U(x,0,z) = 0, \quad 0 < x < L, \quad 0 < z < L'', \tag{6.40d}$$

$$\frac{\partial U(x,L',z)}{\partial y} = 0, \quad 0 < x < L, \quad 0 < z < L'', \tag{6.40e}$$

$$U(x,y,0) = f(x,y), \quad 0 < x < L, \quad 0 < y < L', \tag{6.40f}$$

$$U(x,y,L'') = g(x,y), \quad 0 < x < L, \quad 0 < y < L'. \tag{6.40g}$$

The problem describes steady-state temperature $U(x,y,z)$ in the box of Figure 6.9, where two faces ($x = 0$ and $y = L'$) are insulated, two faces ($x = L$ and $y = 0$) are held at temperature zero, and the remaining faces have prescribed nonzero temperatures $f(x,y)$ and $g(x,y)$. The problem is said to be homogeneous because the PDE is homogeneous, and all boundary conditions are homogeneous except those on a single pair of opposite faces.

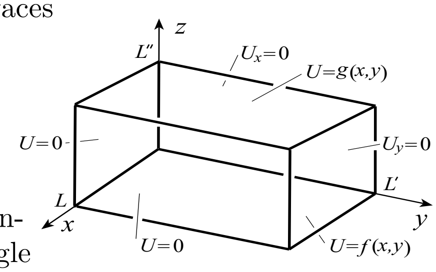


Figure 6.9

When a function with variables separated, $U(x,y,z) = X(x)Y(y)Z(z)$ is substituted into the PDE, separation gives

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda^2 = \text{constant independent of } x, y, \text{ and } z.$$

Combined with boundary conditions 6.40b,c, this yields

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X'(0) = 0 = X(L).$$

Eigenvalues of this Sturm-Liouville are $\lambda_n^2 = (2n-1)^2\pi^2/(4L^2)$, with normalized eigenfunctions $X_n(x) = \sqrt{2/L}\cos\lambda_n x$ (see Table 5.1).

Further separation in $Y(y)$ and $Z(z)$ leads to

$$\frac{Y''}{Y} = -\frac{Z''}{Z} + \lambda_n^2 = -\mu^2 = \text{constant independent of } y \text{ and } z.$$

This equation, along with boundary conditions 6.40d,e gives

$$Y'' + \mu^2 Y = 0, \quad 0 < y < L',$$
$$Y(0) = 0 = Y'(L').$$

Eigenpairs of this Sturm-Liouville system are $\mu_m^2 = (2m-1)^2\pi^2/(4L'^2)$ and $Y_m(y) = \sqrt{2/L'}\sin\mu_m y$.

Finally, the ordinary differential equation

$$Z'' - (\lambda_n^2 + \mu_m^2)Z = 0$$

has general solution $Z(z) = A\cosh\sqrt{\lambda_n^2+\mu_m^2}z + B\sinh\sqrt{\lambda_n^2+\mu_m^2}z$. We have now determined that separated functions

$$X_n(x)Y_m(y)Z(z) = X_n(x)Y_m(y)[A\cosh\sqrt{\lambda_n^2+\mu_m^2}z + B\sinh\sqrt{\lambda_n^2+\mu_m^2}z]$$

for positive integers n and m and arbitrary constants A and B satisfy PDE 6.40a and boundary conditions 6.40b–e. To accommodate the remaining two boundary conditions we superpose separated functions and take

$$U(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X_n(x)Y_m(y)[A_{mn}\cosh\sqrt{\lambda_n^2+\mu_m^2}z+B_{mn}\sinh\sqrt{\lambda_n^2+\mu_m^2}z],\tag{6.41a}$$

in which case conditions 6.40f,g require

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}X_n(x)Y_m(y),\quad 0<x<L,\quad 0<y<L',$$

and

$$g(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X_n(x)Y_m(y)[A_{mn}\cosh\sqrt{\lambda_n^2+\mu_m^2}L''+B_{mn}\sinh\sqrt{\lambda_n^2+\mu_m^2}L''],$$
$$0<x<L,\quad 0<y<L'.$$

Successive multiplications of these equations by eigenfunctions in x and y and integrations with respect to x and y lead to the following expressions for A_{mn} and B_{mn},

$$A_{mn}=\int_0^{L'}\int_0^{L}f(x,y)X_n(x)Y_m(y)dx\,dy,\tag{6.41b}$$

and

$$B_{mn}=\frac{1}{\sinh\sqrt{{\lambda_n}^2+{\mu_m}^2}L''}\left[\int_0^{L'}\int_0^{L}g(x,y)X_n(x)Y_m(y)\,dx\,dy - A_{mn}\cosh\sqrt{\lambda_n^2+\mu_m^2}L''\right].\tag{6.41c}$$

In Section '`The Multidimensional Eigenvalue Problem`', solutions like 6.38 and 6.41 are approached from a different point of view.

EXERCISES 6.4

Part A Heat Conduction

1. A thin rectangle occupying the region $0\le x\le L$, $0\le y\le L'$ has its top and bottom faces insulated. At time $t=0$, its temperature is described by the function $f(x,y)$. Find its temperature for $t>0$, if all four edges $x=0$, $y=0$, $x=L$, and $y=L'$ are maintained at 0°C.

2. Repeat Exercise 1 if edges $x=0$ and $y=L'$ are insulated.

3. (a) Repeat Exercise 1 if edges $y=0$ and $y=L'$ are insulated.
(b) Simplify the solution if the initial temperature is a function only of x.

4. Repeat Exercise 1 if heat is transferred to an environment at temperature 0°C along the edge $x=L$ (according to Newton's law of cooling).

5. A block of metal occupies the region $0\le x\le L$, $0\le y\le L'$, $0\le z\le L''$. The surfaces $y=0$, $y=L'$, and $z=L''$ are insulated, and faces $x=0$, $x=L$, and $z=0$ are held at temperature 0°C. If the temperature of the block is initially a constant U_0 throughout, find the temperature in the block thereafter.

6. Repeat Exercise 5 if the face $z = 0$ is insulated, and heat is transferred to the surrounding medium, at temperature zero, according to Newton's law of cooling on the face $z = L''$.

7. Repeat Exercise 5 if face $z = 0$ is insulated.

8. Repeat Exercise 5 if face $y = L'$ is held at temperature 0°C.

9. In this exercise we prove a result for homogeneous heat conduction problems in two or three space variables that uses solutions of one-dimensional problems provided the initial temperature distribution is the product of one-dimensional functions. In particular, show that the solution of the two-dimensional problem

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,\\
-l_1\frac{\partial U}{\partial x} + h_1 U &= 0, \quad x = 0, \quad 0 < y < L', \quad t > 0,\\
l_2\frac{\partial U}{\partial x} + h_2 U &= 0, \quad x = L, \quad 0 < y < L', \quad t > 0,\\
-l_3\frac{\partial U}{\partial y} + h_3 U &= 0, \quad y = 0, \quad 0 < x < L, \quad t > 0,\\
l_4\frac{\partial U}{\partial y} + h_4 U &= 0, \quad y = L', \quad 0 < x < L, \quad t > 0,\\
U(x, y, 0) &= f(x)g(y), \quad 0 < x < L, \quad 0 < y < L',
\end{aligned}$$

is the product of the solutions of the one-dimensional problems

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\
-l_1\frac{\partial U}{\partial x} + h_1 U &= 0, \quad x = 0, \quad t > 0,\\
l_2\frac{\partial U}{\partial x} + h_2 U &= 0, \quad x = L, \quad t > 0,\\
U(x, 0) &= f(x), \quad 0 < x < L;
\end{aligned}$$

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial y^2}, \quad 0 < y < L', \quad t > 0,\\
-l_3\frac{\partial U}{\partial y} + h_3 U &= 0, \quad y = 0, \quad t > 0,\\
l_4\frac{\partial U}{\partial y} + h_4 U &= 0, \quad y = L', \quad t > 0,\\
U(y, 0) &= g(y), \quad 0 < y < L'.
\end{aligned}$$

This result is easily extended to heat conduction problems in x, y, z, and t. In addition, it can sometimes be generalized to other coordinate systems (see Exercise 19 in Section 9.1).

10. (a) Use the result of Exercise 9, together with those of Exercise 1 in Section 6.2 and Example 4.2 in Section 4.2 to solve the following heat conduction problem:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,\\
U_x(0, y, t) &= 0, \quad 0 < y < L', \quad t > 0,\\
U_x(L, y, t) &= 0, \quad 0 < y < L', \quad t > 0,\\
U_y(x, 0, t) &= 0, \quad 0 < x < L, \quad t > 0,\\
U(x, L', t) &= 0, \quad 0 < x < L, \quad t > 0,\\
U(x, y, 0) &= x(L' - y), \quad 0 < x < L, \quad 0 < y < L',
\end{aligned}$$

(b) Solve the problem in part (a) by separation of variables. Are the solutions identical?

Part B Vibrations

11. (a) A membrane is stretched tightly over the square $0 \leq x, y \leq L$. If all four edges are clamped on the xy-plane and the membrane is released from rest at an initial displacement $f(x, y)$, find its subsequent displacements.

(b) Simplify the solution when

$$f(x, y) = \frac{(L - 2|x - L/2|)(L - 2|y - L/2|)}{32L}.$$

12. (a) A membrane is stretched tightly over the rectangle $0 \leq x \leq L$, $0 \leq y \leq L'$. Edges $x = 0$ and $x = L$ are clamped on the xy-plane, but $y = 0$ and $y = L'$ are free to move vertically. If the membrane is released from rest at time $t = 0$ from a position described by $f(x, y)$, determine subsequent displacements of the membrane.

(b) Simplify the solution when $f(x, y) = (L - 2|x - L/2|)/(32L)$.

13. Equations 6.38 describe displacements of a rectangular membrane with edges fixed on the xy-plane when oscillations are initiated by releasing the membrane from rest at a prescribed displacement. Find nodal curves for the mode $2A_{mn}/\sqrt{LL'} \sin(n\pi x/L) \sin(m\pi y/L') \cos c\pi\sqrt{n^2/L^2 + m^2/L'^2}t$.

14. Is there a result analogous to that in Exercise 9 for the vibration problem of displacements in a membrane?

Part C Potential, Steady-state Heat Conduction

15. Find the potential inside the rectangular parallelopiped $0 \leq x \leq L$, $0 \leq y \leq L'$, $0 \leq z \leq L''$ if faces $x = 0$, $y = 0$, $x = L$, and $y = L'$ are all held at potential zero while faces $z = 0$ and $z = L''$ are maintained at potentials $f(x, y)$ and $g(x, y)$, respectively.

16. Repeat Exercise 15 if faces $x = 0$ and $x = L$ are held at potentials $h(y, z)$ and $k(y, z)$, the other four faces remaining unchanged.

17. Find the steady-state temperature distribution inside a cube $0 \leq x, y, z \leq L$ if faces $x = 0$ and $z = L$ are insulated, faces $y = 0$ and $y = L$ are held at temperature zero, and heat is added to faces $x = L$ and $z = 0$ at constant rates q and Q W/m^2, respectively.

§6.5 The Multi-dimensional Eigenvalue Problem

In Section 6.4 we demonstrated that successively separating off Cartesian variables in homogeneous problems leads to the Sturm-Liouville systems of Section 5.2. When the problem is an initial boundary value one, as opposed to a boundary value problem, there remains an ODE for the time dependence of the unknown function. An alternative procedure is first to separate off the time dependence, leaving what is called the multi-dimensional eigenvalue problem. To illustrate, suppose that the unknown function V in the homogeneous PDE

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial T} + sV \tag{6.42}$$

is separated into a spatial part, which we designate by W, and a time-dependent part, $T(t)$; that is, we set $V = WT(t)$. (We have purposely not expressed W as a function of coordinates because what we are about to do is independent of the particular choice of coordinate system.) When this product representation for V is substituted into the above PDE, the time dependence contained in T may be separated from the spatial dependence of W:

$$\frac{\nabla^2 W}{W} = \frac{pT'' + qT' + sT}{T} = -\lambda^2 = \text{constant independent of all variables.}$$

It follows that $T(t)$ must satisfy the ODE

$$pT'' + qT' + (s + \lambda^2)T = 0$$

and W must satisfy the Helmholtz equation

$$\nabla^2 W + \lambda^2 W = 0.$$

When PDE 6.42 is accompanied by homogeneous boundary conditions on V, these become homogeneous boundary conditions for W,

$$\nabla^2 W + \lambda^2 W = 0, \tag{6.43a}$$

$$\text{Homogeneous boundary conditions.} \tag{6.43b}$$

This is called a **multi-dimensional eigenvalue problem**. For certain eigenvalues λ^2, there exist nontrivial solutions of problem 6.43 called eigenfunctions. Properties of eigenvalues and eigenfunctions of this eigenvalue problem parallel those of Sturm-Liouville systems in Chapter 5, but important differences do exist. We consider one example here and give general discussions and further examples in the exercises.

When boundary conditions 6.43b are of Dirichlet type on the edges of a rectangle $0 \leq x \leq L$, $0 \leq y \leq L'$, problem 6.43 takes the form

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \lambda^2 W = 0, \quad 0 < x < L, \quad 0 < y < L', \tag{6.44a}$$

$$W(0, y) = 0, \quad 0 < y < L', \tag{6.44b}$$

$$W(L, y) = 0, \quad 0 < y < L', \tag{6.44c}$$

$$W(x, 0) = 0, \quad 0 < x < L, \tag{6.44d}$$

$$W(x, L') = 0, \quad 0 < x < L. \tag{6.44e}$$

To solve this problem, we separate $W(x,y) = X(x)Y(y)$. This results in the Sturm-Liouville systems

$$X'' + \mu^2 X = 0, \quad 0 < x < L, \qquad Y'' + (\lambda^2 - \mu^2)Y = 0, \quad 0 < y < L',$$
$$X(0) = 0 = X(L); \qquad Y(0) = 0 = Y(L'),$$

solutions of which are

$$X_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \quad \text{corresponding to eigenvalues } \mu_n^2 = n^2\pi^2/L^2,$$

and

$$Y_m(y) = \sqrt{\frac{2}{L'}} \sin \frac{m\pi y}{L'} \quad \text{corresponding to eigenvalues } \lambda^2 - \mu_n^2 = m^2\pi^2/L'^2.$$

In other words, eigenvalues of problem 6.44 are $\lambda_{mn}^2 = n^2\pi^2/L^2 + m^2\pi^2/L'^2$, with corresponding eigenfunctions

$$W_{mn}(x,y) = \frac{2}{\sqrt{LL'}} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{L'}. \tag{6.45}$$

It is straightforward to show that these functions are orthonormal on the rectangle with respect to the weight function $p(x,y) = 1$; that is,

$$\int_0^L \int_0^{L'} W_{mn}(x,y) W_{lk}(x,y)\, dy dx = \begin{cases} 1, & \text{if } n = k \text{ and } m = l \\ 0, & \text{otherwise.} \end{cases} \tag{6.46}$$

Furthermore, suppose we are given a function $f(x,y)$ that is, along with its first partial derivatives, piecewise continuous on the rectangle $0 \le x \le L$, $0 \le y \le L'$. For fixed y, $f(x,y)$ and $\partial f(x,y)/\partial x$ are piecewise continuous functions of x, and we may therefore express $f(x,y)$ in terms of $X_n(x)$; that is, the eigenfunction expansion of $f(x,y)$ as a function of x is

$$\frac{f(x+,y) + f(x-,y)}{2} = \sum_{n=1}^{\infty} d_n(y) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \tag{6.47a}$$

where the functions $d_n(y)$ are defined by

$$d_n(y) = \int_0^L f(x,y) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx. \tag{6.47b}$$

Equations 6.47 are valid provided $f(x,y)$ is continuous in y at the chosen value of y. When this is not the case, these equations must be replaced by appropriate limiting expressions. Because $d_n(y)$ is itself piecewise continuous, with a piecewise continuous first derivative, it may be expanded in terms of $Y_m(y)$,

$$\frac{d_n(y+) + d_n(y-)}{2} = \sum_{m=1}^{\infty} c_{mn} \sqrt{\frac{2}{L'}} \sin \frac{m\pi y}{L'} \tag{6.48a}$$

where

$$c_{mn} = \int_0^{L'} d_n(y) \sqrt{\frac{2}{L'}} \sin \frac{m\pi y}{L'} dy. \tag{6.48b}$$

We combine these expressions to write

$$\begin{aligned}\left[\frac{f(x+,y+)+f(x-,y+)}{2}\right] &+ \left[\frac{f(x+,y-)+f(x-,y-)}{2}\right]\\ &= \sum_{n=1}^{\infty} d_n(y+)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} d_n(y-)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\\ &= \sum_{n=1}^{\infty}[d_n(y+)+d_n(y-)]\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\\ &= \sum_{n=1}^{\infty}\left(2\sum_{m=1}^{\infty} c_{mn}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\right)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\\ &= 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\frac{2}{\sqrt{LL'}}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}.\end{aligned}$$

In other words, the function $f(x,y)$ has been expressed in terms of the orthonormal eigenfunctions of eigenvalue problem 6.44,

$$\begin{aligned}&\frac{f(x+,y+)+f(x-,y+)+f(x+,y-)+f(x-,y-)}{4}\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}W_{mn}(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\frac{2}{\sqrt{LL'}}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'},\end{aligned} \tag{6.49a}$$

where

$$\begin{aligned}C_{mn} &= \int_0^L\int_0^{L'} f(x,y)W_{mn}(x,y)\,dy\,dx\\ &= \int_0^L\int_0^{L'} f(x,y)\frac{2}{\sqrt{LL'}}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}dy\,dx,\end{aligned} \tag{6.49b}$$

and this result is valid for $0 < x < L$, $0 < y < L'$.

We have illustrated with this example that for the multi-dimensional eigenvalue problem, we should expect multi-subscripted eigenvalues, orthogonal eigenfunctions, and multi-dimensional eigenfunction expansions. This is illustrated further in the exercises.

When solving homogeneous initial boundary value problems by separation of variables, there is always the choice of separating off the time dependence first or last. The solution will ultimately be the same for either approach, but the steps differ in arriving at this solution. Let us illustrate with the heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{6.50a}$$

$$U(0,y,t) = 0, \quad 0 < y < L', \quad t > 0, \tag{6.50b}$$

$$U(L,y,t) = 0, \quad 0 < y < L', \quad t > 0, \tag{6.50c}$$

$$U(x,0,t) = 0, \quad 0 < x < L, \quad t > 0, \tag{6.50d}$$

$$U(x,L',t) = 0, \quad 0 < x < L, \quad t > 0, \tag{6.50e}$$

$$U(x,y,0) = f(x,y), \quad 0 < x < L, \quad 0 < y < L'. \tag{6.50f}$$

If the x- and y-dependences of a separated function $U(x,y,t)=X(x)Y(y)T(t)$ are separated off first (as was done in Section 6.4), Sturm-Liouville systems in $X(x)$ and $Y(y)$ are obtained,

$$X''+\lambda^2X=0,\quad 0<x<L,\qquad\qquad Y''+\mu^2Y=0,\quad 0<y<L',$$
$$X(0)=0=X(L);\qquad\qquad Y(0)=0=Y(L').$$

Eigenpairs of these systems are

$$\lambda_n^2=\frac{n^2\pi^2}{L^2},\quad X_n(x)=\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L};\qquad \mu_m^2=\frac{m^2\pi^2}{L'^2},\quad Y_m(y)=\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}.$$

What remains is an ODE in $T(t)$, namely,

$$T'+k(\lambda_n^2+\mu_m^2)T=0,\quad t>0,$$

with general solution $T(t)=Ae^{-k(\lambda_n^2+\mu_m^2)t}$. To satisfy the initial condition, separated functions are superposed in the form

$$U(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k(\lambda_n^2+\mu_m^2)t}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'},\tag{6.51a}$$

and the initial temperature $f(x,y)$ at $t=0$ then requires

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sqrt{\frac{2}{L}}\sin\frac{m\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'},\quad 0<x<L,\quad 0<y<L'.$$

To find the A_{mn}, we multiply both sides of this equation by $\sqrt{2/L}\sin(k\pi x/L)$ and integrate with respect to x, and then we multiply by $\sqrt{2/L'}\sin(j\pi y/L')$ and integrate with respect to y. Orthogonality gives

$$A_{mn}=\int_0^{L'}\int_0^{L}f(x,y)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}dx\,dy.\tag{6.51b}$$

Alternatively, we can separate time off first by setting $U(x,y,t)=W(x,y)T(t)$. The ODE

$$T'+k\lambda^2T=0$$

is obtained along with eigenvalue problem 6.44. Eigenpairs are $\lambda_{mn}^2=n^2\pi^2/L^2+m^2\pi^2/L'^2$ and $W_{mn}(x,y)=(2/\sqrt{LL'})\sin(n\pi x/L)\sin(m\pi y/L')$. The solution for $T(t)$ is $T(t)=Ae^{-k\lambda_{mn}^2t}$. Superposition of separated functions gives

$$U(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k\lambda_{mn}^2t}W_{mn}(x,y),\tag{6.52a}$$

and the initial condition 6.50f requires

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}W_{mn}(x,y),\quad 0<x<L,\quad 0<y<L'.$$

But then, A_{mn} are the Fourier coefficients in the eigenfunction expansion of $f(x,y)$ in terms of the $W_{mn}(x,y)$,

$$A_{mn} = \int_0^{L'} \int_0^L f(x,y) W_{mn}(x,y)\, dx dy. \tag{6.52b}$$

Solutions 6.51 and 6.52 are identical; it is only the way in which we regard the initial conditions that differs in our arriving at the solution.

EXERCISES 6.5

In Exercises 1–3 we prove some general results concerning eigenvalue problem 6.43 in the xy-plane. Results in three space variables are analogous.

1. Prove the following result corresponding to Theorem 5.1 in Chapter 5. All eigenvalues of the multi-dimensional eigenvalue problem

$$\nabla^2 W + \lambda^2 W = 0, \quad (x,y) \text{ in } A,$$
$$l\frac{\partial W}{\partial n} + hW = 0, \quad (x,y) \text{ on } \beta(A), \quad h > 0, \quad l > 0,$$

are real, and eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the unit weight function.

2. Use eigenvalue problem 6.44 (with $L' = 2L$) to illustrate that a multi-dimensional eigenvalue problem can have linearly independent eigenfunctions corresponding to the same eigenvalue. (Contrast this with Exercise 14 in Section 5.1 for Sturm-Liouville systems.)

3. Show that all eigenvalues of the multi-dimensional eigenvalue problem in Exercise 1 are non-negative and that $\lambda = 0$ is an eigenvalue only when the boundary condition is Neumann. In this case, what is the eigenfunction corresponding to $\lambda = 0$?

In Exercises 4–8 find eigenvalues and orthonormal eigenfunctions of eigenvalue problem 6.43 on the rectangle A: $0 \le x \le L$, $0 \le y \le L'$ for the given boundary conditions.

4. $W(0,y) = 0,\ 0 < y < L'$; $\quad W_x(L,y) = 0,\ 0 < y < L'$; $\quad W(x,0) = 0,\ 0 < x < L$; $W(x,L') = 0,\ 0 < x < L$

5. $W(0,y) = 0,\ 0 < y < L'$; $\quad W(L,y) = 0,\ 0 < y < L'$; $\quad W_y(x,0) = 0,\ 0 < x < L$; $W_y(x,L') = 0,\ 0 < x < L$

6. $W_x(0,y) = 0,\ 0 < y < L'$; $\quad W(L,y) = 0,\ 0 < y < L'$; $\quad W(x,0) = 0,\ 0 < x < L$; $W_y(x,L') = 0,\ 0 < x < L$

7. $W(0,y) = 0,\ 0 < y < L'$; $\quad W(L,y) = 0,\ 0 < y < L'$; $\quad W_y(x,0) = 0,\ 0 < x < L$; $lW_y(x,L') + hW(x,L') = 0,\ 0 < x < L$

8. $-l_1 W_x(0,y) + h_1 W(0,y) = 0,\ 0 < y < L'$; $\quad l_2 W_x(L,y) + h_2 W(L,y) = 0,\ 0 < y < L'$; $-l_3 W_y(x,0) + h_3 W(x,0) = 0,\ 0 < x < L$; $\quad l_4 W_y(x,L') + h_4 W(x,L') = 0,\ 0 < x < L$

In Exercises 9–11 use the multi-dimensional eigenvalue problem approach to solve the initial boundary value problem.

9. Exercise 11(a) in Section 6.4

10. Exercise 12(a) in Section 6.4

11. Exercise 5 in Section 6.4

12. (a) Show that the Rayleigh quotient for an eigenvalue λ_{mn} of the multi-dimensional eigenvalue problem of Exercise 1 can be expressed in terms of its corresponding normalized eigenfunction $W_{mn}(x,y)$ according to

$$\lambda_{mn}^2 = \iint_A |\nabla W_{mn}|^2 \, dA - \oint_{\beta(A)} W_{mn} \frac{\partial W_{mn}}{\partial n} \, ds.$$

(b) What form does the Rayleigh quotient take when the boundary condition is Dirichlet or Neumann?

§6.6 Properties of Parabolic Partial Differential Equations

We now return to a difficulty posed in Chapter 4. In what sense are the series obtained in Chapters 4 and 6 "solutions" of their respective problems? In arriving at each series solution, we superposed an infinity of functions satisfying a linear, homogeneous PDE and linear, homogeneous boundary and/or initial conditions. Because of the questionable validity of this step (superposition principle 1 in Section 4.1 endorses only finite linear combinations), we have called each series a formal solution. It is now incumbent on us to verify that each formal solution is indeed a valid solution of its (initial) boundary value problem. Unfortunately, it is not possible to prove general results that encompass all problems solved by means of separation of variables and generalized Fourier series; on the other hand, the situation is not so bad that every problems is its own special case. Techniques exist that verify formal solutions for large classes of problems. In this section and Sections 6.7 and 6.8, we illustrate techniques that work when separation of variables leads to the Sturm-Liouville systems in Table 5.1. At the same time, we take the opportunity to develop properties of solutions of parabolic, hyperbolic and elliptic PDEs. Time-dependent heat conduction problems are manifested in parabolic equations; vibrations invariably involve hyperbolic equations; and potential problems give rise to elliptic equations.

We choose to illustrate the situation for parabolic PDEs with the heat conduction problem in equation 6.2 of Section 6.2,

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{6.53a}$$

$$U_x(0,t) = 0, \quad t > 0, \tag{6.53b}$$

$$\kappa\frac{\partial U(L,t)}{\partial x} + \mu U(L,t) = 0, \quad t > 0, \tag{6.53c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{6.53d}$$

(See Exercise 1 for verification when both boundary conditions are Robin.) The formal solution of problem 6.53 is

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(x) \qquad \text{where} \quad c_n = \int_0^L f(x)X_n(x)\,dx. \tag{6.54}$$

Eigenfunctions are $X_n(x) = N^{-1}\cos\lambda_n x$, where $2N^2 = L + (\mu/\kappa)[\lambda_n^2 + (\mu/\kappa)^2]$, and eigenvalues are defined by the equation $\tan\lambda L = \mu/(\kappa\lambda)$.

We shall show by direct substitution that the function $U(x,t)$ defined by series 6.54 does indeed satisfy problem 6.53.

When coefficients c_n are calculated according to the formula in equation 6.54, the series $\sum_{n=1}^{\infty} c_n X_n(x)$ converges to $f(x)$ for $0 < x < L$ (provided $f(x)$ is piecewise smooth for $0 \le x \le L$). Since this series is $U(x,0)$, it follows that initial condition 6.53d is satisfied if $f(x)$ is piecewise smooth on $0 \le x \le L$, provided that at any point of discontinuity of $f(x)$, $f(x)$ is defined by $f(x) = [f(x+) + f(x-)]/2$.

To verify 6.53a–c, is not quite so simple. We first show that series 6.54 converges for all $0 \le x \le L$ and $t > 0$ and can be differentiated with respect to either x or t. Because eigenfunctions $X_n(x)$ are uniformly bounded (see Theorem 5.2 in Section 5.2), there exists a constant M such that for all $n \ge 1$ and $0 \le x \le L$,

$|X_n(x)| \le N^{-1} \le M$. Further, since $f(x)$ is piecewise continuous on $0 \le x \le L$, it is also bounded thereon; that is, $|f(x)| \le K$, for some constant K. These two results imply that the coefficients c_n defined by 6.54 are bounded by

$$|c_n| \le \int_0^L |f(x)||X_n(x)|\,dx \le KML. \tag{6.55}$$

It follows that for any x in $0 \le x \le L$, and any time $t \ge t_0 > 0$,

$$\sum_{n=1}^{\infty} |c_n e^{-k\lambda_n^2 t} X_n(x)| \le KM^2 L \sum_{n=1}^{\infty} (e^{-kt_0})^{\lambda_n^2}.$$

Figure 5.3 indicates that the n^{th} eigenvalue $\lambda_n \ge (n-1)\pi/L$. Combine this with the fact that $e^{-kt_0} < 1$, and we may write, for $0 \le x \le L$ and $t \ge t_0 > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} |c_n e^{-k\lambda_n^2 t} X_n(x)| &\le KM^2 L \sum_{n=1}^{\infty} (e^{-kt_0})^{(n-1)^2\pi^2/L^2} \\ &\le KM^2 L \sum_{n=1}^{\infty} [(e^{-kt_0})^{\pi^2/L^2}]^{n-1} = KM^2 L \sum_{n=1}^{\infty} r^{n-1},\end{aligned} \tag{6.56}$$

and the geometric series on the right converges, since $r = e^{-kt_0\pi^2/L^2} < 1$. According to the Weierstrass M-test (Theorem 3.3 in Section 3.4), series 6.54 converges absolutely and uniformly with respect to x and t for $0 \le x \le L$ and $t \ge t_0 > 0$. Because $t_0 > 0$ is arbitrary, it also follows that series 6.54 converges absolutely for $0 \le x \le L$ and $t > 0$.

Term-by-term differentiation of series 6.54 with respect to t gives

$$\sum_{n=1}^{\infty} -k\lambda_n^2 c_n e^{-k\lambda_n^2 t} X_n(x). \tag{6.57}$$

Since $\lambda_n \le n\pi/L$ (see, once again, Figure 5.3), it follows that for all $0 \le x \le L$ and $t \ge t_0 > 0$,

$$\sum_{n=1}^{\infty} |-k\lambda_n^2 c_n e^{-k\lambda_n^2 t} X_n(x)| \le \frac{kKM^2\pi^2}{L} \sum_{n=1}^{\infty} n^2 r^{n-1}. \tag{6.58}$$

Because the series $\sum_{n=1}^{\infty} n^2 r^{n-1}$ converges, we conclude that series 6.57 converges absolutely and uniformly with respect to x and t for $0 \le x \le L$ and $t \ge t_0 > 0$. As a result, series 6.57 represents $\partial U/\partial t$ for $0 \le x \le L$ and $t \ge t_0 > 0$. (Theorem 3.7 in Section 3.4). But, once again, the fact that t_0 is arbitrary implies that we may write

$$\frac{\partial U}{\partial t} = \sum_{n=1}^{\infty} -k\lambda_n^2 c_n e^{-k\lambda_n^2 t} X_n(x) \tag{6.59}$$

for $0 \le x \le L$ and $t > 0$.

Term-by-term differentiation of series 6.54 with respect to x gives

$$\sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n'(x) = \sum_{n=1}^{\infty} c_n(-\lambda_n) e^{-k\lambda_n^2 t} N^{-1} \sin \lambda_n x. \tag{6.60}$$

Since $N^{-1} \leq M$, we have, for $0 \leq x \leq L$ and $t \geq t_0 > 0$,

$$\sum_{n=1}^{\infty} |c_n e^{-k\lambda_n^2 t} X_n'(x)| \leq \sum_{n=1}^{\infty} (KML)(\lambda_n M) e^{-k\lambda_n^2 t_0}$$

$$\leq KM^2 L \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) r^{n-1} = KM^2 \pi \sum_{n=1}^{\infty} n r^{n-1}. \quad (6.61)$$

Because the series $\sum_{n=1}^{\infty} nr^{n-1}$ converges, series 6.60 likewise converges absolutely and uniformly. Consequently, series 6.54 may be differentiated term-by-term to yield, for $0 \leq x \leq L$ and $t > 0$,

$$\frac{\partial U}{\partial x} = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n'(x). \quad (6.62)$$

A similar analysis shows that for $0 \leq x \leq L$ and $t > 0$,

$$\frac{\partial^2 U}{\partial x^2} = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n''(x) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t}[-\lambda_n{}^2 X_n(x)]. \quad (6.63)$$

Expressions 6.59 and 6.63 for $\partial U/\partial t$ and $\partial^2 U/\partial x^2$ clearly indicate that $U(x,t)$ satisfies PDE 6.53a. Finally, expressions 6.62 and 6.54 for $\partial U/\partial x$ and $U(x,t)$ indicate that

$$\frac{\partial U(0,t)}{\partial x} = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n'(0) = 0,$$

(since $X_n'(0) = 0$), and

$$\kappa \frac{\partial U(L,t)}{\partial x} + \mu U(L,t) = \kappa \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n'(L) + \mu \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(L)$$

$$= \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t}[\kappa X_n'(L) + \mu X_n(L)] = 0,$$

(since $X_n(x)$ satisfies $\kappa X_n'(L) + \mu X_n(L) = 0$).

We have now verified that the formal solution $U(x,t)$ defined by series 6.54 satisfies equations 6.53a–d. Clearly demonstrated was the dependence of our verification on properties of the Sturm-Liouville system associated with 6.53. Indeed, indispensable were the facts that eigenvalues satisfied the inequalities $(n-1)\pi/L \leq \lambda_n \leq n\pi/L$ and that eigenfunctions were uniformly bounded. Without a knowledge of these properties, verification of the formal solution would have been impossible. Although series 6.54 satisfies problem 6.53, verification of 6.54 as the solution of the heat conduction problem described by 6.53 is not complete. To illustrate why, consider the function defined by

$$U(x,t) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} b_n e^{-k\lambda_n^2 t} X_n(x), & 0 \leq x \leq L, \quad t > 0 \\ f(x), & 0 \leq x \leq L, \quad t = 0 \end{cases}, \quad (6.64)$$

where $\{b_n\}$ is a completely arbitrary, but bounded, sequence and $X_n(x)$ are the eigenfunctions in 6.54. The above procedure can once again be used to verify that function 6.64 also satisfies 6.53a–c; in addition, it satisfies 6.53d. This means that, as stated, problem 6.53 is not well posed; it does not have a unique solution. It cannot therefore be an adequate description of the physical problem following equation 6.2 in Section 6.2 — temperature in a rod of uniform cross section and insulated sides that at time $t = 0$ has temperature $f(x)$. For time $t > 0$, the end $x = 0$ is also insulated and heat is exchanged at the other end with an environment at temperature zero. In actual fact, 6.53 does have a unique solution, provided we demand that the solution satisfy certain continuity conditions. Our immediate objective, then, is to discover what these conditions are; once we find them, we can then verify that 6.54 is the one and only solution of 6.53.

Continuity conditions for $U(x, t)$ depend on the class of functions permitted for $f(x)$. To simplify discussions, suppose we permit only functions $f(x)$ that are continuous for $0 \le x \le L$ and have piecewise continuous first derivatives. Physically this is realistic; continuity of $f(x)$ implies that the initial temperature distribution in the rod must be continuous. Because $f'(x)$ is proportional to heat flux across cross sections of the rod, piecewise continuity of $f'(x)$ implies that initially there can be no infinite surges of heat.

With $f(x)$ continuous, it is reasonable, physically, to demaind that $U(x, t)$ be continuous for $0 \le x \le L$ and $t \ge 0$. (Were $f(x)$ assumed only piecewise continuous, continuity of $U(x, t)$ for $t = 0$ would be inappropriate.) The fact that $U(x, t)$ must satisfy PDE 6.53a suggests that we demand that $\partial U/\partial t$, $\partial U/\partial x$, and $\partial^2 U/\partial x^2$ all be continuous for $0 < x < L$ and $t > 0$. Boundary conditions 6.53b,c suggest that we require continuity of $\partial U/\partial x$ for $x = 0$, $t > 0$ and for $x = L$, $t > 0$ also. Because there are no heat sources (or sinks) at the ends of the rod, it follows that $\partial U/\partial t$ should be continuous at $x = 0$ and $x = L$ for $t > 0$. For a similar reason, $\partial^2 U/\partial x^2$ should also be continuous at $x = 0$ and $x = L$ for $t > 0$. We now show that these conditions guarantee a unique solution of problem 6.53; that is, we show that (when $f(x)$ is continuous and $f'(x)$ is piecewise continuous for $0 \le x \le L$) there is one and only one solution $U(x, t)$ of 6.53 that also satisfies

$$U(x, t) \text{ be continuous for } 0 \le x \le L \text{ and } t \ge 0; \tag{6.53e}$$

$$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial t}, \quad \frac{\partial^2 U}{\partial x^2} \text{ be continuous for } 0 \le x \le L \text{ and } t > 0. \tag{6.53f}$$

Suppose to the contrary, that there exist two solutions $U_1(x, t)$ and $U_2(x, t)$ satisfying 6.53a–f. The difference $U(x, t) = U_1(x, t) - U_2(x, t)$ must also satisfy 6.53a,b,c,e,f, but initial condition 6.53d is replaced by the homogeneous condition $U(x, 0) = 0$, $0 < x < L$. To show that $U_1(x, t) \equiv U_2(x, t)$, we show that $U(x, t) \equiv 0$. To do this, we multiply PDE 6.53a by $U(x, t)$ and integrate with respect to x from $x = 0$ to $x = L$,

$$\int_0^L \frac{\partial U}{\partial t} U(x, t)\, dx = k \int_0^L \frac{\partial^2 U}{\partial x^2} U(x, t)\, dx, \quad t > 0.$$

Integration by parts on the right gives, for $t > 0$,

$$0 = \int_0^L \frac{1}{2} \frac{\partial [U(x, t)]^2}{\partial t} dx - k \left\{ U(x, t) \frac{\partial U}{\partial x} \right\}_0^L + k \int_0^L \left(\frac{\partial U}{\partial x} \right)^2 dx$$

$$= \frac{1}{2}\int_0^L \frac{\partial(U^2)}{\partial t}dx - kU(L,t)\frac{\partial U(L,t)}{\partial x} + kU(0,t)\frac{\partial U(0,t)}{\partial x} + k\int_0^L \left(\frac{\partial U}{\partial x}\right)^2 dx. \quad (6.65)$$

Substitutions from boundary conditions 6.53b,c yield

$$0 = \frac{1}{2}\int_0^L \frac{\partial(U^2)}{\partial t}dx + k\int_0^L \left(\frac{\partial U}{\partial x}\right)^2 dx + \frac{k\mu[U(L,t)]^2}{\kappa}, \quad t > 0. \quad (6.66)$$

Because the last two terms are clearly nonnegative, we must have

$$\int_0^L \frac{\partial(U^2)}{\partial t}dx = \frac{\partial}{\partial t}\int_0^L [U(x,t)]^2 dx \leq 0, \quad t > 0;$$

that is, the definite integral of $[U(x,t)]^2$ must be a nonincreasing function of t. But, because $U(x,t)$ satisfies the condition $U(x,0) = 0$, $0 < x < L$, the definite integral of $[U(x,t)]^2$ at $t = 0$ has value zero,

$$\int_0^L [U(0,t)]^2 dx = 0.$$

In other words, as a function of t, for $t \geq 0$, the definite integral of $[U(x,t)]^2$ is nonnegative, is nonincreasing, and has value zero at $t = 0$. It must therefore be identically equal to zero:

$$\int_0^L [U(x,t)]^2 dx \equiv 0, \quad t \geq 0.$$

Because the integrand is continuous and nonnegative, we conclude that $U(x,t) \equiv 0$ for $0 \leq x \leq L$ and $t \geq 0$; that is, $U_1(x,t) \equiv U_2(x,t)$.

We have shown then, that for the class of initial temperature distributions $f(x)$ that are continuous and have piecewise continuous first derivatives, conditions 6.53e,f attached to 6.53a–d yield a problem with a unique solution; there is one and only one solution satisfying 6.53a–f. To establish that 6.54 is the one and only one solution of problem 6.53, we must verify that it satisfies 6.53e,f. In verifying 6.54 as a solution of 6.53a–d, we proved that series 6.59, 6.62, and 6.63 converge uniformly for $0 \leq x \leq L$ and $t \geq t_0 > 0$ for arbitrary t_0. This implies that $\partial U/\partial t$, $\partial U/\partial x$, and $\partial^2 U/\partial x^2$ are all continuous functions for $0 \leq x \leq L$ and $t > 0$ (see Theorem 3.5 in Section 3.4). This establishes 6.53f. To verify 6.53e, we assume, for simplicity, that $f(x)$ satisfies the boundary conditions of the Sturm-Liouville system associated with the problem, namely $f'(0) = 0$ and $\kappa f'(L) + \mu f(L) = 0$. In this case, Theorem 5.4 in Section 5.3 indicates that the generalized Fourier series $\sum_{n=1}^{\infty} c_n X_n(x)$ of $f(x)$ converges uniformly to $f(x)$ for $0 \leq x \leq L$. Because the functions $e^{-k\lambda_n^2 t}$ are uniformly bounded for $t \geq 0$ and for each such t, the sequence $\{e^{-k\lambda_n^2 t}\}$ is nonincreasing, it follows by Abel's test (Theorem 3.4 in Section 3.4) that series 6.54 converges uniformly for $0 \leq x \leq L$ and $t \geq 0$. The temperature function $U(x,t)$ as defined by 6.54 must therefore be continuous for $0 \leq x \leq L$ and $t \geq 0$.

Verification of 6.54 as the solution to the heat conduction problem described by 6.53 is now complete.

An important point to notice here is that even though the initial temperature distribution may have discontinuities in its first derivative $f'(x)$, the solution of problem 6.53 has continuous first derivatives for $0 \le x \le L$ and $t > 0$. In fact, it has continuous derivative of all orders for $0 \le x \le L$ and $t > 0$. This means that the heat equation immediately smooths out discontinuities of $f'(x)$ and its derivatives. Even if $f(x)$ itself were piecewise continuous, discontinuities would immediately be smoothed out by the heat equation. We shall see that this is also true for elliptic equations, but not for hyperbolic ones.

The method used to verify that problem 6.53a–f has a unique solution is applicable to much more general problems. Consider, for example, the three-dimensional heat conduction problem

$$\frac{\partial U}{\partial t} = k\nabla^2 U + \frac{kg(x,y,z,t)}{\kappa}, \quad (x,y,z) \text{ in } V, \quad t > 0, \tag{6.67a}$$

$$U(x,y,z,t) = F(x,y,z,t), \quad (x,y,z) \text{ on } \beta(V), \quad t > 0, \tag{6.67b}$$

$$U(x,y,z,0) = f(x,y,z), \quad (x,y,z) \text{ in } V. \tag{6.67c}$$

In Exercise 2 it is proved that there cannot be more than one solution $U(x,y,z,t)$ that satisfies the conditions

$U(x,y,z,t)$ be continuous for (x,y,z) in $\overline{V}$ and $t \ge 0$, (6.67d)

First partial derivatives of $U(x,y,z,t)$ with respect to x, y, z, and t and second partial derivatives with respect to x, y, and z be continuous for (x,y,z) in $\overline{V}$ and $t > 0$, (6.67e)

where $\overline{V}$ is the closed region consisting of V and its boundary $\beta(V)$.

Heat conduction problems satisfy what are called maximum and minimum principles. We state and prove the one-dimensional situation here; three-dimensional principles are proved in Exercise 5. Temperature in a rod with insulated sides, when there is no internal heat generation and when the initial temperature distribution is $f(x)$, must satisfy the one-dimensional heat equation

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{6.68a}$$

and the initial condition

$$U(x,0) = f(x), \quad 0 \le x \le L. \tag{6.68b}$$

By taking a closed interval in 6.68b, we are assuming compatibility between the initial temperature distribution $f(x)$ at $x = 0$ and $x = L$ and the boundary temperatures when $t = 0$. Boundary conditions have not been enunciated because maximum and minimum principles are independent of boundary conditions being Dirichlet, Neumann, or Robin. Let U_M be the largest of the following three numbers:

$$U_1 = \text{maximum value of } f(x) \text{ for } 0 \le x \le L,$$
$$U_2 = \text{maximum value of } U(0,t) \text{ for } 0 \le t \le T,$$
$$U_3 = \text{maximum value of } U(L,t) \text{ for } 0 \le t \le T,$$

where T is some given value of t. In other words, U_M is the maximum of the initial temperature of the rod and that found (or applied) at the ends of the rod up to time T. The **maximum principle** states that $U(x,t) \leq U_M$ for all $0 \leq x \leq L$ and $0 \leq t \leq T$; that is, at no point in the rod during the time interval $0 \leq t \leq T$ can the temperature ever exceed U_M. To prove this result, we define a function $V(x,t) = U(x,t) + \epsilon x^2$, $0 \leq x \leq L$, $0 \leq t \leq T$, where $\epsilon > 0$ is a very small number. Because U satisfies PDE 6.68a, we can say that for $0 < x < L$ and $0 < t < T$,

$$\frac{\partial V}{\partial t} - k\frac{\partial^2 V}{\partial x^2} = \frac{\partial U}{\partial t} - k\left(\frac{\partial^2 U}{\partial x^2} + 2\epsilon\right) = -2k\epsilon < 0. \tag{6.69}$$

Assuming that $U(x,t)$ is continuous, so also is $V(x,t)$, and therefore $V(x,t)$ must take on a maximum in the closed rectangle $\overline{A}$ of Figure 6.10. This value must occur either on the edge of the rectangle or at an interior point (x^*,t^*). In the latter case, $V(x,t)$ must necessarily have a relative maximum at (x^*,t^*), and therefore $\partial V/\partial t = \partial V/\partial x = 0$ and $\partial^2 V/\partial x^2 \leq 0$ at (x^*,t^*). But then $\partial V/\partial t - k\partial^2 V/\partial x^2 \geq 0$ at (x^*,t^*), contradicting inequality 6.69. Hence, the maximum value of V must occur on the boundary of $\overline{A}$. It cannot occur along $t = T$, for, in this case, $\partial V/\partial t \geq 0$ at the point and $\partial^2 V/\partial x^2$ would still be nonpositive. Once again, inequality 6.69 would be violated. Consequently, the maximum value of V on $\overline{A}$ must occur on one of the three boundaries $t = 0$, $x = 0$, or $x = L$. Since $U \leq U_M$ on these three lines, it follows that $V \leq U_M + \epsilon L^2$ on these lines and therefore in $\overline{A}$. But because $U(x,t) \leq V(x,t)$, we can state that, in $\overline{A}$, $U(x,t) \leq U_M + \epsilon L^2$. Since ϵ can be made arbitrarily small, it follows that U_M must be the maximum value of U for $0 \leq x \leq L$ and $0 \leq t \leq T$.

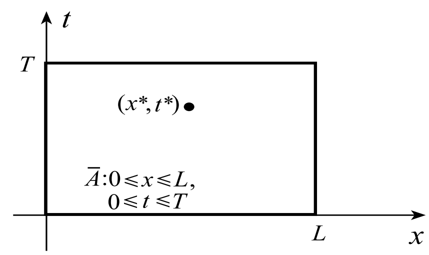


Figure 6.10

When this result is applied to $-U$, the **minimum principle** is obtained — at no point in the rod during the interval $0 \leq t \leq T$ can the temperature ever be less than the minimum of the initial temperature of the rod and that found (or applied) at the ends of the rod up to time T.

We mention one final property of heat conduction problems, which, unfortunately, is not demonstrable with the series solutions of Chapters 4 and 6. (It is illustrated for infinite rods in Case 2 of solution 11.35 in Section 11 and for finite rods in solution 10.44 of Section 10.4.) When heat is added to any part of an object, its effect is instantaneously felt throughout the whole object. For instance, suppose that the initial temperature $f(x)$ of the rod in problem 6.53 is identically equal to zero, and at $t = 0$ a small amount of heat is added to either end of the rod or over some cross section of the rod. Instantaneously, the temperature of every point of the rod rises. The increase may be extremely small, but nonetheless, every point in the rod has a positive temperature for arbitrarily small $t > 0$, and this is true for arbitrarily large L. In other words, heat has been propagated infinitely fast from the source point to all other points in the rod. This is a result of the macroscopic derivation of the heat equation in Section 2.2. On a microscopic level, it would be necessary to take into account the moment of inertia of the molecules transmitting heat, and this would lead to a finite speed for propagation of heat.

EXERCISES 6.6

1. (a) What is the formal series solution of the one-dimensional heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$-l_1\frac{\partial U}{\partial x} + h_1 U = 0, \quad x = 0, \quad t > 0,$$
$$l_2\frac{\partial U}{\partial x} + h_2 U = 0, \quad x = L, \quad t > 0,$$
$$U(x,0) = f(x), \quad 0 < x < L?$$

(b) Use a technique similar to verification of formal solution 6.54 for problem 6.53 to verify that the formal solution in part (a) satisfies the four equations there when $f(x)$ is piecewise smooth on $0 \le x \le L$.
(c) Assuming further that $f(x)$ is continuous on $0 \le x \le L$, show that there is one and only one solution of the problem in part (a) that also satisfies continuity conditions 6.53e,f.
(d) Verify that the formal solution in part (a) satisfies 6.53e,f when $f(x)$ satisfies the boundary conditions of the associated Sturm-Liouville system.

2. Use Green's first identity (see Appendix C) to verify that there cannot be more than one solution to problem 6.67.

3. Repeat Exercise 2 if the boundary condition on $\beta(V)$ is of Robin type.

4. Can you repeat Exercise 2 if the boundary condition on $\beta(V)$ is of Neumann type?

5. In this exercise we prove three-dimensional maximum and minimum principles. Let $U(x,y,z,t)$ be the continuous solution of the homogeneous three-dimensional heat conduction equation in some open region V,

$$\frac{\partial U}{\partial t} = k\nabla^2 U, \quad (x,y,z) \text{ in } V, \quad t > 0,$$

which also satisfies the initial condition

$$U(x,y,z,0) = f(x,y,z), \quad (x,y,z) \text{ in } \overline{V},$$

where $\overline{V}$ is the closed region consisting of V and its boundary $\beta(V)$. Let U_M be the maximum value of $f(x,y,z)$ and the value of U on $\beta(V)$ for $0 \le t \le T$, T some given time.
(a) Define a function $W(x,y,z,t) = U(x,y,z,t) + \epsilon(x^2+y^2+z^2)$, where $\epsilon > 0$ is a very small number. Show that

$$\frac{\partial W}{\partial t} - k\nabla^2 W < 0$$

for (x,y,z) in V and $0 < t < T$, and use this fact to verify that W cannot have a relative maximum for a point (x,y,z) in V and a time $0 < t < T$.
(b) Prove the maximum principle that $U(x,y,z,t) \le U_M$ for (x,y,z) in $\overline{V}$ and $0 \le t \le T$.
(c) What is the minimum principle for this situation?

§6.7 Properties of Elliptic Partial Differential Equations

Verification that formal solutions of boundary value problems do indeed satisfy the elliptic PDEs and boundary conditions from which they were derived are similar to those for parabolic (heat) problems. We illustrate with the following Dirichlet problem for Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L', \tag{6.70a}$$

$$V(0, y) = 0, \quad 0 < y < L', \tag{6.70b}$$

$$V(L, y) = 0, \quad 0 < y < L', \tag{6.70c}$$

$$V(x, 0) = 0, \quad 0 < x < L, \tag{6.70d}$$

$$V(x, L') = f(x), \quad 0 < x < L. \tag{6.70e}$$

Separation leads to the formal solution

$$V(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{L} X_n(x), \tag{6.71a}$$

where

$$A_n = \frac{1}{\sinh(n\pi L'/L)} \int_0^L f(x) X_n(x)\, dx, \tag{6.71b}$$

and $X_n(x) = \sqrt{2/L} \sin(n\pi x/L)$.

Theorem 5.2 in Section 5.2 guarantees that boundary condition 6.70e is satisfied when $f(x)$ is piecewise smooth on $0 \le x \le L$ (provided $f(x)$ is defined as the average of right- and left-hand limits at any point of discontinuity). Boundary conditions 6.70b–d are clearly satisfied by solution 6.71. To verify that $V(x, y)$ as defined by 6.71 satisfies PDE 6.70a, we first note that when $f(x)$ is piecewise continuous, it is necessarily bounded ($|f(x)| \le K$). Combine this with the fact that $|X_n(x)| \le \sqrt{2/L}$, and we obtain

$$|A_n| \le \frac{1}{|\sinh(n\pi L'/L)|} \int_0^L |f(x)||X_n(x)|\, dx \le \frac{K\sqrt{2/L}(L)}{\sinh(n\pi L'/L)} = \frac{\sqrt{2L}K}{\sinh(n\pi L'/L)}. \tag{6.72}$$

With this result, we may write, for any x in $0 \le x \le L$, and any y in $0 \le y \le y_0 < L'$,

$$\begin{aligned} \sum_{n=1}^{\infty} \left| A_n \sinh \frac{n\pi y}{L} X_n(x) \right| &\le \sum_{n=1}^{\infty} \frac{\sqrt{2L}K}{\sinh(n\pi L'/L)} \sinh \frac{n\pi y}{L} \sqrt{\frac{2}{L}} \\ &= 2K \sum_{n=1}^{\infty} \frac{\sinh(n\pi y/L)}{\sinh(n\pi L'/L)} \le 2K \sum_{n=1}^{\infty} e^{-n\pi(L'-y)/L} \\ &\le 2K \sum_{n=1}^{\infty} e^{-n\pi(L'-y_0)/L} = 2K \sum_{n=1}^{\infty} [e^{-\pi(L'-y_0)/L}]^n \\ &= 2K \sum_{n=1}^{\infty} r^n, \end{aligned} \tag{6.73}$$

a convergent geometric series since $r = e^{-\pi(L'-y_0)/L} < 1$. Consequently, according to the Weierstrass M-test, series 6.71 converges absolutely and uniformly with respect to x and y for $0 \le x \le L$ and $0 \le y \le y_0 < L'$. Because y_0 is arbitrary, series 6.71 converges absolutely for $0 \le x \le L$ and $0 \le y < L'$. In addition, series 6.71 represents a continuous function for $0 \le x \le L$ and $0 \le y < L'$. Thus, even though $f(x)$ may have discontinuities, the solution of Laplace's equation must be a continuous function. In other words, Laplace's equation smooths out discontinuities in boundary data.

Term-by-term differentiation of series 6.71 with respect to x gives

$$\sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{L} X_n'(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{L}} \frac{n\pi}{L} A_n \sinh \frac{n\pi y}{L} \cos \frac{n\pi x}{L}. \tag{6.74}$$

It follows that, for $0 \le x \le L$ and $0 \le y \le y_0 < L'$,

$$\sum_{n=1}^{\infty} \left| A_n \sinh \frac{n\pi y}{L} X_n'(x) \right| \le \frac{2K\pi}{L} \sum_{n=1}^{\infty} nr^n. \tag{6.75}$$

Because $\sum_{n=1}^{\infty} nr^n$ converges, series 6.74 converges absolutely and uniformly. Thus, series 6.71 may be differentiated term-by-term to yield, for $0 \le x \le L$ and $0 \le y < L'$,

$$\frac{\partial V}{\partial x} = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{L} X_n'(x). \tag{6.76}$$

Similarly, for $0 \le x \le L$ and $0 \le y < L'$,

$$\frac{\partial^2 V}{\partial x^2} = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{L} X_n''(x). \tag{6.77}$$

Term-by-term differentiation of 6.71 with respect to y gives

$$\frac{\pi}{L} \sum_{n=1}^{\infty} nA_n \cosh \frac{n\pi y}{L} X_n(x). \tag{6.78}$$

Using inequality 6.72 and the fact that $|X_n(x)| \le \sqrt{2/L}$, we may write

$$\frac{\pi}{L} \sum_{n=1}^{\infty} \left| nA_n \cosh \frac{n\pi y}{L} X_n(x) \right| \le \frac{2K\pi}{L} \sum_{n=1}^{\infty} \frac{n \cosh(n\pi y/L)}{\sinh(n\pi L'/L)}. \tag{6.79}$$

Now N can always be chosen sufficiently large that $\sinh(n\pi L'/L) \ge (1/4)e^{n\pi L'/L}$, whenever $n \ge N$. For such N, and $0 \le x \le L$ and $0 \le y \le y_0 < L'$,

$$\begin{aligned} \frac{\pi}{L} \sum_{n=N}^{\infty} \left| nA_n \cosh \frac{n\pi y}{L} X_n(x) \right| &\le \frac{2K\pi}{L} \sum_{n=N}^{\infty} \frac{ne^{n\pi y/L}}{(1/4)e^{n\pi L'/L}} \\ &= \frac{8K\pi}{L} \sum_{n=N}^{\infty} ne^{-n\pi(L'-y)/L} \le \frac{8K\pi}{L} \sum_{n=N}^{\infty} ne^{-n\pi(L'-y_0)/L} \\ &= \frac{8K\pi}{L} \sum_{n=N}^{\infty} nr^n, \end{aligned} \tag{6.80}$$

where $r = e^{-\pi(L'-y_0)/L}$. Since the series $\sum_{n=1}^{\infty} nr^n$ converges, it follows that series 6.78 converges absolutely and uniformly for $0 \le x \le L$ and $0 \le y \le y_0 < L'$. Thus, series 6.71 may be differentiated term-by-term with respect to y to yield, for $0 \le x \le L$ and $0 \le y < L'$,

$$\frac{\partial V}{\partial y} = \frac{\pi}{L}\sum_{n=1}^{\infty} nA_n \cosh\frac{n\pi y}{L} X_n(x). \tag{6.81}$$

For the same values of x and y, we also obtain

$$\frac{\partial^2 V}{\partial y^2} = \frac{\pi^2}{L^2}\sum_{n=1}^{\infty} n^2 A_n \sinh\frac{n\pi y}{L} X_n(x). \tag{6.82}$$

Because $X_n''(x) = (-n^2\pi^2/L^2)X_n(x)$, expressions 6.77 and 6.82 clearly indicate that $V(x,y)$ satisfies Laplace's equation 6.70a. We have shown, therefore, that series solution 6.71 satisfies problem 6.70.

In order to guarantee a unique solution of problem 6.70, continuity conditions must also accompany the problem. We show that when $f(x)$ is a continuous function with a continuous first derivative $f'(x)$ and a piecewise continuous second derivative $f''(x)$, for which $f(0) = f(L) = 0$, appropriate conditions are

$$V, \quad \frac{\partial V}{\partial x}, \quad \text{and} \quad \frac{\partial V}{\partial y} \quad \text{be continuous for } 0 \le x \le L \text{ and } 0 \le y \le L'; \tag{6.70f}$$

second partial derivatives of $V(x,y)$ be continuous for

$$0 < x < L, \quad 0 < y < L'. \tag{6.70g}$$

Suppose, to the contrary, that there exist two solutions $V_1(x,y)$ and $V_2(x,y)$ satisfying problem 6.70. The difference $V(x.y) = V_1(x,y) - V_2(x,y)$ must also satisfy 6.70, but with 6.70e replaced by the homogeneous condition $V(x,L') = 0$, $0 < x < L$. If we multiply PDE 6.70a by $V(x,y)$, integrate over the rectangle R: $0 < x < L$, $0 < y < L'$, and use Green's first identity (Appendix C), we obtain

$$0 = \iint_R V\nabla^2 V\, dA = \oint_{\beta(R)} V\frac{\partial V}{\partial n} ds - \iint_R |\nabla V|^2\, dA, \tag{6.83}$$

where $\partial V/\partial n$ is the directional derivative of V outwardly normal to $\beta(R)$. Since $V \equiv 0$ on $\beta(R)$,

$$0 = -\iint_R |\nabla V|^2\, dA.$$

But this result requires $\nabla V \equiv 0$ in R, and therefore $V(x,y)$ must be constant in R. Because V is constant in R, vanishes on $\beta(R)$, and is continuous for $0 \le x \le L$, $0 \le y \le L'$, it follows that $V(x,y) \equiv 0$. In other words, conditions 6.70f,g guarantee a unique solution of problem 6.70.

Once again, we point out that Laplace's equation, like the heat equation, smooths out discontinuities. Even when the boundary data function $f(x)$ has discontinuities in its second derivative, 6.70g demands that second derivatives of $V(x,y)$ be continuous for $0 < x < L$, $0 < y < L'$.

We now establish that solution 6.71 of problem 6.70a–e also satisfies conditions 6.70f,g. The facts that series 6.77 and 6.82 converge uniformly for $0 \le x \le L$

and $0 \le y \le y_0 < L'$ and y_0 is arbitrary imply that $\partial^2 V/\partial x^2$ and $\partial^2 V/\partial y^2$ are continuous for $0 \le x \le L$ and $0 \le y < L'$. To verify 6.70f, we use Theorem 3.4 in Section 3.4. First, note that with continuity of $f(x)$ and $f(0) = f(L) = 0$, the Fourier series of $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi L'}{L} X_n(x), \tag{6.84}$$

converges uniformly to $f(x)$ on $0 \le x \le L$ (see Theorem 5.4 in Section 5.3). Series 6.71 can be obtained from this series by multiplying the n^{th} term by

$$Y_n(y) = \frac{\sinh(n\pi y/L)}{\sinh(n\pi L'/L)}.$$

These functions are uniformly bounded for $0 \le y \le L'$. For fixed y in $0 \le y \le L'$, the derivative of $Y_n(y)$ as a function of a continuous variable n is

$$\frac{\partial Y_n}{\partial n} = \frac{(\pi y/L)\sinh(n\pi L'/L)\cosh(n\pi y/L) - (\pi L'/L)\sinh(n\pi y/L)\cosh(n\pi L'/L)}{\sinh^2(n\pi L'/L)}.$$

Thus,

$$\begin{aligned}
\frac{L}{\pi}\sinh^2\left(\frac{n\pi L'}{L}\right)\frac{\partial Y_n}{\partial n} &= y\sinh\frac{n\pi L'}{L}\cosh\frac{n\pi y}{L} - L'\sinh\frac{n\pi y}{L}\cosh\frac{n\pi L'}{L} \\
&= \frac{y}{2}\left[\sinh\frac{n\pi(L'+y)}{L} + \sinh\frac{n\pi(L'-y)}{L}\right] \\
&\qquad - \frac{L'}{2}\left[\sinh\frac{n\pi(y+L')}{L} + \sinh\frac{n\pi(y-L')}{L}\right] \\
&= \frac{L'+y}{2}\sinh\frac{n\pi(L'-y)}{L} - \frac{L'-y}{2}\sinh\frac{n\pi(L'+y)}{L} \\
&= \frac{L'+y}{2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left[\frac{n\pi(L'-y)}{L}\right]^{2m+1} \\
&\qquad - \frac{L'-y}{2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left[\frac{n\pi(L'+y)}{L}\right]^{2m+1} \\
&= \frac{(L'+y)(L'-y)}{2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}[(L'-y)^{2m} - (L'+y)^{2m}]\left(\frac{n\pi}{L}\right)^{2m+1},
\end{aligned}$$

which is clearly nonpositive. Thus, for each fixed y in $0 \le y \le L'$, the sequence $\{Y_n(y)\}$ is nonincreasing, and by Theorem 3.4 in Section 3.4, series 6.71 converges uniformly for $0 \le x \le L$ and $0 \le y \le L'$. This series therefore defines a continuous function $V(x, y)$ on $0 \le x \le L$, $0 \le y \le L'$.

Because $f'(x)$ is continuous (and $f''(x)$ is piecewise continuous), the Fourier (cosine) series

$$f'(x) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L'}{L} X_n'(x) = \frac{\sqrt{2}\pi}{L^{3/2}}\sum_{n=1}^{\infty} nA_n \sinh\frac{n\pi L'}{L}\cos\frac{n\pi x}{L}$$

converges uniformly to $f'(x)$ for $0 \le x \le L$ (see exercise 9(c) in Section 3.4). Since series 6.76 for $\partial V/\partial x$ can be obtained from this series by multiplying the n^{th} term by $Y_n(y)$, it follows that series 6.76 converges uniformly to $\partial V/\partial x$ for $0 \le x \le L$ and $0 \le y \le L'$ and that $\partial V/\partial x$ is continuous thereon.

Finally, we must show that $\partial V/\partial y$ as defined by series 6.81 is continuous. Because the above series for $f'(x)$ is uniformly convergent for $0 \le x \le L$, it follows (by setting $x = 0$) that the series

$$\sum_{n=1}^{\infty} \left| nA_n \sinh \frac{n\pi L'}{L} \right|$$

is convergent. Consequently, the series

$$\sum_{n=1}^{\infty} nA_n \sinh \frac{n\pi L'}{L} X_n(x)$$

converges absolutely and uniformly for $0 \le x \le L$. Series 6.81 for $\partial V/\partial y$ can be obtained from this series by multiplying the n^{th} term by

$$Z_n(y) = \frac{\cosh(n\pi y/L)}{\sinh(n\pi L'/L)}.$$

These functions are uniformly bounded for $0 \le y \le L'$, and, furthermore

$$\begin{aligned}
[Z_n(y)]^2 = \frac{\cosh^2(n\pi y/L)}{\sinh^2(n\pi L'/L)} &= \frac{1}{\sinh^2(n\pi L'/L)} + \left[\frac{\sinh(n\pi y/L)}{\sinh(n\pi L'/L)}\right]^2 \\
&= \frac{1}{\sinh^2(n\pi L'/L)} + [Y_n(y)]^2.
\end{aligned}$$

For fixed y in $0 \le y \le L'$, the sequence $\{Y_n(y)\}$ is nonincreasing, as is the sequence $\{1/\sinh^2(n\pi L'/L)\}$. Consequently, the same can be said for $\{Z_n(y)\}$, and it follows by Theorem 3.4 in Section 3.4 that series 6.81 converges uniformly for $0 \le x \le L$ and $0 \le y \le L'$. Thus, $\partial V/\partial y$ must be continuous thereon, and this completes the proof that solution 6.71 satisfies conditions 6.70f,g.

The method used to verify that problem 6.70a–g has a unique solution is applicable to much more general problems. Consider, for example, the three-dimensional boundary value problem

$$\nabla^2 U = F(x,y,z), \quad (x,y,z) \text{ in } V, \tag{6.85a}$$

$$l\frac{\partial U}{\partial n} + hU = f(x,y,z), \quad (x,y,z) \text{ on } \beta(V), \tag{6.85b}$$

$$U \text{ and its first derivatives continuous in } \overline{V} \tag{6.85c}$$

$$\text{Second derivatives of } U \text{ continuous in } V, \tag{6.85d}$$

where $\overline{V}$ is the closed, finite region consisting of V and its boundary, and $l \ge 0$ and $h \ge 0$ are constants. In Exercise 2, it is shown that when $F(x,y,z)$ and $f(x,y,z)$ are continuous, and $h \ne 0$, there cannot be more than one solution of this problem, and when $h = 0$, the solution is unique to an additive constant (that is, if U is a solution, then all solutions are of the form $U + C$, C a constant). Uniqueness also results when different parts of $\beta(V)$ are subjected to different types of boundary conditions. For U not to be unique, the boundary condition must be Neumann on all of $\beta(V)$.

Maximum and Minimum Principles

Maximum and minimum principles for elliptic problems are important theoretically and practically. We verify three-dimensional principles here. The **maximum principle for Poisson's equation** is as follows:

Theorem 6.2 If $U(x,y,z)$ is a continuous solution of Poisson's equation 6.85a in a finite region V, and $F(x,y,z) \geq 0$ in V, then at no point in V can the value of $U(x,y,z)$ exceed the maximum value of U on $\beta(V)$.

Proof We let U_M be the maximum value of U on $\beta(V)$ and define a function $W(x,y,z) = U(x,y,z) + \epsilon(x^2+y^2+z^2)$ in $\overline{V}$, where $\epsilon > 0$ is a very small number. Because U satisfies PDE 6.85a, we can say that in V,

$$\nabla^2 W = \nabla^2 U + 6\epsilon = F(x,y,z) + 6\epsilon > 0. \tag{6.86}$$

Because W is continuous in $\overline{V}$, it must attain an absolute maximum therein. Suppose this maximum occurs at a point (x^*, y^*, z^*) in the interior V (which therefore must be a relative maximum). It follows, then, that

$$\frac{\partial W}{\partial x} = \frac{\partial W}{\partial y} = \frac{\partial W}{\partial z} = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial x^2} \leq 0, \quad \frac{\partial^2 W}{\partial y^2} \leq 0, \quad \frac{\partial^2 W}{\partial z^2} \leq 0,$$

all at (x^*, y^*, z^*). Because the last three inequalities contradict 6.86, the maximum of W must occur on $\beta(V)$.

Since $U \leq U_M$ on $\beta(V)$, $W \leq U_M + \epsilon R^2$ on $\beta(V)$, where R is the radius of a sphere centred at the origin that contains V (such a sphere must exist when V is bounded). Since the maximum value of W must occur on $\beta(V)$, we can state further that $W \leq U_M + \epsilon R^2$ for all (x,y,z) in $\overline{V}$. But because $U(x,y,z) \leq W(x,y,z)$ in $\overline{V}$, it follows that in $\overline{V}$, $U(x,y,z) \leq U_M + \epsilon R^2$. Since ϵ can be made arbitrarily small, we conclude that $U(x,y,z) \leq U_M$ in $\overline{V}$, and the proof is complete. ∎

When $U(x,y,z)$ is a solution of Laplace's equation, the above maximum principle certainly holds. In addition, the principle may also be applied to $-U$, resulting in a minimum principle. In other words, we have the following **maximum-minimum principle for Laplace's equation**.

Theorem 6.3 If a continuous solution of Laplace's equation $\nabla^2 U = 0$ in a finite region V satisfies the condition that $U_m \leq U \leq U_M$ on $\beta(V)$, then $U_m \leq U \leq U_M$ in V also.

This principle provides an alternative, and very simple, proof for uniqueness of solutions to problem 6.85 on finite regions when the boundary condition is Dirichlet. If U_1 and U_2 are solutions of Poisson's equation 6.85a and a Dirichlet condition $U = f(x,y,z)$ on $\beta(V)$, then $U = U_1 - U_2$ is a solution of Laplace's equation $\nabla^2 V = 0$ subject to $U = 0$ on $\beta(V)$. But, according to the maximum-minimum principle for Laplace's equation, U must then be identically equal to zero in V; that is, $U_1 \equiv U_2$.

These principles seem natural and evident in physical settings. For example, the boundary value problem

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{F(x,y)}{\tau}, \quad (x,y) \text{ in } A,$$

$$l\frac{\partial z}{\partial n} + hz = f(x,y), \quad (x,y) \text{ on } \beta(A)$$

describes static deflections of a membrane subjected to a force per unit area with vertical component $F(x,y)$. The two-dimensional maximum principle for Poisson's equation states that if the vertical force is always negative (or zero), then at no point can the membrane have a deflection that exceeds the maximum value on its edge. Furthermore, if there is no external force on the membrane, the maximum-minimum principle for Laplace's equation states that deflections at all points of the membrane must be between maximum and minimum boundary deflections.

When $-F(x,y)/\tau$ is replaced by $-g(x,y)/\kappa$, the problem describes steady-state temperature in a plate insulated top and bottom with internal heat sources (or sinks) described by $g(x,y)$. Poisson's principle implies that when $g(x,y) \le 0$, so that heat is being extracted at every point, then at no point in the plate can the temperature exceed its maximum value on the boundary. In addition, if $g(x,y) \equiv 0$, maximum and minimum temperatures must occur on the boundary. If this were not the case, heat would flow away from the point of maximum temperature in all directions and a steady-state situation would not exist.

Consistency Conditions

Boundary value problems subject to Neumann boundary conditions must satisfy consistency conditions. They are restrictions on the nonhomogeneities in the PDE and boundary conditions which must be satisfied if there are to be solutions of the boundary value problem. We saw one of these in Exercise 9 of Section 2.1. The two-dimensional version stated that solutions of the boundary value problem

$$\nabla^2 V = F(x,y), \quad (x,y) \text{ in } R, \tag{6.87a}$$

$$\frac{\partial V}{\partial n} = G(x,y), \quad (x,y) \text{ on } \beta(R), \tag{6.87b}$$

must satisfy

$$\oint_{\beta(R)} G(x,y)\,ds = \iint_R F(x,y)\,dA. \tag{6.88}$$

We saw interpretations of this condition for heat conduction problems in Exercise 24 of Section 2.2, and vibration problems in Exercise 8 of Section 2.4.

EXERCISES 6.7

1. (a) What is the formal series solution of the two-dimensional potential problem

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0, \quad 0 < x < L, \quad 0 < y < L', \\
-l_1 \frac{\partial V}{\partial x} + h_1 V &= 0, \quad x = 0, \quad 0 < y < L', \\
l_2 \frac{\partial V}{\partial x} + h_2 V &= 0, \quad x = L, \quad 0 < y < L', \\
V(x,0) &= 0, \quad 0 < x < L, \\
V(x,L') &= f(x), \quad 0 < x < L?
\end{aligned}$$

(b) Use a technique similar to verification of formal solution 6.71 for problem 6.70 to verify that the formal solution in part (a) satisfies the five equations there when $f(x)$ is piecewise smooth on $0 \le x \le L$.

(c) Assuming further that $f(x)$ and $f'(x)$ are continuous and that $f''(x)$ is piecewise continuous on $0 \leq x \leq L$, show that there is one and only one solution of the problem in part (a) that also satisfies continuity conditions 6.70f,g.

(d) Verify that the formal solution in part (a) satisfies 6.70f,g when $f(x)$ satisfies the boundary conditions of the associated Sturm-Liouville system. Omit a proof of continuity of $\partial V/\partial x$ and $\partial V/\partial y$.

2. Use Green's first identity (see Appendix C) to verify that there cannot be more than one solution to problem 6.85 except when $h = 0$, in which case the solution is unique to an additive constant.

3. Verify that a solution of Laplace's equation in a volume V of space cannot have a relative maximum or minimum in V.

§6.8 Properties of Hyperbolic Partial Differential Equations

Verification of formal solutions of initial boundary value problems involving hyperbolic PDEs requires a different approach from that used for parabolic and elliptic equations in the previous two sections. To see why, consider the initial boundary value problem for displacements of a string with fixed ends, released from rest at some initial position $f(x)$,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{6.89a}$$

$$y(0,t) = 0, \quad t > 0, \tag{6.89b}$$

$$y(L,t) = 0, \quad t > 0, \tag{6.89c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{6.89d}$$

$$y_t(x,0) = 0, \quad 0 < x < L. \tag{6.89e}$$

If we assume that $f(x)$ is continuous and $f'(x)$ is piecewise continuous for $0 \le x \le L$, and that $f(0) = f(L) = 0$, the formal solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) \cos \frac{n\pi ct}{L} \qquad \text{where} \quad c_n = \int_0^L f(x) X_n(x)\, dx, \tag{6.90}$$

and $X_n(x) = \sqrt{2/L} \sin(n\pi x/L)$. A function $f(x)$ satisfying these requirements was considered in Figure 2.31a of Section 2.7. Figures 2.31b–f in Section 2.7 and Figures 2.49a–h in Section 2.11 illustrate that the discontinuity in $f'(x)$ is propagated in both directions along the string at speed c. In other words, the solution $y(x,t)$ could not possibly satisfy PDE 6.89a. Likewise, discontinuities in the second derivative of $f(x)$ are also propagated at speed c. In order, therefore, for solution 6.90 to satisfy 6.89a pointwise for $0 < x < L$ and $t > 0$, it is necessary to place very stringent conditions on $f(x)$. Suppose, for the moment, that we assume that $f(x)$, $f'(x)$, and $f''(x)$ are all continuous for $0 \le x \le L$ and that $f(0) = f(L) = 0$.

Verification of formal solution 6.54 to heat conduction problem 6.53 involved a detailed analysis of convergence of 6.54 and its term-by-term derivatives with respect to x and t. A similar analysis ensued for potential problem 6.70. This type of analysis is inappropriate for problem 6.89. For instance, how do we show that 6.90 converges for $0 \le x \le L$ and $t > 0$, knowing only that the c_n are bounded? To circumvent this difficulty, we use d'Alembert's representation of 6.90,

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \left[\sin \frac{n\pi(x+ct)}{L} + \sin \frac{n\pi(x-ct)}{L} \right] \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)]. \end{aligned} \tag{6.91}$$

For this solution to define $y(x,t)$ for $0 \le x \le L$ and $t \ge 0$, $f(x)$ is extended as an odd, $2L$-periodic function. This extension immediately implies that 6.91 satisfies boundary conditions 6.89b,c and initial condition 6.89e. Initial condition 6.89d is clearly satisfied. With continuity of $f''(x)$, it is a straightforward application of chain rules to verify 6.89a.

We now show that problem 6.89 has a unique solution when $y(x,t)$ is also required to satisfy the condition

$$y(x,t) \text{ and its first and second partial derivatives be continuous for } 0 \le x \le L \text{ and } t \ge 0. \tag{6.89f}$$

Suppose, to the contrary, that $y_1(x,t)$ and $y_2(x,t)$ are two solutions of 6.89a–f. Their difference $y(x,t) = y_1(x,t) - y_2(x,t)$ must then satisfy 6.89a,b,c,e,f, but 6.89d is replaced by the homogeneous initial condition $y(x,0) = 0$, $0 < x < L$. If we multiply PDE 6.89a by $\partial y/\partial t$ and integrate with respect to x from $x = 0$ to $x = L$,

$$\int_0^L \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial t}dx = \int_0^L c^2\frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial t}dx, \quad t > 0.$$

Integration by parts on the right gives

$$\begin{aligned}\frac{1}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)^2 dx &= c^2\left\{\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\right\}_0^L - c^2\int_0^L \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x\partial t}dx \\ &= c^2\left\{\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\right\}_0^L - c^2\int_0^L \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right)^2 dx, \quad t > 0. \end{aligned} \tag{6.92}$$

With the ends of the string fixed on the x-axis, it follows that $\partial y(0,t)/\partial t = \partial y(L,t)/\partial t = 0$, and therefore this equation reduces to

$$0 = \frac{1}{2}\int_0^L \left[\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)^2 + c^2\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right)^2\right]dx, \quad t > 0. \tag{6.93}$$

When this equation is antidifferentiated with respect to time, the result is

$$\frac{1}{2}\int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right]dx = K, \quad t > 0, \tag{6.94}$$

where K is a constant. To evaluate K, we take limits of each term in this equation as $t \to 0^+$. Because $\partial y/\partial t$ and $\partial y/\partial x$ are assumed continuous (condition 6.89f),

$$\lim_{t\to 0^+}\frac{\partial y(x,t)}{\partial t} = \frac{\partial y(x,0)}{\partial t} = 0, \quad 0 < x < L$$

(initial condition 6.89e). Furthermore, because $y(x,0) = y_1(x,0) - y_2(x,0) = 0$, we find that

$$\lim_{t\to 0^+}\frac{\partial y(x,t)}{\partial x} = \frac{\partial y(x,0)}{\partial x} = 0, \quad 0 < x < L.$$

With these results, limits as $t \to 0^+$ in equation 6.94 show that $K = 0$, and therefore, for $t \ge 0$, we may write

$$\int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right]dx = 0. \tag{6.95}$$

Since each term in this equation is continuous and nonnegative, it follows that each must vanish separately; that is, we must have $\partial y/\partial x = \partial y/\partial t = 0$ for $0 \le x \le L$ and $t \ge 0$. These imply that $y(x,t)$ is constant for $0 \le x \le L$ and $t \ge 0$, and this constant must be zero since $y(x,0) = 0$. Thus, $y(x,t) \equiv 0$, and the solution of problem 6.89 is unique.

That solution 6.91 satisfies continuity condition 6.89f is an immediate consequence of the assumption that $f''(x)$ is continuous for $0 \le x \le L$.

In Section 6.6 we saw that discontinuities in the initial temperature function were smoothed out by the heat equation. Discontinuities in boundary data were also smoothed out by Laplace's equation. This is not the case for hyperbolic equations; a distinguishing property of hyperbolic equations is that discontinuities in initial data are propagated by the solution. We have already seen this with the discontinuity in $f'(x)$ for $f(x)$ in Figure 2.31a. The discontinuity in $f'(x)$ is propagated in both directions along the string at speed c; it is not smoothed out. For a small time t (before the disturbance reaches the ends of the string), the discontinuity is found at positions $x = L/2 \pm ct$, that is, at points given by $x \pm ct = L/2$. But these are equations of characteristic curves for the one-dimensional wave equation (see Example 2.7 in Section 2.8). We have illustrated, therefore, that discontinuities in derivatives of initial data are propagated along characteristic curves of hyperbolic equations. These characteristics are shown in Figure 6.11. At time $t = L/(2c)$, the discontinuities reach the ends of the string for the first time, whereupon they are reflected to travel once again along the string. By drawing a horizontal line, say $t = t_0$, to intersect the broken lines in this figure, we obtain the positions of the discontinuities at time t_0. Intersections with a vertical line $x = x_0$ give the times at which the discontinuities pass through the point x_0 on the string.

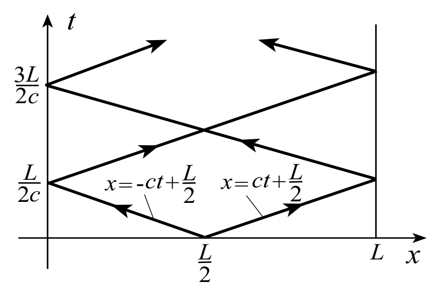


Figure 6.11

The formal solution of problem 6.89 when $f(x)$ is as shown in Figure 2.31a is still defined by 6.90 or, more compactly, by 6.91. It is not, however, a function that satisfies 6.89a for all $0 < x < L$ and $t > 0$. It satisfies 6.89a at all points (x,t) in Figure 6.11 that are not on the characteristics $x = L/2 \pm ct$ and their reflections.

EXERCISES 6.8

1. (a) What is the formal series solution of the vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$y_x(0,t) = 0, \quad t > 0,$$
$$y_x(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L?$$

Express this solution in closed form.

(b) Verify that the formal solution in part (a) satisfies the five equations in (a) when $f(x)$, $f'(x)$, and $f''(x)$ are continuous on $0 \le x \le L$ and $f'(0) = f'(L) = 0$.
(c) Show that there is a unique solution to the problem in part (a) that also satisfies continuity condition 6.89f.
(d) Verify that the formal solution in part (a) satisfies 6.89f.

2. (a) What is the formal series solution of vibration problem 6.89 if initial conditions 6.89d,e are replaced by

$$y(x,0) = 0, \qquad y_t(x,0) = g(x), \qquad 0 < x < L?$$

Express the formal solution in closed form when $g(x)$ and $g'(x)$ are continuous for $0 \le x \le L$ and $g(0) = g(L) = 0$.
(b) Verify that the formal solution in part (a) satisfies 6.89a–c and the initial conditions there.
(c) Show that there is a unique solution to the problem in part (a) that also satisfies continuity condition 6.89f.
(d) Verify that the formal solution in part (a) satisfies 6.89f.

CHAPTER 7 FINITE FOURIER TRANSFORMS AND NONHOMOGENEOUS PROBLEMS

§7.1 Finite Fourier Transforms

In Section 4.3 we used transformations and variation of constants to solve nonhomogeneous (initial) boundary value problems. These problems were relatively straightforward, principally because they contained only one spatial variable. When nonhomogeneities were time independent, the solution was represented as the sum of steady-state and transient parts. The steady-state portion was determined by an ODE, and the transient portion satisfied a homogeneous problem. (In problems with two or three spatial variables, the steady-state part will satisfy two- or three-dimensional boundary value problems.) When nonhomogeneities were time dependent, the method of variation of constants had to be used. The corresponding homogeneous problem was solved, and arbitrary constants were then replaced by functions of time.

In this chapter we present an alternative technique for solving nonhomogeneous (initial) boundary value problems, namely, *finite Fourier transforms.* They handle time-dependent and time-independent nonhomogeneities in exactly the same way and adapt to problems in higher dimensions very easily.

Theorem 5.2 of Section 5.2 states that every Sturm-Liouville system

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) - q(x)]y = 0, \quad a < x < b, \tag{7.1a}$$

$$-l_1 y'(a) + h_1 y(a) = 0, \tag{7.1b}$$

$$l_2 y'(b) + h_2 y(b) = 0 \tag{7.1c}$$

has an infinity of eigenvalues λ_n $(n = 1, 2, \ldots)$ and corresponding orthonormal eigenfunctions $y_n(x)$. Furthermore, if $f(x)$ is a piecewise smooth function on the interval $a \le x \le b$, then on the open interval $a < x < b$, $f(x)$ can be expressed in a (generalized) Fourier series

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \qquad \text{where} \quad c_n = \int_a^b p(x) f(x) y_n(x)\, dx. \tag{7.2}$$

Equality holds only if $f(x)$ is defined as $[f(x+)+f(x-)]/2$ at points of discontinuity of $f(x)$.

We say that the Sturm-Liouville system defines an **integral transform** that associates with a function $f(x)$ a sequence of constants $\{c_n\}$ defined by the integral in equation 7.2. This sequence of constants is called the **finite Fourier transform** of $f(x)$, associated with the Sturm-Liouville system, and is given the notation $\{\tilde{f}(\lambda_n)\}$, where, therefore,

$$\tilde{f}(\lambda_n) = \int_a^b p(x) f(x) y_n(x)\, dx. \tag{7.3a}$$

We often say somewhat loosely, that $\tilde{f}(\lambda_n)$ is the transform of $f(x)$ rather than the sequence $\{\tilde{f}(\lambda_n)\}$ of the $\tilde{f}(\lambda_n)$. If $f(x)$ is piecewise smooth on $a \le x \le b$, then (for $a < x < b$), series 7.2 becomes

$$f(x) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_n) y_n(x). \tag{7.3b}$$

This series is called the **inverse transform** corresponding to system 7.3a; it defines a function $f(x)$ that has $\{\tilde{f}(\lambda_n)\}$ as its finite Fourier transform.

Example 7.1 Find the Fourier transform of the function $f(x) = 2x^2$, $0 \le x \le L$, associated with the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X'(0) = X(L) = 0.$$

Does the inverse transform converge to $f(x)$ for $0 \le x \le L$?

Solution According to Table 5.1 in Section 5.2, eigenvalues of the Sturm-Liouville system are $\lambda_n^2 = (2n-1)^2\pi^2/(4L^2)$ with orthonormal eigenfunctions $X_n(x) = \sqrt{2/L}\cos[(2n-1)\pi x/(2L)]$. The finite Fourier transform of $f(x)$ is

$$\tilde{f}(\lambda_n) = \int_0^L 2x^2 \sqrt{\frac{2}{L}} \cos\frac{(2n-1)\pi x}{2L} dx,$$

and integration by parts leads to

$$\tilde{f}(\lambda_n) = \frac{4\sqrt{2}L^{5/2}(-1)^{n+1}}{\pi^3}\left[\frac{\pi^2}{2n-1} - \frac{8}{(2n-1)^3}\right].$$

The inverse transform is

$$\sum_{n=1}^{\infty} \tilde{f}(\lambda_n) X_n(x) = \sum_{n=1}^{\infty} \frac{4\sqrt{2}L^{5/2}(-1)^{n+1}}{\pi^3}\left[\frac{\pi^2}{2n-1} - \frac{8}{(2n-1)^3}\right]\sqrt{\frac{2}{L}}\cos\frac{(2n-1)\pi x}{2L}.$$

Since $f(x) = 2x^2$ is continuous for $0 \le x \le L$, this series is guaranteed to converge to $f(x)$ for $0 < x < L$. Since the sum of the series is zero at $x = L$, it does not converge to $f(x)$ at $x = L$. Using the facts that $\sum_{n=1}^{\infty} (-1)^{n+1}/(2n-1) = \pi/4$ and $\sum_{n=1}^{\infty} (-1)^{n+1}/(2n-1)^3 = \pi^3/32$, the sum of the series at $x = 0$ is zero, as is the value of $f(x)$. Hence, the inverse transform converges to $f(x)$ for $0 \le x < L$.•

Once a Sturm-Liouville system is stipulated, the finite Fourier transform of a function $f(x)$ is unique; that is, integral 7.3a defines a unique sequence of constants $\{\tilde{f}(\lambda_n)\}$ for $f(x)$. (Obviously, if the Sturm-Liouville system is changed, then the finite Fourier transform of the same function $f(x)$ changes.)

Let us for the moment fix on a particular Sturm-Liouville system on the interval $a \le x \le b$. Many functions can have the same transform. For example, the functions $f_1(x)$ and $f_2(x)$ in Figure 7.1, which differ only in their values at x_1, x_2 and x_3, have the same transform. Of all functions with the same transform, inverse transform 7.3b defines a continuous function with $\tilde{f}(\lambda_n)$ as transform, if such a function exists. If no such function exists, the inverse transform defines a function $f(x)$ that has only finite jump discontinuities, the value of the function at any discontinuity being the average of its left- and right-hand limits.

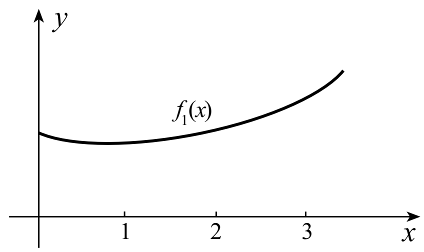


Figure 7.1a

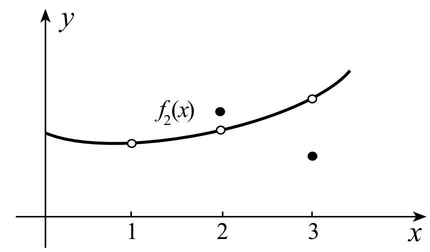


Figure 7.1b

For example, eigenvalues and orthonormal eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = 0 = X(L),$$

are $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$. The finite Fourier transform of a function $f(x)$ defined on $0 \le x \le L$ is $\{\tilde{f}(\lambda_n)\}$, where

$$\tilde{f}(\lambda_n) = \int_0^L f(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx.$$

In particular, if $f(x) = x$, integration by parts gives

$$\tilde{f}(\lambda_n) = \frac{\sqrt{2L^3}}{n\pi}(-1)^{n+1}.$$

Because $f(x) = x$ is continuous for $0 \le x \le L$, the inverse transform

$$\sum_{n=1}^\infty \tilde{f}(\lambda_n)X_n(x) = \sum_{n=1}^\infty \frac{\sqrt{2L^3}}{n\pi}(-1)^{n+1}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} = \frac{2L}{\pi}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

returns the original function x on the interval $0 < x < L$; that is, we can write

$$x = \frac{2L}{\pi}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

Because the series also converges to $f(x) = x$ at $x = 0$, but not at $x = L$, we write finally that

$$x = \frac{2L}{\pi}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}, \quad 0 \le x < L.$$

With respect to the same Sturm-Liouville system, the finite Fourier transform for the function $g(x)$ in Figure 7.2a is

$$\tilde{g}(\lambda_n) = \int_0^L g(x)X_n(x)\,dx = \int_0^{L/2}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx = \frac{\sqrt{2L}}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right).$$

On the interval $0 < x < L$, the inverse transform 7.3b defines a function

$$k(x) = \sum_{n=1}^\infty \tilde{g}(\lambda_n)X_n(x) = \sum_{n=1}^\infty \frac{\sqrt{2L}}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi/2)}{n}\sin\frac{n\pi x}{L}.$$

This function is identical to $g(x)$ except at $x = L/2$, where its value is 1/2. In addition because the series converges to zero at $x = 0$ and $x = L$, we may write

$$k(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi/2)}{n}\sin\frac{n\pi x}{L}, \quad 0 \le x \le L,$$

where $k(x)$ is the function in Figure 7.2b.

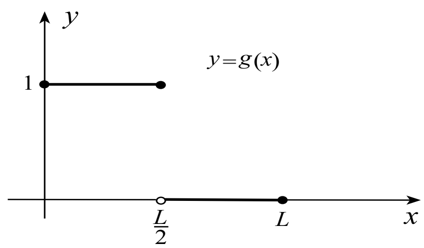


Figure 7.2a

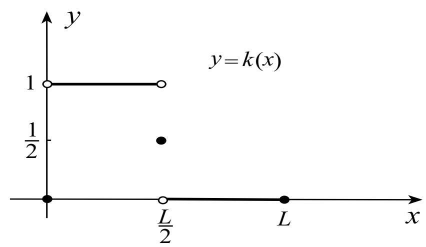


Figure 7.2b

Readers who have studied Laplace transforms should continue with this paragraph; readers unfamiliar with this transform should proceed to the next paragraph. When Laplace transforms are used to solve an ordinary differential equation for a function $y(t)$, the procedure is as follows: take Laplace transforms of all terms in the ODE and solve the resulting algebraic equation for the transform $\tilde{y}(s)$ of $y(t)$. Then use tables of Laplace transforms to invert $\tilde{y}(s)$ to find $y(t)$. The procedure for finite Fourier transforms will be somewhat the same, as we shall see, except that there are, in general, no tables of transforms to help with both processes, taking the transform and taking the inverse transform. The reason for this is that there is an infinity of Sturm-Liouville systems, and with each one, there would be tables of transforms and their inverses. As a result, the processes are much less structured; they depend on which finite Fourier transform is being utilized.

When solving (initial) boundary value problems by finite Fourier transforms, an integral part of the process is to find the inverse finite Fourier transform of a given sequence $\{\tilde{f}(\lambda_n)\}$ of constants for an unknown function $f(x)$. We can always write that the inverse function is

$$f(x) = \sum_{n=1}^{\infty}\tilde{f}(\lambda_n)X_n(x),$$

where the $X_n(x)$ are eigenfunctions of the associated Sturm-Liouville system, but can we find $f(x)$ in closed form? This can sometimes be a daunting task. We illustrate in the following example.

Example 7.2 Given that the finite Fourier transform of a function $f(x)$ with respect to the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = 0 = X(L),$$

is

$$\tilde{f}(\lambda_n) = \frac{\sqrt{2L}[1+(-1)^{n+1}] + (2L)^{3/2}(-1)^{n+1}}{n\pi},$$

find $f(x)$ in closed form.

Solution Eigenvalues of the Sturm-Liouville system are $\lambda_n^2 = n^2\pi^2/L^2$ with normalized eigenfunctions $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$. When $g(x) = x$,

$$\tilde{g}(\lambda_n) = \int_0^L x\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx = \sqrt{\frac{2}{L}}\left\{\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right\}_0^L = \frac{\sqrt{2L^3}(-1)^{n+1}}{n\pi}.$$

In addition, for $k(x) = 1$,

$$\tilde{k}(\lambda_n) = \int_0^L \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx = \sqrt{\frac{2}{L}}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_0^L = \frac{\sqrt{2L}[1+(-1)^{n+1}]}{n\pi}.$$

Since $\tilde{f}(\lambda_n) = 2\tilde{g}(\lambda_n) + \tilde{k}(\lambda_n)$, it follows that $f(x) = 2g(x) + k(x) = 2x + 1$.●

EXERCISES 7.1

In Exercises 1–10 find the finite Fourier transform of the function $f(x)$, defined on the interval $0 \le x \le L$, with respect to the given Sturm-Liouville system.

1. $f(x) = x^2 - 2x$; $X'' + \lambda^2 X = 0,\ X(0) = X'(L) = 0$

2. $f(x) = 5$; $X'' + \lambda^2 X = 0,\ X(0) = X(L) = 0$

3. $f(x) = 5$; $X'' + \lambda^2 X = 0,\ X'(0) = X'(L) = 0$

4. $f(x) = x$; $X'' + \lambda^2 X = 0,\ X(0) = 0,\ l_2X'(L) + h_2X(L) = 0$

5. $f(x) = L - x$; $X'' + \lambda^2 X = 0,\ X'(0) = 0, l_2X'(L) + h_2X(L) = 0$

6. $f(x) = \sin x$; $X'' + \lambda^2 X = 0,\ X'(0) = X(L) = 0$

7. $f(x) = e^x$; $X'' + \lambda^2 X = 0,\ X'(0) = X'(L) = 0$

8. $f(x) = \begin{cases} x^2, & 0 \le x \le L/2 \\ 0, & L/2 < x \le L \end{cases}$; $X'' + \lambda^2 X = 0,\ X(0) = X'(L) = 0$

9. $f(x) = \sin(\pi x/L)\cos(\pi x/L)$; $X'' + \lambda^2 X = 0,\ X(0) = X(L) = 0$

10. $f(x) = 1$; $X'' + 2X' + \lambda^2 X = 0,\ X'(0) = X'(L) = 0$

In Exercises 11–14 find, in closed form, the inverse finite Fourier transform for $\tilde{f}(\lambda_n)$ with respect to the given Sturm-Liouville system.

11. $\tilde{f}(\lambda_n) = (-1)^{n+1}(2L)^{3/2}/(n\pi)$; $X'' + \lambda^2 X = 0,\ X(0) = X(L) = 0$

12. $\tilde{f}(\lambda_n) = \dfrac{3\sqrt{2}L^{5/2}(-1)^n}{n\pi} + \dfrac{6\sqrt{2}L^{5/2}[1+(-1)^{n+1}]}{n^3\pi^3}$; $X'' + \lambda^2 X = 0,\ X(0) = X(L) = 0$

13. $\tilde{f}(\lambda_n) = \begin{cases} 2\sqrt{2}L, & n = 0 \\ 0, & n > 0 \end{cases}$; $X'' + \lambda^2 X = 0,\ X'(0) = X'(L) = 0$

14. $\tilde{f}(\lambda_n) = \dfrac{(2L-1)\sqrt{2/L}(-1)^{n+1}}{\lambda_n} - \dfrac{2\sqrt{2/L}}{{\lambda_n}^2}$; $X'' + \lambda^2 X = 0,\ X'(0) = X(L) = 0$

§7.2 Nonhomogeneous Problems in Two Variables

We now show how finite Fourier transforms can be used to solve (initial) boundary value problems. Every initial boundary value problem that we have solved by separation of variables can also be solved using transforms. There is little advantage, however, in using transforms for homogeneous problems; their power is realized when the PDE and/or the boundary conditions are nonhomogeneous. Nonetheless, we choose to introduce the method with problem 4.9 of Section 4.2, a problem with homogeneous PDE and homogeneous boundary conditions. We do this because the application of finite Fourier transforms to initial boundary value problems always follows the same pattern whether the problem is homogeneous or nonhomogeneous. As a result, we can clearly illustrate the technique in a homogeneous problem without the added complications due to nonhomogeneities.

Separation of variables on

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{7.4a}$$

$$y(0,t) = 0, \quad t > 0, \tag{7.4b}$$

$$y(L,t) = 0, \quad t > 0, \tag{7.4c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{7.4d}$$

$$y_t(x,0) = 0, \quad 0 < x < L, \tag{7.4e}$$

determines separated functions $y(x,t) = X(x)T(t)$ which satisfy 7.4a,b,c,e. The result is a Sturm-Liouville system in $X(x)$ and an ordinary differential equation in $T(t)$,

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \tag{7.5a}$$

$$X(0) = 0, \tag{7.5b}$$

$$X(L) = 0; \tag{7.5c}$$

$$T'' + c^2\lambda^2 T = 0, \quad t > 0, \tag{7.6a}$$

$$T'(0) = 0. \tag{7.6b}$$

From these, separated functions take the form $C\sqrt{2/L}\sin(n\pi x/L)\cos(n\pi ct/L)$ for arbitrary C. The solution of problem 7.4 is obtained by superposing these functions

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}, \tag{7.7a}$$

and imposing condition 7.4d to give

$$c_n = \int_0^L f(x)\sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} dx. \tag{7.7b}$$

To solve this problem by finite Fourier transforms, we note that the transform associated with Sturm-Liouville system 7.5 is

$$\tilde{f}(\lambda_n) = \int_0^L f(x) X_n(x)\, dx, \tag{7.8}$$

where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenvalues and orthonormal eigenfunctions. If we apply this transform to both sides of PDE 7.4a,

$$\int_0^L \frac{\partial^2 y}{\partial t^2} X_n(x)\,dx = c^2 \int_0^L \frac{\partial^2 y}{\partial x^2} X_n(x)\,dx. \tag{7.9}$$

We interchange orders of integration with respect to x and differentiation with respect to t on the left side of this equation. Integration by parts on the right, together with the fact that $X_n(0) = X_n(L) = 0$, gives

$$\frac{\partial^2}{\partial t^2} \int_0^L y X_n(x)\,dx = c^2 \left\{ \frac{\partial y}{\partial x} X_n \right\}_0^L - c^2 \int_0^L \frac{\partial y}{\partial x} X_n'\,dx = -c^2 \int_0^L \frac{\partial y}{\partial x} X_n'\,dx. \tag{7.10}$$

The integral on the left of this equation is the definition of $\tilde{y}(\lambda_n, t)$, the finite Fourier transform of $y(x,t)$. Integration by parts once again on the right yields

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \tilde{y}(\lambda_n, t) &= -c^2 \left\{ y X_n' \right\}_0^L + c^2 \int_0^L y X_n''\,dx \\ &= -c^2 y(L,t) X_n'(L) + c^2 y(0,t) X_n'(0) + c^2 \int_0^L y X_n''\,dx. \end{aligned} \tag{7.11}$$

Now, boundary conditions 7.4b,c imply that the first two terms on the right vanish. Further, equation 7.5a may be used to replace X_n'' with $-\lambda_n^2 X_n$, with the result

$$\frac{\partial^2 \tilde{y}(\lambda_n, t)}{\partial t^2} = c^2 \int_0^L y(-\lambda_n^2 X_n)\,dx = -c^2 \lambda_n^2 \int_0^L y X_n\,dx = -c^2 \lambda_n^2 \tilde{y}(\lambda_n, t). \tag{7.12}$$

Because $\tilde{y}(\lambda_n, t)$ is a function of only one variable, t, and a parameter, λ_n, the partial derivative may be replaced by an ordinary derivative,

$$\frac{d^2 \tilde{y}}{dt^2} = -c^2 \lambda_n^2 \tilde{y}. \tag{7.13a}$$

This is an ordinary differential equation for $\tilde{y}(\lambda_n, t)$. When we take finite Fourier transforms of initial conditions 7.4d,e, we obtain initial conditions for this ODE,

$$\tilde{y}(\lambda_n, 0) = \tilde{f}(\lambda_n), \tag{7.13b}$$

$$\frac{d\tilde{y}(\lambda_n, 0)}{dt} = 0. \tag{7.13c}$$

What the finite Fourier transform has done is replace initial boundary value problem 7.4 for $y(x,t)$ with initial value problem 7.13 for $\tilde{y}(\lambda_n, t)$; a PDE has been reduced to an ODE. In actual fact, 7.13 is an infinite system of ODEs ($n = 1, 2, \ldots$), but because all differential equations have exactly the same form, solving one solves them all.

A general solution of ODE 7.13a is

$$\tilde{y}(\lambda_n, t) = A_n \cos c\lambda_n t + B_n \sin c\lambda_n t, \tag{7.14}$$

where A_n and B_n are constants. Initial conditions 7.13b,c require these constants to satisfy

$$A_n = \tilde{f}(\lambda_n), \qquad 0 = c\lambda_n B_n, \tag{7.15}$$

and therefore

$$\tilde{y}(\lambda_n, t) = \tilde{f}(\lambda_n) \cos c\lambda_n t. \tag{7.16}$$

The inverse transform defines the solution of problem 7.4 as

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} \tilde{y}(\lambda_n, t) X_n(x) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_n) \cos c\lambda_n t X_n(x) \\ &= \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \tilde{f}(\lambda_n) \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}, \end{aligned} \tag{7.17}$$

a solution identical to that obtained by separation of variables.

Briefly, the transform technique replaces the PDE in $y(x,t)$ with an ODE in its transform $\tilde{y}(\lambda_n, t)$. Once the ODE is solved, the inverse transform gives $y(x,t)$. A number of aspects of the method deserve special mention:

1. Not just any finite Fourier transform will yield a solution to this initial boundary value problem. It must be the transform associated with the Sturm-Liouville system 7.5; that is, it must be the transform associated with the Sturm-Liouville system that would result if separation of variables were applied to the problem (see Exercise 2). In nonhomogeneous problems, we use the transform associated with the Sturm-Liouville system that would result were separation used on the corresponding homogeneous problem. Apparently then, to use transforms efficiently, it is advantageous to quickly recognize the Sturm-Liouville system that would result were we to use separation of variables.
2. Boundary conditions on $y(x,t)$ are incorporated in the simplification leading to the ordinary differential equation in $\tilde{y}(\lambda_n, t)$.
3. Initial conditions on $y(x,t)$ are converted by the transform into initial conditions on $\tilde{y}(\lambda_n, t)$.
4. Finite Fourier transforms always give a solution in the form of an infinite series (the inverse transform). It may happen that part or all of the solution is the generalized Fourier series of a simple function. In particular, when nonhomogeneities are time independent, part of the solution is always representable in closed form. Considerable ingenuity may be required to discover this function. The next example illustrates this point.

It is probably fair to say that the transform technique applied to the above problem is more involved than the separation method. This is in agreement with our earlier statement that the transform method shows its true versatility in problems with nonhomogeneous PDEs and/or boundary conditions. To illustrate this, consider problem 4.35 of Section 4.3, where gravity introduces a nonhomogeneity into the PDE,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0, \quad (g = 9.81), \tag{7.18a}$$

$$y(0,t) = 0, \quad t > 0, \tag{7.18b}$$

$$y(L,t) = 0, \quad t > 0, \tag{7.18c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{7.18d}$$

$$y_t(x,0) = 0, \quad 0 < x < L. \tag{7.18e}$$

In Section 4.3 we expressed the solution in the form $y(x,t) = z(x,t) + \psi(x)$, where $\psi(x) = [g/(2c^2)](x^2 - Lx)$ is the solution of the corresponding static deflection problem. The function $z(x,t)$ must then satisfy the homogeneous problem

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{7.19a}$$

$$z(0,t) = 0, \quad t > 0, \tag{7.19b}$$

$$z(L,t) = 0, \quad t > 0, \tag{7.19c}$$

$$z(x,0) = f(x) + \frac{g}{2c^2}(Lx - x^2), \quad 0 < x < L, \tag{7.19d}$$

$$z_t(x,0) = 0, \quad 0 < x < L. \tag{7.19e}$$

Separation of variables on problem 7.19 gives

$$z(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}, \tag{7.20a}$$

where

$$c_n = \frac{2}{L} \int_0^L \left[f(x) + \frac{g}{2c^2}(Lx - x^2) \right] \sin \frac{n\pi x}{L} dx. \tag{7.20b}$$

The final solution is

$$y(x,t) = z(x,t) + \frac{g}{2c^2}(x^2 - Lx). \tag{7.21}$$

Consider now the finite Fourier transform technique applied to this problem. The transform associated with this problem is again 7.8, where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenpairs of 7.5 (this being the Sturm-Liouville system that would result were separation of variables used on the corresponding homogeneous problem). If we apply the transform to PDE 7.18a,

$$\int_0^L \frac{\partial^2 y}{\partial t^2} X_n(x)\,dx = \int_0^L \left(c^2 \frac{\partial^2 y}{\partial x^2} - g \right) X_n(x)\,dx. \tag{7.22}$$

Integration by parts on the right, along with the fact that $X_n(0) = X_n(L) = 0$, gives

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \int_0^L y X_n\,dx &= c^2 \left\{ \frac{\partial y}{\partial x} X_n \right\}_0^L - c^2 \int_0^L \frac{\partial y}{\partial x} X_n'\,dx - g\tilde{1} \\ &= -c^2 \int_0^L \frac{\partial y}{\partial x} X_n'\,dx - g\tilde{1}, \end{aligned} \tag{7.23}$$

where $\tilde{1}$ is the transform of the function identically equal to unity,

$$\tilde{1} = \int_0^L X_n(x)\,dx = \int_0^L \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx = \frac{\sqrt{2L}}{n\pi}[1 + (-1)^{n+1}]. \tag{7.24}$$

Integration by parts again and boundary conditions 7.18b,c yield

$$\frac{\partial^2}{\partial t^2} \tilde{y}(\lambda_n, t) = -c^2 \Big\{ y X_n' \Big\}_0^L + c^2 \int_0^L y X_n''\,dx - g\tilde{1} = c^2 \int_0^L y(-\lambda_n^2 X_n)\,dx - g\tilde{1}$$

or,

$$\frac{d^2\tilde{y}}{dt^2} + c^2\lambda_n^2\tilde{y} = -g\tilde{1}. \tag{7.25a}$$

This is an ordinary differential equation for $\tilde{y}(\lambda_n, t)$. Transforms of initial conditions 7.18d,e require $\tilde{y}(\lambda_n, t)$ to satisfy the initial conditions

$$\tilde{y}(\lambda_n, 0) = \tilde{f}(\lambda_n), \tag{7.25b}$$

$$\frac{d\tilde{y}(\lambda_n, 0)}{dt} = 0. \tag{7.25c}$$

A general solution of the ODE is

$$\tilde{y}(\lambda_n, t) = A_n \cos c\lambda_n t + B_n \sin c\lambda_n t - \frac{g\tilde{1}}{c^2\lambda_n^2}, \tag{7.26}$$

where A_n and B_n are constants. The initial conditions require these constants to satisfy

$$\tilde{f}(\lambda_n) = A_n - \frac{g\tilde{1}}{c^2\lambda_n^2}, \qquad 0 = c\lambda_n B_n, \tag{7.27}$$

and therefore

$$\tilde{y}(\lambda_n, t) = \left[\tilde{f}(\lambda_n) + \frac{g\tilde{1}}{c^2\lambda_n^2}\right]\cos c\lambda_n t - \frac{g\tilde{1}}{c^2\lambda_n^2}. \tag{7.28}$$

The inverse transform now defines the solution of problem 7.18 as

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} \tilde{y}(\lambda_n, t) X_n(x) = \sum_{n=1}^{\infty} X_n(x)\left\{\left[\tilde{f}(\lambda_n) + \frac{g\tilde{1}}{c^2\lambda_n^2}\right]\cos c\lambda_n t - \frac{g\tilde{1}}{c^2\lambda_n^2}\right\} \\ &= \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left\{\left[\tilde{f}(\lambda_n) + \frac{g\tilde{1}}{c^2\lambda_n^2}\right]\cos c\lambda_n t - \frac{g\tilde{1}}{c^2\lambda_n^2}\right\}. \end{aligned} \tag{7.29}$$

To show that this solution is identical to that obtained by separation, we calculate that for $\psi(x) = [g/(2c^2)](x^2 - Lx)$,

$$\tilde{\psi}(\lambda_n) = \int_0^L \psi(x) X_n(x)\, dx = \frac{-g}{c^2 n^3 \pi^3}\sqrt{2L^5}[1 + (-1)^{n+1}] = \frac{-g\tilde{1}}{c^2\lambda_n^2}. \tag{7.30}$$

Consequently, the last term of the series in 7.29 can be expressed as

$$\sum_{n=1}^{\infty}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\left(\frac{-g\tilde{1}}{c^2\lambda_n^2}\right) = \sum_{n=1}^{\infty}\tilde{\psi}(\lambda_n)X_n(x) = \psi(x).$$

Solution 7.29 can therefore be written in the form

$$y(x,t) = \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}[\tilde{f}(\lambda_n) - \tilde{\psi}(\lambda_n)]\cos\frac{n\pi ct}{L} + \psi(x), \tag{7.31}$$

which is clearly identical to that obtained by separation.

The transform method applied to this problem with a nonhomogeneous PDE is essentially the same as when applied to the homogeneous problem 7.4. This is the advantage of the transform method; it does not require homogeneous PDEs or boundary conditions. To illustrate the method applied to nonhomogeneous boundary conditions, we consider Example 4.5 in Section 4.3,

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{7.32a}$$

$$U(0,t) = U_0, \quad t > 0, \tag{7.32b}$$

$$U(L,t) = U_L, \quad t > 0, \tag{7.32c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{7.32d}$$

The finite Fourier transform for this problem is once again 7.3, where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are eigenpairs of Sturm-Liouville system 7.5 (obtained by separation when 7.32b,c are homogeneous). If we apply this transform to PDE 7.32a,

$$\int_0^L \frac{\partial U}{\partial t} X_n(x)\,dx = k\int_0^L \frac{\partial^2 U}{\partial x^2} X_n(x)\,dx. \tag{7.33}$$

Integration by parts on the right, together with the fact that $X_n(0) = X_n(L) = 0$, gives

$$\frac{\partial}{\partial t}\int_0^L UX_n\,dx = k\left\{\frac{\partial U}{\partial x}X_n\right\}_0^L - k\int_0^L \frac{\partial U}{\partial x}X_n'\,dx = -k\int_0^L \frac{\partial U}{\partial x}X_n'\,dx.$$

Another integration by parts yields

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{U}(\lambda_n,t) &= -k\Big\{UX_n'\Big\}_0^L + k\int_0^L UX_n''\,dx \\ &= -k[U(L,t)X_n'(L) - U(0,t)X_n'(0)] + k\int_0^L U(-\lambda_n^2 X_n)\,dx,\end{aligned}$$

in which we may use boundary conditions 7.32b,c,

$$\begin{aligned}\frac{d\tilde{U}}{dt} &= -kU_L\sqrt{\frac{2}{L}}\lambda_n(-1)^n + kU_0\sqrt{\frac{2}{L}}\lambda_n - k\lambda_n^2\tilde{U} \\ &= -k\lambda_n^2\tilde{U} + k\sqrt{\frac{2}{L}}\lambda_n[U_0 + U_L(-1)^{n+1}].\end{aligned} \tag{7.34a}$$

Accompanying this ODE in $\tilde{U}(\lambda_n,t)$ is the transform of initial condition 7.32d,

$$\tilde{U}(\lambda_n,0) = \tilde{f}(\lambda_n). \tag{7.34b}$$

A general solution of ODE 7.34a is

$$\tilde{U}(\lambda_n,t) = A_n e^{-k\lambda_n^2 t} + \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}], \tag{7.35}$$

where A_n is a constant. Initial condition 7.34b requires

$$\tilde{f}(\lambda_n) = A_n + \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}], \tag{7.36}$$

and therefore

$$\tilde{U}(\lambda_n, t) = e^{-k\lambda_n^2 t}\left\{\tilde{f}(\lambda_n) - \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}]\right\} + \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}]. \tag{7.37}$$

Inverse transform 7.3b defines the solution of problem 7.32 as

$$\begin{aligned} U(x,t) &= \sum_{n=1}^{\infty} \tilde{U}(\lambda_n, t) X_n(x) \\ &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \left\{ e^{-n^2\pi^2 kt/L^2} \left[\tilde{f}(\lambda_n) - \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}]\right] \right. \\ &\qquad \left. + \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}] \right\}. \end{aligned} \tag{7.38}$$

To show that this solution is identical to that obtained by separation of variables in Example 4.5 in Section 4.3, we calculate that for $\psi(x) = U_0 + (U_L - U_0)x/L$,

$$\tilde{\psi}(\lambda_n) = \int_0^L \psi(x) X_n(x)\, dx = \lambda_n^{-1}[U_0 + U_L(-1)^{n+1}]. \tag{7.39}$$

Solution 7.38 can therefore be written in the form

$$U(x,t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} e^{-n^2\pi^2 kt/L^2}[\tilde{f}(\lambda_n) - \tilde{\psi}(\lambda_n)] \sin\frac{n\pi x}{L} + \psi(x), \tag{7.40}$$

identical to that obtained by separation of variables.

If we set $x = 0$ and $x = L$ in solution 7.38, we obtain $U(0,t) = U(L,t) = 0$, whereas $x = 0$ and $x = L$ in solution 7.40 give $U(0,t) = U_0$ and $U(L,t) = U_L$, provided we define $\psi(0) = U_0$ and $\psi(L) = U_L$. In other words, the function in 7.38 does not satisfy boundary conditions 7.32b,c, but 7.40 does. This is because the series expansion of $\psi(x)$ in 7.38 is a Fourier sine series, and as such it converges to the odd extension of $\psi(x)$ to a function of period $2L$. At $x = 0$ and $x = L$, this extension (see Figure 7.3) is discontinuous, and the series therefore converges to the average value of the right and left limits, namely zero. For any other value of x between 0 and L, solutions 7.38 and 7.40 give identical results.

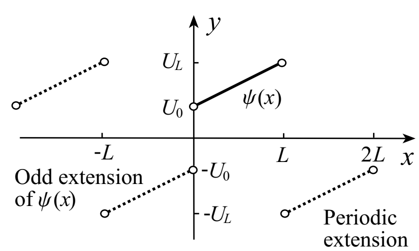


Figure 7.3

Parts of a finite Fourier transform solution that can be expressed in closed form should always be represented. The above example suggests that in so doing, values of the closed form portion at end points should be defined as limiting values. An additional reason for extracting closed form portions is that the rate of convergence of the remaining series is enhanced. To illustrate this, suppose, for simplicity, that $f(x) = 0$ in problem 7.32. The finite Fourier transform solution 7.38 with $f(x) = 0$ is

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[U_0 + U_L(-1)^{n+1}](1 - e^{-n^2\pi^2 kt/L^2})\sin\frac{n\pi x}{L}. \tag{7.41}$$

Solution 7.40 with $f(x) = 0$ is

$$U(x,t) = U_0 + \frac{(U_L - U_0)x}{L} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[U_0 + U_L(-1)^{n+1}]e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}. \tag{7.42}$$

Terms in solution 7.42 have the factor $1/n$, but each is multiplied by the exponential $e^{-n^2\pi^2 kt/L^2}$ that decreases rapidly to zero for large n and/or t. In solution 7.41, $1 - e^{-n^2\pi^2 kt/L^2}$ replaces $e^{-n^2\pi^2 kt/L^2}$. For large n or t, these factors approach unity, not zero. Convergence is much slower.

In the remainder of this section we consider two additional problems that have more general nonhomogeneities.

Example 7.3 Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{7.43a}$$

$$U(0,t) = f_1(t), \quad t > 0, \tag{7.43b}$$

$$U_x(L,t) = -\kappa^{-1}f_2(t), \quad t > 0, \tag{7.43c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{7.43d}$$

Described is a rod of length L with insulated sides that at time $t = 0$ has temperature $f(x)$. For $t > 0$, the temperature of its left end is a prescribed $f_1(t)$, and heat is transferred across the right end at a rate $f_2(t)$. When $f_2(t)$ is positive, heat is being removed from the rod, and when it is negative, heat is being added.

Solution If separation of variables is applied to the associated homogeneous problem (with $f_1(t) = f_2(t) = 0$), the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad X(0) = 0 = X'(L),$$

results. Eigenvalues are $\lambda_n^2 = (2n-1)^2\pi^2/(4L^2)$, with corresponding eigenfunctions $X_n(x) = \sqrt{2/L}\sin\lambda_n x$. If we apply the finite Fourier transform associated with this system to PDE 7.43a,

$$\int_0^L \frac{\partial U}{\partial t}X_n(x)\,dx = k\int_0^L \frac{\partial^2 U}{\partial x^2}X_n(x)\,dx.$$

Integration by parts on the right gives

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^L UX_n\,dx &= k\left\{\frac{\partial U}{\partial x}X_n\right\}_0^L - k\int_0^L \frac{\partial U}{\partial x}X_n'\,dx \\ &= kU_x(L,t)X_n(L) - k\Big\{UX_n'\Big\}_0^L + k\int_0^L UX_n''\,dx \\ &= k\left[U_x(L,t)X_n(L) + U(0,t)X_n'(0) + \int_0^L -\lambda_n^2 X_n U\,dx\right].\end{aligned}$$

When we use boundary conditions 7.43b,c, we may write

$$\frac{d\tilde{U}}{dt} = k[-\kappa^{-1} f_2(t) X_n(L) + f_1(t) X_n'(0) - \lambda_n^2 \tilde{U}(\lambda_n, t)].$$

Thus, $\tilde{U}(\lambda_n, t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\lambda_n^2 \tilde{U} = A(\lambda_n, t), \tag{7.44a}$$

where

$$\begin{aligned} A(\lambda_n, t) &= k[-\kappa^{-1} f_2(t) X_n(L) + f_1(t) X_n'(0)] \\ &= k\sqrt{\frac{2}{L}}[(-1)^n \kappa^{-1} f_2(t) + \lambda_n f_1(t)], \end{aligned} \tag{7.44b}$$

subject to the transform of 7.43d,

$$\tilde{U}(\lambda_n, 0) = \tilde{f}(\lambda_n). \tag{7.44c}$$

A general solution of ODE 7.44a is

$$\tilde{U}(\lambda_n, t) = e^{-k\lambda_n^2 t} \int A(\lambda_n, t) e^{k\lambda_n^2 t} dt.$$

In order to incorporate initial condition 7.44c with its arbitrary function $\tilde{f}(\lambda_n)$, it is advantageous to express this solution as a definite integral,

$$\begin{aligned} \tilde{U}(\lambda_n, t) &= e^{-k\lambda_n^2 t} \left[\int_0^t A(\lambda_n, u) e^{k\lambda_n^2 u} du + C_n \right] \\ &= C_n e^{-k\lambda_n^2 t} + \int_0^t A(\lambda_n, u) e^{k\lambda_n^2 (u-t)} du. \end{aligned}$$

Condition 7.44c now requires $\tilde{f}(\lambda_n) = C_n$, and therefore

$$\tilde{U}(\lambda_n, t) = \tilde{f}(\lambda_n) e^{-k\lambda_n^2 t} + \int_0^t A(\lambda_n, u) e^{k\lambda_n^2 (u-t)} du. \tag{7.45}$$

The solution to problem 7.43 is defined by the inverse finite Fourier transform,

$$\begin{aligned} U(x, t) &= \sum_{n=1}^{\infty} \tilde{U}(\lambda_n, t) X_n(x) \\ &= \sum_{n=1}^{\infty} \left[\tilde{f}(\lambda_n) e^{-k\lambda_n^2 t} + \int_0^t A(\lambda_n, u) e^{k\lambda_n^2 (u-t)} du \right] \sqrt{\frac{2}{L}} \sin \lambda_n x. \end{aligned} \tag{7.46}$$

As a specific example, suppose the rod is initially at temperature zero ($f(x) \equiv 0$), its right end is insulated ($f_2(t) \equiv 0$), and its left end is held at constant temperature 100°C. According to equations 7.44b and 7.45,

$$\tilde{U}(\lambda_n, t) = \int_0^t k\sqrt{\frac{2}{L}} \lambda_n (100) e^{k\lambda_n^2 (u-t)} du = \frac{100\sqrt{2/L}}{\lambda_n}(1 - e^{-k\lambda_n^2 t}),$$

and hence,

$$U(x,t) = \sum_{n=1}^{\infty} \frac{100\sqrt{2/L}}{\lambda_n}(1 - e^{-k\lambda_n^2 t})\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}.$$

The solution may be simplified by noting that when $g(x) = 100$,

$$\tilde{g}(\lambda_n) = \int_0^L 100\sqrt{\frac{2}{L}}\sin\lambda_n x\,dx = \frac{100\sqrt{2/L}}{\lambda_n}.$$

Thus,

$$g(x) = 100 = \sum_{n=1}^{\infty}\frac{100\sqrt{2/L}}{\lambda_n}\sqrt{\frac{2}{L}}\sin\lambda_n x,$$

and it follows that

$$U(x,t) = 100 - \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\pi^2 kt/(4L^2)}}{2n-1}\sin\frac{(2n-1)\pi x}{2L}.$$

This function is plotted for various values of t in Figure 7.4 (assuming a thermal diffusivity of $k = 12 \times 10^{-6}$ m^2/s). Notice, in particular, that each curve is horizontal at $x = L$, a consequence of the insulation there.●

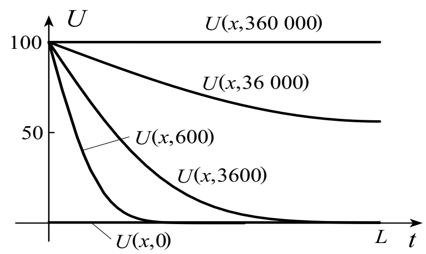


Figure 7.4

Example 7.4 A taut string has one end, at $x = 0$, fixed on the x-axis while the other end, at $x = L$, is forced to undergo periodic vertical motion described by $g(t) = A\sin\omega t$, $t \geq 0$ (A a constant). If the string is initially at rest on the x-axis, find its subsequent displacement.

Solution The initial boundary value problem for displacements $y(x,t)$ of points on the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{7.47a}$$

$$y(0,t) = 0, \quad t > 0, \tag{7.47b}$$

$$y(L,t) = g(t), \quad t > 0, \tag{7.47c}$$

$$y(x,0) = 0, \quad 0 < x < L, \tag{7.47d}$$

$$y_t(x,0) = 0, \quad 0 < x < L. \tag{7.47e}$$

The finite Fourier transform associated with x is

$$\tilde{f}(\lambda_n) = \int_0^L f(x)X_n(x)\,dx,$$

where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$. Application of the transform to PDE 7.47a leads to the following ODE in $\tilde{y}(\lambda_n, t)$,

$$\frac{d^2\tilde{y}}{dt^2} + c^2\lambda_n^2\tilde{y} = -c^2 X_n'(L)g(t) \tag{7.48a}$$

subject to

$$\tilde{y}(\lambda_n, 0) = \tilde{y}'(\lambda_n, 0) = 0. \tag{7.48b}$$

(Details are left to the reader). Variation of parameters on problem 7.48 gives the solution in the form

$$\tilde{y}(\lambda_n, t) = \frac{-cX_n'(L)}{\lambda_n} \int_0^t g(u) \sin c\lambda_n(t-u)\, du. \tag{7.49}$$

This is a general formula valid for any function $g(t)$ whatsoever. In this problem, $g(t) = A\sin\omega t$, so that $\tilde{y}(\lambda_n, t)$ could be obtained by evaluation of integral 7.49. (Try it.) Alternatively, if we return to ODE 7.48a, a general solution when $g(t) = A\sin\omega t$ is

$$\tilde{y}(\lambda_n, t) = B_n \cos c\lambda_n t + D_n \sin c\lambda_n t - \frac{Ac^2 X_n'(L)}{c^2\lambda_n^2 - \omega^2} \sin\omega t, \tag{7.50}$$

provided $\omega \neq c\lambda_n$ for any integer n. Initial conditions 7.48b imply that

$$0 = B_n, \qquad 0 = c\lambda_n D_n - \frac{Ac^2\omega X_n'(L)}{c^2\lambda_n^2 - \omega^2},$$

from which

$$\tilde{y}(\lambda_n, t) = \frac{Ac\omega X_n'(L)}{\lambda_n(c^2\lambda_n^2 - \omega^2)} \sin c\lambda_n t - \frac{Ac^2 X_n'(L)}{c^2\lambda_n^2 - \omega^2} \sin\omega t. \tag{7.51}$$

Thus,

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} \tilde{y}(\lambda_n, t) X_n(x) \\ &= \sum_{n=1}^{\infty} \frac{AcX_n'(L)}{c^2\lambda_n^2 - \omega^2} \left(\frac{\omega}{\lambda_n} \sin c\lambda_n t - c\sin\omega t \right) X_n(x) \\ &= 2cA \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2c^2 - \omega^2 L^2} \left(\omega L \sin\frac{n\pi ct}{L} - n\pi c\sin\omega t \right) \sin\frac{n\pi x}{L}. \end{aligned} \tag{7.52a}$$

This is the solution of problem 7.47, provided $\omega \neq c\lambda_n$; that is, provided ω is not equal to a natural frequency of the vibrating string. If this solution is separated into two series,

$$\begin{aligned} y(x,t) = 2\omega cLA &\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2c^2 - \omega^2L^2} \sin\frac{n\pi ct}{L} \sin\frac{n\pi x}{L} \\ &+ 2\pi c^2 A \sin\omega t \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{n^2\pi^2c^2 - \omega^2L^2} \sin\frac{n\pi x}{L}, \end{aligned}$$

it is not unreasonable to expect that the second series, since it is void of t, is the Fourier sine series for some function. Indeed, it is straightforward to show that the series represents $(2\pi c^2)^{-1} \sin(\omega x/c)/\sin(\omega L/c)$. In other words, the solution can be expressed in the simplified form

$$y(x,t) = \frac{A\sin(\omega x/c)\sin\omega t}{\sin(\omega L/c)} + 2\omega cLA\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2c^2 - \omega^2L^2}\sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}. \quad (7.52b)$$

(Undoubtedly, the question in your mind must be: How did we arrive at the function $(2\pi c^2)^{-1}\sin(\omega x/c)/\sin(\omega L/c)$? The answer is experience, or in conjunction with other techniques like Laplace transforms.)

We now investigate what happens when ω is equal to a natural frequency of the vibrating string; that is, suppose $\omega = m\pi c/L$ for some integer m. When $n \neq m$, solution 7.51 of 7.48 is unchanged. But for $n = m$, $\tilde{y}(\lambda_m, t)$ must satisfy

$$\frac{d^2\tilde{y}}{dt^2} + c^2\lambda_m^2\tilde{y} = -c^2X_m'(L)A\sin c\lambda_m t, \quad (7.53a)$$

$$\tilde{y}(\lambda_m, 0) = \tilde{y}'(\lambda_m, 0) = 0. \quad (7.53b)$$

A general solution of the differential equation is

$$\tilde{y}(\lambda_m, t) = B_m\cos c\lambda_m t + D_m\sin c\lambda_m t + \frac{AcX_m'(L)}{2\lambda_m}t\cos c\lambda_m t. \quad (7.54)$$

The initial conditions imply that

$$0 = B_m, \qquad 0 = c\lambda_m D_m + \frac{AcX_m'(L)}{2\lambda_m},$$

from which

$$\tilde{y}(\lambda_m, t) = \frac{-AX_m'(L)}{2\lambda_m^2}\sin c\lambda_m t + \frac{AcX_m'(L)}{2\lambda_m}t\cos c\lambda_m t. \quad (7.55)$$

In other words, when $\omega = c\lambda_m = m\pi c/L$, the sequence $\{\tilde{y}(\lambda_n, t)\}$ remains unchanged except for the m^{th} term. The inverse transform now gives

$$y(x,t) = \tilde{y}(\lambda_m, t)X_m(x) + \sum_{\substack{n=1 \\ n\neq m}}^{\infty}\tilde{y}(\lambda_n, t)X_n(x),$$

and substitutions from 7.55 and 7.51 lead to

$$\begin{aligned} y(x,t) = {}& \frac{A(-1)^m}{L}\left(ct\cos\frac{m\pi ct}{L} - \frac{L}{m\pi}\sin\frac{m\pi ct}{L}\right)\sin\frac{m\pi x}{L} \\ &+ \frac{2A}{\pi}\sum_{\substack{n=1 \\ n\neq m}}^{\infty}\frac{(-1)^n}{n^2 - m^2}\left(m\sin\frac{n\pi ct}{L} - n\sin\frac{m\pi ct}{L}\right)\sin\frac{n\pi x}{L}. \quad (7.56) \end{aligned}$$

For large t, the first term in solution 7.56 becomes unbounded. This phenomenon is known as **resonance**. When the forcing frequency (ω) is equal to one of the natural frequencies ($c\lambda_n$) of the vibrating system, oscillations become excessive and destroy the system.●

Further instances of resonance are discussed in Exercises 26–36 and 39. In some applications, resonance is disastrous; in others, resonance (of a slightly different nature) is exactly what is desired.

In this section we have dealt with initial boundary problems. Finite Fourier transforms can also be used to solve nonhomogeneous boundary value problems in

Cartesian coordinates x and y. We have already suggested (see Section 6.3) that when nonhomogeneities occur only in boundary conditions, the problem can easily be solved by subdivision into homogeneous problems. In other words, finite Fourier transforms need only be used to accommodate nonhomogeneities in the PDE. This is illustrated in Exercises 48–51.

EXERCISES 7.2

Use finite Fourier transforms to solve all problems in this set of exercises. Wherever possible, express solutions, or parts of solutions, in closed form.

Part A Heat Conduction

1. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature $f(x)$, $0 \le x \le L$, at time $t = 0$. For time $t > 0$, the end $x = 0$ is held at 0°C and the end $x = L$ is held at constant temperature U_L°C. What is the temperature in the rod for $0 < x < L$ and $t > 0$?

2. We have claimed that to solve an initial boundary value problem with finite Fourier transforms, it is necessary to use the transform associated with the Sturm-Liouville system that would result were separation of variables used on the corresponding homogeneous problem. To illustrate this, apply the finite Fourier transform associated with Sturm-Liouville system 5.2 of Chapter 5 to Exercise 1. Show that an insoluble problem in $\tilde{U}(\lambda_n, t)$ is obtained.

3. Solve Exercise 1 in Section 4.3.

4. Solve Exercise 6 in Section 4.3.

5. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature $U_0(1 - x/L)$, where U_0 is a constant. For time $t > 0$, the end $x = 0$ is maintained at temperature U_0 and end $x = L$ is insulated. Find the temperature in the rod for $0 < x < L$ and $t > 0$.

6. Solve the initial boundary value problem for temperature in a homogeneous, isotropic rod with insulated sides, and ends held at temperature zero. Heat generation is defined at position x and time t by $g(x, t)$, and the initial temperature of the rod is described by $f(x)$.

7. Repeat Exercise 6 if the ends of the rod are insulated.

8. (a) Show that finite Fourier transforms applied to Exercise 5 of Section 4.3 when $k \neq L^2/(n^2\pi^2)$ leads to the following solution,

$$U(x,t) = 200\sum_{n=1}^{\infty}\left\{\left[\frac{[1+(-1)^{n+1}}{n\pi} + \frac{n\pi k(-1)^n}{n^2\pi^2 k - L^2}\right]e^{-n^2\pi^2 kt/L^2} + \frac{n\pi k(-1)^{n+1}}{n^2\pi^2 k - L^2}e^{-t}\right\}\sin\frac{n\pi x}{L}.$$

(b) Simplify this solution by finding the transform of the function $f(x) = x$, and using the following partial fraction decomposition on the last term

$$\frac{1}{n(n^2\pi^2 k - L^2)} = \frac{-1/L^2}{n} + \frac{n\pi^2 k/L^2}{n^2\pi^2 k - L^2}.$$

(c) Solve the problem when $k = L^2/(m^2\pi^2)$ for some positive integer m.

9. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature zero throughout. For time $t > 0$, there is located at cross section $x = b$ $(0 < b < L)$ a plane heat source of constant strength g. If the ends $x = 0$ and $x = L$ of the rod are kept at zero temperature, the initial boundary value problem for temperature in the rod is

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial t^2} + \frac{kg}{\kappa}\delta(x-b), \quad 0 < x < L, \quad t > 0,\\
U(0,t) &= 0, \quad t > 0,\\
U(L,t) &= 0, \quad t > 0,\\
U(x,0) &= 0, \quad 0 < x < L,
\end{aligned}$$

where $\delta(x - b)$ is the Dirac delta function. Solve this problem for $U(x,t)$.

10. Solve Exercise 16 in Section 4.3.

11. Solve Exercise 8 in Section 4.3.

12. Repeat Exercise 5 if the temperature of the end $x = 0$ is $U_0 e^{-\alpha t}$ ($\alpha > 0$ a constant). Assume that (a) $\alpha \neq (2n-1)^2\pi^2 k/(4L^2)$ for any integer n, and (b) $\alpha = (2m-1)^2\pi^2 k/(4L^2)$ for some integer m.

13. If the ends $x = 0$ and $x = L$ of the thin-wire problem in Exercise 4 of Section 6.2 are kept at constant temperatures U_0 and U_L, respectively, and the initial temperature is zero throughout, show that

$$\begin{aligned}
U(x,t) &= \frac{U_0 \sinh\sqrt{h/k}(L-x) + U_L \sinh\sqrt{h/k}x}{\sinh\sqrt{h/k}L}\\
&\quad - 2k\pi e^{-ht}\sum_{n=1}^{\infty}\frac{n[U_0 + (-1)^{n+1}U_L]}{hL^2 + n^2\pi^2 k}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}.
\end{aligned}$$

14. Repeat Exercise 5 if heat is added uniformly over the end $x = L$ at a constant rate q W/m^2.

15. (a) A cylindrical, homogeneous, isotropic rod with insulated sides is initially at constant temperature U_0 throughout. For time $t > 0$, the right end, $x = L$, continues to be held at temperature U_0. Heat is added uniformly over the left end, $x = 0$, at a constant rate q W/m^2 for the first t_0 seconds, and the end is insulated thereafter. Find the temperature in the rod for $0 < x < L$ and $0 < t < t_0$.
(b) Assuming that $U(x,t)$ must be continuous at time t_0, find $U(x,t)$ for $0 < x < L$ and $t > t_0$.
(c) What is the steady-state solution?

16. Repeat Exercise 15 if the end $x = L$ is insulated.

17. Find a formula for the solution of the general one-dimensional heat conduction problem

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial t^2} + \frac{kg(x,t)}{\kappa}, \quad 0 < x < L, \quad t > 0,\\
-l_1\frac{\partial U}{\partial x} + h_1 U &= f_1(t), \quad x = 0, \quad t > 0,\\
l_2\frac{\partial U}{\partial x} + h_2 U &= f_2(t), \quad x = L, \quad t > 0,\\
U(x,0) &= f(x), \quad 0 < x < L.
\end{aligned}$$

18. The general thin-wire problem (see Exercise 41 in Section 2.2) is

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} - h(U - U_m) + \frac{kg(x,t)}{\kappa}, \quad 0 < x < L, \quad t > 0,\\
-l_1\frac{\partial U}{\partial x} + h_1 U &= f_1(t), \quad x = 0, \quad t > 0,\\
l_2\frac{\partial U}{\partial x} + h_2 U &= f_2(t), \quad x = L, \quad t > 0,\\
U(x,0) &= f(x), \quad 0 < x < L.
\end{aligned}
$$

(a) Show that the change of dependent variable $\overline{U}(x,t) = e^{ht}U(x,t)$ leads to the initial boundary value problem

$$
\begin{aligned}
\frac{\partial \overline{U}}{\partial t} &= k\frac{\partial^2 \overline{U}}{\partial x^2} + \left[hU_m + \frac{kg(x,t)}{\kappa}\right]e^{ht}, \quad 0 < x < L, \quad t > 0,\\
-l_1\frac{\partial \overline{U}}{\partial x} + h_1\overline{U} &= e^{ht}f_1(t), \quad x = 0, \quad t > 0,\\
l_2\frac{\partial \overline{U}}{\partial x} + h_2\overline{U} &= e^{ht}f_2(t), \quad x = L, \quad t > 0,\\
\overline{U}(x,0) &= f(x), \quad 0 < x < L.
\end{aligned}
$$

(b) Use the result of Exercise 17 to find $\overline{U}(x,t)$ and hence $U(x,t)$.

Vibrations

19. Solve Exercise 17 in Section 4.3.

20. Solve Exercise 19 in Section 4.3.

21. Solve Exercise 18 in Section 4.3.

22. Solve Exercise 22 in Section 4.3.

23. Solve Exercise 23 in Section 4.3.

24. (a) Find a series solution for displacements of the bar in Exercise 20 of Section 4.3.
(b) Find a closed form representation for $y(x,t)$.
(c) Evaluate $y(L,t)$ and draw its graph as a function of t to illustrate the motion of the end $x = L$ of the bar. *Hint*: See Exercise 20 in Section 3.2.

25. A horizontal elastic bar of natural length L lies along the x-axis between $x = 0$ and $x = L$. At time $t = 0$, it is stretched so that displacements of cross sections at positions x are given by the function kx, $k > 0$ a constant, $0 \le x \le L$. The bar is released from rest at this position. If a constant force per unit area F acts parallel to the bar on the end $x = 0$, find subsequent displacements of cross sections of the bar.

26. (a) Solve parts (a) and (b) of Exercise 21 in Section 4.3.
(b) Discuss the resonant case.

27. (a) A horizontal elastic bar is originally at rest and unstrained along the x-axis between $x = 0$ and $x = L$. For time $t > 0$, the left end is fixed and the right end is subjected to an elongating force per unit area $F_0 \sin\omega t$ parallel to the bar. Find a series representation for displacements of cross sections of the bar in the nonresonant case.
(b) Simplify the solution by finding the finite Fourier transform of the function $\sin(\omega x/c)$.

(c) Discuss the resonant case.

28. (a) A taut string initially at rest along the x-axis has its end at $x = 0$ fixed on the x-axis. The end $x = L$ is forced to undergo periodic vertical motion $A \sin \omega t$, $t \geq 0$ (A and ω constants). Find displacements for points of the string.

(b) Discuss the resonant case.

In Exercises 29–36 deteremine frequencies of the applied force that will produce resonance. Do not determine the solution to the initial boundary value problem, only the frequencies.

29. A taut string with one end $x = 0$ fixed on the x-axis and the other end $x = L$ free to slide vertically is initially at rest along the x-axis. An external force $F_0 \sin \omega t$, $t \geq 0$, per unit x-length acts at every point on the string.

30. A taut string with both ends free to slide vertically is initially at rest along the x-axis. An external force $F_0 \sin \omega t$, $t \geq 0$, per unit x-length acts at every point on the string. (Find the solution $y(x,t)$ in this case.)

31. A horizontal elastic bar of natural length L lies along the x-axis between $x = 0$ and $x = L$. Its left end is fixed at $x = 0$, and a force per unit area $F = F_0 \sin \omega t$ acts parallel to the bar on the end $x = L$.

32. The bar in Exercise 31 if the end $x = 0$ is free.

33. The bar in Exercise 31 if the end $x = L$ has a prescribed displacement $A_0 \sin \omega t$.

34. The bar in Exercise 31 if the end $x = 0$ is free and the end $x = L$ has a prescribed displacement $A_0 \sin \omega t$.

35. The bar in Exercise 31 if the ends $x = 0$ and $x = L$ have prescribed displacements $A_0 \sin \omega t$ and $B_0 \sin \phi t$, respectively.

36. The bar in Exercise 31 if the ends $x = 0$ and $x = L$ are subjected to forces $F_0 \sin \omega t$ and $G_0 \sin \phi t$ (per unit area), respectively.

37. An elastic bar of natural length L is clamped along its length, turned to the vertical position, and hung from its end $x = 0$. At time $t = 0$, the clamp is removed and gravity is therefore permitted to act on the bar.

(a) Show that vertical displacements of cross sections of the bar are given by

$$y(x,t) = \frac{gx(2L-x)}{2c^2} - \frac{16gL^2}{c^2\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos \frac{(2n-1)\pi ct}{2L} \sin \frac{(2n-1)\pi x}{2L}.$$

(b) Find a closed-form solution for $y(x,t)$. (Hint: See Exercise 20 in Section 3.2.)

(c) Draw a graph of $y(L,t)$. Does the end $x = L$ of the bar oscillate about its equilibrium position, that is, the position of the lower end of the bar if the bar were to hang motionless under its own weight? (See Exercise 15 in Section 2.3.) Is the motion simple harmonic?

38. (a) Find displacements in the bar of Exercise 37 if the top of the bar is attached to a spring with constant k. Let $x = 0$ correspond to the top end of the bar when the spring is in the unstretched position.

(b) Does the lower end of the bar oscillate about its equilibrium position? (See Exercise 16 in Section 2.3.)

39. Repeat Example 7.4 if a damping force $-\beta \partial y/\partial t$, proportional to velocity, acts at every point on the string. Assume that $\beta < 2\pi\rho c/L$. Can resonance with unbounded oscillations occur?

40. (a) The ends of a taut string are fixed at $x = 0$ and $x = L$ on the x-axis. The string is initially at rest along the axis and then is allowed to drop under its own weight. Find a series representation for the displacement of the string.
(b) Show that the solution in part (a) can be expressed in the closed form

$$y(x,t) = \psi(x) - \frac{1}{2}[\psi(x+ct) + \psi(x-ct)],$$

where $\psi(x)$ is the function $g(x^2 - Lx)/(2c^2)$ for $0 \le x \le L$, and is extended as an odd function of period $2L$.

41. Repeat Exercise 40 if the string has an initial displacement $f(x)$.

42. The ends of a taut string are looped around smooth vertical supports at $x = 0$ and $x = L$. If the string falls from rest along the x-axis, and a constant vertical force F_0 acts on the loop at $x = L$, find displacements of the string. Take gravity into account.

43. A motionless, horizontal beam has its ends simply supported at $x = 0$ and $x = L$. At time $t = 0$, a concentrated force of magnitude A is suddenly applied at the midpoint.
(a) If the weight per unit length of the beam is negligible compared to A, show that the initial boundary value problem for transverse displacements $y(x,t)$ is

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} + c^2\frac{\partial^4 y}{\partial x^4} &= -\frac{A}{\rho}\delta(x - L/2), \quad 0 < x < L, \quad t > 0,\\
y(0,t) &= y(L,t) = 0, \quad t > 0,\\
y_{xx}(0,t) &= y_{xx}(L,t) = 0, \quad t > 0,\\
y(x,0) &= y_t(x,0) = 0, \quad 0 < x < L,
\end{aligned}$$

where $c^2 = EI/\rho$.
(b) Solve this problem using the finite Fourier transform associated with Sturm-Liouville system 5.1 of Chapter 5.

44. Find a formula for the solution of the general one-dimensional vibration problem

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \quad 0 < x < L, \quad t > 0,\\
-l_1\frac{\partial y}{\partial x} + h_1 y &= f_1(t), \quad x = 0, \quad t > 0,\\
l_2\frac{\partial y}{\partial x} + h_2 y &= f_2(t), \quad x = L, \quad t > 0,\\
y(x,0) &= f(x), \quad 0 < x < L,\\
y_t(x,0) &= g(x), \quad 0 < x < L.
\end{aligned}$$

45. The end $x = 0$ of a horizontal elastic bar of length L is kept fixed, and the other end has a mass m attached to it. The mass m is then subjected to a horizontal periodic force $F = F_0 \sin \omega t$. If the bar is initially unstrained and at rest, set up the initial boundary value problem for longitudinal displacements in the bar. Can we solve this problem with finite Fourier transforms?

46. (a) A taut string of length L is at rest along the x-axis. At time $t = 0$, a concentrated force is placed at one of the nodes of one of the modes of vibration of the string. Show that when

ends of the string satisfy homogeneous Robin conditions, this mode does not contribute to the motion of the string.
(b) Is the result in part (a) true if the string has a nonzero initial displacement and/or velocity?

47. (a) A taut string with ends at $x = 0$ and $x = L$ fixed on the x-axis is at rest along the x-axis. At time $t = 0$, a concentrated force is placed at the midpoint of the string. Show that the displacement of the string contains only odd harmonics.
(b) Is the result in part (a) true if the string has a nonzero initial displacement and/or velocity?

Part C Potential, Steady-state Heat Conduction, Static Deflections of Membranes

48. A charge distribution with density $\sigma(x, y)$ coulombs per cubic metre occupies the volume R in space bounded by the planes $x = 0$, $x = L$, $y = 0$, and $y = L'$, and these planes are all held at potential zero.
(a) Use finite Fourier transforms to find the potential $V(x, y)$ in R when σ is constant. Find two series, one by transforming the x-variable and the other by transforming the y-variable.
(b) If $\sigma = \sigma(x)$ is a function of x only, find $V(x, y)$.
(c) Find $V(x, y)$ if $\sigma = xy$.

49. A uniform charge distribution of density σ coulombs per cubic metre occupies the volume R bounded by the planes $x = 0$, $x = L$, $y = 0$, and $y = L'$. If the electrostatic potential on the planes $x = 0$, $y = 0$, and $y = L'$ is zero and that on $x = L$ is $f(y)$, find the potential in R.

50. Repeat 49 when planes $x = 0$, $x = L$, and $y = L'$ are held at zero potential and $y = 0$ is at potential $g(x)$.

51. Repeat 49 when planes $x = L$, and $y = L'$ are held at zero potential and $x = 0$ and $y = 0$ are at potentials $f(y)$ and $g(x)$, respectively.

52. The following problem describes steady-state temperature in a rectangular plate with constant heat generation, two sides of the plate insulated, and two sides at temperature zero:

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= -C, \quad 0 < x < L, \quad 0 < x < L', \\
U_x(0, y) &= 0, \quad 0 < y < L', \\
U(L, y) &= 0, \quad 0 < y < L', \\
U_y(x, 0) &= 0, \quad 0 < x < L, \\
U(x, L') &= 0, \quad 0 < x < L.
\end{aligned}$$

Use a finite Fourier transform with respect to x, to find $U(x, t)$.

53. Use a finite Fourier transform with respect to x to find a formula for the solution of the two-dimensional Dirichlet boundary value problem

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= F(x, y), \quad 0 < x < L, \quad 0 < x < L', \\
V(0, y) &= f_1(y), \quad 0 < y < L', \\
V(L, y) &= f_2(y), \quad 0 < y < L', \\
V(x, 0) &= g_1(x), \quad 0 < x < L, \\
V(x, L') &= g_2(x), \quad 0 < x < L.
\end{aligned}$$

54. An alternative to a single series solution of the problem in Exercise 53 is a double series produced by taking finite Fourier transforms wiwth respect to x and y. Find this solution.

55. (a) The boundary value problem for steady-state temperature in Exercise 20 of Section 6.3 could only be solved by separation of variables when boundary conditions were $U(r,0) = k_1$ and $U(r,\alpha) = k_2$, where k_1 and k_2 were constants. Find the ordinary differential equation that must be satisfied by the transform of the temperature function when a finite Fourier transform associated with the θ-variable is applied to the PDE.

(b) Find steady-state temperature in the plate when boundary conditions are as in part (a). Simplify the solution as much as possible.

(c) Find the solution when $U(r,0) = r$ and $U(r,\alpha) = 0$.

56. We suggested at the end of this section that two-dimensional boundary value problems on rectangles with four nonhomogeneous boundary conditions and homogeneous PDEs can be subdivided into two problems, each of which has two homogeneous and two nonhomogeneous boundary conditions. There is an exception to this, namely the Neumann problem. For example, the Neumann problem associated with Laplace's equation is

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0, \quad 0 < x < L, \quad 0 < y < L', \\
\frac{\partial V(0,y)}{\partial x} &= f_1(y), \quad 0 < y < L', \\
\frac{\partial V(L,y)}{\partial x} &= f_2(y), \quad 0 < y < L', \\
\frac{\partial V(x,0)}{\partial y} &= g_1(x), \quad 0 < x < L, \\
\frac{\partial y(x,L')}{\partial y} &= g_2(x), \quad 0 < x < L,
\end{aligned}$$

where the nonhomogeneities must satisfy the consistency condition

$$\int_0^L [g_2(x) - g_1(x)]\,dx + \int_0^{L'} [f_2(y) - f_1(y)]\,dy = 0.$$

Our previous suggestion would indicate that $V(x,y)$ should be set equal to $V(x,y) = V_1(x,y) + V_2(x,y)$ where V_1 and V_2 satisfy Laplace's equation on the rectangle and the following boundary conditions:

$$\begin{aligned}
\frac{\partial V_1(0,y)}{\partial x} &= f_1(y), \quad 0 < y < L', & \frac{\partial V_2(0,y)}{\partial x} &= 0, \quad 0 < y < L', \\
\frac{\partial V_1(L,y)}{\partial x} &= f_2(y), \quad 0 < y < L', & \frac{\partial V_2(L,y)}{\partial x} &= 0, \quad 0 < y < L', \\
\frac{\partial V_1(x,0)}{\partial y} &= 0, \quad 0 < x < L, & \frac{\partial V_2(x,0)}{\partial y} &= g_1(x), \quad 0 < x < L, \\
\frac{\partial V_1(x,L')}{\partial y} &= 0, \quad 0 < x < L; & \frac{\partial V_2(x,L')}{\partial y} &= g_2(x), \quad 0 < x < L.
\end{aligned}$$

But these Neumann problems must satisfy the consistency conditions

$$\int_0^{L'} [f_2(y) - f_1(y)]\,dy = 0, \qquad \int_0^L [g_2(x) - g_1(x)]\,dx = 0.$$

The difficulty is that the combined consistency condition on f_1, f_2, g_1 and g_2 may not imply these separately. In general, then, solutions for V_1 and V_2 may not exist. With finite Fourier transforms, this difficulty presents no problem. Find $V(x, y)$ using such a transform.

§7.3 Higher-Dimensional Problems in Cartesian Coordinates

To solve nonhomogeneous initial boundary value problems in three and four variables, we can once again remove space variables from the problem with finite Fourier transforms, leaving an ODE in the transform of the function regarded as a function of time. There are two ways to do this. Successive finite Fourier transforms, each a transform in only one space variable, can be applied to the PDE. This corresponds to successively separating off space variables in homogeneous problems. Alternatively, multi-dimensional finite Fourier transforms associated with multi-dimensional eigenvalue problems (see Section 6.5) can be introduced. We take the former approach. To illustrate, consider the following initial boundary value problem.

Example 7.5 Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{7.57a}$$

$$U(0, y, t) = U_1, \quad 0 < y < L', \quad t > 0, \tag{7.57b}$$

$$U(L, y, t) = 0, \quad 0 < y < L', \quad t > 0, \tag{7.57c}$$

$$U(x, 0, t) = U_2, \quad 0 < x < L, \quad t > 0, \tag{7.57d}$$

$$U_y(x, L', t) = 0, \quad 0 < x < L, \quad t > 0, \tag{7.57e}$$

$$U(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L'. \tag{7.57f}$$

Described is a horizontal plate that is insulated top and bottom and along the edge $y = L'$. Initially the temperature is zero throughout the plate, and for $t > 0$, faces $x = 0$, $x = L$, and $y = 0$ are held at constant temperatures U_1, 0, and U_2, respectively.

Solution The finite Fourier transform associated with the x-variable is

$$\tilde{f}(\lambda_n) = \int_0^L f(x) X_n(x)\, dx, \tag{7.58}$$

where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenpairs of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = X(L) = 0.$$

This is the system that would result were separation of variables applied to problem 7.57 with homogeneous boundary conditions. If we apply this transform to PDE 7.57a, and use integration by parts,

$$\int_0^L \frac{\partial U}{\partial t} X_n\, dx = k\int_0^L \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) X_n\, dx$$
$$= k\frac{\partial^2}{\partial y^2}\int_0^L U X_n\, dx + k\left\{\frac{\partial U}{\partial x} X_n\right\}_0^L - k\int_0^L \frac{\partial U}{\partial x} X_n'\, dx.$$

Since $X_n(0) = X_n(L) = 0$,

$$\frac{\partial}{\partial t}\int_0^L UX_n\,dx = k\frac{\partial^2\tilde{U}(\lambda_n,y,t)}{\partial y^2} - k\Big\{UX_n'\Big\}_0^L + k\int_0^L UX_n''\,dx.$$

Boundary conditions 7.57b,c and the fact that $X_n'' = -\lambda_n^2 X_n$ now give

$$\frac{\partial\tilde{U}}{\partial t} = k\frac{\partial^2\tilde{U}}{\partial y^2} + kU_1X_n'(0) + k\int_0^L U(-\lambda_n^2 X_n)\,dx.$$

Thus, $\tilde{U}(\lambda_n,y,t)$ must satisfy the PDE

$$\frac{\partial\tilde{U}}{\partial t} = k\frac{\partial^2\tilde{U}}{\partial y^2} + kU_1X_n'(0) - k\lambda_n^2\tilde{U}, \quad 0<y<L', \quad t>0, \tag{7.59a}$$

subject to the transforms of conditions 7.57d,e,f,

$$\tilde{U}(\lambda_n,0,t) = U_2\tilde{1}_n, \quad t>0, \tag{7.59b}$$

$$\tilde{U}_y(\lambda_n,L',t) = 0, \quad t>0, \tag{7.59c}$$

$$\tilde{U}(\lambda_n,y,0) = 0, \quad 0<y<L', \tag{7.59d}$$

where

$$\tilde{1}_n = \int_0^L X_n\,dx = \int_0^L \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx = \frac{\sqrt{2L}[1+(-1)^{n+1}]}{n\pi}. \tag{7.59e}$$

The finite Fourier transform associated with the y-variable in problem 7.59 is

$$\tilde{f}(\mu_m) = \int_0^{L'} f(y)Y_m(y)\,dy, \tag{7.60}$$

where $\mu_m^2 = (2m-1)^2\pi^2/(4L'^2)$ and $Y_m(y) = \sqrt{2/L'}\sin[(2m-1)\pi y/(2L')]$ are the eigenpairs of the Sturm-Liouville system

$$Y'' + \mu^2 Y = 0, \quad 0<y<L',$$
$$Y(0) = Y'(L') = 0.$$

If we apply this transform to PDE 7.59a,

$$\int_0^{L'}\frac{\partial\tilde{U}}{\partial t}Y_m\,dy = k\int_0^{L'}\frac{\partial^2\tilde{U}}{\partial y^2}Y_m\,dy + \int_0^{L'}[kU_1X_n'(0) - k\lambda_n^2\tilde{U}]Y_m\,dy,$$

and use integration by parts,

$$\frac{\partial\tilde{\tilde{U}}(\lambda_n,\mu_m,t)}{\partial t} - kU_1X_n'(0)\tilde{1}_m + k\lambda_n^2\tilde{\tilde{U}} = k\left\{\frac{\partial\tilde{U}}{\partial y}Y_m\right\}_0^{L'} - k\int_0^{L'}\frac{\partial\tilde{U}}{\partial y}Y_m'\,dy,$$

where

$$\tilde{1}_m = \int_0^{L'} Y_m\,dy = \int_0^{L'}\sqrt{\frac{2}{L'}}\sin\frac{(2m-1)\pi y}{2L'}dy = \frac{2\sqrt{2L'}}{(2m-1)\pi}.$$

Since $Y_m(0)=0$ and $\partial\tilde{U}(\lambda_n,L',t)/\partial y = 0$,

$$\frac{\partial\tilde{\tilde{U}}}{\partial t} - kU_1X_n'(0)\tilde{1}_m + k\lambda_n^2\tilde{\tilde{U}} = -k\Big\{\tilde{U}Y_m'\Big\}_0^{L'} + k\int_0^{L'}\tilde{U}Y_m''\,dy.$$

Boundary condition 7.59b and the facts that $Y_m'(L') = 0$ and $Y_m'' = -\mu_m^2 Y_m$ yield

$$\frac{\partial \tilde{\tilde{U}}}{\partial t} - kU_1 X_n'(0)\tilde{1}_m + k\lambda_n^2 \tilde{\tilde{U}} = kU_2 Y_m'(0)\tilde{1}_n + k\int_0^{L'} \tilde{U}(-\mu_m^2 Y_m)\, dy$$

or,

$$\frac{d\tilde{\tilde{U}}}{dt} + k(\lambda_n^2 + \mu_m^2)\tilde{\tilde{U}} = k[U_2 Y_m'(0)\tilde{1}_n + U_1 X_n'(0)\tilde{1}_m]. \tag{7.61a}$$

Accompanying this ODE in $\tilde{\tilde{U}}(\lambda_n, \mu_m, t)$ is the transform of initial condition 7.59d,

$$\tilde{\tilde{U}}(\lambda_n, \mu_m, 0) = 0. \tag{7.61b}$$

Because the right side of ODE 7.61a is a constant with respect to t, a general solution of this ODE is

$$\tilde{\tilde{U}}(\lambda_n, \mu_m, t) = A_{mn} e^{-k(\lambda_n^2+\mu_m^2)t} + \frac{U_2 Y_m'(0)\tilde{1}_n + U_1 X_n'(0)\tilde{1}_m}{\lambda_n^2 + \mu_m^2},$$

where the A_{mn} are constants. Initial condition 7.61b requires

$$0 = A_{mn} + \frac{U_2 Y_m'(0)\tilde{1}_n + U_1 X_n'(0)\tilde{1}_m}{\lambda_n^2 + \mu_m^2},$$

and therefore

$$\tilde{\tilde{U}}(\lambda_n, \mu_m, t) = \frac{U_2 Y_m'(0)\tilde{1}_n + U_1 X_n'(0)\tilde{1}_m}{\lambda_n^2 + \mu_m^2}[1 - e^{-k(\lambda_n^2+\mu_m^2)t}]. \tag{7.62}$$

To find $U(x, y, t)$ we now invert transforms 7.58 and 7.60,

$$U(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{\tilde{U}}(\lambda_n, \mu_m, t) Y_m(y) X_n(x).$$

Substitutions for $\tilde{\tilde{U}}(\lambda_n, \mu_m, t)$, $Y_m(y)$, and $X_n(x)$ lead to

$$U(x, y, t) = 8\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\{1 - e^{-[n^2/L^2 + (2m-1)^2/(4L'^2)]\pi^2 kt}\} \sin\frac{n\pi x}{L} \sin\frac{(2m-1)\pi y}{2L'}, \tag{7.63a}$$

where

$$B_{mn} = \frac{[1 + (-1)^{n+1}](2m-1)^2 L^2 U_2 + 4(2n-1)^2 L'^2 U_1}{n(2m-1)[4n^2\pi^2 L'^2 + (2m-1)^2\pi^2 L^2]}. \tag{7.63b}$$

Terms vanish for even n, and we therefore rewrite the solution displaying only nonzero terms,

$$U(x, y, t) = 16\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\{1 - e^{-[(2n-1)^2/L^2 + (2m-1)^2/(4L'^2)]\pi^2 kt}\} \sin\frac{(2n-1)\pi x}{L} \sin\frac{(2m-1)\pi y}{2L'}, \tag{7.64a}$$

where

$$B_{mn} = \frac{(2m-1)^2 L^2 U_2 + 4n^2 L'^2 U_1}{(2n-1)(2m-1)[4(2n-1)^2\pi^2 L'^2 + (2m-1)^2\pi^2 L^2]}. \tag{7.64b}$$

As a second example, we consider a boundary value problem in three dimensions.

Example 7.6 Find the potential inside the region bounded by the planes $x = 0$, $x = L$, $y = 0$, $y = L'$, $z = 0$, and $z = L''$ if all such planes are held at potential zero and the region contains a uniform charge distribution with density σ coulombs per cubic metre.

Solution The boundary value problem for potential $V(x, y, z)$ in the region is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\sigma}{\epsilon}, \quad 0 < x < L, \quad 0 < y < L', \quad 0 < z < L'', \tag{7.65a}$$

$$V(0, y, z) = 0, \quad 0 < y < L', \quad 0 < z < L'', \tag{7.65b}$$

$$V(L, y, z) = 0, \quad 0 < y < L', \quad 0 < z < L'', \tag{7.65c}$$

$$V(x, 0, z) = 0, \quad 0 < x < L, \quad 0 < z < L'', \tag{7.65d}$$

$$V(x, L', z) = 0, \quad 0 < x < L, \quad 0 < z < L'', \tag{7.65e}$$

$$V(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L', \tag{7.65f}$$

$$V(x, y, L'') = 0, \quad 0 < x < L, \quad 0 < y < L'. \tag{7.65g}$$

The finite Fourier transform associated with the x-variable is

$$\tilde{f}(\lambda_n) = \int_0^L f(x) X_n(x)\, dx, \tag{7.66}$$

where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenpairs of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = X(L) = 0.$$

When we apply this transform to the PDE and use integration by parts,

$$\begin{aligned}
\int_0^L \left(\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\sigma}{\epsilon} \right) X_n\, dx &= -\int_0^L \frac{\partial^2 V}{\partial x^2} X_n\, dx \\
&= -\left\{ \frac{\partial V}{\partial x} X_n \right\}_0^L + \int_0^L \frac{\partial V}{\partial x} X_n'\, dx \\
&\qquad \text{[and since } X_n(0) = X_n(L) = 0\text{]} \\
&= \left\{ V X_n' \right\}_0^L - \int_0^L V X_n''\, dx \\
&\qquad \text{[and since } V(L, y, z) = V(0, y, z) = 0\text{]} \\
&= -\int_0^L V(-\lambda_n^2 X_n)\, dx = \lambda_n^2 \tilde{V}(\lambda_n, y, z).
\end{aligned}$$

Thus, $\tilde{V}(\lambda_n, y, z)$ must satisfy the PDE

$$\frac{\partial^2 \tilde{V}}{\partial y^2} + \frac{\partial^2 \tilde{V}}{\partial z^2} - \lambda_n^2 \tilde{V} = -\frac{\sigma}{\epsilon} \tilde{1}_n, \quad 0 < y < L', \quad 0 < z < L'', \tag{7.67a}$$

subject to the boundary conditions

$$\tilde{V}(\lambda_n, 0, z) = 0, \quad 0 < z < L'', \tag{7.67b}$$
$$\tilde{V}(\lambda_n, L', z) = 0, \quad 0 < z < L'', \tag{7.67c}$$
$$\tilde{V}(\lambda_n, y, 0) = 0, \quad 0 < y < L', \tag{7.67d}$$
$$\tilde{V}(\lambda_n, y, L'') = 0, \quad 0 < y < L', \tag{7.67e}$$

and

$$\tilde{1}_n = \int_0^L X_n\, dx = \frac{\sqrt{2L}[1 + (-1)^{n+1}]}{n\pi}. \tag{7.67f}$$

To eliminate y from this problem, we use the finite Fourier transform

$$\tilde{f}(\mu_m) = \int_0^{L'} f(y) Y_m(y)\, dy, \tag{7.68}$$

where $\mu_m^2 = m^2\pi^2/L'^2$ and $Y_m(y) = \sqrt{2/L'} \sin(m\pi y/L')$ are the eigenpairs of the Sturm-Liouville system

$$Y'' + \mu^2 Y = 0, \quad 0 < y < L',$$
$$Y(0) = Y(L') = 0.$$

Application of this transform to the PDE yields

$$\begin{aligned}
\int_0^{L'} \left(\frac{\partial^2 \tilde{V}}{\partial z^2} - \lambda_n^2 \tilde{V} + \frac{\sigma}{\epsilon} \tilde{1}_n \right) Y_m\, dy &= -\int_0^{L'} \frac{\partial^2 \tilde{V}}{\partial y^2} Y_m\, dy \\
&= -\left\{ \frac{\partial \tilde{V}}{\partial y} Y_m \right\}_0^{L'} + \int_0^{L'} \frac{\partial \tilde{V}}{\partial y} Y_m'\, dy \\
&\qquad [\text{and since } Y_m(0) = Y_m(L') = 0] \\
&= \left\{ \tilde{V} Y_m' \right\}_0^{L'} - \int_0^{L'} \tilde{V} Y_m''\, dy \\
&\qquad [\text{and since } \tilde{V}(\lambda_n, 0, z) = \tilde{V}(\lambda_n, L', z) = 0] \\
&= -\int_0^{L'} \tilde{V}(-\mu_m^2 Y_m)\, dy = \mu_m^2 \tilde{\tilde{V}}(\lambda_n, \mu_m, z).
\end{aligned}$$

Thus, $\tilde{\tilde{V}}(\lambda_n, \mu_m, z)$ must satisfy the ODE

$$\frac{d^2 \tilde{\tilde{V}}}{dz^2} - (\lambda_n^2 + \mu_m^2) \tilde{\tilde{V}} = -\frac{\sigma}{\epsilon} \tilde{\tilde{1}}_{nm}, \quad 0 < z < L'', \tag{7.69a}$$

subject to

$$\tilde{\tilde{V}}(\lambda_n, \mu_m, 0) = 0, \tag{7.69b}$$
$$\tilde{\tilde{V}}(\lambda_n, \mu_m, L'') = 0, \tag{7.69c}$$

and

$$\tilde{\tilde{1}}_{nm} = \int_0^{L'} \tilde{1}_n Y_m \, dy = \frac{2\sqrt{LL'}[1+(-1)^{n+1}][1+(-1)^{m+1}]}{mn\pi^2}. \tag{7.69d}$$

A general solution of the ODE is

$$\tilde{\tilde{V}}(\lambda_m, \mu_m, z) = A_{mn} \cosh \sqrt{\lambda_n^2 + \mu_m^2} z + B_{mn} \sinh \sqrt{\lambda_n^2 + \mu_m^2} z + \frac{(\sigma/\epsilon)\tilde{\tilde{1}}_{nm}}{\lambda_n^2 + \mu_m^2}. \tag{7.70}$$

Boundary conditions 7.69b,c require

$$0 = A_{mn} + \frac{(\sigma/\epsilon)\tilde{\tilde{1}}_{nm}}{\lambda_n^2 + \mu_m^2},$$

$$0 = A_{mn} \cosh \sqrt{\lambda_n^2 + \mu_m^2} L'' + B_{mn} \sinh \sqrt{\lambda_n^2 + \mu_m^2} L'' + \frac{(\sigma/\epsilon)\tilde{\tilde{1}}_{nm}}{\lambda_n^2 + \mu_m^2}.$$

When these are solved for A_{mn} and B_{mn} and substituted into 7.70, $\tilde{\tilde{V}}(\lambda_n, \mu_m, z)$ simplifies to

$$\tilde{\tilde{V}}(\lambda_n, \mu_m, z) = \frac{-(\sigma/\epsilon)\tilde{\tilde{1}}_{nm}}{(\lambda_n^2 + \mu_m^2) \sinh \sqrt{\lambda_n^2 + \mu_m^2} L''} [\sinh \sqrt{\lambda_n^2 + \mu_m^2} (L'' - z)$$
$$+ \sinh \sqrt{\lambda_n^2 + \mu_m^2} z - \sinh \sqrt{\lambda_n^2 + \mu_m^2} L'']. \tag{7.71}$$

The solution of problem 7.65 is therefore

$$V(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{\tilde{V}}(\lambda_n, \mu_m, z) X_n(x) Y_m(y). \tag{7.72}$$

EXERCISES 7.3

Part A Heat Conduction

1. An isotropic, homogeneous, horizontal plate has its top and bottom faces insulated. Edges $x = 0$, $x = L$, $y = 0$, and $y = L'$ are all held at constant temperatures U_1, U_2, U_3, and U_4, respectively, for time $t > 0$. If the temperature of the plate at time $t = 0$ is $f(x, y)$, $0 \le x \le L$, $0 \le y \le L'$, find its temperature thereafter.

2. (a) Solve the following heat conduction problem:

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,$$
$$U(0, y, t) = U_1, \quad 0 < y < L', \quad t > 0,$$
$$U(L, y, t) = U_2, \quad 0 < y < L', \quad t > 0,$$
$$U_y(x, 0, t) = \kappa_1^{-1}\phi_1, \quad 0 < x < L, \quad t > 0,$$
$$U_y(x, L', t) = -\kappa_2^{-1}\phi_2, \quad 0 < x < L, \quad t > 0,$$
$$U(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L',$$

where U_1, U_2, ϕ_1, and ϕ_2 are constants. Interpret the problem physically.
(b) What is the solution when $\phi_1 = \phi_2 = 0$?

3. Repeat Exercise 2(a) when U_1, U_2, ϕ_1, and ϕ_2 are functions of time.

4. Find a formula for the solution of the general two-dimensional heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{kg(x,y,t)}{\kappa}, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,$$
$$-l_1\frac{\partial U}{\partial x} + h_1 U = f_1(y,t), \quad x = 0, \quad 0 < y < L', \quad t > 0,$$
$$l_2\frac{\partial U}{\partial x} + h_2 U = f_2(y,t), \quad x = L, \quad 0 < y < L', \quad t > 0,$$
$$-l_3\frac{\partial U}{\partial y} + h_3 U = f_3(x,t), \quad y = 0, \quad 0 < x < L, \quad t > 0,$$
$$l_4\frac{\partial U}{\partial y} + h_4 U = f_4(x,t), \quad y = L', \quad 0 < x < L, \quad t > 0,$$
$$U(x,y,0) = f(x,y), \quad 0 < x < L, \quad 0 < y < L'.$$

Part B Vibrations

5. A rectangular membrane of side lengths L and L' has its edges fixed on the xy-plane. If it is released from rest at a displacement given by $f(x,y)$, find subsequent displacements of the membrane if gravity is taken into account.

6. A square membrane of side length L, which is initially at rest on the xy-plane, has its edges fixed on the xy-plane. If a periodic force per unit area $A\cos\omega t$, (A a constant), acts at every point in the membrane for $t > 0$, find displacements in the membrane. Assume that $\omega \neq c\pi\sqrt{n^2+m^2}/L$ for any positive integers m and n.

7. Repeat Exercise 6 if $\omega = \sqrt{2}\pi c/L$.

8. Repeat Exercise 6 if $\omega = \sqrt{17}\pi c/L$.

9. Repeat Exercise 6 if $\omega = \sqrt{65}\pi c/L$.

10. Repeat Exercise 6 if $\omega = \sqrt{10}\pi c/L$.

11. Repeat Exercise 6 if $\omega = \sqrt{130}\pi c/L$.

12. Find a formula for the solution of the general two-dimensional vibration problem

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{F(x,y,t)}{\rho}, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,$$
$$-l_1\frac{\partial z}{\partial x} + h_1 z = f_1(y,t), \quad x = 0, \quad 0 < y < L', \quad t > 0,$$
$$l_2\frac{\partial z}{\partial x} + h_2 z = f_2(y,t), \quad x = L, \quad 0 < y < L', \quad t > 0,$$
$$-l_3\frac{\partial z}{\partial y} + h_3 z = f_3(x,t), \quad y = 0, \quad 0 < x < L, \quad t > 0,$$
$$l_4\frac{\partial z}{\partial y} + h_4 z = f_4(x,t), \quad y = L', \quad 0 < x < L, \quad t > 0,$$
$$z(x,y,0) = g(x,y), \quad 0 < x < L, \quad 0 < y < L',$$
$$z_t(x,y,0) = h(x,y), \quad 0 < x < L, \quad 0 < y < L'.$$

CHAPTER 8 SPECIAL FUNCTIONS

§8.1 Introduction

In Chapters 4–7, discussions have been confined to (initial) boundary value problems expressed in Cartesian coordinates (with the exception of Laplace's equation in polar coordinates in Section 6.3). When separation of variables, finite Fourier transforms, and Laplace transforms are applied to initial boundary value problems in polar, cylindrical, and spherical coordinates, new functions arise, namely, Bessel functions and Legendre functions. In Sections 8.3 and 8.5 we introduce these functions as solutions of ordinary differential equations, as this is how they arise in the context of PDEs. Bessel's differential equation and Legendre's differential equation are homogeneous, second-order, linear differential equations with variable coefficients. The most general form of such an equation is

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0. \tag{8.1}$$

A point x_0 is said to be an **ordinary point** of this differential equation when the functions $Q(x)/P(x)$ and $R(x)/P(x)$ have convergent Taylor series about x_0; otherwise, x_0 is called a **singular point**. When x_0 is an ordinary point of equation 8.1, there exist two independent solutions $y_1(x)$ and $y_2(x)$ both with Taylor series convergent in some interval $|x - x_0| < \delta$. A general solution of the differential equation valid in this interval is $c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants.

When x_0 is a singular point of equation 8.1, independent solutions in the form of power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ about x_0 may not exist. In this case, it is customary to search for solutions in the form

$$(x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r}, \tag{8.2}$$

called **Frobenius** solutions. Solutions of this type may or may not exist, depending on the severity of the singularity. A singular point x_0 is said to be **regular** if

$$(x - x_0)\frac{Q(x)}{P(x)} \qquad \text{and} \qquad (x - x_0)^2\frac{R(x)}{P(x)}$$

both have Taylor series expansions about x_0. Otherwise, x_0 is said to be an **irregular** singular point.

When x_0 is a regular singular point of equation 8.1, a Frobenius solution always leads to a quadratic equation for the unknown index r. Depending on the nature of the roots of this quadratic, called the **indicial equation**, three situations arise; they are summarized in the following theorem.

Theorem 8.1 Let r_1 and r_2 be the indicial roots for a Frobenius solution of differential equation 8.1 about a regular singular point x_0. To find linearly independent solutions of the differential equation, it is necessary to consider the cases in which the difference $r_1 - r_2$ is not an integer, is zero, or is a positive integer (assuming $r_1 \geq r_2$).

Case 1: $r_1 \neq r_2$ and $r_1 - r_2 \neq$ integer

In this case, two linearly independent solutions

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=1}^{\infty} a_n (x - x_0)^n \quad \text{with } a_0 = 1 \tag{8.3a}$$

$$y_2(x) = (x - x_0)^{r_2} \sum_{n=1}^{\infty} b_n (x - x_0)^n \quad \text{with } b_0 = 1 \tag{8.3b}$$

always exist.

Case 2: $r_1 = r_2 = r$

In this case, one Frobenius solution

$$y_1(x) = (x - x_0)^{r} \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with } a_0 = 1 \tag{8.4a}$$

is obtained. A second (independent) solution exists in the form

$$y_2(x) = y_1(x) \ln (x - x_0) + (x - x_0)^{r} \sum_{n=1}^{\infty} A_n (x - x_0)^n, \quad x > x_0. \tag{8.4b}$$

Case 3: $r_1 - r_2 =$ positive integer

In this case, one Frobenius solution can always be obtained from the larger root r_1,

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with } a_0 = 1. \tag{8.5a}$$

The smaller root r_2 may yield no solution, one solution, or a general solution. In the event that it yields no solution, a second (independent) solution can always be found in the form

$$y_2(x) = A y_1(x) \ln (x - x_0) + (x - x_0)^{r_2} \sum_{n=0}^{\infty} A_n (x - x_0)^n \text{ with } A_0 = 1,\ x > x_0. \tag{8.5b}$$

In all cases, a general solution of the differential equation is $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

§8.2 Gamma Function

The gamma function is a generalization of the factorial operation to noninteger values. For $\nu > 0$, it is defined by the convergent improper integral

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x}\, dx. \tag{8.6}$$

Integration by parts yields the recursive formula

$$\Gamma(\nu+1) = \nu\Gamma(\nu), \quad \nu > 0. \tag{8.7a}$$

With this formula, and the fact that the gamma function is well tabulated in many references for $1 \le \nu < 2$, $\Gamma(\nu)$ can be calculated quickly for all $\nu > 0$. We note, in particular, that

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1, \tag{8.8}$$

and hence for ν a positive integer,

$$\Gamma(\nu+1) = \nu!. \tag{8.9}$$

Example 8.1 Evaluate $\Gamma(4.2)$.

Solution With recursive formula 8.7a,

$$\Gamma(4.2) = 3.2\Gamma(3.2) = (3.2)(2.2)\Gamma(2.2) = (3.2)(2.2)(1.2)\Gamma(1.2).$$

But from tables $\Gamma(1.2) = 0.918\,169$, and therefore

$$\Gamma(4.2) = (3.2)(2.2)(1.2)(0.918\,169) = 7.7567.\bullet$$

If $\nu \le 0$, the improper integral in equation 8.6 diverges (at $x = 0$), so that the integral cannot be used to define $\Gamma(\nu)$ for $\nu \le 0$. Instead we reverse recursive formula 8.7a,

$$\Gamma(\nu) = \frac{\Gamma(\nu+1)}{\nu}, \tag{8.10b}$$

and iterate to define

$$\Gamma(\nu) = \frac{\Gamma(\nu+k)}{\nu(\nu+1)(\nu+2)\cdots(\nu+k-1)}, \tag{8.13}$$

where k is chosen such that $1 < \nu + k < 2$. With equation 8.13 as the definition of $\Gamma(\nu)$ for $\nu < 1$, $\Gamma(\nu)$ is now defined for all ν except $\nu = 0, -1, -2, \ldots$, and its graph is as shown in Figure 8.1.

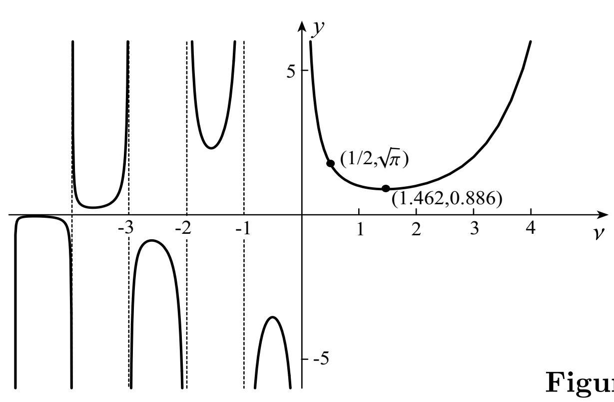


Figure 8.1

Example 8.2 Evaluate $\Gamma(-2.3)$.

Solution We use equation 8.13b to write

$$\Gamma(-2.3) = \frac{\Gamma(-1.3)}{-2.3} = \frac{\Gamma(-0.3)}{(-2.3)(-1.3)} = \frac{\Gamma(0.7)}{(-2.3)(-1.3)(-0.3)} = \frac{\Gamma(1.7)}{(-2.3)(-1.3)(-0.3)(0.7)}.$$

But from tables, $\Gamma(1.7) = 0.908\,639$, and therefore

$$\Gamma(-2.3) = \frac{0.908\,639}{(-2.3)(-1.3)(-0.3)(0.7)} = -1.4471.\bullet$$

EXERCISES 8.2

1. Use tables for the gamma function, or otherwise, to evaluate:
(a) $\Gamma(6)$ (b) $\Gamma(3.4)$ (c) $\Gamma(4.16)$ (d) $\Gamma(-0.8)$ (e) $\Gamma(-3.2)$ (f) $\Gamma(-2.44)$

2. Show that

$$\int_0^\infty x^\nu e^{-\alpha x}\,dx = \frac{\Gamma(\nu+1)}{\alpha^{\nu+1}}, \quad \nu > -1, \quad \alpha > 0.$$

3. (a) To evaluate the integral

$$I = \int_{-\infty}^{\infty} e^{-kx^2}\,dx = 2\int_0^\infty e^{-kx^2}\,dx,$$

we write

$$\frac{I^2}{4} = \left(\int_0^\infty e^{-kx^2}\,dx\right)\left(\int_0^\infty e^{-ky^2}\,dy\right) = \int_0^\infty\int_0^\infty e^{-k(x^2+y^2)}\,dy\,dx$$

and transform the double integral into polar coordinates. Show that $I = \sqrt{\pi/k}$.

(b) By equation 8.6,

$$\Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x}\,dx.$$

Set $x = y^2$ to show that

$$\Gamma(1/2) = 2\int_0^\infty e^{-y^2}\,dy,$$

and use the result of part (a) to obtain $\Gamma(1/2) = \sqrt{\pi}$.

4. Prove that for n a positive integer,

$$\Gamma(n + 1/2) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}.$$

§8.3 Bessel Functions

Bessel functions arise when separation of variables is applied to initial boundary value problems expressed in polar, cylindrical, and spherical coordinates. They are solutions of the linear, homogeneous, second-order ODE

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \quad \nu \geq 0, \tag{8.14}$$

called **Bessel's differential equation of order** ν. When we assume a Frobenius solution $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ ($x = 0$ being a regular singular point for the differential equation), we obtain the indicial equation

$$r^2 - \nu^2 = 0, \tag{8.15a}$$

and the following equations defining coefficients,

$$a_1[(r+1)^2 - \nu^2] = 0, \tag{8.15b}$$

$$a_n[(n+r)^2 - \nu^2] + a_{n-2} = 0, \quad n \geq 2. \tag{8.15c}$$

For the nonnegative indicial root $r = \nu$, we must choose $a_1 = 0$, and iteration of equation 8.15c yields, for $n > 0$,

$$a_{2n+1} = 0, \tag{8.16a}$$

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(\nu+1)(\nu+2)\cdots(\nu+n)}. \tag{8.16b}$$

If we choose $a_0 = 1/[2^\nu \Gamma(\nu+1)]$, the particular solution of Bessel's differential equation corresponding to the indicial root $r = \nu$ is denoted by

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n} \tag{8.17}$$

and is called the **Bessel function of the first kind of order** ν. Because this series converges for all x, $J_\nu(x)$ is a solution of Bessel's differential equation for all $x \geq 0$.

When ν is a nonnegative integer, the gamma function can be expressed as a factorial:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+\nu)!} \left(\frac{x}{2}\right)^{2n}, \qquad \nu = 0, 1, 2, \ldots . \tag{8.18}$$

Graphs of $J_\nu(x)$ for $\nu = 0, 1, 2$ are shown in Figure 8.2.

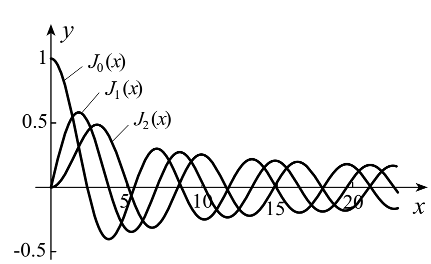


Figure 8.2

To obtain a second independent solution of Bessel's differential equation, three cases arise, depending on whether ν is not an integer, ν is zero, or ν is a positive integer.

Case 1: ν is not an integer.

We could iterate recursive relation 8.15c with the negative indicial root $r = -\nu$ (see Exercise 1), but there is a more direct route to the same solution. We examine the function obtained by replacing ν by $-\nu$ in $J_\nu(x)$,

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n}. \tag{8.19}$$

It is clear that this function also satisfies Bessel's differential equation (since the differential equation involves only ν^2). Further, it is independent of $J_\nu(x)$, since $J_\nu(0) = 0$, and $\lim_{x\to 0^+} J_{-\nu}(x) = \infty$. Thus, if ν is not an integer, a general solution of Bessel's differential equation is

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x), \tag{8.20}$$

which certainly is valid for $x > 0$ (and may or may not be valid for $x < 0$, depending on the value of ν). In the special case that ν is one-half an odd integer (1/2,3/2,5/2, etc.), the indicial roots differ by an integer, and this general solution is generated by the negative indicial root alone. The solutions in this case are called **spherical Bessel functions** (see Exercise 6).

Case 2: $\nu = 0$

When $\nu = 0$, the indicial roots are equal, and a solution of Bessel's differential equation of order zero,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0, \tag{8.21}$$

independent of

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}, \tag{8.22}$$

can be found in the form

$$y(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} A_n x^n$$

(see Case 2 of Theorem 8.1 in Section 8.1). Substitution of this solution into Bessel's differential equation leads to

$$2xJ_0' + \sum_{n=1}^{\infty} n(n-1)A_n x^n + \sum_{n=1}^{\infty} nA_n x^n + \sum_{n=1}^{\infty} A_n x^{n+2} = 0.$$

When $J_0'(x)$ is calculated using equation 8.22 and the remaining three summations are combined, the result is

$$A_1 x + 4A_2 x^2 + \sum_{n=3}^{\infty} (n^2 A_n + A_{n-2})x^n + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!2^{2n-2}} x^{2n} = 0.$$

Evidently, $A_1 = 0$, and if n is odd, the recursive formula

$$n^2 A_n + A_{n-2} = 0$$

yields $A_{2n+1} = 0$ for $n > 0$. From the terms in x^2, $A_2 = 1/4$, and from those in x^{2n}, $n \geq 2$,

$$(2n)^2 A_{2n} + A_{2n-2} + \frac{(-1)^n}{n!(n-1)!2^{2n-2}} = 0. \tag{8.23}$$

Iteration of this result gives

$$A_{2n} = \frac{(-1)^{n+1}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right), \quad n \geq 1. \tag{8.24}$$

With the notation

$$\phi(n) = \sum_{r=1}^{n} \frac{1}{r}, \tag{8.25}$$

we obtain the independent solution

$$y(x) = J_0(x) \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\phi(n)}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \tag{8.26}$$

called **Neumann's Bessel function (of the second kind) of order zero**. The series converges for all x, but the logarithm term restricts the function to $x > 0$. Any linear combination of this solution and $J_0(x)$,

$$aJ_0(x) + by(x)$$

constitutes a general solution of Bessel's differential equation of order zero. Often taken are

$$a = A + \frac{2B}{\pi}(\gamma - \ln 2), \qquad b = \frac{2}{\pi}B,$$

where γ is Euler's constant, defined by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right), \tag{8.27}$$

and A and B are arbitrary constants. In this case, a general solution of Bessel's differential equation of order zero is

$$y(x) = AJ_0(x) + BY_0(x), \tag{8.28a}$$

where

$$Y_0(x) = \frac{2}{\pi}\left\{J_0(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\phi(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right\}. \tag{8.28b}$$

Solution $Y_0(x)$ is called **Weber's Bessel function (of the second kind) of order zero**.

Case 3: ν is a positive integer

When ν is a positive integer, the indicial roots differ by an integer, and we find that $r = -\nu$ once again yields $J_\nu(x)$ (see Exercise 2). A second solution can be found in the form

$$y(x) = AJ_\nu(x)\ln x + \sum_{n=0}^{\infty} A_n x^{n-\nu} \tag{8.29}$$

(see Case 3 in Theorem 8.1 of Section 8.1). Substitution of this series into Bessel's differential equation 8.14 gives

$$2AxJ_\nu' + \sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)A_n x^{n-\nu} + \sum_{n=0}^{\infty}(n-\nu)A_n x^{n-\nu} + (x^2-\nu^2)\sum_{n=0}^{\infty} A_n x^{n-\nu} = 0,$$

and, if this equation is multiplied by x^ν and the summations are combined,

$$(1-2\nu)A_1 x + \sum_{n=2}^{\infty}[n(n-2\nu)A_n + A_{n-2}]x^n + \sum_{n=0}^{\infty}\frac{(-1)^n A(2n+\nu)}{n!(n+\nu)!2^{2n+\nu-1}}x^{2n+2\nu} = 0.$$

Evidently, $A_1 = 0$, and if n is odd, the recursive formula

$$n(n-2\nu)A_n + A_{n-2} = 0$$

requires $A_{2n+1} = 0$ for $n > 0$. Since this recursive formula is also valid for even n and $0 < n < 2\nu$, iteration gives

$$A_{2n} = \frac{A_0(\nu-n-1)!}{2^{2n}n!(\nu-1)!}, \quad 0 < n < \nu. \tag{8.30}$$

From the coefficient of $x^{2\nu}$,

$$A_{2\nu-2} + \frac{A\nu}{\nu!2^{\nu-1}} = 0,$$

which can be used with $n = \nu - 1$ in equation 8.30 to get

$$A = \frac{-A_0}{2^{\nu-1}(\nu-1)!}. \tag{8.31}$$

From the terms in $x^{2n+2\nu}$, $n > 0$,

$$2n(2n+2\nu)A_{2n+2\nu} + A_{2n+2\nu-2} + \frac{(-1)^n A(2n+\nu)}{n!(n+\nu)!2^{2n+\nu-1}} = 0.$$

Iteration of this result gives

$$A_{2n+2\nu} = \frac{(-1)^{n+1}A[\phi(n)+\phi(n+\nu)]}{n!(n+\nu)!2^{2n+\nu+1}}, \quad n > 0, \tag{8.32a}$$

provided we make the choice

$$A_{2\nu} = \frac{-A\phi(\nu)}{2^{\nu+1}\nu!}. \tag{8.32b}$$

Finally, then, the solution is

$$y(x) = AJ_\nu(x)\ln x + x^{-\nu}\left[\sum_{n=0}^{\nu-1}\frac{A_0(\nu-n-1)!}{n!(\nu-1)!}\left(\frac{x}{2}\right)^{2n} - \frac{A\phi(\nu)}{2^{\nu+1}\nu!}x^{2\nu} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}A[\phi(n)+\phi(n+\nu)]}{n!(n+\nu)!2^{2n+\nu+1}}x^{2n+2\nu}\right]. \tag{8.33}$$

The particular solution obtained by setting $A_0 = -2^{\nu-1}(\nu-1)!$ is

$$y(x) = J_\nu(x)\ln x - \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu}\sum_{n=0}^{\nu-1}\frac{(\nu-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n} - \frac{1}{2}\left(\frac{x}{2}\right)^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^n[\phi(n)+\phi(n+\nu)]}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2n}, \tag{8.34}$$

where we have adopted the convention that $\phi(0) = 0$. This solution is called **Neumann's Bessel function (of the second kind) of order** ν. Any linear combination of this solution and $J_\nu(x)$,

$$aJ_\nu(x) + by(x),$$

constitutes a general solution of Bessel's differential equation of order ν, when ν is a positive integer. Often taken are a and b as in the $\nu = 0$ case, in which case a general solution of Bessel's differential equation of positive integer order ν is

$$y(x) = AJ_\nu(x) + BY_\nu(x), \tag{8.35a}$$

where

$$Y_\nu(x) = \frac{2}{\pi}\left\{J_\nu(x)\left[\ln\left(\frac{x}{2}\right)+\gamma\right] - \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu}\sum_{n=0}^{\nu-1}\frac{(\nu-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n} - \frac{1}{2}\left(\frac{x}{2}\right)^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^n[\phi(n)+\phi(n+\nu)]}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2n}\right\}. \tag{8.35b}$$

The solution $Y_\nu(x)$ is called **Weber's Bessel function (of the second kind) of order** ν.

Notice that in the special case that $\nu = 0$, $Y_\nu(x)$ reduces to $Y_0(x)$, provided we stipulate that the first sum vanish. Graphs of $Y_0(x)$ and $Y_1(x)$ are shown in Figure 8.3.

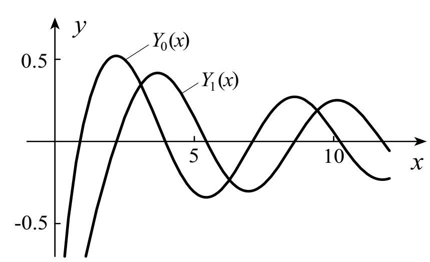


Figure 8.3

For nonnegative integer values of ν, a general solution of Bessel's differential equation has been obtained in the form $y(x) = AJ_\nu(x)+BY_\nu(x)$, and, for noninteger

ν, the solution is $y(x) = AJ_\nu(x) + BJ_{-\nu}(x)$. This situation is not completely satisfactory because the second solution is defined differently, depending on whether ν is an integer. To provide uniformity of formalism and numerical tabulation, a form of the second solution valid for all orders is sometimes preferable. Such a form is contained in

$$Y_\nu(x) = \frac{1}{\sin\nu\pi}[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)], \quad \nu \neq \text{integer}, \tag{8.36a}$$

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x), \quad n = \text{integer}. \tag{8.36b}$$

If ν is not an integer, $Y_\nu(x)$ is simply a linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$, and since $J_\nu(x)$ and $Y_\nu(x)$ must therefore be independent,

$$AJ_\nu(x) + BY_\nu(x) \tag{8.37}$$

is a general solution of Bessel's differential equation. It can be shown that as ν approaches n, $Y_\nu(x)$ is also given by 8.28b or 8.35b. Consequently, a general solution of Bessel's differential equation 8.14 is 8.37, where $J_\nu(x)$ is given by 8.17 and $Y_\nu(x)$ is given by 8.36. When ν is an integer, $Y_\nu(x)$ is also given by 8.28b or 8.35b.

Recurrence Relations

Bessel functions of lower orders are well tabulated. With recurrence relations, it is then possible to evaluate Bessel functions of higher orders. We now develop some of these relations.

Using series 8.17,

$$\begin{aligned}
J_{\nu-1}(x) + J_{\nu+1}(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\nu)}\left(\frac{x}{2}\right)^{2n+\nu-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\nu+2)}\left(\frac{x}{2}\right)^{2n+\nu+1}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\nu)}\left(\frac{x}{2}\right)^{2n+\nu-1} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu-1}\\
&= \frac{1}{\Gamma(\nu)}\left(\frac{x}{2}\right)^{\nu-1} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!\,\Gamma(n+\nu+1)}[-(n+\nu)+n]\left(\frac{x}{2}\right)^{2n+\nu-1}\\
&= \frac{\nu}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu-1} + \sum_{n=1}^{\infty}\frac{(-1)^n\nu}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu-1}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\nu}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu-1}\\
&= \frac{2\nu}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu} = \frac{2\nu}{x}J_\nu(x).
\end{aligned}$$

Thus, we have the recurrence relation

$$J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x), \quad \nu \geq 1, \tag{8.38}$$

which allows evaluation of Bessel functions of higher order by means of Bessel functions of lower order.

In addition to this functional relation, there exist many relationships among the Bessel functions and their derivatives. A derivation similar to the above yields

$$2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x), \quad \nu \geq 1, \tag{8.39}$$

(see Exercise 5). This result combines with recurrence relation 8.38 to give

$$J'_\nu(x) = -\frac{\nu}{x}J_\nu(x) + J_{\nu-1}(x), \quad \nu \geq 1, \tag{8.40}$$

and

$$J'_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x), \quad \nu \geq 0. \tag{8.41}$$

Further, multiplication of these equations by x^ν and $x^{-\nu}$, respectively, implies that

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x), \quad \nu \geq 1, \tag{8.42}$$

and

$$\frac{d}{dx}[x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x), \quad \nu \geq 0. \tag{8.43}$$

The results in equations 8.38–8.43 are also valid for $Y_\nu(x)$.

Zeros of Bessel Functions

Zeros of Bessel functions play an important role in Sturm-Liouville systems involving Bessel's differential equation (see Section 8.4). We shall show that $J_\nu(x)$ has an infinite number of positive zeros and that these zeros cannot be contained in an interval of finite length; that is, there must be arbitrarily large zeros of $J_\nu(x)$. (The result will also be valid for $Y_\nu(x)$, but our interest is in $J_\nu(x)$, and we shall therefore deal directly with $J_\nu(x)$.) We begin by changing dependent variables in Bessel's differential equation 8.14 according to $R = \sqrt{x}y(x)$ for $x > 0$ (see Exercise 7). The result is

$$\frac{d^2R}{dx^2} + \left(1 + \frac{1/4 - \nu^2}{x^2}\right)R = 0, \quad x > 0, \tag{8.44}$$

and $R(x) = \sqrt{x}J_\nu(x)$ is a solution of this equation. When $0 < \epsilon < 1$, the differential equation

$$\frac{d^2R}{dx^2} + \epsilon^2 R = 0, \quad x > 0, \tag{8.45}$$

has general solution $R(x) = A\sin(\epsilon x + \phi)$, where A and ϕ $(0 < \phi < \pi)$ are arbitrary constants, and this solution has an infinity of positive zeros, $x = (n\pi - \phi)/\epsilon$, where $n > 0$.

According to the Sturm Comparison Theorem 5.8 in Section 5.3, if $1 + (1/4 - \nu^2)/x^2$ is geater than or equal to ϵ^2, every solution of equation 8.44 has a zero between every consecutive pair of zeros of $A\sin(\epsilon x + \phi)$. But

$$1 + \frac{1/4 - \nu^2}{x^2} > \epsilon^2 \tag{8.46}$$

if, and only if,

$$x^2 > \frac{\nu^2 - 1/4}{1 - \epsilon^2}.$$

When $0 \le \nu \le 1/2$, this is valid for all $x > 0$. When $\nu > 1/2$, this is valid for all $x > x_0$ if $x_0 = \sqrt{(\nu^2 - 1/4)/(1 - \epsilon^2)}$. In other words, it is always possible to find an interval $x > x_0 \ge 0$ on which inequality 8.46 is valid. On this interval then, $R(x)$, and therefore $J_\nu(x)$ has at least one zero between every consecutive pair of zeros of $A \sin(\epsilon x + \phi)$. Since the zeros $x = (n\pi - \phi)/\epsilon$ of $A \sin(\epsilon x + \phi)$ become indefinitely large with increasing n, it follows that $J_\nu(x)$ must also have arbitrarily large zeros. The first five zeros of $J_0(x)$, $J_1(x)$, and $J_2(x)$ are shown in Figure 8.2.

EXERCISES 8.3

1. Show that when ν is not an integer, solution 8.19 of Bessel's differential equation can be obtained from the negative indicial root.

2. Show that when ν is a positive integer, the solution obtained from the negative indicial root $r = -\nu$ is $J_\nu(x)$.

3. Use series 8.18 to find values of the following, correct to four decimals: (a) $J_0(0.4)$ (b) $J_0(1.3)$ (c) $J_1(0.8)$ (d) $J_1(3.6)$ (e) $J_2(3.6)$ (f) $J_2(6.2)$ (g) $J_3(4.1)$ (h) $J_4(2.9)$

4. Calculate the following using recurrence relation 8.38 and tabulated values of J_0 and J_1: (a) $J_2(3.6)$ (b) $J_2(6.2)$ (c) $J_3(4.1)$ (d) $J_4(2.9)$

5. Verify identity 8.39.

6. Bessel functions of the first kind of order $\pm(n + 1/2)$, n a nonnegative integer, are called **spherical Bessel functions**. They can be expressed in terms of sines and cosines.

(a) Use series 8.17 and the result of Exercise 4 in Section 8.2 to show that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

(b) Use properties 8.42 and 8.43 to show that

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n [x^{-\nu} J_\nu(x)] = (-1)^n x^{-\nu-n} J_{\nu+n}(x), \qquad \left(\frac{1}{x}\frac{d}{dx}\right)^n [x^{\nu} J_\nu(x)] = x^{\nu-n} J_{\nu-n}(x),$$

where the left sides mean to apply the operator $x^{-1} d/dx$ successively n times.

(c) Prove that for $n = 0, 1, 2, \ldots$,

$$J_{n+1/2}(x) = (-1)^n \sqrt{\frac{2}{\pi}} x^{n+1/2} \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right), \quad J_{1/2-n}(x) = \sqrt{\frac{2}{\pi}} x^{n-1/2} \left(\frac{1}{x}\frac{d}{dx}\right)^n (\sin x).$$

7. Show that the change of dependent variable $R(x) = \sqrt{x}\, y(x)$ transforms Bessel's differential equation into equation 8.44.

8. Show that the function $e^{x(t-1/t)/2}$ can be expressed as the product of the series

$$e^{x(t-1/t)/2} = \left[\sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k \frac{t^k}{k!}\right] \left[\sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n \frac{t^{-n}}{n!}\right]$$

and that the product can be rearranged into the form

$$e^{x(t-1/t)/2} = J_0(x) + \sum_{m=1}^{\infty} [J_m(x)t^m + (-1)^m J_m(x)t^{-m}].$$

Because of this, $e^{x(t-1/t)/2}$ is said to be a **generating function** for $J_m(x)$, m a nonnegative integer.

9. Use integration by parts and the facts that $d[xJ_1(x)]/dx = xJ_0(x)$ and $dJ_0(x)/dx = -J_1(x)$ (see identities 8.42 and 8.41) to derive the reduction formula

$$\int x^n J_0(x)\,dx = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int x^{n-2} J_0(x)\,dx, \qquad n \geq 2.$$

10. (a) The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+\nu^2)y = 0, \quad \nu \geq 0,$$

is called **Bessel's modified differential equation of order** ν. Show that the change of independent variable $z = ix$ reduces it to Bessel's differential equation of order ν.

(b) Verify that the function $I_\nu(x) = i^{-\nu}J_\nu(ix)$, called the **modified Bessel function of the first kind of order** ν, is a solution of Bessel's modified differential equation. Find the Maclaurin series for $I_\nu(x)$ to illustrate why the factor $i^{-\nu}$ is included in its definition.

(c) Sketch graphs of $I_0(x)$ and $I_1(x)$ for $x \geq 0$.

(d) A second (linearly independent) solution of the modified equation is called the **modified Bessel function of the second kind of order** ν. Its definition is analogous to definition 8.36 for $Y_\nu(x)$:

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi}[I_{-\nu}(x) - I_\nu(x)], \quad \nu \neq \text{integer},$$
$$K_n(x) = \lim_{\nu\to n} K_\nu(x), \quad n = \text{integer}.$$

It can be shown that this definition leads to the following expressions for $K_\nu(x)$ when ν is an integer:

$$K_0(x) = -I_0(x)\left[\ln\left(\frac{x}{2}\right)+\gamma\right] + \sum_{n=1}^{\infty}\frac{\phi(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n},$$

$$K_\nu(x) = (-1)^{\nu+1}I_\nu(x)\left[\ln\left(\frac{x}{2}\right)+\gamma\right] + \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu}\sum_{n=0}^{\nu-1}\frac{(-1)^n(\nu-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n}$$
$$+ \frac{1}{2}\left(-\frac{x}{2}\right)^{\nu}\sum_{n=0}^{\infty}\frac{[\phi(n)+\phi(n+\nu)]}{n!(\nu+n)!}\left(\frac{x}{2}\right)^{2n}, \quad \nu > 0.$$

Express $K_\nu(x)$ in terms of $J_\nu(ix)$ and $Y_\nu(ix)$ when ν is an integer.

(e) Show that $K_\nu(x)$ is unbounded near $x = 0$ when ν is an integer.

In the remaining exercises we develop useful integral representations of, and integrals involving, Bessel functions of the first kind.

11. (a) Substitute $t = e^{\theta i}$ into the generating function of Exercise 8 to obtain the equations

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{m=1}^{\infty} J_{2m}(x)\cos 2m\theta, \qquad \sin(x\sin\theta) = 2\sum_{m=0}^{\infty} J_{2m+1}(x)\sin(2m+1)\theta.$$

(b) Use the facts that the sets of functions $\left\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\cos 2n\theta\right\}$ and $\left\{\sqrt{\frac{2}{\pi}}\sin(2n+1)\theta\right\}$ are orthonormal on the interval $0 \le \theta \le \pi$ to show that for $n \ge 0$ an integer,

$$J_{2n}(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\cos 2n\theta\, d\theta \quad \text{and} \quad J_{2n+1}(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin(2n+1)\theta\, d\theta.$$

(c) Show that when n is odd,

$$\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta = 0,$$

and that when n is even,

$$\int_0^\pi \sin(x\sin\theta)\sin n\theta\, d\theta = 0.$$

(d) Combine these results to show that

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\, d\theta.$$

12. (a) Use an analysis similar to that in Exercise 11 to show that for $n \ge 0$ an integer,

$$J_{2n}(x) = \frac{(-1)^n}{\pi}\int_0^\pi \cos(x\cos\theta)\cos 2n\theta\, d\theta, \qquad J_{2n+1}(x) = \frac{(-1)^n}{\pi}\int_0^\pi \sin(x\cos\theta)\cos(2n+1)\theta\, d\theta.$$

Hint: Set $t = ie^{\theta i}$ in the generating function for Bessel functions.

(b) Verify that the results in part (a) are contained in

$$J_n(x) = \frac{(-i)^n}{\pi}\int_0^\pi e^{ix\cos\theta}\cos n\theta\, d\theta.$$

13. In this exercise we verify the following integral representation for Bessel functions

$$J_n(x) = \frac{2}{\sqrt{\pi}\,\Gamma(n+1/2)}\left(\frac{x}{2}\right)^n \int_0^{\pi/2} \cos(x\sin\theta)\cos^{2n}\theta\, d\theta.$$

Let $I_n = \int_0^{\pi/2} \cos(x\sin\theta)\cos^{2n}\theta\, d\theta$, expand $\cos(x\sin\theta)$ in a Maclaurin series, and integrate term-by-term to obtain

$$I_n = \sum_{m=0}^\infty \frac{(-1)^m x^{2m}}{(2m)!}\int_0^{\pi/2} \sin^{2m}\theta\, \cos^{2n}\theta\, d\theta.$$

Now use the integration formula $\int_0^{\pi/2} \sin^{2m}\theta\, \cos^{2n}\theta\, d\theta = \frac{\Gamma(m+1/2)\,\Gamma(n+1/2)}{2\Gamma(m+n+1)}$ to verify the required integral representation of $J_n(x)$.

14. Multiply the power series representation of $J_0(bx)$ by e^{-ax} and interchange orders of integration and summation to show that when $a > 0$ and $b > 0$ are constants,

$$\int_0^\infty e^{-ax} J_0(bx)\, dx = \sum_{n=0}^\infty \frac{(-1)^n (2n)! b^{2n}}{2^{2n}(n!)^2 a^{2n+1}}.$$

Now show that the right side of this equation is the binomial expansion for $1/\sqrt{a^2+b^2}$, and hence that

$$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}.$$

15. Use Exercise 14 to prove that

$$\int_0^\infty J_0(bx)\,dx = \frac{1}{b} \qquad \text{and} \qquad \int_0^\infty J_0(x)\,dx = 1.$$

16. Although the integral in Exercise 14 was derived on the basis that a and b were positive constants with $a > b$, assume that a and b are arbitrary constants and that a can be replaced by ai to show that:

(a) if $b > a > 0$,

$$\int_0^\infty J_0(bx)\cos ax\,dx = \frac{1}{\sqrt{b^2-a^2}},$$
$$\int_0^\infty J_0(bx)\sin ax\,dx = 0;$$

(b) if $a > b > 0$,

$$\int_0^\infty J_0(bx)\sin ax\,dx = \frac{1}{\sqrt{a^2-b^2}},$$
$$\int_0^\infty J_0(bx)\cos ax\,dx = 0.$$

These are often called **Weber's discontinuous integrals**.

17. (a) Use mathematical induction, or otherwise, to show that when n is a nonnegative integer and a and b are positive constants,

$$\int_0^\infty x^n e^{-ax} J_n(bx)\,dx = \frac{(2b)^n\Gamma(n+1/2)}{\sqrt{\pi}(a^2+b^2)^{n+1/2}}.$$

(b) Prove that

$$\int_0^\infty x^{n+1} e^{-ax} J_n(bx)\,dx = \frac{2a(2b)^n\Gamma(n+3/2)}{\sqrt{\pi}(a^2+b^2)^{n+3/2}}.$$

18. Use part (a) of Exercise 17 with $n = 1$ to verify that

$$\int_0^\infty e^{-ax} J_1(bx)\,dx = \frac{1}{b} - \frac{a}{b\sqrt{a^2+b^2}}.$$

19. Use the result of Exercise 16 to show that when $a > 0$ and $b > 0$,

$$\int_0^\infty \frac{1}{x} J_0(bx)\sin ax\,dx = \begin{cases} \operatorname{Sin}^{-1}\left(\dfrac{a}{b}\right), & a < b \\ \dfrac{\pi}{2}, & a > b. \end{cases}$$

20. (a) Substitute for $J_n(bx)$ from Exercise 12 and reverse the order of integration to show that for $a > 0$, $b > 0$, and n a nonnegative integer,

$$\int_0^\infty e^{-ax} J_n(bx)\,dx = \frac{(-i)^n}{\pi}\int_0^\pi \frac{\cos n\theta}{a - ib\cos\theta}\,d\theta.$$

(b) Evaluate the contour integral $\oint_C \frac{z^n}{bz^2 + 2iaz + b}\,dz$ around the unit circle C in the complex plane to show that

$$\int_0^{\pi} \frac{\cos n\theta}{a - ib\cos\theta}\,d\theta = \frac{\pi i^n}{\sqrt{a^2+b^2}}\left(\frac{\sqrt{a^2+b^2}-a}{b}\right)^n,$$

and hence verify that

$$\int_0^{\infty} e^{-ax}J_n(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}\left(\frac{\sqrt{a^2+b^2}-a}{b}\right)^n.$$

Is Exercise 14 a special case of this when $n = 0$?

21. (a) Although the integral in Exercise 20 was derived on the basis that a and b were positive constants, assume that a can be replaced by ai to show that when $a > b$:

$$\int_0^{\infty} J_n(bx)\cos ax\,dx = -\sin\frac{n\pi}{2}\frac{1}{\sqrt{a^2-b^2}}\left(\frac{a-\sqrt{a^2-b^2}}{b}\right)^n,$$

$$\int_0^{\infty} J_n(bx)\sin ax\,dx = \cos\frac{n\pi}{2}\frac{1}{\sqrt{a^2-b^2}}\left(\frac{a-\sqrt{a^2-b^2}}{b}\right)^n.$$

(b) Why do we not derive results for the case that $a < b$ as in Exercise 16 when $n = 0$?

§8.4 Sturm-Liouville Systems and Bessel's Differential Equation

When separation of variables is applied to initial boundary value problems in polar, cylindrical, and spherical coordinates (and we shall do this in Chapter 9), both regular and singular Sturm-Liouville systems in the radial coordinate r occur. Regular systems take the form

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda^2 r - \frac{\nu^2}{r}\right)R = 0, \quad 0 < r_1 < r < r_2, \tag{8.47a}$$

$$-l_1 R'(r_1) + h_1 R(r_1) = 0, \tag{8.47b}$$

$$l_2 R'(r_2) + h_2 R(r_2) = 0, \tag{8.47c}$$

where l_1, l_2, h_1, h_2, and ν are nonnegative constants. Eigenvalues have been represented as λ^2, since 8.47 is a proper Sturm-Liouville system (the eigenvalues of which must be nonnegative). More important to our discussions is the singular system

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda^2 r - \frac{\nu^2}{r}\right)R = 0, \quad 0 < r < a, \tag{8.48a}$$

$$lR'(a) + hR(a) = 0, \tag{8.48b}$$

where l, h and ν are nonnegative constants.

Properties of system 8.47 are a straightforward application of the general theory in Section 5.1. Although we make limited use of the results, we include a brief discussion; the notation introduced and some of the results obtained are useful in the discussion of singular system 8.48.

We begin by making a change of independent variable $x = \lambda r$ in differential equation 8.47a. Since $d/dr = \lambda d/dx$, the resulting differential equation is

$$\lambda\frac{d}{dx}\left(x\frac{dR}{dx}\right) + \left(\lambda x - \frac{\lambda}{x}\nu^2\right)R = 0,$$

or,

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - \nu^2)R = 0, \tag{8.49}$$

Bessel's differential equation of order ν. According to equation 8.37, a general solution of this equation is

$$R = AJ_\nu(x) + BY_\nu(x), \tag{8.50}$$

where A and B are arbitrary constants and J_ν and Y_ν are Bessel functions of the first and second kind of order ν. Consequently, a general solution of differential equation 8.47a is

$$R(\lambda, r) = AJ_\nu(\lambda r) + BY_\nu(\lambda r). \tag{8.51}$$

If we let J'_ν denote the derivative of J_ν with respect to its argument; that is, if

$$J'_\nu(x) = \frac{d}{dx}J_\nu(x), \qquad \text{then} \qquad \frac{d}{dr}J_\nu(\lambda r) = \lambda J'_\nu(\lambda r).$$

With this notation, boundary conditions 8.47b,c require

$$-l_1\lambda[AJ'_\nu(\lambda r_1)+BY'_\nu(\lambda r_1)]+h_1[AJ_\nu(\lambda r_1)+BY_\nu(\lambda r_1)]=0, \tag{8.52a}$$
$$l_2\lambda[AJ'_\nu(\lambda r_2)+BY'_\nu(\lambda r_2)]+h_2[AJ_\nu(\lambda r_2)+BY_\nu(\lambda r_2)]=0. \tag{8.52b}$$

From the second of these,

$$B=-A\left[\frac{\lambda l_2 J'_\nu(\lambda r_2)+h_2 J_\nu(\lambda r_2)}{\lambda l_2 Y'_\nu(\lambda r_2)+h_2 Y_\nu(\lambda r_2)}\right],$$

which, substituted into the first yields

$$\frac{-\lambda l_1 J'_\nu(\lambda r_1)+h_1 J_\nu(\lambda r_1)}{-\lambda l_2 J'_\nu(\lambda r_2)+h_2 J_\nu(\lambda r_2)}=\frac{-\lambda l_1 Y'_\nu(\lambda r_1)+h_1 Y_\nu(\lambda r_1)}{\lambda l_2 Y'_\nu(\lambda r_2)+h_2 Y_\nu(\lambda r_2)}. \tag{8.53}$$

This is the eigenvalue equation, the equation defining eigenvalues of Sturm-Liouville system 8.47. Because values of λ will depend on the value of ν in differential equation 8.47a, we denote eigenvalues of equation 8.53 by $\lambda_{\nu n}$ $(n=1,2,\ldots)$ (although, in fact, $(\lambda_{\nu n})^2$ are the eigenvalues of the Sturm-Liouville system). Corresponding orthonormal eigenfunctions can be expressed in the form

$$R_{\nu n}(r)=\frac{1}{N}\left[\frac{J_\nu(\lambda_{\nu n}r)}{\lambda_{\nu n}l_2 J'_\nu(\lambda_{\nu n}r_2)+h_2 J_\nu(\lambda_{\nu n}r_2)}-\frac{Y_\nu(\lambda_{\nu n}r)}{\lambda_{\nu n}l_2 Y'_\nu(\lambda_{\nu n}r_2)+h_2 Y_\nu(\lambda_{\nu n}r_2)}\right], \tag{8.54a}$$

where the normalizing factor N^{-1} is given by

$$N^2=\int_{r_1}^{r_2} r\left[\frac{J_\nu(\lambda_{\nu n}r)}{\lambda_{\nu n}l_2 J'_\nu(\lambda_{\nu n}r_2)+h_2 J_\nu(\lambda_{\nu n}r_2)}-\frac{Y_\nu(\lambda_{\nu n}r)}{\lambda_{\nu n}l_2 Y'_\nu(\lambda_{\nu n}r_2)+h_2 Y_\nu(\lambda_{\nu n}r_2)}\right]^2 dr. \tag{8.54b}$$

This integral is evaluated in Exercise 1. We end our discussion of system 8.47 by noting that according to Theorem 5.2 in Section 5.2, functions of r can be expressed in terms of the orthonormal eigenfunctions $R_{\nu n}(r)$. Indeed, when $f(r)$ is piecewise smooth for $r_1\le r\le r_2$, we find that at any point in the open interval $r_1<r<r_2$,

$$\frac{f(r+)+f(r-)}{2}=\sum_{n=1}^{\infty}c_n R_{\nu n}(r) \quad \text{where} \quad c_n=\int_{r_1}^{r_2} r f(r) R_{\nu n}(r)\,dr. \tag{8.55}$$

This is often called the **Fourier-Bessel series** for $f(r)$. It is important to remember that ν has been fixed throughout this discussion; that is, for a fixed value of $\nu\ge 0$, there is a sequence of eigenvalues $\{\lambda_{\nu n}^2\}$ of system 8.47 together with corresponding orthonormal eigenfunctions $R_{\nu n}(r)$ and an eigenfunction expansion 8.55. Changing the value of ν results in another set of eigenpairs and a new eigenfunction expansion.

More important for our discussions is singular Sturm-Liouville system 8.48; we consider it in detail. The system is singular because no boundary condition exists at $r=0$. Notice also that $q(r)=-\nu^2/r$ is not continuous at $r=0$.

We are not really justified in denoting eigenvalues of a singular system by λ^2, since we cannot yet be sure that eigenvalues are nonnegative. However, because we shall show shortly that all eigenvalues must indeed be nonnegative, and because use of λ^2 has the immediate advantage of avoiding square roots in subsequent discussions, it is convenient to adopt this notation. Since the coefficient function of $R'(r)$ vanishes at $r=0$, the corollary to Theorem 5.1 in Section 5.1 indicates that a boundary condition at $r=0$ is unnecessary for that theorem. Examination of the proof of the theorem also indicates that continuity of $q(r)$ at $r=0$ is unnecessary. Consequently, eigenvalues of this singular system are real and corresponding eigenfunctions are orthogonal. As in the discussion of system 8.47, the change of independent variable $x = \lambda r$ leads to the general solution

$$R = AJ_\nu(\lambda r) + BY_\nu(\lambda r) \tag{8.56}$$

of differential equation 8.48a. Because $Y_\nu(\lambda r)$ is unbounded near $r = 0$, B must be set equal to zero, and we take

$$R = AJ_\nu(\lambda r). \tag{8.57}$$

Boundary condition 8.48b yields the eigenvalue equation

$$l\lambda J_\nu'(\lambda a) + hJ_\nu(\lambda a) = 0, \tag{8.58}$$

where once again the prime in the first term indicates differentiation of J_ν with respect to its argument.

Because the Sturm-Liouville system is singular, we cannot quote the results of Theorem 5.2 in Section 5.2; we must verify that the theorem is valid for this system. We first show that there is an infinity of eigenvalues, all of which are positive (except when $\nu = h = 0$, in which case zero is also an eigenvalue). We divide our discussion into three cases, depending on whether $l = 0$, $h = 0$, or $hl \neq 0$.

Case 1: $l = 0$

In this case, we set $h = 1$, and from equation 8.58 eigenvalues are defined by

$$J_\nu(\lambda a) = 0; \tag{8.59}$$

that is, eigenvalues are the zeros of Bessel function $J_\nu(x)$ divided by a. In Section 8.3 we verified that Bessel functions have an infinity of positive zeros.

Case 2: $h = 0$

In this case, we set $l = 1$, and eigenvalues are defined by the equation

$$J_\nu'(\lambda a) = 0; \tag{8.60}$$

that is, eigenvalues are critical values of Bessel function $J_\nu(x)$ divided by a. Since $J_\nu(x)$ has a continuous first derivative, Rolle's theorem from elementary calculus indicates that between every pair of zeros of $J_\nu(x)$, there is at least one point at which its derivative vanishes. Hence, equation 8.60 has an infinity of positive solutions. (The first few positive critical values of $J_0(x)$, $J_1(x)$, and $J_2(x)$ are shown in Figure 8.2.)

Case 3: $hl \neq 0$

In this case, eigenvalues are defined by equation 8.58. If we set $x = \lambda a$, eigenvalues are roots of the equation

$$Q(x) = xJ_\nu'(x) + \frac{ah}{l}J_\nu(x) = 0 \tag{8.61}$$

divided by a. When x_j and x_{j+1} are consecutive positive zeros of $J_\nu(x)$, $Q(x)$ has one sign at x_j and the opposite sign at x_{j+1}. Because $Q(x)$ is continuous, it must have at least one zero between x_j and x_{j+1}. It follows, therefore, that equation 8.61 must have an infinity of positive solutions.

We have shown that each of the eigenvalue equations 8.58, 8.59, and 8.60 has an infinity of positive solutions λ. These solutions define positive eigenvalues λ^2 of the singular Sturm-Liouville system. To show that the system can have no negative eigenvalues, we set $\lambda = i\phi$ (ϕ real and not equal to zero). Equation 8.58 with $\lambda = i\phi$ then reads

$$il\phi J_\nu'(i\phi a) + hJ_\nu(i\phi a) = 0.$$

If we replace J_ν' by J_ν and $J_{\nu+1}$ according to equation 8.41, this equation becomes

$$(ah + \nu l)J_\nu(i\phi a) - i\phi al J_{\nu+1}(i\phi a) = 0.$$

We now express $J_\nu(i\phi a)$ and $J_{\nu+1}(i\phi a)$ in terms of their power series; the result is

$$0 = \left(\frac{i\phi a}{2}\right)^\nu \left[(ah + \nu l)\sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+\nu+1)}\left(\frac{\phi a}{2}\right)^{2n} + \frac{\phi^2 a^2 l}{2}\sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+\nu+2)}\left(\frac{\phi a}{2}\right)^{2n}\right].$$

Because $ah + \nu l \geq 0$ and $l \geq 0$, and both series contain only positive terms, there can be no solution ϕ. Thus, all eigenvalues of equation 8.58 must be nonnegative.

We now show that $\lambda = 0$ is an eigenvalue only when $h = \nu = 0$. Since the eigenfunction corresponding to an eigenvalue λ is always $J_\nu(\lambda r)$, it is clear that the eigenfunction will be identically zero if $\lambda = 0$ is an eigenvalue, except when $\nu = 0$ (when $\nu = 0$, the eigenfunction corresponding to $\lambda = 0$ is $J_0(0) = 1$). Because $J_0(0) \neq 0$ and $J_0'(0) = 0$, it follows that $\lambda = 0$ is an eigenvalue of equation 8.60 but not of 8.58 or 8.59. Thus, there is only one possibility for a zero eigenvalue — both h and ν must be equal to zero.

One last point remains to be cleared up. If ν is such that J_ν is defined for negative arguments, then for every positive solution λ of equations 8.58, 8.59, and 8.60, $-\lambda$ is also a solution. However, the power series expansion for J_ν clearly indicates that the eigenfunction $J_\nu(-\lambda r)$ is, except for a multiplicative constant, identical to $J_\nu(\lambda r)$. Thus, negative solutions of the eigenvalue equations lead to the same eigenvalues λ^2 of the Sturm-Liouville system and the same eigenfunctions.

We have now shown that singular Sturm-Liouville system 8.48 has an infinity of eigenvalues, all of which are positive (except when $\nu = h = 0$, in which case zero is also an eigenvalue). If we denote these eigenvalues by $\lambda_{\nu n}$ ($n = 1, 2, \ldots$), then from equation 8.57, corresponding orthonormal eigenfunctions are

$$R_{\nu n}(r) = \frac{1}{N}J_\nu(\lambda_{\nu n} r) \quad \text{where} \quad N^2 = \int_0^a r[J_\nu(\lambda_{\nu n} r)]^2\,dr. \tag{8.62}$$

To avoid direct integration of J_ν, we note that any function R satisfying differential equation 8.48a also satisfies

$$0 = 2rR'(rR')' + \left(\lambda^2 r - \frac{\nu^2}{r}\right)2rRR' = \frac{d}{dr}(rR')^2 + (\lambda^2 r^2 - \nu^2)\frac{d}{dr}(R^2).$$

Integration of this equation with respect to r from $r = 0$ to $r = a$ gives

$$0 = \left\{ (rR')^2 - \nu^2 R^2 \right\}_0^a + \lambda^2 \int_0^a r^2 \frac{d}{dr}(R^2)\, dr$$
$$= \left\{ (rR')^2 - \nu^2 R^2 \right\}_0^a + \lambda^2 \left\{ aR^2 \right\}_0^a - \lambda^2 \int_0^a 2rR^2\, dr,$$

and when this is solved for the remaining integral,

$$2\lambda^2 \int_0^a rR^2\, dr = \left\{ (rR')^2 - \nu^2 R^2 + \lambda^2 r^2 R^2 \right\}_0^a.$$

If we now replace λ with $\lambda_{\nu n}$ and R with solution $J_\nu(\lambda_{\nu n} r)$ of 8.48a,

$$2\lambda_{\nu n}^2 \int_0^a r[J_\nu(\lambda_{\nu n} r)]^2\, dr = a^2 \lambda_{\nu n}^2 [J_\nu'(\lambda_{\nu n} a))]^2 + (\lambda_{\nu n}^2 a^2 - \nu^2)[J_\nu(\lambda_{\nu n} a)]^2,$$

from which

$$2N^2 = 2\int_0^a r[J_\nu(\lambda_{\nu n} r)]^2\, dr = a^2 [J_\nu'(\lambda_{\nu n} a)]^2 + \left(a^2 - \frac{\nu^2}{\lambda_{\nu n}^2} \right) [J_\nu(\lambda_{\nu n} a)]^2$$
$$= a^2 \left[\frac{-hJ_\nu(\lambda_{\nu n} a)}{\lambda_{\nu n} l} \right]^2 + \left[a^2 - \left(\frac{\nu}{\lambda_{\nu n}} \right)^2 \right] [J_\nu(\lambda_{\nu n} a)]^2$$
$$= a^2 \left[1 - \left(\frac{\nu}{\lambda_{\nu n} a} \right)^2 + \left(\frac{h}{\lambda_{\nu n} l} \right)^2 \right] [J_\nu(\lambda_{\nu n} a)]^2.$$

In summary, orthonormal eigenfunctions of singular system 8.48 are

$$R_{\nu n}(r) = \frac{1}{N} J_\nu(\lambda_{\nu n} r), \tag{8.63a}$$

where

$$2N^2 = a^2 \left[1 - \left(\frac{\nu}{\lambda_{\nu n} a} \right)^2 + \left(\frac{h}{\lambda_{\nu n} l} \right)^2 \right] [J_\nu(\lambda_{\nu n} a)]^2 \tag{8.63b}$$

and eigenvalues $\lambda_{\nu n}$ are defined by the equation $l\lambda J_\nu'(\lambda a) + hJ_\nu(\lambda a) = 0$. There are three possible boundary conditions at $r = a$, depending on whether $l = 0$, $h = 0$, or $lh \neq 0$. The results for all three cases are listed in Table 8.1.

Condition at $r = a$	Eigenvalue Equation	$NR_{\nu n}$	$2N^2$
$hl \neq 0$	$l\lambda J_\nu'(\lambda a) + hJ_\nu(\lambda a) = 0$	$J_\nu(\lambda_{\nu n} r)$	$a^2 \left[1 - \left(\frac{\nu}{\lambda_{\nu n} a} \right)^2 + \left(\frac{h}{\lambda_{\nu n} l} \right)^2 \right] [J_\nu(\lambda_{\nu n} a)]^2$
$h = 0$	$J_\nu'(\lambda a) = 0$	$J_\nu(\lambda_{\nu n} r)$	$a^2 \left[1 - \left(\frac{\nu}{\lambda_{\nu n} a} \right)^2 \right] [J_\nu(\lambda_{\nu n} a)]^2$
$l = 0$	$J_\nu(\lambda a) = 0$	$J_\nu(\lambda_{\nu n} r)$	$a^2 [J_\nu'(\lambda_{\nu n} a)]^2 = a^2 [J_{\nu+1}(\lambda_{\nu n} a)]^2$

Table 8.1

According to the following theorem, piecewise smooth functions of r can be expanded in Fourier Bessel series of these eigenfunctions.

Theorem 8.2 If a function $f(r)$ is piecewise smooth on the interval $0 \le r \le a$, then for each r in $0 < r < a$,

$$\frac{f(r+) + f(r-)}{2} = \sum_{n=1}^{\infty} c_n R_{\nu n}(r) \quad \text{where} \quad c_n = \int_0^a r f(r) R_{\nu n}(r)\, dr. \tag{8.64}$$

The extra r in the integrand is the weight function $p(r) = r$ for Sturm-Liouville system 8.48. It is no coincidence that this is the same r that appears in evaluation of double integrals in polar coordinates.

Example 8.3 Find the Fourier Bessel series for the function $f(r) = r^2$ in terms of the eigenfunctions of Sturm-Liouville system 8.48 when $a = 1$, $l = 0$, $h = 1$, and $\nu = 0$. Plot the fifth and tenth partial sums of the expansion.

Solution Orthonormal eigenfunctions of

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2 r R = 0, \quad 0 < r < 1,$$
$$R(1) = 0,$$

are given in line three of Table 8.1 with $\nu = 0$. When we suppress the first subscript $\nu = 0$ in the eigenfunctions $R_{\nu n}$ and the eigenvalues $\lambda_{\nu n}$, we have

$$R_n(r) = \frac{\sqrt{2} J_0(\lambda_n r)}{J_1(\lambda_n)},$$

where eigenvalues λ_n are solutions of $J_0(\lambda) = 0$. The Fourier Bessel series of r^2 is

$$r^2 = \sum_{n=1}^{\infty} c_n R_n(r),$$

where

$$c_n = \int_0^1 r^3 R_n(r)\, dr = \frac{\sqrt{2}}{J_1(\lambda_n)} \int_0^1 r^3 J_0(\lambda_n r)\, dr.$$

To evaluate this integral, we first set $x = \lambda_n r$, in which case

$$c_n = \frac{\sqrt{2}}{J_1(\lambda_n)} \int_0^{\lambda_n} \left(\frac{x}{\lambda_n}\right)^3 J_0(x) \frac{dx}{\lambda_n} = \frac{\sqrt{2}}{\lambda_n^4 J_1(\lambda_n)} \int_0^{\lambda_n} x^3 J_0(x)\, dx.$$

We now use the reduction formula in Exercise 9 of Section 8.3,

$$\begin{aligned}
c_n &= \frac{\sqrt{2}}{\lambda_n^4 J_1(\lambda_n)} \left[\left\{x^3 J_1(x) + 2x^2 J_0(x)\right\}_0^{\lambda_n} - 4\int_0^{\lambda_n} x J_0(x)\, dx\right] \\
&= \frac{\sqrt{2}}{\lambda_n^4 J_1(\lambda_n)} \left[\lambda_n^3 J_1(\lambda_n) - 4\int_0^{\lambda_n} \frac{d}{dx}[x J_1(x)]\, dx\right] \qquad \text{(see identity 8.42 with } \nu = 1\text{)} \\
&= \frac{\sqrt{2}}{\lambda_n^4 J_1(\lambda_n)} [\lambda_n^3 J_1(\lambda_n) - 4\lambda_n J_1(\lambda_n)] = \frac{\sqrt{2}(\lambda_n^2 - 4)}{\lambda_n^3}.
\end{aligned}$$

Consequently,

$$r^2 = \sum_{n=1}^{\infty} \frac{\sqrt{2}(\lambda_n^2 - 4)}{\lambda_n^3} \frac{\sqrt{2}J_0(\lambda_n r)}{J_1(\lambda_n)} = 2\sum_{n=1}^{\infty} \frac{\lambda_n^2 - 4}{\lambda_n^3 J_1(\lambda_n)} J_0(\lambda_n r), \quad 0 < r < 1.$$

The first ten eigenvalues of the Sturm-Liouville system, obtained from the eigenvalue equation $J_0(\lambda) = 0$, are

$$\lambda_1 = 2.40483, \quad \lambda_2 = 5.52008, \quad \lambda_3 = 8.65373, \quad \lambda_4 = 11.7915, \quad \lambda_5 = 14.9309,$$

$$\lambda_6 = 18.0711, \quad \lambda_7 = 21.2116, \quad \lambda_8 = 24.3525, \quad \lambda_9 = 27.4935, \quad \lambda_{10} = 30.6346.$$

Using these, the fifth and tenth partial sums of the series are plotted in Figures 8.4a,b. Needless to say, many more terms are required to obtain a reasonable approximation.●

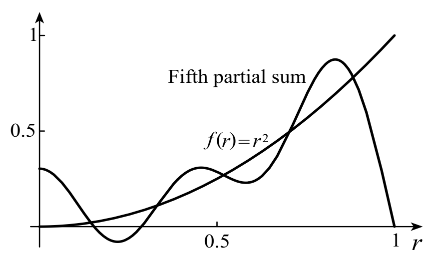


Figure 8.4a

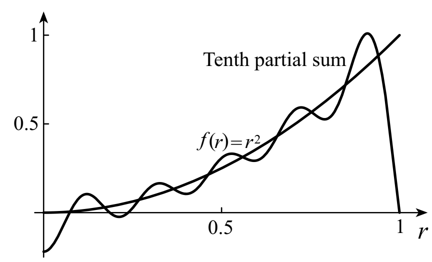


Figure 8.4b

Finding coefficients in Fourier Bessel series is far more formidable than finding coefficients in the generalized Fourier series of Section 5.2, where orthonormal eigenfunctions consisted of trigonometric functions. With enough perseverance we could find the generalized Fourier series for any polynomial whatsoever. Such is not the case for Fourier Bessel series. For instance, in calculating coefficients for the Fourier Bessel series of r^2 in Example 8.3, the reduction formula from Exercise 9 of Section 8.3 was indispensable. Unfortunately, it yields c_n in closed form only when r is raised to an even power (see Exercise 5 when the power is 1). When $\nu \neq 0$, calculation of coefficients in closed form is usually impossible. Exercise 2 deals with an exceptional case when r is raised to power ν.

EXERCISES 8.4

1. Use the following argument to evaluate the normalizing factor N^{-1} in equation 8.54b.

(a) Show that any solution of differential equation 8.47a also satisfies

$$\frac{d}{dr}(rR')^2 + (\lambda^2 r^2 - \nu^2)\frac{d}{dr}(R^2) = 0.$$

(b) Integrate this equation from r_1 to r_2 to obtain

$$2\lambda^2 \int_{r_1}^{r_2} rR^2\, dr = \Big\{(rR')^2 + (\lambda^2 r^2 - \nu^2)R^2\Big\}_{r_1}^{r_2}.$$

(c) Use boundary conditions 8.47b,c to write this expression in the form

$$2\lambda^2 \int_{r_1}^{r_2} rR^2\,dr = [r_2R(r_2)]^2 \left[\lambda^2 - \left(\frac{\nu}{r_2}\right)^2 + \left(\frac{h_2}{l_2}\right)^2\right] - [r_1R(r_1)]^2 \left[\lambda^2 - \left(\frac{\nu}{r_1}\right)^2 + \left(\frac{h_1}{l_1}\right)^2\right].$$

(d) Substitute $\lambda = \lambda_{\nu n}$ and $R = R_{\nu n}$ (from 8.54a, without the normalizing factor N^{-1}) to obtain an expression for N^{-1}.

2. Find the Fourier Bessel series for the function r^ν ($\nu \geq 1$) in terms of the eigenfunctions of Sturm-Liouville system 8.48 when (a) $l = 0$ and (b) $h = 0$.

3. Find the Fourier Bessel series for the function $f(r) = 1$ in terms of the eigenfunctions of Sturm-Liouville system 8.48 when $\nu = 0$.

4. Plot the tenth partial sum of the series in Exercise 3 in the case that $a = 1$ and $l = 0$.

5. Show that it is not possible to find coefficients in the Fourier Bessel series of the function $f(r) = r$ in closed form using the orthonormal eigenfunctions in Example 8.3.

6. Show that eigenpairs for the singular Sturm-Liouville system

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda^2 r - \frac{\nu^2}{r}\right)R = 0, \quad 0 < r < a, \quad \nu > 0,$$
$$lR'(a) - hR(a) = 0,$$

where $l > 0$ and $h > 0$, are also given in the first line of Table 8.1 (with h replaced by $-h$).

7. (a) Use a Frobenius series to obtain the general solution

$$R(r) = \begin{cases} A/r + B, & \lambda = 0 \\ (A\cos\lambda r + B\sin\lambda r)/r, & \lambda \neq 0 \end{cases}$$

of the differential equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda^2 r^2 R = 0, \quad 0 < r < a.$$

(b) Show that the change of dependent variable $Z(r) = \sqrt{\lambda r}R(r)$ leads to the differential equation

$$\frac{d}{dr}\left(r\frac{dZ}{dr}\right) + \left(\lambda^2 r - \frac{1/4}{r}\right)Z = 0, \quad 0 < r < a,$$

and the solutions in part (a).

8. The singular Sturm-Liouville system

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda^2 r^2 R = 0, \quad 0 < r < a,$$
$$lR'(a) + hR(a) = 0,$$

associated with the differential equation in Exercise 7 arises when separation of variables is applied to heat conduction problems in a sphere when temperature is a function of only radial distance r and time. Use the general solution of the differential equation derived in Exercise 7 to confirm the tabulated results below for the three cases $l = 0$, $h = 0$, and $lh \neq 0$. Illustrate the eigenvalues graphically when $hl \neq 0$ and $h = 0$. Assume that $ah < l$ in line 1.

Condition at $r = a$	Eigenvalue Equation	NR_{mn}	$2N^2$
$hl \neq 0$	$\tan \lambda a = \dfrac{\lambda a}{1 - ha/l}$	$\dfrac{1}{r} \sin \lambda_n r$	$a\left[1 + \dfrac{ha/l - 1}{\lambda_n^2 a^2 + (1 - ha/l)^2}\right]$
$h = 0$	$\tan \lambda a = \lambda a$	$1 \quad , \; n = 0$ $\dfrac{1}{r} \sin \lambda_n r, \; n > 0$	$\dfrac{2a^3}{3} \quad , \; n = 0$ $\dfrac{a^3 \lambda_n^2}{1 + \lambda_n^2 a^2}, \; n > 0$
$l = 0$	$\sin \lambda a = 0$ $\lambda_n = \dfrac{n\pi}{a}, \quad n = 1, 2, \ldots$	$\dfrac{1}{r} \sin \dfrac{n\pi r}{a}$	a

9. The singular Sturm-Liouville system

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + [\lambda^2 r^2 - m(m+1)]R = 0, \quad 0 < r < a,$$
$$lR'(a) + hR(a) = 0,$$

where $m \geq 0$ is an integer and $l \geq 0$ and $h \geq 0$ arises when separation of variables is applied to heat conduction problems in a sphere when temperature is a function of radial distance r, angle θ, and time. Use the change of variable in part (b) of Exercise 7 to confirm the tabulated results below for the three cases $l = 0$, $h = 0$, and $lh \neq 0$.

Condition at $r = a$	Eigenvalue Equation	NR_{mn}	$2N^2$
$hl \neq 0$	$0 = 2\lambda a J'_{m+1/2}(\lambda a)$ $+ \left(\dfrac{2ha - l}{l}\right) J_{m+1/2}(\lambda a)$	$\dfrac{J_{m+1/2}(\lambda_{mn} r)}{\sqrt{r}}$	$a^2\left[1 - \left(\dfrac{m + 1/2}{\lambda_{mn} a}\right)^2 + \left(\dfrac{2ha/l - 1}{2\lambda_{mn} a}\right)^2\right][J_{m+1/2}(\lambda_{mn} a)]^2$
$h = 0$	$0 = 2\lambda a J'_{m+1/2}(\lambda r)$ $- J_{m+1/2}(\lambda a)$	$\dfrac{J_{m+1/2}(\lambda_{mn} r)}{\sqrt{r}}$	$a^2\left[1 - \left(\dfrac{m + 1/2}{\lambda_{mn} a}\right)^2 + \left(\dfrac{1}{2\lambda_{mn} a}\right)^2\right][J_{m+1/2}(\lambda_{mn} a)]^2$
$l = 0$	$0 = J_{m+1/2}(\lambda a)$	$\dfrac{J_{m+1/2}(\lambda_{mn} r)}{\sqrt{r}}$	$a^2[J'_{m+1/2}(\lambda_{mn} a)]^2 = a^2[J_{m+3/2}(\lambda_{mn} a)]^2$

10. The Sturm-Liouville system

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \lambda^2 r^2 R = 0, \quad r_1 < r < r_2,$$

$$-l_1R'(r_1) + h_1R(r_1) = 0$$
$$l_2R'(r_2) + h_2R(r_2) = 0,$$

where l_1, l_2, h_1, and h_2 are nonnegative constants, is associated with the differential equation in Exercise 7. It arises when separation of variables is applied to heat conduction problems in spherical shells when temperature is a function of radial distance r and time. Use the general solution of the differential equation derived in Exercise 7 to confirm the tabulated results below for the four cases shown.

Condition at $r = r_1$	Condition at $r = r_2$	Eigenvalue Equation	NR_n	$2N^2$
$h_1 = 0$ $(l_1 = 1)$	$h_2 = 0$ $(l_2 = 1)$	$\sin\lambda(r_2 - r_1) = 0$ $\lambda_n = \dfrac{n\pi}{r_2 - r_1},\ n = 0, 1, 2, \ldots$	$\cos\lambda_n r$	$r_2 - r_1\ (n \neq 0)$ $2(r_2 - r_1)\ (n = 0)$
$h_1 = 0$ $(l_1 = 1)$	$l_2 = 0$ $(h_2 = 1)$	$\tan\lambda(r_2 - r_1) = -\lambda r_1$	$\dfrac{1}{r}\sin\lambda_n(r_2 - r)$	$r_2 - r_1 + \dfrac{r_1}{1 + \lambda_n^2 r_1^2}$
$l_1 = 0$ $(h_1 = 1)$	$h_2 = 0$ $(l_2 = 1)$	$\tan\lambda(r_2 - r_1) = \lambda r_2$	$\dfrac{1}{r}\sin\lambda_n(r - r_1)$	$r_2 - r_1$
$l_1 = 0$ $(h_1 = 1)$	$l_2 = 0$ $(h_2 = 1)$	$\sin\lambda(r_2 - r_1) = 0$ $\lambda_n = \dfrac{n\pi}{r_2 - r_1},\ n = 1, 2, \ldots$	$\dfrac{1}{r}\sin\dfrac{n\pi(r - r_1)}{r_2 - r_1}$	$r_2 - r_1$

§8.5 Legendre Functions

Legendre functions arise when separation of variables is applied to (initial) boundary value problems expressed in spherical coordinates. They are solutions of the linear, homogeneous, second-order differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \tag{8.65}$$

called **Legendre's differential equation**. If we assume a power series solution $y(x) = \sum_{k=0}^{\infty} a_k x^k$ ($x = 0$ being an ordinary point of the differential equation), we obtain arbitrary a_0 and a_1 and the recurrence relation

$$a_k = -\frac{(n-k+2)(n+k-1)}{k(k-1)}a_{k-2}, \quad k \geq 2. \tag{8.66}$$

Iteration of this result leads to the general solution

$$\begin{aligned} y(x) = a_0 &\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+2)\cdots(n-2)n(n+1)(n+3)\cdots(n+2k-1)}{(2k)!}x^{2k}\right] \\ &+ a_1\left[x + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+1)\cdots(n-3)(n-1)(n+2)(n+4)\cdots(n+2k)}{(2k+1)!}x^{2k+1}\right], \end{aligned} \tag{8.67}$$

which converges for $|x| < 1$.

When n is a nonnegative integer, one of these series reduces to a polynomial while the other remains an infinite series. In particular, if n is an even integer, all terms in the first series vanish for $2k > n$, and if n is odd, all terms in the second series vanish for $2k + 1 > n$. Thus, in either case, the solution defines a polynomial of degree n. To express these polynomials compactly, we reverse recurrence relation 8.66 to write

$$a_{k-2} = -\frac{k(k-1)}{(n-k+2)(n+k-1)}a_k$$

and iterate to obtain

$$a_{n-2k} = \frac{(-1)^k n(n-1)(n-2)\cdots(n-2k+1)}{2^k k!(2n-1)(2n-3)\cdots(2n-2k+1)}a_n. \tag{8.68}$$

When we choose $a_n = (2n)!/[2^n(n!)^2]$, this becomes

$$a_{n-2k} = \frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!}, \quad k = 1, 2, \ldots, \lfloor n/2 \rfloor, \tag{8.69}$$

where $\lfloor n/2 \rfloor$ denotes the integer part of $n/2$. With this choice for a_n, the particular polynomial solution of Legendre's differential equation 8.65 is called the **Legendre polynomial of degree** n, denoted by

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!}x^{n-2k}. \tag{8.70}$$

The first five Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2 - 1}{2}, \quad P_3(x) = \frac{5x^3 - 3x}{2}, \quad P_4(x) = \frac{35x^4 - 30x^2 + 3}{8}.$$

The remaining solution of differential equation 8.65 for n a nonnegative integer is in the form of an infinite series valid for $|x| < 1$. When n is even, and a_1 is chosen as $(-1)^{n/2}2^n[(n/2)!]^2/n!$, the series solution is denoted by

$$Q_n(x) = \frac{(-1)^{n/2}2^n[(n/2)!]^2}{n!}\Bigg[x + \sum_{k=1}^{\infty} \frac{(-1)^k(n-2k+1)\cdots(n-3)(n-1)(n+2)(n+4)\cdots(n+2k)}{(2k+1)!}x^{2k+1}\Bigg]. \tag{8.71a}$$

When n is odd, and a_0 is set equal to $(-1)^{(n+1)/2}2^{n-1}\{[(n-1)/2]!\}^2/n!$, the series solution is

$$Q_n(x) = \frac{(-1)^{(n+1)/2}2^{n-1}\{[(n-1)/2]!\}^2}{n!}\Bigg[1 + \sum_{k=1}^{\infty} \frac{(-1)^k(n-2k+2)\cdots(n-2)n(n+1)(n+3)\cdots(n+2k-1)}{(2k)!}x^{2k}\Bigg]. \tag{8.71b}$$

These solutions are called **Legendre functions of the second kind of order** n. Closed-form representations are discussed in Exercise 10; they are unbounded near $x = \pm 1$.

In summary, a general solution of Legendre's differential equation 8.65 for n a nonnegative integer is

$$y(x) = AP_n(x) + BQ_n(x), \tag{8.72}$$

where A and B are arbitrary constants. Legendre polynomials $P_n(x)$ are given by equation 8.70, and Legendre functions $Q_n(x)$ of the second kind are defined by 8.71. Our discussions concentrate on Legendre polynomials.

Generating Function for Legendre Polynomials

When the binomial expansion is applied to the function $(1 - 2xt + t^2)^{-1/2}$,

$$\frac{1}{(1 - 2xt + t^2)^{1/2}} = 1 + \sum_{m=1}^{\infty} \frac{(1/2)(3/2)\cdots(1/2 + m - 1)}{m!}(2xt - t^2)^m,$$

and the binomial theorem is then used on $(2xt - t^2)^m$,

$$\frac{1}{(1 - 2xt + t^2)^{1/2}} = 1 + \sum_{m=1}^{\infty} \frac{(1)(3)(5)\cdots(2m-1)}{2^m m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (2x)^{m-k}t^{m+k}.$$

Terms in t^n occur when $k+m=n$, and since k ranges from 0 to m, it follows that the coefficient of t^n is

$$\sum_{m=\lfloor (n+1)/2 \rfloor}^{n} \frac{(1)(3)(5)\cdots(2m-1)}{2^m m!}(-1)^{n-m}\binom{m}{n-m}(2x)^{2m-n}.$$

If we set $k=n-m$ in this summation, the coefficient of t^n is

$$\sum_{k=n-\lfloor (n+1)/2 \rfloor}^{0} \frac{(1)(3)(5)\cdots(2n-2k-1)}{2^{n-k}(n-k)!}(-1)^{k}\binom{n-k}{k}(2x)^{n-2k},$$

and this immediately reduces to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!}x^{n-2k};$$

that is,

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!}x^{n-2k}\right]t^n. \tag{8.73}$$

The coefficient of t^n is $P_n(x)$, and we say that $(1-2xt+t^2)^{-1/2}$ is a generating function for $P_n(x)$,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{8.74}$$

Recurrence Relations

When we differentiate equation 8.74 with respect to t,

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}, \tag{8.75}$$

from which

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Equating coefficients of like powers of t gives the recurrence relation

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0, \quad n\geq 1, \tag{8.76}$$

which permits evaluation of Legendre polynomials of higher orders in terms of those of lower orders. Useful relations among the derivatives of Legendre polynomials also exist. Differentiation of equation 8.74 with respect to x gives

$$\frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)t^n,$$

which, together with equation 8.75, implies that

$$t\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} P_n'(x)t^n. \tag{8.77}$$

Equating coefficients yields

$$xP_n'(x) - P_{n-1}'(x) - nP_n(x) = 0, \quad n \geq 1. \tag{8.78}$$

Differentiation of recurrence relation 8.76 gives

$$(n+1)P_{n+1}'(x) - (2n+1)P_n(x) - (2n+1)xP_n'(x) + nP_{n-1}'(x) = 0, \quad n \geq 1. \tag{8.79}$$

Elimination of $P_n'(x)$ between the last two equations yields

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x), \quad n \geq 1 \tag{8.80}$$

and, in addition,

$$P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x), \quad n \geq 0. \tag{8.81}$$

We now show that $P_n(x)$ is a constant multiple of $d^n(x^2-1)^n/dx^n$. We first note that

$$\frac{d}{dx}(x^2-1)^n = 2nx(x^2-1)^{n-1} \quad \Longrightarrow \quad (x^2-1)\frac{d}{dx}(x^2-1)^n = 2nx(x^2-1)^n.$$

Differentiation of this equation $n+1$ times with Leibniz's rule* gives

$$\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k}{dx^k}(x^2-1)\frac{d^{n-k+2}}{dx^{n-k+2}}(x^2-1)^n = 2n\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k}{dx^k}x\frac{d^{n-k+1}}{dx^{n-k+1}}(x^2-1)^n,$$

but only the first three terms on the left and the first two terms on the right do not vanish. When these terms are written out and rearranged,

$$(1-x^2)\frac{d^2}{dx^2}\left[\frac{d^n}{dx^n}(x^2-1)^n\right] - 2x\frac{d}{dx}\left[\frac{d^n}{dx^n}(x^2-1)^n\right] + n(n+1)\left[\frac{d^n}{dx^n}(x^2-1)^n\right] = 0.$$

This equation indicates that the function $d^n(x^2-1)^n/dx^n$ satisfies Legendre's differential equation 8.65. Since the function is a polynomial in x, it follows that

$$P_n(x) = A\frac{d^n}{dx^n}(x^2-1)^n.$$

To obtain the constant A, we equate coefficients of x^n on each side,

$$\frac{(2n)!}{2^n(n!)^2} = A(2n)(2n-1)\cdots(n+1).$$

Thus, $A = 1/(2^n n!)$, and we obtain what is called **Rodrigues' formula**,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{8.82}$$

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* Leibniz's rule for the n^{th} derivative of a product is

$$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{k=0}^{n}\binom{n}{k}\left[\frac{d^k}{dx^k}f(x)\right]\left[\frac{d^{n-k}}{dx^{n-k}}g(x)\right].$$

Rodrigues' formula is useful in the evaluation of definite integrals involving Legendre's polynomials. In addition, it quickly yields values for $P_n(\pm 1)$. With $x^2 - 1$ in factored form, Leibniz's rule gives

$$\begin{aligned}
P_n(\pm 1) &= \frac{1}{2^n n!}\left[\frac{d^n}{dx^n}(x^2-1)^n\right]_{|x=\pm 1} \\
&= \frac{1}{2^n n!}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}(x+1)^n\frac{d^{n-k}}{dx^{n-k}}(x-1)^n\right]_{|x=\pm 1}.
\end{aligned}$$

The only term in the summation that does not involve $x - 1$ occurs when $k = 0$, and therefore

$$P_n(1) = \frac{\binom{n}{0}2^n n!}{2^n n!} = 1. \tag{8.83a}$$

Similarly, because $k = n$ is the only term without a factor $x + 1$,

$$P_n(-1) = \frac{\binom{n}{n}n!(-2)^n}{2^n n!} = (-1)^n. \tag{8.83b}$$

Associated Legendre Functions

Legendre's associated differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0, \tag{8.84}$$

where m is some given nonnegative integer. When $m = 0$, it reduces to Legendre's differential equation 8.65. It is straightforward to show (see Exercise 9) that when $y(x)$ is a solution of 8.65, $(1-x^2)^{m/2}d^my/dx^m$ is a solution of 8.84. This means that a general solution of 8.84 is

$$y(x) = (1-x^2)^{m/2}\left[A\frac{d^mP_n(x)}{dx^m} + B\frac{d^mQ_n(x)}{dx^m}\right], \tag{8.85}$$

where $P_n(x)$ are Legendre polynomials and $Q_n(x)$ are Legendre functions of the second kind. The functions

$$P_{mn}(x) = (1-x^2)^{m/2}\frac{d^mP_n(x)}{dx^m}, \tag{8.86a}$$

$$Q_{mn}(x) = (1-x^2)^{m/2}\frac{d^mQ_n(x)}{dx^m}, \tag{8.86b}$$

are called **associated Legendre functions of degree n and order m of the first and second kind**. Since $P_n(x)$ is a polynomial of degree n, it follows that $P_{mn}(x)$ is nonvanishing only when $n \geq m$.

EXERCISES 8.5

1. Calculate the first seven Legendre polynomials using (a) equation 8.82 (b) equation 8.70.

2. Show that Legendre polynomials $P_n(x)$ are even when n is even and odd when n is odd.

3. Use $P_0(x) = 1$, $P_1(x) = x$, and recurrence relation 8.76 to obtain $P_2(x)$, $P_3(x)$, $P_4(x)$, $P_5(x)$, and $P_6(x)$.

4. Prove the following:

(a) $P_{2n+1}(0) = 0$

(b) $P_{2n}(0) = \dfrac{(-1)^n(2n)!}{2^{2n}(n!)^2}$

(c) $P'_{2n}(0) = 0$

(d) $P'_{2n+1}(0) = \dfrac{(-1)^n(2n+1)!}{2^{2n}(n!)^2}$

(e) $P'_n(1) = \dfrac{n(n+1)}{2}$

(f) $P'_n(-1) = \dfrac{(-1)^{n-1}n(n+1)}{2}$

5. Verify the following identities for Legendre polynomials:

(a) $nP_{n-1}(x) - P'_n(x) + xP'_{n-1}(x) = 0, \quad n \geq 1$

Hint: Show that the generating function for $P_n(x)$ satisfies

$$t\frac{\partial}{\partial t}\left(\frac{t}{\sqrt{1-2xt+t^2}}\right) + (tx-1)\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1-2xt+t^2}}\right) = 0.$$

(b) $(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x), \quad n \geq 1$

(c) $nP_n(x) = nxP_{n-1}(x) + (x^2-1)P'_{n-1}(x), \quad n \geq 1$

6. Verify that when $f(x)$ has continuous derivatives of orders up to and including n,

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n\,dx.$$

7. Verify the following results:

(a) $\displaystyle\int_{-1}^{1} P_n(x)\,dx = \begin{cases} 2, & n=0 \\ 0, & n \neq 0 \end{cases}$

(b) $\displaystyle\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & n \neq m \\ 2/(2n+1), & n = m \end{cases}$ Hint: Use Exercise 6

(c) $\displaystyle\int_{-1}^{1} xP_n(x)P'_n(x)\,dx = \frac{2n}{2n+1}, \quad n \geq 0$

(d) $\displaystyle\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}, \quad n \geq 1$

(e) $\displaystyle\int_{-1}^{1} P_n(x)P'_{n+1}(x)\,dx = 2, \quad n \geq 1$

(f) $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = \begin{cases} 0, & m < n \\ \dfrac{2^{n+1}(n!)^2}{(2n+1)!}, & m = n \\ 0, & m-n>0 \text{ is odd} \\ \dfrac{2^{n+1}m!\left(\frac{m+n}{2}\right)!}{(m+n+1)!\left(\frac{m-n}{2}\right)!}, & m-n>0 \text{ is even} \end{cases}$

Hint: Use Exercise 6.

8. Verify that

(a) $\displaystyle\int_{0}^{1} P_n(x)\,dx = \begin{cases} 1, & n=0 \\ 0, & n>0 \text{ even} \\ \dfrac{(-1)^{(n-1)/2}(n-1)!}{2^n\left(\dfrac{n+1}{2}\right)!\left(\dfrac{n-1}{2}\right)!}, & n \text{ odd} \end{cases}$

(b) $$\int_0^1 xP_n(x)\,dx = \begin{cases} 0, & n \ge 3 \text{ odd} \\ 1/2, & n = 0 \\ 1/3, & n = 1 \\ \dfrac{(-1)^{(n-2)/2}(n-2)!}{2^n\left(\dfrac{n-2}{2}\right)!\left(\dfrac{n+2}{2}\right)!}, & n \ge 2 \text{ even} \end{cases}$$

9. Verify that whenever $y(x)$ is a solution of Legendre's differential equation 8.65, then the function $(1-x^2)^{m/2}d^m y/dx^m$ is a solution of Legendre's associated equation 8.84.

10. (a) Use series 8.71a,b to show that

$$Q_0(x) = \text{Tanh}^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad Q_1(x) = xQ_0(x) - 1.$$

(b) Assuming that the $Q_n(x)$ also satisfy recurrence relation 8.76, express $Q_2(x)$, $Q_3(x)$, and $Q_4(x)$ in terms of $Q_0(x)$.

(c) Express $Q_n(x)$ $(n = 2, 3, 4)$ in terms of $Q_0(x)$ and $P_n(x)$.

11. Prove the following recurrence relations for $P_{mn}(x)$:

(a) $P_{m+1,n+1}(x) - P_{m+1,n-1}(x) = (2n+1)\sqrt{1-x^2}P_{mn}(x)$

(b) $xP_{m+1,n}(x) - P_{m+1,n-1}(x) = (n-m)\sqrt{1-x^2}P_{mn}(x)$

(c) $(n-m+1)P_{m,n+1}(x) - (2n+1)xP_{mn}(x) + (n+m)P_{m,n-1}(x) = 0$

§8.6 Sturm-Liouville Systems and Legendre's Differential Equation

When separation of variables is applied to (initial) boundary value problems expressed in spherical coordinates, the following singular Sturm-Liouville system often results:

$$\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{\partial\phi}\right) + \left(\lambda\sin\phi - \frac{m^2}{\sin\phi}\right)\Phi = 0, \quad 0 < \phi < \pi, \tag{8.87}$$

where m is some given nonnegative integer. The system is singular because there are no boundary conditions and also because $q(\phi) = -m^2/\sin\phi$ is not continuous at $\phi = 0$ and $\phi = \pi$. Because the coefficient $\sin\phi$ of $d\Phi/d\phi$ vanishes at $\phi = 0$ and $\phi = \pi$, the corollary to Theorem 5.1 of Section 5.1 indicates that boundary conditions at $\phi = 0$ and $\phi = \pi$ are unnecessary for that theorem. Examination of the proof of the theorem also indicates that continuity of $q(\phi)$ at $\phi = 0$ and $\phi = \pi$ is not necessary. Consequently, eigenvalues of this singular system are real, and corresponding eigenfunctions are orthogonal.

With a change of independent variable $\mu = \cos\phi$, and $d/d\mu = -(\sin\phi)^{-1}d/d\phi$, differential equation 8.87 is replaced by

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Phi}{d\phi}\right] + \left(\lambda - \frac{m^2}{1-\mu^2}\right)\Phi = 0,$$

or,

$$(1-\mu^2)\frac{d^2\Phi}{d\mu^2} - 2\mu\frac{d\Phi}{d\mu} + \left(\lambda - \frac{m^2}{1-\mu^2}\right)\Phi = 0, \quad -1 < \mu < 1, \tag{8.88}$$

Legendre's associated differential equation. When λ is set equal to $n(n+1)$, where $n \ge m$ is an integer, this equation has general solution

$$\Phi = AP_{mn}(\mu) + BQ_{mn}(\mu), \tag{8.89}$$

where A and B are arbitrary constants and P_{mn} and Q_{mn} are associated Legendre functions of degree n and order m of the first and second kind. Since $Q_{mn}(\mu)$ is unbounded near $\mu = \pm 1$, bounded solutions are

$$\Phi = AP_{mn}(\mu). \tag{8.90}$$

In other words, $\lambda_{mn} = n(n+1)$, where $n \ge m$, are eigenvalues of this singular Sturm-Liouville system with corresponding orthonormal eigenfunctions

$$\Phi_{mn}(\phi) = \Phi(\lambda_{mn}, \phi) = \frac{1}{N}P_{mn}(\cos\phi), \tag{8.91a}$$

where

$$N^2 = \int_0^\pi \sin\phi[P_{mn}(\cos\phi)]^2\,d\phi = \int_{-1}^1 [P_{mn}(\mu)]^2\,d\mu. \tag{8.91b}$$

To evaluate N, we proceed as follows. Since

$$P_{mn}(\mu) = (1-u^2)^{m/2}\frac{d^m}{d\mu^m}P_n(\mu),$$

where $P_n(\mu)$ is the Legendre polynomial of degree n, differentiation with respect to μ yields

$$\frac{d}{d\mu}P_{mn}(\mu) = -\mu m(1-\mu^2)^{m/2-1}\frac{d^m}{d\mu^m}P_n(\mu) + (1-\mu^2)^{m/2}\frac{d^{m+1}}{d\mu^{m+1}}P_n(\mu).$$

Multiplication of this result by $(1-\mu^2)^{1/2}$ gives

$$\begin{aligned}(1-u^2)^{1/2}\frac{d}{d\mu}P_{mn}(\mu) &= \frac{-\mu m(1-\mu^2)^{m/2}}{(1-\mu^2)^{1/2}}\frac{d^m}{d\mu^m}P_n(\mu) + (1-\mu^2)^{(m+1)/2}\frac{d^{m+1}}{d\mu^{m+1}}P_n(\mu)\\ &= \frac{-\mu m}{(1-\mu^2)^{1/2}}P_{mn}(\mu) + P_{m+1,n}(\mu).\end{aligned}$$

When this equation is solved for $P_{m+1,n}(\mu)$, squared, and integrated between the limits $\mu = \pm 1$,

$$\begin{aligned}\int_{-1}^{1}(P_{m+1,n})^2\,d\mu = \int_{-1}^{1}(1-\mu^2)\left(\frac{d}{d\mu}P_{mn}\right)^2 d\mu + 2m\int_{-1}^{1}\mu P_{mn}\frac{d}{d\mu}P_{mn}\,d\mu\\ + m^2\int_{-1}^{1}\frac{\mu^2}{1-\mu^2}(P_{mn})^2\,d\mu.\end{aligned}$$

Integration by parts on the first two integrals on the right gives

$$\begin{aligned}\int_{-1}^{1}(P_{m+1,n})^2\,d\mu &= \left\{(1-\mu^2)\frac{dP_{mn}}{d\mu}P_{mn}\right\}_{-1}^{1} - \int_{-1}^{1}P_{mn}\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_{mn}}{d\mu}\right]d\mu\\ &\quad + 2m\left\{\frac{\mu}{2}(P_{mn})^2\right\}_{-1}^{1} - 2m\int_{-1}^{1}\frac{1}{2}(P_{mn})^2\,d\mu + m^2\int_{-1}^{1}\frac{\mu^2}{1-\mu^2}(P_{mn})^2\,d\mu\\ &= \int_{-1}^{1}P_{mn}\left\{-\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_{mn}}{d\mu}\right] - mP_{mn} + \frac{m^2\mu^2}{1-\mu^2}P_{mn}\right\}d\mu,\end{aligned}$$

since $P_{mn}(\pm 1) = 0$ for $m > 0$; $P_{0n}(1) = P_n(1) = 1$, and $P_{0n}(-1) = P_n(-1) = (-1)^n$. Now, using Legendre's associated differential equation 8.84, we obtain

$$\begin{aligned}\int_{-1}^{1}(P_{m+1,n})^2\,d\mu &= \int_{-1}^{1}P_{mn}\left\{\left[\frac{-m^2}{1-mu^2} + n(n+1)\right]P_{mn} - mP_{mn} + \left(\frac{m^2}{1-\mu^2} - m^2\right)P_{mn}\right\}d\mu\\ &= \int_{-1}^{1}(P_{mn})^2[n(n+1) - m - m^2]\,d\mu\end{aligned}$$

or,

$$\int_{-1}^{1}(P_{mn})^2\,d\mu = \frac{1}{(n-m)(n+m+1)}\int_{-1}^{1}(P_{m+1,n})^2\,d\mu.$$

Iteration of this result on m from m to n gives

$$\int_{-1}^{1}(P_{mn})^2\,d\mu = \frac{(n+m)!}{(n-m)!(2n)!}\int_{-1}^{1}(P_{nn})^2\,d\mu. \tag{8.92}$$

Now,

$$P_{nn} = (1-\mu^2)^{n/2}\frac{d^n}{d\mu^n}P_n = (1-\mu^2)^{n/2}\frac{d^n}{d\mu^n}\left[\frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n\right]$$
$$= \frac{(1-\mu^2)^{n/2}}{2^n n!}\frac{d^{2n}}{d\mu^{2n}}(\mu^2-1)^n = \frac{(2n)!}{2^n n!}(1-\mu^2)^{n/2},$$

and substitution of this into equation 8.92 yields

$$\int_{-1}^{1}(P_{mn})^2\,d\mu = \frac{(n+m)!}{(n-m)!(2n)!}\frac{[(2n)!]^2}{2^{2n}(n!)^2}\int_{-1}^{1}(1-\mu^2)^n\,d\mu.$$

In elementary calculus (see also Exercise 7(a) in Section 8.5), it is shown that

$$\int_{-1}^{1}(1-\mu^2)^n\,d\mu = \frac{2^{2n+1}(n!)^2}{(2n+1)!},$$

and therefore

$$N^2 = \int_{-1}^{1}[P_{mn}(\mu)]^2\,d\mu = \frac{(n+m)!}{(n-m)!(2n)!}\frac{[(2n)!]^2}{2^{2n}(n!)^2}\frac{2^{2n+1}(n!)^2}{(2n+1)!} = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}. \tag{8.93}$$

In summary, orthonormal eigenfunctions of singular Sturm-Liouville system 8.87 are

$$\Phi_{mn}(\phi) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}P_{mn}(\cos\phi), \tag{8.94}$$

corresponding to eigenvalues $\lambda_{mn} = n(n+1)$, where n is an integer greater than or equal to m.

Because the Sturm-Liouville system is singular, we cannot quote the results of Theorem 5.2 in Section 5.2. We have already shown that there is an infinite number of eigenvalues, all of which are positive, except when $m = 0$, in which case $\lambda = 0$ is also an eigenvalue. According to the following theorem, piecewise smooth functions can be expanded in Fourier Legendre series in terms of these eigenfunctions.

Theorem 8.3 If a function $f(\phi)$ is piecewise smooth on the interval $0 \le \phi \le \pi$, then for each ϕ in $0 < \phi < \pi$,

$$\frac{f(\phi+) + f(\phi-)}{2} = \sum_{n=m}^{\infty} c_n\Phi_{mn}(\phi) \quad \text{where} \quad c_n = \int_0^{\pi}\sin\phi\, f(\phi)\Phi_{mn}(\phi)\,d\phi. \tag{8.95}$$

Example 8.4 Find the Fourier Legendre series for the function

$$f(\phi) = \begin{cases} 1, & 0 \le \phi < \pi/2 \\ 0, & \phi = \pi/2 \\ -1, & \pi/2 < \phi \le \pi \end{cases}$$

in terms of the eigenfunctions of Sturm-Liouville system 8.87 when $m = 0$.

Solution Orthonormal eigenfunctions of

$$\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + \lambda\sin\phi\,\Phi = 0, \quad 0 < \phi < \pi,$$

are Legendre polynomials

$$\Phi_{0n}(\phi) = \sqrt{\frac{2n+1}{2}} P_n(\cos\phi), \quad n \geq 0.$$

The Fourier Legendre series of $f(\phi)$ is

$$f(\phi) = \sum_{n=0}^{\infty} c_n \Phi_{0n}(\phi) \quad \text{where} \quad c_n = \int_0^{\pi} \sin\phi\, f(\phi) \Phi_{0n}(\phi)\, d\phi.$$

When we set $\mu = \cos\phi$,

$$\sqrt{\frac{2}{2n+1}} c_n = \int_1^{-1} f[\phi(\mu)] P_n(\mu)(-d\mu) = -\int_{-1}^{0} P_n(\mu)\, d\mu + \int_0^1 P_n(\mu)\, d\mu$$

$$= \begin{cases} 0, & n \text{ even} \\ 2\displaystyle\int_0^1 P_n(\mu)\, d\mu, & n \text{ odd} \end{cases}$$

$$= \begin{cases} 0, & n \text{ even} \\ \dfrac{(-1)^{(n-1)/2}(n-1)!}{2^{n-1}\left(\dfrac{n+1}{2}\right)!\left(\dfrac{n-1}{2}\right)!}, & n \text{ odd} \end{cases}$$

(see Exercise 8 in Section 8.5). Consequently,

$$f(\phi) = \sum_{n=1}^{\infty} \sqrt{\frac{4n-1}{2}} \frac{(-1)^{n-1}(2n-2)!}{2^{2n-2} n!(n-1)!} \sqrt{\frac{4n-1}{2}} P_{2n-1}(\cos\phi)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-2)!(4n-1)}{2^{2n-1} n!(n-1)!} P_{2n-1}(\cos\phi). \bullet$$

EXERCISES 8.6

In Exercises 1–4, find the Fourier Legendre series for the function in terms of the eigenfunctions of Sturm-Liouville system 8.87 when $m = 0$.

1. $f(\phi) = \begin{cases} 1, & 0 \leq \phi < \pi/2 \\ 0, & \pi/2 < \phi < \pi \end{cases}$

2. $f(\phi) = \cos^4\phi$

3. $f(\phi) = \begin{cases} \cos\phi, & 0 \leq \phi \leq \pi/2 \\ 0, & \pi/2 < \phi \leq \pi \end{cases}$

4. $f(\phi) = \begin{cases} \cos\phi, & 0 \leq \phi \leq \pi/2 \\ -\cos\phi, & \pi/2 < \phi \leq \pi \end{cases}$

5. Find eigenvalues and orthonormal eigenfunctions of the singular Sturm-Liouville system

$$\frac{d}{d\phi}\left(\sin\phi \frac{d\Phi}{d\phi}\right) + \lambda \sin\phi\, \Phi = 0, \quad 0 < \phi < \pi/2,$$

$$\Phi(\pi/2) = 0.$$

6. Repeat Exercise 5 if the boundary condition is $\Phi'(\pi/2) = 0$.

CHAPTER 9 PROBLEMS IN POLAR, CYLINDRICAL AND SPHERICAL COORDINATES

§9.1 Homogeneous Problems in Polar, Cylindrical, and Spherical Coordinates

In Section 6.3, separation of variables was used to solve homogeneous boundary value problems expressed in polar coordinates. With the results of Chapter 8, we are in a position to tackle boundary value problems in cylindrical and spherical coordinates and initial boundary value problems in all three coordinate systems. Homogeneous problems are discussed in this section; nonhomogeneous problems are discussed in Section 9.2.

We begin with the following heat conduction problem.

Example 9.1 An infinitely long cylinder of radius a is initially at temperature $f(r) = a^2 - r^2$, and for time $t > 0$, the boundary $r = a$ is insulated. Find the temperature in the cylinder for $t > 0$.

Solution With the initial temperature a function of r and the surface of the cylinder insulated, temperature in the cylinder is a function $U(r,t)$ of r and t only. It satisfies the initial boundary value problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < a, \quad t > 0, \tag{9.1a}$$

$$\frac{\partial U(a,t)}{\partial r} = 0, \quad t > 0, \tag{9.1b}$$

$$U(r,0) = a^2 - r^2, \quad 0 < r < a. \tag{9.1c}$$

When a function $U(r,t) = R(r)T(t)$ with variables separated is substituted into the PDE, and the equation is divided by kRT, the result is

$$\frac{T'}{kT} = \frac{R''}{R} + \frac{R'}{rR} = \alpha = \text{constant independent of } r \text{ and } t.$$

This equation and boundary condition 9.1b yield the Sturm-Liouville system

$$(rR')' - \alpha rR = 0, \quad 0 < r < a, \tag{9.2a}$$

$$R'(a) = 0. \tag{9.2b}$$

This singular system was discussed in Section 8.4 (see Table 8.1 with $\nu = 0$). If we set $\alpha = -\lambda^2$, eigenvalues are defined by the equation $J_1(\lambda a) = 0$, and normalized eigenfunctions are

$$R_n(r) = \frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}, \quad n \geq 0. \tag{9.3}$$

(For simplicity of notation, we have dropped the zero subscript on R_{0n} and λ_{0n}.)

The differential equation

$$T' + k\lambda_n^2 T = 0 \tag{9.4}$$

has general solution

$$T(t) = Ce^{-k\lambda_n^2 t}. \tag{9.5}$$

In order to satisfy initial condition 9.1c, we superpose separated functions and take

$$U(r,t) = \sum_{n=0}^{\infty} C_n e^{-k\lambda_n^2 t} R_n(r), \tag{9.6}$$

where the C_n are constants. Condition 9.1c requires these constants to satisfy

$$a^2 - r^2 = \sum_{n=0}^{\infty} C_n R_n(r), \quad 0 < r < a. \tag{9.7}$$

Thus, the C_n are coefficients in the Fourier Bessel series of $a^2 - r^2$, and, according to equation 8.64 in Section 8.4,

$$C_n = \int_0^a r(a^2 - r^2) R_n(r)\, dr = \frac{\sqrt{2}}{aJ_0(\lambda_n a)} \int_0^a r(a^2 - r^2) J_0(\lambda_n r)\, dr.$$

To evaluate this integral when $n > 0$, we set $u = \lambda_n r$, in which case

$$\begin{aligned} C_n &= \frac{\sqrt{2}}{aJ_0(\lambda_n a)} \int_0^{\lambda_n a} \left(\frac{a^2 u}{\lambda_n} - \frac{u^3}{\lambda_n^3} \right) J_0(u) \frac{du}{\lambda_n} \\ &= \frac{\sqrt{2}}{\lambda_n^4 a J_0(\lambda_n a)} \int_0^{\lambda_n a} (a^2 \lambda_n^2 u - u^3) J_0(u)\, du. \end{aligned}$$

For the term involving u^3, we use the reduction formula in Exercise 9 of Section 8.3,

$$C_n = \frac{\sqrt{2}}{\lambda_n^4 a J_0(\lambda_n a)} \left[a^2 \lambda_n^2 \int_0^{\lambda_n a} u J_0(u)\, du - \left\{ u^3 J_1(u) \right\}_0^{\lambda_n a} - \left\{ 2u^2 J_0(u) \right\}_0^{\lambda_n a} + 4 \int_0^{\lambda_n a} u J_0(u)\, du \right].$$

If we recall the eigenvalue equation $J_1(\lambda a) = 0$, and equation 8.42 in Section 8.3 with $\nu = 1$, we may write

$$\begin{aligned} C_n &= \frac{\sqrt{2}}{\lambda_n^4 a J_0(\lambda_n a)} \left[-2\lambda_n^2 a^2 J_0(\lambda_n a) + (a^2 \lambda_n^2 + 4) \int_0^{\lambda_n a} \frac{d}{du} [u J_1(u)]\, du \right] \\ &= \frac{\sqrt{2}}{\lambda_n^4 a J_0(\lambda_n a)} \left[-2\lambda_n^2 a^2 J_0(\lambda_n a) + (a^2 \lambda_n^2 + 4) \left\{ u J_1(u) \right\}_0^{\lambda_n a} \right] \\ &= \frac{-2\sqrt{2} a}{\lambda_n^2}. \end{aligned}$$

When $n = 0$, the eigenfunction is $R_0(r) = \sqrt{2}/a$, and

$$C_0 = \int_0^a r(a^2 - r^2) R_0(r)\, dr = \frac{\sqrt{2}}{a} \left\{ \frac{a^2 r^2}{2} - \frac{r^4}{4} \right\}_0^a = \frac{\sqrt{2} a^3}{4}.$$

The solution of problem 9.1 is therefore

$$U(r,t) = \frac{\sqrt{2}a^3}{4}\left(\frac{\sqrt{2}}{a}\right) + \sum_{n=1}^{\infty} \frac{-2\sqrt{2}a}{\lambda_n^2} e^{-k\lambda_n^2 t} \frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}$$
$$= \frac{a^2}{2} - 4\sum_{n=1}^{\infty} \frac{e^{-k\lambda_n^2 t}}{\lambda_n^2} \frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}. \tag{9.8}$$

Notice that for large t, the limit of this solution is $a^2/2$, and this is the average value of $a^2 - r^2$ over the circle $r \le a$.•

In the following heat conduction problem, we add angular dependence to the temperature function.

Example 9.2 An infinitely long rod with semicircular cross section is initially ($t = 0$) at a constant nonzero temperature throughout. For $t > 0$, its flat side is held at temperature 0°C while its round side is insulated. Find temperature in the rod for $t > 0$.

Solution Temperature in that half of the rod for which $x < 0$ in Figure 9.1 is identical to that in the half for which $x \ge 0$; no heat crosses the $x = 0$ plane. As a result, the temperature function $U(r, \theta, t)$ (and it is independent of z) must satisfy the initial boundary value problem

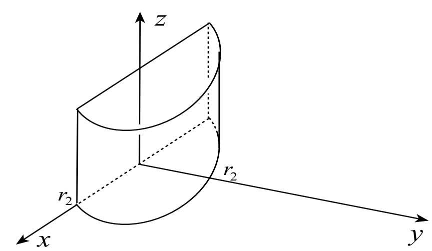


Figure 9.1

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right), \quad 0 < r < a, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0, \tag{9.9a}$$
$$U(r, 0, t) = 0, \quad 0 < r < a, \quad t > 0, \tag{9.9b}$$
$$U_\theta\left(r, \frac{\pi}{2}, t\right) = 0, \quad 0 < r < a, \quad t > 0, \tag{9.9c}$$
$$U_r(a, \theta, t) = 0, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0, \tag{9.9d}$$
$$U(r, \theta, 0) = U_0, \quad 0 < r < a, \quad 0 < \theta < \frac{\pi}{2}. \tag{9.9e}$$

(In Exercise 4, the problem is solved for $0 < \theta < \pi$ with the condition $U(r, \pi, t) = 0$ in place of 9.9c.)

When a function with variables separated, $U(r, \theta, t) = R(r)H(\theta)T(t)$, is substituted into the PDE,

$$RHT' = k(R''HT + r^{-1}R'HT + r^{-2}RH''T)$$

or,

$$-\frac{H''}{H} = \frac{r^2R''}{R} + \frac{rR'}{R} - \frac{r^2T'}{kT} = \alpha = \text{constant independent of } r,\ \theta, \text{ and } t.$$

When boundary conditions 9.9b,c are imposed on the separated function, a Sturm-Liouville system in $H(\theta)$ results,

$$H'' + \alpha H = 0, \quad 0 < \theta < \pi/2, \tag{9.10a}$$
$$H(0) = 0 = H'(\pi/2). \tag{9.10b}$$

This system was discussed in Section 5.2. If we set $\alpha = \nu^2$, then according to Table 5.1, eigenvalues are $\nu_m^2 = (2m-1)^2$ $(m = 1, 2, \ldots)$, with orthonormal eigenfunctions

$$H_m(\theta) = \frac{2}{\sqrt{\pi}} \sin(2m-1)\theta. \tag{9.11}$$

Continued separation of the equation in $R(r)$ and $T(t)$ gives

$$\frac{R'' + r^{-1}R'}{R} - \frac{\nu_m^2}{r^2} = \frac{T'}{kT} = \beta = \text{constant independent of } r \text{ and } t.$$

Boundary condition 9.9d leads to the Sturm-Liouville system

$$(rR')' + \left[-\beta r - \frac{(2m-1)^2}{r}\right] R = 0, \quad 0 < r < a, \tag{9.12a}$$

$$R'(a) = 0. \tag{9.12b}$$

This is singular Sturm-Liouville system 8.48 of Section 8.4. If we set $\beta = -\lambda^2$, eigenvalues λ_{mn} are defined by the equation

$$J'_{2m-1}(\lambda a) = 0 \tag{9.13}$$

with corresponding eigenfunctions

$$R_{mn}(r) = \frac{1}{N} J_{2m-1}(\lambda_{mn} r), \tag{9.14a}$$

where

$$2N^2 = a^2 \left[1 - \left(\frac{2m-1}{\lambda_{mn} a}\right)^2\right] [J_{2m-1}(\lambda_{mn} a)]^2. \tag{9.14b}$$

The differential equation

$$T' = -k\lambda_{mn}^2 T \tag{9.15}$$

has general solution

$$T(t) = Ce^{-k\lambda_{mn}^2 t}. \tag{9.16}$$

To satisfy initial condition 9.9e, we superpose separated functions and take

$$U(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-k\lambda_{mn}^2 t} R_{mn}(r) H_m(\theta), \tag{9.17}$$

where C_{mn} are constants. The initial condition requires these constants to satisfy

$$U_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} R_{mn}(r) H_m(\theta), \quad 0 < r < a, \quad 0 < \theta < \pi/2. \tag{9.18}$$

If we multiply this equation by $H_i(\theta)$ and integrate with respect to θ from $\theta = 0$ to $\theta = \pi/2$, orthogonality of the eigenfunctions $H_m(\theta)$ gives

$$\sum_{n=1}^{\infty} C_{in}R_{in}(r) = \int_0^{\pi/2} U_0 H_i(\theta)\,d\theta = U_0 \int_0^{\pi/2} \frac{2}{\sqrt{\pi}} \sin{(2i-1)\theta}\,d\theta$$
$$= \frac{2U_0}{\sqrt{\pi}} \left\{ \frac{-1}{2i-1} \cos{(2i-1)\theta} \right\}_0^{\pi/2} = \frac{2U_0}{(2i-1)\sqrt{\pi}}.$$

But this equation implies that the C_{in} are Fourier Bessel coefficients for the function $2U_0/[(2i-1)\sqrt{\pi}]$; that is,

$$C_{in} = \int_0^a \frac{2U_0}{(2i-1)\sqrt{\pi}} rR_{in}(r)\,dr.$$

Thus, the solution of problem 9.9 for $0 \le \theta \le \pi/2$ is 9.17, where

$$C_{mn} = \frac{2U_0}{(2m-1)\sqrt{\pi}} \int_0^a rR_{mn}(r)\,dr. \tag{9.19}$$

For an angle θ between $\pi/2$ and π, we should evaluate $U(r, \pi-\theta, t)$. Since

$$H_m(\pi-\theta) = \frac{2}{\sqrt{\pi}} \sin{(2m-1)(\pi-\theta)} = \frac{2}{\sqrt{\pi}} \sin{(2m-1)\theta},$$

it follows that $U(r, \pi-\theta, t) = U(r, \theta, t)$. Hence, solution 9.17 is valid for $0 \le \theta \le \pi$.•

Our next example is a vibration problem.

Example 9.3 Solve the initial boundary value problem

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \right), \quad 0 < r < a, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{9.20a}$$
$$z(a, \theta, t) = 0, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{9.20b}$$
$$z(r, \theta, 0) = f(r, \theta), \quad 0 < r < a, \quad -\pi < \theta \le \pi, \tag{9.20c}$$
$$z_t(r, \theta, 0) = 0, \quad 0 < r < a, \quad -\pi < \theta \le \pi. \tag{9.20d}$$

Described is a membrane stretched over the circle $r \le a$ that has an initial displacement $f(r, \theta)$ and zero initial velocity. Boundary condition 9.20b states that the edge of the membrane is fixed on the xy-plane.

Solution When a separated function $z(r, \theta, t) = R(r)H(\theta)T(t)$, is substituted into the PDE,

$$RHT'' = c^2(R''HT + r^{-1}R'HT + r^{-2}RH''T)$$

or,

$$-\frac{H''}{H} = r^2 \left(\frac{R'' + r^{-1}R'}{R} - \frac{T''}{c^2T} \right) = \alpha = \text{constant independent of } r,\ \theta, \text{ and } t.$$

Since the solution and its first derivative with respect to θ must be 2π-periodic in θ, it follows that $H(\theta)$ must satisfy the periodic Sturm-Liouville system

$$H'' + \alpha H = 0, \quad -\pi < \theta \le \pi, \tag{9.21a}$$
$$H(-\pi) = H(\pi), \tag{9.21b}$$
$$H'(-\pi) = H'(\pi). \tag{9.21c}$$

This system was discussed in Chapter 5 (Example 5.2 and equation 5.20). Eigenvalues are $\alpha = m^2$, m a nonnegative integer, with orthonormal eigenfunctions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}}\sin m\theta, \quad \frac{1}{\sqrt{\pi}}\cos m\theta. \tag{9.22}$$

Continued separation of the equation in $R(r)$ and $T(t)$ gives

$$\frac{R'' + r^{-1}R'}{R} - \frac{m^2}{r^2} = \frac{T''}{c^2T} = \beta = \text{constant independent of } r \text{ and } t.$$

When boundary condition 9.20b is imposed on the separated function, a Sturm-Liouville system in $R(r)$ results,

$$(rR')' + \left(-\beta r - \frac{m^2}{r}\right)R = 0, \quad 0 < r < a, \tag{9.23a}$$

$$R(a) = 0. \tag{9.23b}$$

This is, once again, singular system 8.48 in Section 8.4. If we set $\beta = -\lambda^2$, eigenvalues λ_{mn} are defined by

$$J_m(\lambda a) = 0, \tag{9.24}$$

with corresponding orthonormal eigenfunctions

$$R_{mn}(r) = \frac{\sqrt{2}J_m(\lambda_{mn}r)}{aJ_{m+1}(\lambda_{mn}a)} \tag{9.25}$$

(see Table 8.1). The differential equation

$$T'' + c^2\lambda_{mn}^2 T = 0 \tag{9.26}$$

has general solution

$$T(t) = d\cos c\lambda_{mn}t + b\sin c\lambda_{mn}t, \tag{9.27}$$

where d and b are constants. Initial condition 9.20d implies that $b = 0$, and hence

$$T(t) = d\cos c\lambda_{mn}t. \tag{9.28}$$

In order to satisfy the final initial condition 9.20c, we superpose separated functions and take

$$z(r,\theta,t) = \sum_{n=1}^{\infty} d_{0n}\frac{R_{0n}(r)}{\sqrt{2\pi}}\cos c\lambda_{0n}t + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{mn}(r)\left(d_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + f_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right)\cos c\lambda_{mn}t, \tag{9.29}$$

where d_{mn} and f_{mn} are constants. Condition 9.20c requires these constants to satisfy

$$f(r,\theta) = \sum_{n=1}^{\infty} d_{0n}\frac{R_{0n}(r)}{\sqrt{2\pi}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{mn}(r)\left(d_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + f_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right) \tag{9.30}$$

for $0 < r < a$, $-\pi < \theta \le \pi$. If we multiply this equation by $(1/\sqrt{\pi})\cos i\theta$ and integrate with respect to θ from $\theta = -\pi$ to $\theta = \pi$, orthogonality of the eigenfunctions in θ gives

$$\int_{-\pi}^{\pi} f(r,\theta)\frac{\cos i\theta}{\sqrt{\pi}}d\theta = \sum_{n=1}^{\infty} d_{in}R_{in}(r).$$

Multiplication of this equation by $rR_{ij}(r)$ and integration with respect to r from $r = 0$ to $r = a$ yields (because of orthogonality of the R_{ij} for fixed i)

$$\int_0^a\int_{-\pi}^{\pi} rf(r,\theta)R_{ij}\frac{\cos i\theta}{\sqrt{\pi}}d\theta\,dr = d_{ij};$$

that is

$$d_{mn} = \int_{-\pi}^{\pi}\int_0^a rR_{mn}\frac{\cos m\theta}{\sqrt{\pi}}f(r,\theta)\,dr\,d\theta. \tag{9.31a}$$

Similarly,

$$f_{mn} = \int_{-\pi}^{\pi}\int_0^a rR_{mn}\frac{\sin m\theta}{\sqrt{\pi}}f(r,\theta)\,dr\,d\theta, \tag{9.31b}$$

and

$$d_{0n} = \int_{-\pi}^{\pi}\int_0^a rR_{0n}\frac{f(r,\theta)}{\sqrt{2\pi}}\,dr\,d\theta. \tag{9.31c}$$

The solution of problem 9.20 is therefore 9.29, where d_{mn} and f_{mn} are defined by 9.31.●

Coefficients d_{mn} and f_{mn} in this example were calculated by first using orthogonality of the trigonometric eigenfunctions and then using orthogonality of the $R_{mn}(r)$. An alternative procedure is to determine the multi-dimensional eigenfunctions for problem 9.20. This approach is discussed in Exercise 27.

Our final example on separation is a potential problem.

Example 9.4 Find the potential interior to a sphere when the potential is $f(\phi,\theta)$ on the sphere.

Solution The boundary value problem for the potential $V(r,\phi,\theta)$ is

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 V}{\partial\theta^2} = 0,$$

$$0 < r < a, \quad 0 < \phi < \pi, \quad -\pi < \theta \le \pi, \tag{9.32a}$$

$$V(a,\phi,\theta) = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi. \tag{9.32b}$$

When a function with variables separated, $V(r,\phi,\theta) = R(r)\Phi(\phi)H(\theta)$, is substituted into PDE 9.32a,

$$R''\Phi H + \frac{2}{r}R'\Phi H + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\,R\Phi' H) + \frac{R\Phi H''}{r^2\sin^2\phi} = 0$$

or,

$$r^2\sin^2\phi\left[\frac{R''}{R} + \frac{2R'}{rR} + \frac{1}{r^2\sin\phi\,\Phi}\frac{d}{d\phi}(\sin\phi\,\Phi')\right] = -\frac{H''}{H}$$

$$= \alpha = \text{constant independent of } r,\ \phi,\ \text{and } \theta.$$

Because $V(r,\phi,\theta)$ must be 2π-periodic in θ, as must its first derivative with respect to θ, it follows that $H(\theta)$ must satisfy the periodic Sturm-Liouville system

$$H'' + \alpha H = 0, \quad -\pi < \theta \le \pi, \tag{9.33a}$$
$$H(-\pi) = H(\pi), \tag{9.33b}$$
$$H'(-\pi) = H'(\pi). \tag{9.33c}$$

This is Sturm-Liouville system 9.21 with eigenvalues $\alpha = m^2$ and orthonormal eigenfunctions

$$\frac{1}{\sqrt{2\pi}}, \qquad \frac{1}{\sqrt{\pi}}\cos m\theta, \qquad \frac{1}{\sqrt{\pi}}\sin m\theta.$$

Continued separation of the equation in $R(r)$ and $\Phi(\phi)$ gives

$$\frac{r^2R''}{R} + \frac{2rR'}{R} = \frac{m^2}{\sin^2\phi} - \frac{1}{\Phi\sin\phi}\frac{d}{d\phi}(\sin\phi\,\Phi') = \beta = \text{constant independent of } r \text{ and } \phi.$$

Thus, $\Phi(\phi)$ must satisfy the singular Sturm-Liouville system

$$\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + \left(\beta\sin\phi - \frac{m^2}{\sin\phi}\right)\Phi = 0, \quad 0 < \phi < \pi. \tag{9.34}$$

According to the results of Section 8.6, eigenvalues are $\beta = n(n+1)$, where $n \ge m$ is an integer, with orthonormal eigenfunctions

$$\Phi_{mn}(\phi) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}P_{mn}(\cos\phi). \tag{9.35}$$

The remaining differential equation

$$r^2R'' + 2rR' - n(n+1)R = 0 \tag{9.36}$$

is a Cauchy-Euler equation that can be solved by setting $R(r) = r^s$, s an unknown constant. This results in the general solution

$$R(r) = \frac{C}{r^{n+1}} + Ar^n. \tag{9.37}$$

For $R(r)$ to remain bounded as r approaches zero, we must set $C = 0$. Superposition of separated functions now yields

$$V(r,\phi,\theta) = \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi}}A_{0n}r^n\Phi_{0n}(\phi) + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}r^n\Phi_{mn}(\phi)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right), \tag{9.38}$$

where A_{mn} and B_{mn} are constants. Boundary condition 9.32b requires these constants to satisfy

$$f(\phi,\theta) = \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi}}A_{0n}a^n\Phi_{0n}(\phi) + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}a^n\Phi_{mn}(\phi)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right) \tag{9.39}$$

for $0 \le \phi \le \pi$, $-\pi < \theta \le \pi$. Because of orthogonality of eigenfunctions in ϕ and θ, multiplication by $(1/\sqrt{2\pi})\sin\phi\,\Phi_{0j}(\phi)$ and integration with respect to ϕ and θ give

$$A_{0j} = \frac{1}{a^j}\int_{-\pi}^{\pi}\int_0^{\pi} f(\phi,\theta)\frac{1}{\sqrt{2\pi}}\sin\phi\,\Phi_{0j}(\phi)\,d\phi\,d\theta. \tag{9.40a}$$

Similarly,

$$A_{mn} = \frac{1}{a^n}\int_{-\pi}^{\pi}\int_0^{\pi} f(\phi,\theta)\frac{\cos m\theta}{\sqrt{\pi}}\sin\phi\,\Phi_{mn}(\phi)\,d\phi\,d\theta, \tag{9.40b}$$

$$B_{mn} = \frac{1}{a^n}\int_{-\pi}^{\pi}\int_0^{\pi} f(\phi,\theta)\frac{\sin m\theta}{\sqrt{\pi}}\sin\phi\,\Phi_{mn}(\phi)\,d\phi\,d\theta. \tag{9.40c}$$

Notice that the potential at the centre of the sphere is

$$V(0,\phi,\theta) = \frac{1}{\sqrt{2\pi}}A_{00}\Phi_{00}(\phi) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\pi}^{\pi}\int_0^{\pi} f(\phi,\theta)\frac{1}{\sqrt{2\pi}}\sin\phi\,\Phi_{00}(\phi)\,d\phi\,d\theta\right]\Phi_{00}(\phi).$$

Since $\Phi_{00}(\phi) = 1/\sqrt{2}$,

$$\begin{aligned} V(0,\phi,\theta) &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi} f(\phi,\theta)\sin\phi\,d\phi\,d\theta \\ &= \frac{1}{4\pi a^2}\int_{-\pi}^{\pi}\int_0^{\pi} f(\phi,\theta)a^2\sin\phi\,d\phi\,d\theta, \end{aligned}$$

and this is the average value of $f(\phi,\theta)$ over the surface of the sphere. We can develop an integral formula for the solution analogous to Poisson's integral formula for a circle, equation 6.34. We change variables of integration for the coefficients to α and β, substitute the coefficients into summation 9.38 and interchange orders of integration and summation

$$\begin{aligned} V(r,\phi,\theta) &= \int_0^{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n f(\alpha,\beta)\sin\alpha\,\Phi_{0n}(\phi)\Phi_{0n}(\alpha)\right. \\ &\quad \left. +\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n f(\alpha,\beta)\sin\alpha\,\Phi_{mn}(\phi)\Phi_{mn}(\alpha)(\cos m\theta\cos m\beta+\sin m\theta\sin m\beta)\right]d\beta\,d\alpha \\ &= \frac{1}{\pi}\int_0^{\pi}\int_{-\pi}^{\pi} f(\alpha,\beta)\sin\alpha\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\Phi_{0n}(\phi)\Phi_{0n}(\alpha)\right. \\ &\quad \left. +\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\Phi_{mn}(\phi)\Phi_{mn}(\alpha)\cos m(\theta-\beta)\right]d\beta\,d\alpha. \end{aligned}$$

Let us define

$$S(r,\phi,\theta) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\Phi_{0n}(\phi)\Phi_{0n}(\alpha) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\Phi_{mn}(\phi)\Phi_{mn}(\alpha)\cos m(\theta-\beta).$$

Consider the potential at a point inside the sphere and on the z-axis with spherical coordinates $(r,0,\theta)$, where θ is arbitrary and $0<r<a$. For such a point,

$$S(r,0,\theta) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\Phi_{0n}(0)\Phi_{0n}(\alpha) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\Phi_{mn}(0)\Phi_{mn}(\alpha)\cos m(\theta-\beta).$$

Since

$$\Phi_{0n}(0) = \sqrt{\frac{2n+1}{2}} P_n(1) = \sqrt{\frac{2n+1}{2}}, \qquad \Phi_{mn}(0) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} P_{mn}(1) = 0,$$

$$S(r,0,\theta) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n \left(\frac{2n+1}{2}\right) P_n(\cos\alpha) = \frac{1}{4}\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{a}\right)^n P_n(\cos\alpha).$$

To find a closed value for this summation, we differentiate the generating function 8.74 for Legendre polynomials

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

with respect to t,

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

If we multiply this by $2t$ and add it to the generating function, we obtain

$$\sum_{n=0}^{\infty}(2n+1)P_n(x)t^n = \frac{2t(x-t)}{(1-2xt+t^2)^{3/2}} + \frac{1}{\sqrt{1-2xt+t^2}} = \frac{1-t^2}{(1-2xt+t^2)^{3/2}}.$$

It follows that

$$S(r,0,\theta) = \frac{1}{4}\left[\frac{1-\dfrac{r^2}{a^2}}{\left(1-\dfrac{2r\cos\alpha}{a}+\dfrac{r^2}{a^2}\right)^{3/2}}\right] = \frac{a(a^2-r^2)}{4(a^2-2ar\cos\alpha+r^2)^{3/2}}.$$

Thus,

$$\begin{aligned} V(r,0,\theta) &= \frac{1}{\pi}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{a(a^2-r^2)}{4(a^2-2ar\cos\alpha+r^2)^{3/2}} f(\alpha,\beta)\sin\alpha\, d\beta\, d\alpha \\ &= \frac{a(a^2-r^2)}{4\pi}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{f(\alpha,\beta)\sin\alpha}{(a^2-2ar\cos\alpha+r^2)^{3/2}} d\beta\, d\alpha. \end{aligned}$$

This is the potential at a point $(r,0,\theta)$ on the z-axis. The distance between this point and a point (a,α,β) on the sphere is

$$\sqrt{(a\sin\alpha\cos\beta)^2+(a\sin\alpha\sin\beta)^2+(a\cos\alpha-r)^2} = \sqrt{r^2+a^2-2ar\cos\alpha}.$$

The denominator in the above integral is therefore the cube of the distance from points on the sphere to the point at which the potential is calculated. Since the axes could always be rotated so that the observation point is on the z-axis, it follows that to find the potential at any point with spherical coordinates (r,ϕ,θ) inside the sphere, we need only replace $\sqrt{r^2+a^2-2ar\cos\alpha}$ with the distance from (r,ϕ,θ) to (a,α,β), namely,

$$\begin{aligned} &\sqrt{(r\sin\phi\cos\theta-a\sin\alpha\cos\beta)^2+(r\sin\phi\sin\theta-a\sin\alpha\sin\beta)^2+(r\cos\phi-a\cos\alpha)^2} \\ &\quad= \sqrt{r^2+a^2-2ar[\sin\phi\sin\alpha\cos(\theta-\beta)+\cos\phi\cos\alpha]}. \end{aligned}$$

Thus,

$$V(r,\phi,\theta) = \frac{a(a^2-r^2)}{4\pi}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{f(\alpha,\beta)\sin\alpha}{\{r^2+a^2-2ar[\sin\phi\sin\alpha\cos(\theta-\beta)+\cos\phi\cos\alpha]\}^{3/2}}d\beta\, d\alpha. \tag{9.41}$$

This is called **Poisson's integral formula for a sphere.**•

EXERCISES 9.1

Part A Heat Conduction

1. (a) The initial temperature of an infinitely long cylinder of radius a is $f(r)$. If, for time $t > 0$, the outer surface is held at 0°C, find the temperature in the cylinder.
(b) Simplify the solution in part (a) when $f(r)$ is a constant U_0.
(c) Find the solution when $f(r) = a^2 - r^2$.

2. An infinitely long cylinder of radius a is initially at temperature $f(r)$ and, for time $t > 0$, the boundary $r = a$ is insulated.
(a) Find the temperature $U(r,t)$ in the cylinder.
(b) What is the limit of $U(r,t)$ for large t?

3. A thin circular plate of radius a is insulated top and bottom. At time $t = 0$ its temperature is $f(r,\theta)$. If the temperature of its edge is held at 0°C for $t > 0$, find its interior temperature for $t > 0$.

4. Solve Example 9.2 using the boundary condition $U(r,\pi,t) = 0$ in place of $\partial U(r,\pi/2,t)/\partial\theta = 0$.

5. An infinitely long cylinder is bounded by the surfaces $r = a$, $\theta = 0$, and $\theta = \pi/2$. At time $t = 0$, its temperature is $f(r,\theta)$, and for $t > 0$, all surfaces are held at temperature zero. Find temperature in the cylinder.

6. Repeat Exercise 5 if the flat sides are insulated.

7. Repeat Exercise 5 if the curved side is insulated.

8. Repeat Exercise 5 if all sides are insulated. Show that the limit of the temperature as $t \to \infty$ is the average of $f(r,\theta)$ over the cylinder.

9. A flat plate in the form of a sector of a circle of radius 1 and angle α is insulated top and bottom. At time $t = 0$, the temperature of the plate increases linearly from 0°C at $r = 0$ to a constant value $\overline{U}^\circ$C at $r = 1$ (and is therefore independent of θ). If, for $t > 0$, the rounded edge is insulated and the straight edges are held at temperature 0°C, find the temperature in the plate for $t > 0$. Prove that heat never crosses the line $\theta = \alpha/2$.

10. Find the temperature in the plate of Exercise 9 if the initial temperature is $f(r)$, the straight sides are insulated, and the curved edge is held at temperature 0°C.

11. Repeat Exercise 10 if the initial temperature is a function of r and θ, namely, $f(r,\theta)$.

12. A cylinder occupies the region $r \le a$, $0 \le z \le L$. It has temperature $f(r,z)$ at time $t = 0$. For $t > 0$, its end $z = 0$ is insulated, and the remaining two surfaces are held at temperature 0°C. Find the temperature in the cylinder.

13. Solve Exercise 1(a),(b) if heat is transferred at $r = a$ according to Newton's law of cooling to an environment at temperature zero.

14. (a) A sphere of radius a is initially at temperature $f(r)$ and, for time $t > 0$, the boundary $r = a$ is held at temperature zero. Find the temperature in the sphere for $t > 0$. (You will need the results of Exercise 8 in Section 8.4). Compare the solution to that in Exercise 12 of Section 4.2.
(b) Simplify the solution when $f(r) = U_0$, a constant.
(c) Suppose the sphere has radius 20 cm and is made of steel with $k = 12.4 \times 10^{-6}$. Find the temperature at the centre of the sphere after 10 minutes when $f(r) = U_0$ as in part (b).
(d) Repeat part (c) if the sphere is asbestos with $k = 0.247 \times 10^{-6}$.

15. Repeat parts (a) and (b) of Exercise 14 if the surface of the sphere is insulated. (See Exercise 8 in Section 8.4.) What is the temperature for large t?

16. Repeat parts (a) and (b) of Exercise 14 if the surface transfers heat to an environment at temperature zero according to Newton's law of cooling; that is, take as boundary condition

$$\kappa \frac{\partial U(a,t)}{\partial r} + \mu U(a,t) = 0, \quad t > 0.$$

(Assume that $\mu a < \kappa$ and see Exercise 8 in Section 8.4.)

17. Repeat Exercise 14(a) if the initial temperature is also a function of ϕ. (You will need the results of Exercise 9 in Section 8.4.)

18. (a) Repeat Exercise 14(a) if the initial temperature is also a function of ϕ and the surface of the sphere is insulated. (You will need the results of Exercise 9 in Section 8.4.)
(b) What is the limit of the solution for large t?

19. The result of this exercise is analogous to that in Exercise 9 of Section 6.4. Show that the solution of the homogeneous heat conduction problem

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right), \quad 0 < r < a, \quad 0 < z < L, \quad t > 0,\\
-l_1 \frac{\partial U}{\partial z} + h_1 U &= 0, \quad z = 0, \quad 0 < r < a, \quad t > 0,\\
l_2 \frac{\partial U}{\partial z} + h_2 U &= 0, \quad z = L, \quad 0 < r < a, \quad t > 0,\\
l_3 \frac{\partial U}{\partial r} + h_3 U &= 0, \quad r = a, \quad 0 < z < L, \quad t > 0,\\
U(r,z,0) &= f(r)g(z), \quad 0 < r < a, \quad 0 < z < L,
\end{aligned}$$

where the initial temperature is the product of a function of r and a function of z, is the product of the solutions of the problems

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < a, \quad t > 0,\\
l_3 \frac{\partial U(a,t)}{\partial r} + h_3 U(a,t) &= 0, \quad t > 0,\\
U(r,0) &= f(r), \quad 0 < r < a;
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial z^2}, \quad 0 < z < L, \quad t > 0,\\
-l_1\frac{\partial U(0,t)}{\partial z} + h_1 U(0,t) &= 0, \quad t > 0,\\
l_2\frac{\partial U(L,t)}{\partial z} + h_2 U(L,t) &= 0, \quad t > 0,\\
U(z,0) &= g(z), \quad 0 < z < L.
\end{aligned}$$

20. Solve the heat conduction problem

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right), \quad 0 < r < a, \quad 0 < z < L, \quad t > 0,\\
U_z(r,0,t) &= 0, \quad 0 < r < a, \quad t > 0,\\
U(r,L,t) &= 0, \quad 0 < r < a, \quad t > 0,\\
U_r(a,z,t) &= 0, \quad 0 < z < L, \quad t > 0,\\
U(r,z,0) &= (a^2 - r^2)(L - z), \quad 0 < r < a, \quad 0 < z < L,
\end{aligned}$$

(a) by using the results of Exercise 19, Example 9.1, and Exercise 1(a) in Section 6.2.
(b) by separation of variables.

Part B Vibrations

21. (a) A vibrating circular membrane of radius a is given an initial displacement that is a function only of r, namely, $f(r)$, $0 \le r \le a$, and zero initial velocity. Show that subsequent displacements of the membrane, if its edge $r = a$ is fixed on the xy-plane, are of the form

$$z(r,t) = \frac{\sqrt{2}}{a}\sum_{n=1}^{\infty} A_n \cos c\lambda_n t \frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}.$$

What is A_n?

(b) The first term in the series in part (a), called the **fundamental mode of vibration** for the membrane, is

$$H_1(r,t) = \frac{\sqrt{2}}{a} A_1 \cos c\lambda_1 t \frac{J_0(\lambda_1 r)}{J_1(\lambda_1 a)}.$$

Simplify and describe this mode when $a = 1$. Does $H_1(r,t)$ have nodal curves?

(c) Repeat part (b) for the second mode of vibration.

(d) Are frequencies of higher modes of vibration integer multiples of the frequency of the fundamental mode? Were they for a vibrating string with fixed ends?

22. A circular membrane of radius a has its edge fixed on the xy-plane. In addition, a clamp holds the membrane on the xy-plane along a radial line from the centre to the circumference. If the membrane is released from rest at a displacement $f(r,\theta)$, find subsequent displacements. (For consistency, we require $f(r,\theta)$ to vanish along the clamped radial line.)

23. Simplify the solution in part (a) of Exercise 21 when $f(r) = a^2 - r^2$. (See Example 9.1.)

24. A circular membrane of radius a is parallel to the xy-plane and is falling with constant speed v_0. At time $t = 0$, it strikes the xy-plane. For $t > 0$, the edge of the membrane is fixed on the

xy-plane, but the remainder of the membrane is free to vibrate vertically. Find displacements of the membrane.

25. Equation 9.29 with coefficients defined in 9.31 describes displacements of a circular membrane with fixed edge when oscillations are initiated from rest at some prescribed displacement. In this exercise we examine nodal curves for various modes of vibration.

(a) The first mode of vibration is the term $(d_{01}/\sqrt{2\pi})R_{01}(r)\cos c\lambda_{01}t$. Show that this mode has no nodal curves.

(b) Show that the mode $(d_{02}/\sqrt{2\pi})R_{02}(r)\cos c\lambda_{02}t$ has one nodal curve, a circle.

(c) Show that the mode $(d_{03}/\sqrt{2\pi})R_{03}(r)\cos c\lambda_{03}t$ has two circular nodal curves.

(d) On the basis of parts (a), (b), and (c), what are the nodal curves for the mode $(d_{0n}/\sqrt{2\pi})R_{0n}(r)\cos c\lambda_{0n}t$?

(e) Corresponding to $n = m = 1$, there are two modes, $(d_{11}/\sqrt{\pi})R_{11}(r)\cos c\lambda_{11}t\cos\theta$ and $(f_{11}/\sqrt{\pi})R_{11}(r)\cos c\lambda_{11}t\sin\theta$. Show that each of these modes has only one nodal curve, a straight line.

(f) Find nodal curves for the modes $(d_{12}/\sqrt{\pi})R_{12}(r)\cos c\lambda_{12}t\cos\theta$ and $(f_{12}/\sqrt{\pi})R_{12}(r)\cos c\lambda_{12}t\sin\theta$.

(g) Find nodal curves for the modes $(d_{22}/\sqrt{\pi})R_{22}(r)\cos c\lambda_{22}t\cos 2\theta$ and $(f_{22}/\sqrt{\pi})R_{22}(r)\cos c\lambda_{22}t\sin 2\theta$.

(h) On the basis of parts (e), (f), and (g), what are the nodal curves for the modes $(d_{mn}/\sqrt{\pi})R_{mn}(r)\cos c\lambda_{mn}t\cos m\theta$ and $(f_{mn}/\sqrt{\pi})R_{mn}(r)\cos c\lambda_{mn}t\sin m\theta$?

26. The initial boundary value problem for small horizontal displacements of a suspended cable when gravity is the only force acting on the cable is

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right), \quad 0 < x < L, \quad t > 0,\\ y(L,t) &= 0, \quad t > 0,\\ y(x,0) &= f(x), \quad 0 < x < L,\\ y_t(x,0) &= h(x), \quad 0 < x < L.\end{aligned}$$

(See Exercise 26 in Section 2.3.)

(a) Show that when a new independent variable $z = \sqrt{4x/g}$ is introduced, $y(z,t)$ must satisfy

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= \frac{1}{z}\frac{\partial}{\partial z}\left(z\frac{\partial y}{\partial z}\right), \quad 0 < z < M, \quad t > 0,\\ y(M,t) &= 0, \quad t > 0,\\ y(z,0) &= f(gz^2/4), \quad 0 < z < M,\\ y_t(z,0) &= h(gz^2/4), \quad 0 < z < M,\end{aligned}$$

where $M = \sqrt{4L/g}$.

(b) Solve this problem by separation of variables, and hence find $y(x,t)$.

27. Multidimensional eigenfunctions for problem 9.20 are solutions of the two-dimensional eigenvalue problem

$$\begin{aligned}\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \lambda^2 W = 0, &\quad 0 < r < a, \quad -\pi < \theta \le \pi,\\ W(a,\theta) = 0, &\quad -\pi < \theta \le \pi.\end{aligned}$$

(a) Find eigenfunctions (normalized with respect to the unit weight function over the circle $r \le a$).
(b) Use the eigenfunctions in part (a) to solve problem 9.20.

Part C Potential, Steady-state Heat Conduction, Static Deflections of Membranes

28. (a) Solve the following boundary value problem associated with the Helmholtz equation on a circle

$$\nabla^2 V + k^2 V = 0, \quad 0 < r < a, \quad -\pi < \theta \le \pi \quad (k > 0 \text{ a constant})$$
$$V(a,\theta) = f(\theta), \quad -\pi < \theta \le \pi.$$

(b) Is $V(0,\theta)$ the average value of $f(\theta)$ on $r = a$?
(c) What is the solution when $f(\theta) = 1$?

29. Solve the following problem for potential in a cylinder

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0, \quad 0 < r < a, \quad 0 < z < L,$$
$$V(a,z) = 0, \quad 0 < z < L,$$
$$V(r,0) = 0, \quad 0 < r < a,$$
$$V(r,L) = f(r), \quad 0 < r < a.$$

30. Find the potential inside a cylinder of length L and radius a when potential on the curved surface is zero and potentials on the flat ends are nonzero.

31. (a) Find the steady-state temperature in a cylinder of radius a and length L if the end $z = 0$ is maintained at temperature $f(r)$, the end $z = L$ is kept at temperature zero, and heat is transferred on $r = a$ to a medium at temperature zero according to Newton's law of cooling.
(b) Simplify the solution when $f(r) = U_0$, a constant.

32. The temperature in a semi-infinite cylinder $0 < r < a$, $z > 0$ is in a steady-state situation. Find the temperature if the cylindrical wall is at temperature zero and the temperature of the base $z = 0$ is $f(r)$.

33. Repeat Exercise 32 if the cylindrical wall is insulated.

34. Use separation of variables to find the potential inside a sphere of radius a when the potential on the sphere is a function $f(\phi)$ of ϕ only. Does the solution for Example 9.4 specialize to this result? What is the solution when $f(\phi)$ is a constant function?

35. Show that if the potential on the surface of a sphere is a function $f(\theta)$ of θ only, the potential interior to the sphere is still a function of r, ϕ, and θ.

36. Find the potential interior to a sphere of radius a when the potential must satisfy a Neumann condition on the sphere,

$$\frac{\partial V(a,\phi,\theta)}{\partial r} = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi.$$

37. Find the potential interior to a sphere of radius a when the potential must satisfy a Robin condition on the sphere,

$$l\frac{\partial V(a,\phi,\theta)}{\partial r} + hV(a,\phi,\theta) = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi.$$

38. Find the steady-state temperature inside a hemisphere $r \le a$, $z \ge 0$ when temperature on $z = 0$ is zero and that on $r = a$ is a function of ϕ only. (Hint: See Exercise 5 in Section 8.6.) Simplify the solution when $f(\phi)$ is a constant function.

39. Repeat Exercise 38 if the base of the hemisphere is insulated. (Hint: See Exercise 6 in Section 8.6.)

40. Find the bounded potential outside the hemisphere $r \le a$, $z \ge 0$ when potential on $z = 0$ is zero and that on $r = a$ is a function of ϕ only. (Hint: See the results of Exercise 5 in Section 8.6.)

41. Find the potential interior to a sphere of radius a when the potential on the upper half is a constant V_0 and the potential on the lower half is zero.

42. Use the result of Exercise 41 to find the potential inside a sphere of radius a when potentials on the top and bottom halves are constant values V_0 and V_1, respectively.

43. Find the potential in the region between two concentric spheres when the potential on each sphere is
(a) a constant;
(b) a function of ϕ only (and show that the solution reduces to that in part (a) when the functions are constant;
(c) a function of ϕ and θ (and show that the solution reduces to that in part (b) when the functions depend only on ϕ.

44. (a) Show that the negative of Poisson's integral formula 9.41 is the solution to Laplace's equation exterior to the sphere $r = a$ if $V(r,\phi,\theta)$ is required to vanish at infinity.
(b) Show that if $V(r,\phi,\theta)$ is the solution to the interior problem, then $(a/r)V(a^2/r,\phi,\theta)$ is the solution to the exterior problem. Do this using the result in part (a), and also by checking that the function satisfies the boundary value problem.

45. (a) What is the potential interior to a sphere of radius a when its value on the sphere is a constant V_0?
(b) Determine the potential exterior to a sphere of radius a when its value on the sphere is a constant V_0, and the potential must vanish at infinity. Do this in two ways, using separation of variables, and the result of Exercise 44.

46. What is the potential exterior to a sphere of radius a when the potential must vanish at infinity and satisfy a Neumann condition on the sphere,

$$-\frac{\partial V(a,\phi,\theta)}{\partial r} = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi.$$

47. What is the potential exterior to a sphere of radius a when the potential must vanish at infinity and satisfy a Robin condition on the sphere,

$$-l\frac{\partial V(a,\phi,\theta)}{\partial r} + hV(a,\phi,\theta) = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi.$$

48. Consider the following boundary value problem for steady-state temperature inside a cylinder of length L and radius a when the temperature of each end is zero:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0, \quad 0 < r < a, \quad 0 < z < L,$$
$$U(r,0) = 0, \quad 0 < r < a,$$
$$U(r,L) = 0, \quad 0 < r < a,$$
$$U(a,z) = f(z), \quad 0 < z < L.$$

(a) Verify that separation of variables $U(r,z) = R(r)Z(z)$ leads to a Sturm-Liouville system in $Z(z)$ and the following differential equation in $R(r)$:

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} - \lambda^2 rR = 0, \quad 0 < r < a.$$

(b) Show that the change of variable $x = \lambda r$ leads to Bessel's modified differential equation of order zero,

$$x\frac{d^2R}{dx^2} + \frac{dR}{dx} - xR = 0.$$

(See Exercise 10 in Section 8.3.)

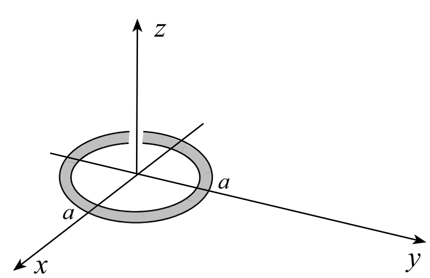
(c) Find functions $R_n(r)$ corresponding to eigenvalues λ_n, and use superposition to solve the boundary value problem.

(d) Simplify the solution in part (c) in the case that $f(z)$ is a constant value U_0.

49. Solve the boundary value problem in Exercise 48 if the ends of the cylinder are insulated.

50. (a) A charge Q is distributed uniformly around a thin ring of radius a in the xy-plane with centre at the origin (figure to the right). Show that potential at every point on the z-axis due to this charge is

$$V = \frac{Q}{4\pi\epsilon_0\sqrt{a^2+r^2}}.$$



(b) The potential at other points in space must be independent of the spherical coordinate θ. Show that $V(r,\phi)$ must be of the form

$$V(r,\phi) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\sqrt{\frac{2n+1}{2}}P_n(\cos\phi).$$

What does this result predict for potential at points on the positive z-axis?

(c) Equate expressions from parts (a) and (b) for V on the positive z-axis and expand $1/\sqrt{a^2+r^2}$ in powers of r/a and a/r to find $V(r,\phi)$.

51. Repeat Exercise 50 in the case that charge Q is distributed uniformly over a disc of radius a in the xy-plane with centre at the origin.

§9.2 Nonhomogeneous Problems in Polar, Cylindrical, and Spherical Coordinates

Nonhomogeneities in problems expressed in polar, cylindrical, and spherical coordinates can be treated in the same way that they were treated in Cartesian coordinates — split off *steady-state* or *static deflection* solutions, or use variation of constants, or use finite Fourier transforms. It is always an advantage to split off steady-state and static deflection solutions in initial boundary value problems, and we will continue to do this whenever nonhomogeneities are time-independent (and it is feasible to do so). Our work in Chapter 7, demonstrated the simplicity of finite Fourier transforms over variation of constants for problems in Cartesian coordinates. The same can be said about nonhomogeneous problems in polar, cylindrical, and spherical coordinates. Finite Fourier transforms are substantially easier than variation of constants. With this in mind, we discuss finite Fourier transforms associated with Sturm-Liouville sytems in Sections 8.4 and 8.6. For the singular system

$$(rR')' + \left(\lambda^2 r - \frac{\nu^2}{r}\right) R = 0, \quad 0 < r < a, \tag{9.42a}$$

$$lR'(a) + hR(a) = 0, \tag{9.42b}$$

with eigenvalues and eigenfunctions in Table 8.1, we define the transform

$$\tilde{f}(\lambda_{\nu n}) = \int_0^a r f(r) R_{\nu n}(r)\, dr, \tag{9.43a}$$

called a finite **Hankel** transform. It associates with a function $f(r)$, the sequence $\{\tilde{f}(\lambda_{\nu n})\}$ of coefficients in the Fourier Bessel series of $f(r)$ in terms of the $R_{\nu n}(r)$. The inverse transform of 9.43a is the Fourier Bessel series

$$f(r) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_{\nu n}) R_{\nu n}(r), \quad 0 < r < a, \tag{9.43b}$$

provided that $f(r)$ is defined as the average of right and left limits at any point of discontinuity. Finite Hankel transforms are used to eliminate the r-variable from initial boundary value problems in polar, cylindrical, and spherical coordinates. Most often this transform involves Bessel functions, but not always. We will continue to call finite Fourier transforms that eliminate r Hankel transforms even when Bessel functions are not present.

With the singular Sturm-Liouville system

$$(\sin\phi\, \Phi')' + \left(\lambda \sin\phi - \frac{m^2}{\sin\phi}\right) \Phi = 0, \quad 0 < \phi < \pi, \tag{9.44}$$

($m \geq 0$ an integer) is associated the **Legendre** transform

$$\tilde{f}(m,n) = \int_0^\pi \sin\phi f(\phi) \Phi_{mn}(\phi)\, d\phi, \tag{9.45a}$$

where eigenvalues are $\lambda_{mn} = n(n+1)$, $n \geq m$ an integer, and Φ_{mn} are normalized associated Legendre functions of the first kind (see equation 8.94 in Section 8.6). The inverse transform is the series

$$f(\phi) = \sum_{n=m}^{\infty} \tilde{f}(m,n)\Phi_{mn}(\phi). \tag{9.45b}$$

This transform removes the ϕ-variable from problems in spherical coordinates.

To complete the set of finite Fourier transforms, we associate a transform with the periodic Sturm-Liouville system

$$H'' + \lambda^2 H = 0, \quad -\pi < \theta \le \pi, \tag{9.46a}$$
$$H(-\pi) = H(\pi), \tag{9.46b}$$
$$H'(-\pi) = H'(\pi), \tag{9.46c}$$

which arises in so many of our problems. Eigenvalues of this system are $\lambda_m^2 = m^2$, m a nonnegative integer, with orthonormal eigenfunctions

$$\frac{1}{\sqrt{2\pi}} \leftrightarrow \lambda_0 = 0; \qquad \frac{1}{\sqrt{\pi}}\cos m\theta, \quad \frac{1}{\sqrt{\pi}}\sin m\theta \leftrightarrow \lambda_m,\ m > 0.$$

Periodic functions $f(\theta)$ may be expressed in terms of these eigenfunctions as ordinary trigonometric Fourier series:

$$f(\theta) = \frac{a_0}{\sqrt{2\pi}} + \sum_{m=1}^{\infty}\left(a_m \frac{\cos m\theta}{\sqrt{\pi}} + b_m \frac{\sin m\theta}{\sqrt{\pi}}\right), \tag{9.47a}$$

where

$$a_0 = \int_{-\pi}^{\pi} \frac{f(\theta)}{\sqrt{2\pi}} d\theta, \quad a_m = \int_{-\pi}^{\pi} f(\theta)\frac{\cos m\theta}{\sqrt{\pi}} d\theta, \quad b_m = \int_{-\pi}^{\pi} f(\theta)\frac{\sin m\theta}{\sqrt{\pi}} d\theta. \tag{9.47b}$$

The complex representation of this series in Exercise 27 of Section 3.1 provides the finite Fourier transform. We may rewrite equation 9.47 in the form

$$f(\theta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} C_m e^{im\theta}, \tag{9.48a}$$

where

$$C_m = \int_{-\pi}^{\pi} f(\theta)e^{-im\theta} d\theta. \tag{9.48b}$$

(We took the liberty in Exercise 27 of Section 3.1 of incorporating the 2π-factor into the series rather than the coefficient C_m. The series representation of $f(\theta)$ is the same in either case.) Associated with this representation is the finite Fourier transform of 2π-periodic functions

$$\tilde{f}(m) = \int_{-\pi}^{\pi} f(\theta)e^{-im\theta} d\theta \tag{9.49a}$$

and its inverse

$$f(\theta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \tilde{f}(m)e^{im\theta}. \tag{9.49b}$$

(The exponentials in equations 9.49 could be interchanged to give an alternative transform; this uses the complex representation of equation 3.15 in Section 3.1.)

The following examples illustrate how these transforms facilitate the solution of nonhomogeneous (initial) boundary value problems.

Example 9.5 A circular plate of radius a is insulated top and bottom. At time $t = 0$, its temperature is 0°C throughout. If, for $t > 0$, all points on the edge of the plate have the same temperature $\overline{U}$, find the temperature in the plate for $t > 0$.

Solution The initial boundary value problem for $U(r, t)$ is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < a, \quad t > 0, \tag{9.50a}$$

$$U(a, t) = \overline{U}, \quad t > 0, \tag{9.50b}$$

$$U(r, 0) = 0, \quad 0 < r < a. \tag{9.50c}$$

For comparison, we solve the problem by splitting off the steady-state solution and by finite Hankel transforms.

Finite Fourier Transforms

To eliminate r from the problem, we use the finite Hankel transform

$$\tilde{f}(\lambda_n) = \int_0^a r f(r) R_n(r)\, dr, \tag{9.51}$$

where $R_n(r) = \sqrt{2}J_0(\lambda_n r)/[aJ_1(\lambda_n a)]$ are eigenfunctions of the Sturm-Liouville system

$$(rR')' + \lambda^2 rR = 0, \quad 0 < r < a, \tag{9.52a}$$

$$R(a) = 0. \tag{9.52b}$$

(This is the system that would result were separation of variables applied to the corresponding homogeneous problem.) Application of the transform to the PDE gives

$$\int_0^a r\frac{\partial U}{\partial t}R_n\, dr = k\int_0^a r\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right)R_n\, dr.$$

An interchange of differentiation with respect to t and integration with respect to r on the left, and integration by parts on the right, yield

$$\begin{aligned}
\frac{\partial \tilde{U}}{\partial t} &= k\left\{r\frac{\partial U}{\partial r}R_n\right\}_0^a + k\int_0^a \frac{\partial U}{\partial r}\left[-\frac{d}{dr}(rR_n) + R_n\right] dr \\
&= -k\int_0^a r\frac{\partial U}{\partial r}R_n'\, dr \qquad \text{(because of 9.52b)} \\
&= -k\Big\{UrR_n'\Big\}_0^a + k\int_0^a U(rR_n')'dr \qquad \text{(by a second integration by parts)} \\
&= -kaR_n'(a)\overline{U} + k\int_0^a U(-\lambda_n^2 rR_n)\, dr \qquad \text{(from 9.50b and 9.52a)} \\
&= -k\overline{U}aR_n'(a) - k\lambda_n^2\tilde{U}.
\end{aligned}$$

Thus, $\tilde{U}(\lambda_n, t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\lambda_n^2 \tilde{U} = -k\overline{U} a R_n'(a) \tag{9.53a}$$

subject to the transform of initial condition 9.50c,

$$\tilde{U}(\lambda_n, 0) = 0. \tag{9.53b}$$

Since the solution of problem 9.53 is

$$\tilde{U}(\lambda_n, t) = \frac{\overline{U} a R_n'(a)}{\lambda_n^2}(-1 + e^{-k\lambda_n^2 t}), \tag{9.54}$$

we obtain

$$\begin{aligned}
U(r,t) &= \sum_{n=1}^{\infty} \tilde{U}(\lambda_n, t) R_n(r) \\
&= \sum_{n=1}^{\infty} \frac{\overline{U} a \sqrt{2} \lambda_n J_0'(\lambda_n a)}{a J_1(\lambda_n a) \lambda_n^2} (e^{-k\lambda_n^2 t} - 1) \frac{\sqrt{2} J_0(\lambda_n r)}{a J_1(\lambda_n a)} \\
&= \frac{2\overline{U}}{a} \sum_{n=1}^{\infty} \frac{-J_1(\lambda_n a)}{\lambda_n [J_1(\lambda_n a)]^2} (e^{-k\lambda_n^2 t} - 1) J_0(\lambda_n r) \\
&= \frac{2\overline{U}}{a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n J_1(\lambda_n a)} (1 - e^{-k\lambda_n^2 t}) J_0(\lambda_n r).
\end{aligned} \tag{9.55}$$

The limit of this temperature function for large t is

$$\lim_{t \to \infty} U(r,t) = \frac{2\overline{U}}{a} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n a)}.$$

The transform $\tilde{1}_n$ of the function $f(r) \equiv 1$ is

$$\begin{aligned}
\tilde{1}_n &= \int_0^a r \frac{\sqrt{2} J_0(\lambda_n r)}{a J_1(\lambda_n a)} dr = \frac{\sqrt{2}}{a J_1(\lambda_n a)} \int_0^{\lambda_n a} \left(\frac{u}{\lambda_n}\right) J_0(u) \left(\frac{du}{\lambda_n}\right) \\
&= \frac{\sqrt{2}}{a \lambda_n^2 J_1(\lambda_n a)} \int_0^{\lambda_n a} \frac{d}{du}[u J_1(u)]\, du \quad \text{(see identity 8.42 in Section 8.3 with } \nu = 1) \\
&= \frac{\sqrt{2}}{a \lambda_n^2 J_1(\lambda_n a)} \Big\{ u J_1(u) \Big\}_0^{\lambda_n a} = \frac{\sqrt{2}}{\lambda_n}.
\end{aligned}$$

Consequently,

$$1 = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{\lambda_n} \frac{\sqrt{2} J_0(\lambda_n r)}{a J_1(\lambda_n a)} = \frac{2}{a} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n a)}, \tag{9.56}$$

and it follows that

$$\lim_{t \to \infty} U(r,t) = \overline{U},$$

as expected. Furthermore, this suggests that we write $U(r,t)$ in the form

$$U(r,t) = \overline{U} - \frac{2\overline{U}}{a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1(\lambda_n a)}e^{-k\lambda_n^2 t}J_0(\lambda_n r). \tag{9.57}$$

Splitting off the Steady-state Solution

Because the nonhomogeneity in boundary condition 9.50b is independent of time, we can split off the steady-state solution. We set $U(r,t) = V(r,t) + \psi(r)$, where $\psi(r)$ is the solution of

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = 0, \quad 0 < r < a, \tag{9.58a}$$

$$\psi(a) = \overline{U}. \tag{9.58b}$$

The only bounded solution of this system is $\psi(r) = \overline{U}$. With this steady-state solution, $V(r,t)$ must satisfy the homogeneous problem

$$\frac{\partial V}{\partial t} = k\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right), \quad 0 < r < a, \quad t > 0, \tag{9.59a}$$

$$V(a,t) = 0, \quad t > 0, \tag{9.59b}$$

$$V(r,0) = -\overline{U}, \quad 0 < r < a. \tag{9.59c}$$

Separation $V(r,t) = R(r)T(t)$ leads to Sturm-Liouville system 9.52 in $R(r)$ and the ODE

$$T' + k\lambda^2 T = 0, \quad t > 0. \tag{9.60}$$

Eigenvalues are defined by $J_0(\lambda a) = 0$, and normalized eigenfunctions are $R_n(r) = \sqrt{2}J_0(\lambda_n r)/[aJ_1(\lambda_n a)]$. Corresponding solutions of differential equation 9.60 are

$$T(t) = Ce^{-k\lambda_n^2 t}. \tag{9.61}$$

Superposition of separated functions yields

$$V(r,t) = \sum_{n=1}^{\infty} C_n e^{-k\lambda_n^2 t}R_n(r), \tag{9.62}$$

and initial condition 9.59c requires

$$-\overline{U} = \sum_{n=1}^{\infty} C_n R_n(r). \tag{9.63}$$

The C_n are therefore Fourier coefficients in the eigenfunction expansion of the function $-\overline{U}$; that is,

$$C_n = \int_0^a r(-\overline{U})R_n(r)\,dr = -\overline{U}\int_0^a r\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}dr = \frac{-\sqrt{2}\,\overline{U}}{\lambda_n}.$$

(This integral was evaluated in the above transform solution.) Consequently,

$$\begin{aligned} U(r,t) &= \overline{U} + \sum_{n=1}^{\infty}\frac{-\sqrt{2}\,\overline{U}}{\lambda_n}e^{-k\lambda_n^2 t}\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)} \\ &= \overline{U} - \frac{2\overline{U}}{a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1(\lambda_n a)}e^{-k\lambda_n^2 t}J_0(\lambda_n r), \end{aligned}$$

the same solution as that obtained by the finite Hankel transform.●

Our next example is a vibration problem.

Example 9.6 A circular membrane of radius a has an initial displacement at time $t = 0$ described by the function $f(r, \theta)$, $0 \le r \le a$, $-\pi < \theta \le \pi$, but no initial velocity. For time $t > 0$, its edge $r = a$ is forced to undergo periodic oscillations described by $A \sin \omega t$, A a constant. (For consistency, we assume that $f(a, \theta) = 0$.) Find its displacement as a function of r, θ, and t.

Solution The initial boundary value problem for $z(r, \theta, t)$ is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \right), \quad 0 < r < a, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{9.64a}$$

$$z(a, \theta, t) = A \sin \omega t, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{9.64b}$$

$$z(r, \theta, 0) = f(r, \theta), \quad 0 < r < a, \quad -\pi < \theta \le \pi, \tag{9.64c}$$

$$z_t(r, \theta, 0) = 0, \quad 0 < r < a, \quad -\pi < \theta \le \pi. \tag{9.64d}$$

To remove θ from the problem, we apply transform 9.49a to the PDE,

$$\int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial t^2} e^{-im\theta} d\theta = c^2 \int_{-\pi}^{\pi} \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \right) e^{-im\theta} d\theta.$$

Integrations with respect to θ and differentiations with respect to t and r may be interchanged, with the result that

$$\frac{\partial^2 \tilde{z}}{\partial t^2} - c^2 \left(\frac{\partial^2 \tilde{z}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{z}}{\partial r} \right) = \frac{c^2}{r^2} \int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial \theta^2} e^{-im\theta} d\theta.$$

Integration by parts on the remaining integral gives

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial \theta^2} e^{-im\theta} d\theta &= \left\{ \frac{\partial z}{\partial \theta} e^{-im\theta} \right\}_{-\pi}^{\pi} + \int_{-\pi}^{\pi} im \frac{\partial z}{\partial \theta} e^{-im\theta} d\theta \\ &= \frac{\partial z(r, \pi, t)}{\partial \theta} \cos(-m\pi) - \frac{\partial z(r, -\pi, t)}{\partial \theta} \cos m\pi + im \int_{-\pi}^{\pi} \frac{\partial z}{\partial \theta} e^{-im\theta} d\theta.\end{aligned}$$

Because $\partial z/\partial\theta$ must be 2π-periodic, it follows that $\partial z(r, \pi, t)/\partial\theta = \partial z(r, -\pi, t)/\partial\theta$, and therefore

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial \theta^2} e^{-im\theta} d\theta &= im \int_{-\pi}^{\pi} \frac{\partial z}{\partial \theta} e^{-im\theta} d\theta \\ &= im \left\{ z e^{-im\theta} \right\}_{-\pi}^{\pi} + im \int_{-\pi}^{\pi} im z e^{-im\theta} d\theta \\ &= im[z(r, \pi, t) \cos(-m\pi) - z(r, -\pi, t) \cos m\pi] - m^2 \int_{-\pi}^{\pi} z e^{-im\theta} d\theta \\ &= -m^2 \tilde{z},\end{aligned}$$

since $z(r, \theta, t)$ must also be 2π-periodic. Consequently, $\tilde{z}(r, m, t)$ must satisfy the PDE

$$\frac{\partial^2 \tilde{z}}{\partial t^2} = c^2 \left(\frac{\partial^2 \tilde{z}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{z}}{\partial r} - \frac{m^2}{r^2} \tilde{z} \right), \quad 0 < r < a, \quad t > 0, \tag{9.65a}$$

subject to the transforms of conditions 9.64b,c,d,

$$\tilde{z}(a,m,t) = A\sin\omega t\,\tilde{1}_m, \quad t>0, \tag{9.65b}$$

$$\tilde{z}(r,m,0) = \tilde{f}(r,m), \quad 0<r<a, \tag{9.65c}$$

$$\tilde{z}_t(r,m,0) = 0, \quad 0<r<a, \tag{9.65d}$$

where

$$\tilde{1}_m = \int_{-\pi}^{\pi} e^{-im\theta}\,d\theta = \begin{cases} 2\pi, & m=0 \\ 0, & m\neq 0. \end{cases} \tag{9.65e}$$

To eliminate r from this problem, we use the finite Hankel transform

$$\tilde{f}(\lambda_{mn}) = \int_0^a r f(r) R_{mn}(r)\,dr, \tag{9.66}$$

where $R_{mn}(r)$ are the orthonormal eigenfunctions of the Sturm-Liouville system

$$(rR')' + \left(\lambda^2 r - \frac{m^2}{r}\right)R = 0, \quad 0<r<a, \tag{9.67a}$$

$$R(a) = 0, \tag{9.67b}$$

(the system that would result were separation performed on problem 9.65 with the homogeneous version of 9.65b). Application of the transform to the PDE and integration by parts give

$$\begin{aligned}
\frac{\partial^2 \tilde{\tilde{z}}}{\partial t^2} &= c^2\int_0^a r\left(\frac{\partial^2\tilde{z}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{z}}{\partial r} - \frac{m^2}{r^2}\tilde{z}\right)R_{mn}\,dr \\
&= c^2\left\{rR_{mn}\frac{\partial\tilde{z}}{\partial r}\right\}_0^a + c^2\int_0^a\left[-\frac{\partial\tilde{z}}{\partial r}(rR_{mn})' + \frac{\partial\tilde{z}}{\partial r}R_{mn} - \frac{m^2}{r}\tilde{z}R_{mn}\right]dr \\
&= c^2\int_0^a\left(-r\frac{\partial\tilde{z}}{\partial r}R'_{mn} - \frac{m^2}{r}\tilde{z}R_{mn}\right)dr \qquad (\text{since } R_{mn}(a)=0) \\
&= c^2\left\{-r\tilde{z}R'_{mn}\right\}_0^a + c^2\int_0^a\left[\tilde{z}(rR'_{mn})' - \frac{m^2}{r}\tilde{z}R_{mn}\right]dr \\
&= -ac^2A\sin\omega t\tilde{1}_mR'_{mn}(a) + c^2\int_0^a\tilde{z}\left[(rR'_{mn})' - \frac{m^2}{r}R_{mn}\right)dr \quad (\text{by } 9.65\text{b}) \\
&= -ac^2A\tilde{1}_mR'_{mn}(a)\sin\omega t + c^2\int_0^a\tilde{z}(-\lambda_{mn}^2 r)R_{mn}\,dr \quad (\text{by } 9.67\text{a}) \\
&= -ac^2\tilde{1}_mAR'_{mn}(a)\sin\omega t - c^2\lambda_{mn}^2\tilde{\tilde{z}}.
\end{aligned}$$

Thus, $\tilde{\tilde{z}}(\lambda_{mn},m,t)$ must satisfy the ODE

$$\frac{d^2\tilde{\tilde{z}}}{dt^2} + c^2\lambda_{mn}^2\tilde{\tilde{z}} = -ac^2A\tilde{1}_m\sin\omega t, \tag{9.68a}$$

subject to transforms of conditions 9.65c,d,

$$\tilde{\tilde{z}}(\lambda_{mn},m,0) = \tilde{\tilde{f}}(\lambda_{mn},m), \tag{9.68b}$$

$$\tilde{\tilde{z}}_t(\lambda_{mn},m,0) = 0. \tag{9.68c}$$

A general solution of this ODE is

$$\tilde{\tilde{z}}(\lambda_{mn}, m, t) = \begin{cases} B_{0n} \cos c\lambda_{0n} t + D_{0n} \sin c\lambda_{0n} t + \dfrac{2\pi a c^2 A R'_{0n}(a) \sin \omega t}{\omega^2 - c^2 \lambda_{0n}^2}, & m = 0 \\ B_{mn} \cos c\lambda_{mn} t + D_{mn} \sin c\lambda_{mn} t, & m \neq 0 \end{cases} \quad (9.69)$$

provided $\omega \neq c\lambda_{0n}$ for any n. Discussion of this special case is given in Exercise 24. Initial conditions 9.68b,c yield

$$\tilde{\tilde{z}}(\lambda_{mn}, m, t) = \begin{cases} \tilde{\tilde{f}}(\lambda_{0n}, 0) \cos c\lambda_{0n} t + \dfrac{2\pi A a c R'_{0n}(a)}{\lambda_{0n}(\omega^2 - c^2 \lambda_{0n}^2)} (c\lambda_{0n} \sin \omega t - \omega \sin c\lambda_{0n} t), & m = 0 \\ \tilde{\tilde{f}}(\lambda_{mn}, m) \cos c\lambda_{mn} t, & m \neq 0. \end{cases}$$

$$(9.70)$$

Inverse transforms now give

$$\begin{aligned} z(r, \theta, t) &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \tilde{\tilde{z}}(\lambda_{mn}, m, t) R_{mn}(r) e^{im\theta} \\ &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[\tilde{\tilde{f}}(\lambda_{0n}, 0) \cos c\lambda_{0n} t + \frac{2\pi A a c R'_{0n}(a)}{\lambda_{0n}(\omega^2 - c^2 \lambda_{0n}^2)} (c\lambda_{0n} \sin \omega t - \omega \sin c\lambda_{0n} t) \right] R_{0n}(r) \\ &\quad + \frac{1}{2\pi} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{n=1}^{\infty} \tilde{\tilde{f}}(\lambda_{mn}, m) \cos c\lambda_{mn} t R_{mn}(r) e^{im\theta}. \end{aligned} \quad (9.71)$$

We reduce the second double summation by noting that $\lambda_{-mn} = \lambda_{mn}$, $R_{-mn}(r) = R_{mn}(r)$, and $\tilde{\tilde{f}}(\lambda_{-mn}, -m) = \overline{\tilde{\tilde{f}}(\lambda_{mn}, m)}$, (the complex conjugate of $\tilde{\tilde{f}}(\lambda_{mn}, m)$). Then

$$\begin{aligned} z(r, \theta, t) &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[\tilde{\tilde{f}}(\lambda_{0n}, 0) \cos c\lambda_{0n} t + \frac{2\pi A a c R'_{0n}(a)}{\lambda_{0n}(\omega^2 - c^2 \lambda_{0n}^2)} (c\lambda_{0n} \sin \omega t - \omega \sin c\lambda_{0n} t) \right] R_{0n}(r) \\ &\quad + \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\tilde{\tilde{f}}(\lambda_{mn}, m) e^{im\theta} + \overline{\tilde{\tilde{f}}(\lambda_{mn}, m)} e^{-im\theta} \right] \cos c\lambda_{mn} t R_{mn}(r) \end{aligned}$$

or,

$$\begin{aligned} z(r, \theta, t) &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[\tilde{\tilde{f}}(\lambda_{0n}, 0) \cos c\lambda_{0n} t + \frac{2\pi A a c R'_{0n}(a)}{\lambda_{0n}(\omega^2 - c^2 \lambda_{0n}^2)} (c\lambda_{0n} \sin \omega t - \omega \sin c\lambda_{0n} t) \right] R_{0n}(r) \\ &\quad + \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \operatorname{Re}[\tilde{\tilde{f}}(\lambda_{mn}, m) e^{im\theta}] \cos c\lambda_{mn} t R_{mn}(r). \bullet \end{aligned} \quad (9.72)$$

Our final example is a potential problem.

Example 9.7 Find the potential inside a sphere if the potential on the sphere is only a function $g(\phi)$ of angle ϕ and the region contains a constant charge with density σ.

Solution The boundary value problem is

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial V}{\partial \phi}\right) = -\frac{\sigma}{\epsilon}, \quad 0 < r < a, \quad 0 < \phi < \pi, \tag{9.73a}$$

$$V(a, \phi) = g(\phi), \quad 0 < \phi < \pi. \tag{9.73b}$$

To remove ϕ from the problem, we use the Legendre transform

$$\tilde{f}(n) = \int_0^\pi \sin\phi f(\phi)\Phi_n(\phi)\, d\phi \tag{9.74}$$

where $\Phi_n(\phi) = \sqrt{(2n+1)/2}P_n(\cos\phi)$ are orthonormal eigenfunctions of the Sturm-Liouville system

$$(\sin\phi\, \Phi')' + n(n+1)\sin\phi\, \Phi = 0, \quad 0 < \phi < \pi, \tag{9.75}$$

(the system that would result were separation of variables applied to the homogeneous version of 9.73a). Application of this transform to the PDE and integration by parts give

$$\begin{aligned}
\frac{d^2\tilde{V}}{dr^2} + \frac{2}{r}\frac{d\tilde{V}}{dr} + \frac{\sigma}{\epsilon}\tilde{1}_n &= \frac{-1}{r^2}\int_0^\pi \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial \phi}\right)\Phi_n(\phi)\, d\phi \\
&= \frac{-1}{r^2}\left[\left\{\sin\phi\frac{\partial V}{\partial\phi}\Phi_n\right\}_0^\pi - \int_0^\pi \sin\phi\frac{\partial V}{\partial \phi}\Phi_n'\, d\phi\right] \\
&= \frac{1}{r^2}\left[\left\{\sin\phi V \Phi_n'\right\}_0^\pi - \int_0^\pi V(\sin\phi\,\Phi_n')'\, d\phi\right] \\
&= \frac{-1}{r^2}\int_0^\pi V[-n(n+1)\sin\phi\,\Phi_n]\, d\phi \quad \text{(by 9.75)} \\
&= \frac{n(n+1)}{r^2}\tilde{V}.
\end{aligned}$$

Thus, $\tilde{V}(r, n)$ must satisfy the ODE

$$\frac{d^2\tilde{V}}{dr^2} + \frac{2}{r}\frac{d\tilde{V}}{dr} - \frac{n(n+1)}{r^2}\tilde{V} = -\frac{\sigma}{\epsilon}\tilde{1}_n, \tag{9.76a}$$

where

$$\tilde{1}_n = \int_0^\pi \sin\phi\,\Phi_n(\phi)\, d\phi = \begin{cases} \sqrt{2}, & n = 0 \\ 0, & n > 0 \end{cases} \tag{9.76b}$$

subject to

$$\tilde{V}(a, n) = \tilde{g}(n). \tag{9.76c}$$

A general solution of the ODE is

$$\tilde{V}(r, n) = \begin{cases} A_0 + \dfrac{B_0}{r} - \dfrac{\sqrt{2}\sigma r^2}{6\epsilon}, & n = 0 \\ A_n r^n + \dfrac{B_n}{r^{n+1}}, & n > 0. \end{cases} \tag{9.77}$$

The only bounded solution satisfying 9.76c is

$$\tilde{V}(r,n) = \begin{cases} \tilde{g}(0) + \dfrac{\sqrt{2}\sigma}{6\epsilon}(a^2 - r^2), & n = 0 \\ \dfrac{\tilde{g}(n)}{a^n} r^n, & n > 0, \end{cases} \tag{9.78}$$

and therefore

$$\begin{aligned} V(r,\phi) &= \sum_{n=0}^{\infty} \tilde{V}(r,n)\Phi_n(\phi) \\ &= \frac{\tilde{g}(0)}{\sqrt{2}} + \frac{\sigma}{6\epsilon}(a^2 - r^2) + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \tilde{g}(n)\Phi_n(\phi) \\ &= \frac{\sigma}{6\epsilon}(a^2 - r^2) + \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \tilde{g}(n)\Phi_n(\phi). \end{aligned} \tag{9.79}$$

In retrospect, notice that $\sigma(a^2 - r^2)/(6\epsilon)$ satisfies 9.73a and a homogeneous 9.73b, while the series satisfies 9.73b with a homogeneous 9.73a.•

EXERCISES 9.2

Part A Heat Conduction

1. Solve Example 9.5 if the temperature of the edge $r = a$ is a function $f(t)$ of time.

2. (a) Solve Example 9.5 if heat is transferred to the plate along its edge $r = a$ at a rate $f_1(t)$ W/m^2 equally all around.
(b) Simplify the solution when $f_1(t) = Q$, a constant. Hint: You will want to consider the finite Hankel transform of the function $h(r) = 2r^2 - a^2$.
(c) Can you solve part (b) by splitting off a steady-state solution?

3. (a) A very long cylinder of radius a is initially at temperature $f(r)$. For time $t > 0$, its edge $r = a$ is held at 0°C. If heat generation within the cylinder is $g(r,t)$, find the temperature for $0 \le r < a$ and $t > 0$.
(b) Simplify the solution in part (a) when $f(r) \equiv 0$ and $g(r,t)$ is constant.
(c) Solve the problem in part (b) by splitting off the steady-state solution.

4. Repeat parts (a) and (b) of Exercise 3 if the boundary $r = a$ is insulated.

5. Repeat Exercise 3 if heat is transferred at $r = a$ to a medium at constant temperature U_m according to Newton's law of cooling.

6. (a) A sphere of radius a is initially at temperature $f(r)$. For $t > 0$, its surface is held at temperature $f_1(t)$, and heat is generated at a rate $g(r,t)$. Find the temperature in the sphere. (See Exercise 8 in Section 8.4 for the appropriate finite Fourier transform.)
(b) Simplify the solution when $f(r) \equiv 0$, $f_1(t) \equiv 0$, and $g(r,t)$ is constant.
(c) Solve the problem in part (b) by splitting off the steady-state solution.
(d) Simplify the solution in part (a) when $f(r) \equiv 0$, $g(r,t) \equiv 0$, and $f_1(t)$ is constant.
(e) Solve the problem in part (d) by splitting off the steady-state solution.

7. (a) A sphere of radius a is initially at temperature $f(r)$. For $t > 0$, heat is added to its surface at a rate $f_1(t)$ W/m^2, and heat is generated at a rate $g(r,t)$ W/m^3. Find the temperature in the sphere. (See Exercise 8 in Section 8.4 for the appropriate finite Fourier transform.)

(b) Simplify the solution when $f(r) \equiv 0$, $g(r,t) \equiv 0$, and $f_1(t)$ is constant.
(c) Can you solve part (b) by splitting off a steady-state solution?

8. A cylinder of length L and radius a is initially at temperature $f(r,z)$, $0 \le r \le a$, $0 \le z \le L$. For time $t > 0$, the face $z = 0$ is insulated, face $z = L$ has a time-dependent temperature $f_1(t)$, and the curved surface $r = a$ has temperature $f_2(t)$. Find the temperature of the cylinder for $t > 0$.

9. A hemisphere $x^2 + y^2 + z^2 \le a^2$, $z \ge 0$, is initially at temperature zero throughout. For time $t > 0$, its base $z = 0$ continues to be held at temperature zero, but the surface of the hemisphere has a time-dependent temperature $f_1(t)$. Find a series representation for temperature inside the hemisphere. (Hint: You will need the eigenfunctions of Exercise 5 in Section 8.6 and Exercise 9 in Section 8.4.)

10. Solve Example 9.5 if the constant temperature on $r = a$ is replaced by $f(\theta) = \sin\theta$.

11. (a) Solve Example 9.5 when the initial temperature of the plate is $f(r,\theta)$.
(b) Does the solution reduce to that of Example 9.5 when $f(r,\theta) = 0$?

12. Solve parts (a) and (b) of Exercise 2 when the initial temperature of the plate is $f(r,\theta)$.

13. Solve Example 9.5 if heat is exchanged with a constant-temperature environment along the edge $r = a$ according to Newton's law of cooling and the initial temperature of the plate is $f(r,\theta)$.

14. A very long cylinder of radius r_1 is at uniform temperature U_i. At time $t = 0$, a long hollow cylinder with inner radius r_1 and outer radius r_2 is fitted tightly over the smaller cylinder. The temperature of the hollow cylinder is uniformly U_o and both cylinders are of the same material. The outer surface of the hollow cylinder is kept at temperature U_o. Find temperatures in the cylinders for $t > 0$.

15. (a) A hollow sphere has inner radius r_1 and outer radius r_2. Its initial temperature is $f(r)$. For time $t > 0$, its inner surface has temperature $f_1(t)$ and its outer surface has temperature $f_2(t)$. Find the temperature of the hollow sphere when $t > 0$ if heat generation $g(r,t)$ is only a function of r and t. (See Exercise 10 in Section 8.4 for the appropriate finite Fourier transform.)
(b) Simplify the solution when $f(r) = f_1(t) = f_2(t) = 0$ and $g(r,t)$ is a constant g.
(c) Solve the problem in part (b) by splitting off the steady-state solution.
(d) Simplify the solution in part (a) when $f(r) = g(r,t) = 0$, $f_1(t) = f_1$, and $f_2(t) = f_2$, where f_1 and f_2 are constants.
(e) Solve the problem in part (d) by splitting off the steady-state solution.

16. (a) A hollow sphere has inner radius r_1 and outer radius r_2. Its initial temperature is $f(r)$. For time $t > 0$, its outer surface has temperature $f_2(t)$ and heat is added to its inner surface at rate $Q(t)$. Find the temperature of the hollow sphere when $t > 0$ if heat generation $g(r,t)$ is only a function of r and t. (See Exercise 10 in Section 8.4 for the appropriate finite Fourier transform.)
(b) Simplify the solution when $g(r,t) = f(r) = f_2(t) = 0$ and $Q(t)$ is a constant Q.
(c) Solve the problem in part (b) by splitting off the steady-state solution.

Part B Vibrations

17. A circular membrane of radius a is initially at rest with displacement $f(r)$. The boundary of the membrane is fixed on the xy-plane. If a constant vertical force per unit area F acts at all points on the membrane, find subsequent displacements of points on the membrane.

18. Repeat Exercise 17 if the force per unit area F is proportional to the square of the distance from the centre of the membrane.

19. Repeat Exercise 24 in Section 9.1 if at the instant the membrane strikes the xy-plane, it is hanging in its equilibrium position were gravity and tension the only forces acting on the membrane. Take gravity into account in the vibrations.

20. (a) Find the displacement of a circular membrane of radius a that is initially ($t = 0$) at rest, but is displaced according to $f(r, \theta)$, the boundary of which is displaced permanently according to $f_1(\theta)$.
(b) Simplify the solution when $f(r, \theta)$ and $f_1(\theta)$ are independent of θ.
(c) Solve the problem in part (b) by splitting off the static deflection solution.

21. Solve the following nonhomogeneous version of Exercise 26 in Section 9.1:

$$\frac{\partial^2 y}{\partial t^2} = g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) + \frac{F(x,t)}{\rho}, \quad 0 < x < L, \quad t > 0,$$
$$y(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = h(x), \quad 0 < x < L.$$

22. A circular membrane of radius a is initially at rest on the xy-plane. For time $t > 0$, its edge is forced to undergo periodic oscillations described by $A \sin \omega t$, A a constant. Use a finite Hankel transform to find its displacement as a function of r and t. Include a discussion of resonance.

23. A circular membrane of radius a is initially at rest on the xy-plane. For time $t > 0$, a periodic vertical force per unit area $F_0 \sin \omega t$, F_0 a constant, acts at every point on the membrane.
(a) If the edge $r = a$ of the plate is fixed on the xy-plane, find a series representiation for displacements of the membrane in the non-resonant case.
(b) Discuss the resonant case.

24. Discuss the solution of Example 9.6 when $\omega = c\lambda_{0k}$ for some k.

25. Do the solutions of Example 9.6 and Exercise 24 reduce to those of Exercise 22 when $f(r, \theta) \equiv 0$?

Part C Potential, Steady-state Heat Conduction, Static Deflections of Membranes

26. A infinitely long solid cylinder is bounded by the planes $\theta = 0$ and $\theta = \beta$ and the curved surface $r = a$ ($0 \le \theta \le \beta$). A constant charge density σ exists inside the cylinder. If the three bounding surfaces are all held at potential zero, find the potential interior to the cylinder. Special consideration is required for the cases (a) $\beta = \pi/2$, (b) π, and (c) $3\pi/2$.

27. An infinite cylinder of radius a has charge density kr^n, $k > 0$ and $n > 0$ constants. If the surface of the cylinder has potential $f(\theta)$, what is the interior potential?

28. A hemisphere $x^2 + y^2 + z^2 \le a^2$, $z \ge 0$, has a constant charge density σ throughout. If potentials on the rounded and flat surfaces are both specified constants, but different ones, find the potential inside. (You will need the results of Exercise 5 in Section 8.6 and Exercise 8 in Section 8.5.)

29. A thin plate is in the shape of a sector of a circle bounded by the lines $\theta = 0$ and $\theta = \beta < \pi$ and the arc $r = a$, $0 \le \theta \le \beta$. Edge $\theta = \beta$ is insulated, as are the top and bottom of the plate. Heat is removed from the plate along the edge $\theta = 0$ at a constant rate $q > 0$ W/m^2. Along the curved edge $r = a$, heat is removed at a constant rate $Q > 0$ W/m^2. Heat is being generated at each point in the plate at a uniform rate of g W/m^3.

(a) Formulate the boundary value problem for steady-state temperature in the plate. (See Exercises 16 and 17 in Section 2.2 for the boundary conditions along $\theta = 0$ and $r = a$.) What condition must q, Q, and g satisfy?

(b) Solve the problem in part (a). To simplify the solution, you will need the finite Fourier transform of the function $h(\theta) = \dfrac{1}{\beta} - \dfrac{\cos(\beta - \theta)}{\sin \beta}$.

30. (a) Repeat Exercise 48 in Section 9.1 if there is a heat source $g(z)$.

(b) Simplify the solution when $f(z) = U_0$, a constant, and $g(z) = G$, also a constant.

CHAPTER 10 LAPLACE TRANSFORMS

§10.1 Introduction

The Laplace transform is a mathematical operation that replaces differentiation problems with algebraic ones, an essential simplification for ordinary and partial differential equations. For partial differential equations, Fourier transforms are associated with space variables; the Laplace transform is associated with time. In this section we give a brief review of the transform and some of its simple properties; the complex inversion integral is developed in Section 10.3, and the transform is applied to initial boundary values problems in Sections 10.2, 10.4, and 10.5.

The Laplace transform of a function $f(t)$ is a function denoted by $\tilde{f}(s)$ or $\mathcal{L}\{f(t)\}(s)$ with values given by

$$\tilde{f}(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{10.1}$$

provided the improper integral converges. When $f(t)$ is piecewise continuous on every finite interval $0 \le t \le T$, and $f(t)$ is of exponential order* α, its Laplace transform exists for $s > \alpha$ (see Exercise 34 for verification).

The Laplace transforms contained in Table 10.1 are fundamental to applications of the transform to ordinary and partial differential equations; more extensive tables are contained in many mathematical references. All entries are straightforward applications of definition 10.1.

$f(t)$	$\tilde{f}(s)$	$f(t)$	$\tilde{f}(s)$
t^n	$\dfrac{n!}{s^{n+1}}$	e^{at}	$\dfrac{1}{s-a}$
$\sin at$	$\dfrac{a}{s^2+a^2}$	$\cos at$	$\dfrac{s}{s^2+a^2}$
$t\sin at$	$\dfrac{2as}{(s^2+a^2)^2}$	$t\cos at$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh at$	$\dfrac{a}{s^2-a^2}$	$\cosh at$	$\dfrac{s}{s^2-a^2}$
$t\sinh at$	$\dfrac{2as}{(s^2-a^2)^2}$	$t\cosh at$	$\dfrac{s^2+a^2}{(s^2-a^2)^2}$

Table 10.1

* A function $f(t)$ is said to be of exponential order α, written $O(e^{\alpha t})$, if there exist constants T and M such that $|f(t)| < Me^{\alpha t}$ for all $t > T$. For example, e^{2t} is $O(e^{2t})$, $\sin t$ is $O(e^{0t})$, and t^n, n a nonnegative integer, is $O(e^{\epsilon t})$ for arbitrarily small $\epsilon > 0$.

When $\tilde{f}(s)$ is the Laplace transform of $f(t)$, we call $f(t)$ the **inverse Laplace transform** of $\tilde{f}(s)$ and write

$$f(t) = \mathcal{L}^{-1}\{\tilde{f}(s)\}(t). \tag{10.2}$$

Because the Laplace transform is an integral transform, $\tilde{f}(s)$ is unique for given $f(t)$, but there exist many functions $f(t)$ having the same transform $\tilde{f}(s)$. For example, the functions

$$f(t) = t^2 \qquad \text{and} \qquad g(t) = \begin{cases} 0, & t = 1 \\ t^2, & t \neq 1, 2 \\ 0, & t = 2 \end{cases}$$

which are identical except for their values at $t = 1$ and $t = 2$, both have the same transform $2/s^3$. What we are saying is that because the Laplace transform is not a one-to-one operation, the inverse transform $\mathcal{L}^{-1}\{\tilde{f}(s)\}$ in equation 10.2 cannot be a true inverse. In Section 10.3 we derive a formula for calculating inverse transforms, and this formula always yields a continuous function $f(t)$, if this is possible. In the event that this is not possible, the formula gives a piecewise continuous function whose value is the average of right- and left-limits at discontinuities, namely, $[f(t+) + f(t-)]/2$. This is reminiscent of equation 5.13 in Section 5.2 for generalized Fourier series. The importance, then, of this formula is that it defines $f(t) = \mathcal{L}^{-1}\{\tilde{f}(s)\}$ in a unique way. Other functions that have the same transform $\tilde{f}(s)$ differ from $f(t)$ only in their values at isolated points; they cannot differ from $f(t)$ over an entire interval $a \leq t \leq b$. In compliance with this anticipated formula, we adopt the procedure of always choosing a continuous function $\mathcal{L}^{-1}\{\tilde{f}(s)\}(t)$ for given $\tilde{f}(s)$ or, when this is not possible, a piecewise continuous function.

The Laplace transform and its inverse are linear operators. Some of their simple properties are summarized below.

One of two shifting properties is

$$\mathcal{L}\{e^{at}f(t)\}(s) = \tilde{f}(s-a), \tag{10.3a}$$

$$\mathcal{L}^{-1}\{\tilde{f}(s-a)\}(t) = e^{at}f(t). \tag{10.3b}$$

(See Exercise 1 for proof of this result.) It states that multiplication by an exponential function e^{at} in the time domain is equivalent to a translation in the s domain. For example, since $\mathcal{L}\{\cos 2t\} = s/(s^2+4)$, property 10.3a implies that

$$\mathcal{L}\{e^{3t}\cos 2t\} = \frac{s-3}{(s-3)^2+4}.$$

The other shifting property is

$$\mathcal{L}\{f(t)h(t-a)\}(s) = e^{-as}\mathcal{L}\{f(t+a)\}(s), \tag{10.4a}$$

$$\mathcal{L}^{-1}\{e^{-as}\tilde{f}(s)\}(t) = f(t-a)h(t-a), \tag{10.4b}$$

where $h(t-a)$ is the Heaviside unit step function. It has value 0 when $t < a$ and value 1 when $t > a$. (See Exercise 2 for proof of these properties.) The second of these implies that multiplication by an exponential function e^{-as} in the s domain is equivalent to a translation in the time domain. Graphs of $f(t)$ and $f(t-a)h(t-a)$ are shown in Figure 10.1.

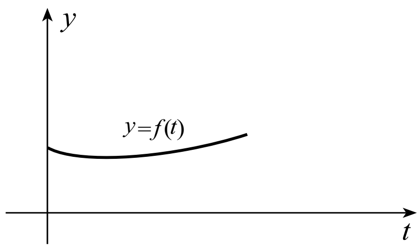


Figure 10.1a

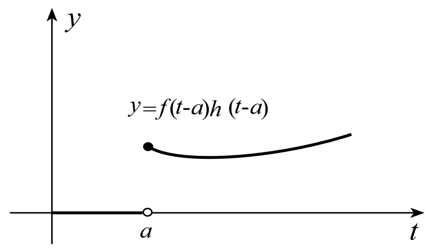


Figure 10.1b

Example 10.1 Find the Laplace transform for the ramp function in Figure 10.2.

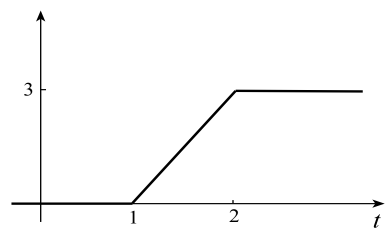


Figure 10.2

Solution The function is continuous, but because it is defined differently on the intervals $0 \le t \le 1$, $1 < t \le 2$, and $t > 2$, it can be represented efficiently in terms of Heaviside functions (except for its values at $t = 1$ and $t = 2$),

$$\begin{aligned} f(t) &= 3(t-1)[h(t-1) - h(t-2)] + 3h(t-2) \\ &= 3(t-1)\,h(t-1) + (6-3t)\,h(t-2). \end{aligned}$$

We can now use property 10.4a to find its Laplace transform,

$$F(s) = e^{-s}\mathcal{L}\{3t\} + e^{-2s}\mathcal{L}\{6 - 3(t+2)\} = \frac{3e^{-s}}{s^2} - e^{-2s}\left(\frac{3}{s^2}\right).\bullet$$

Example 10.2 Find the inverse transform for $F(s) = \dfrac{e^{-s}}{s^2 - s}$.

Solution Partial fractions give $\dfrac{1}{s(s-1)} = \dfrac{1}{s-1} - \dfrac{1}{s}$, and therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = e^t - 1.$$

Property 10.4b now gives $\mathcal{L}^{-1}\{F(s)\} = (e^{t-1} - 1)h(t-1)$.●

Multiplication by t^n

The following theorem indicates that multiplying a function $f(t)$ by t^n results in its transform being differentiated n times.

Theorem 10.1 If $f(t)$ is piecewise continuous on every finite interval and of exponential order α, and n is a positive integer, then

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}[\mathcal{L}\{f(t)\}(s)], \tag{10.5a}$$

$$\mathcal{L}^{-1}\{\tilde{f}^{(n)}(s)\}(t) = (-1)^n t^n \mathcal{L}^{-1}\{\tilde{f}(s)\}(t). \tag{10.5b}$$

(See Exercise 3 for a proof.)

Example 10.3 Confirm the Laplace transform for $t \sin at$ in Table 10.1.

Solution According to equation 10.5a,

$$\mathcal{L}\{t \sin at\} = -\frac{d}{ds}[\mathcal{L}\{\sin at\}] = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}.\bullet$$

Example 10.4 Find the inverse Laplace transform for $\tilde{f}(s) = \dfrac{s+1}{(s^2+2+5)^2}$.

Solution We note that

$$\frac{d}{ds}\left(\frac{1}{s^2+2s+5}\right) = \frac{-(2s+2)}{(s^2+2s+5)^2}.$$

Consequently, using property 10.5b,

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+2s+5)^2}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+2s+5}\right)\right\} = -\frac{1}{2}(-t)\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$
$$= \frac{t}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{t}{2}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{t}{4}e^{-t}\sin 2t.\bullet$$

Periodic Functions

When a function is periodic with period p, the improper integral in equation 10.1 may be replaced by an integral over $0 \le t \le p$:

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt. \tag{10.6}$$

(See Exercise 4 for proof.)

Example 10.5 Find the Laplace transform of the function in Figure 10.3 with period 2.

Solution Using property 10.6,

$$F(s) = \frac{1}{1-e^{-2s}}\int_0^2 (1-t)e^{-st}\,dt.$$

Integration by parts gives

$$F(s) = \frac{1}{1-e^{-2s}}\left\{\frac{(t-1)}{s}e^{-st} + \frac{1}{s^2}e^{-st}\right\}_0^2 = \frac{1+e^{-2s}}{s(1-e^{-2s})} - \frac{1}{s^2}.$$

We can avoid integration by parts by interpreting the integral over the interval $0 \le t \le 2$ as the Laplace transform of the function in Figure 10.4. Its Laplace transform is

$$\begin{aligned}\mathcal{L}\{(1-t)[h(t)-h(t-2)]\} &= \mathcal{L}\{(1-t)\,h(t)\} + \mathcal{L}\{(t-1)\,h(t-2)\}\\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-2s}\mathcal{L}\{t+1\}\\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s^2}+\frac{1}{s}\right).\end{aligned}$$

Hence,

$$F(s) = \frac{1}{1-e^{-2s}}\left[\frac{1}{s} - \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\right] = \frac{1+e^{-2s}}{s(1-e^{-2s})} - \frac{1}{s^2}. \bullet$$

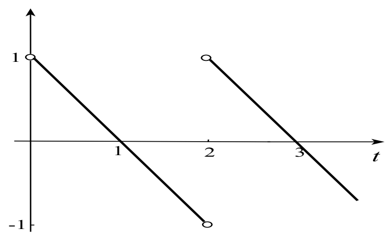


Figure 10.3

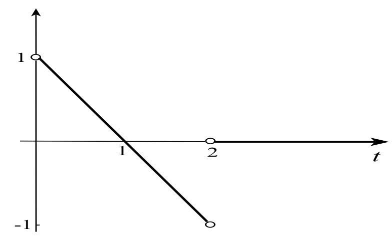


Figure 10.4

The following theorem and its corollary eliminate much of the work when Laplace transforms are applied to initial boundary value problems.

Theorem 10.2 Suppose $f(t)$ is continuous with a piecewise continuous first derivative on every finite interval $0 \le t \le T$. If $f(t)$ is $O(e^{\alpha t})$, then $\mathcal{L}\{f'(t)\}$ exists for $s > \alpha$ and

$$\mathcal{L}\{f'(t)\}(s) = s\tilde{f}(s) - f(0). \tag{10.7a}$$

Proof If t_j, $j = 1, \ldots, n$ denote the discontinuities of $f(t)$ in $0 \le t \le T$, then

$$\int_0^T e^{-st} f'(t)\,dt = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} e^{-st} f'(t)\,dt,$$

where $t_0 = 0$ and $t_{n+1} = T$. Since $f'(t)$ is continuous on each subinterval, we may integrate by parts on each subinterval:

$$\int_0^T e^{-st} f'(t)\,dt = \sum_{j=0}^{n}\left[\left\{e^{-st} f(t)\right\}_{t_j}^{t_{j+1}} + s\int_{t_j}^{t_{j+1}} e^{-st} f(t)\,dt\right].$$

Because $f(t)$ is continuous, $f(t_j+) = f(t_j-)$, $j = 1, \ldots, n$, and therefore

$$\int_0^T e^{-st} f'(t)\,dt = -f(0) + e^{-sT} f(T) + s\int_0^T e^{-st} f(t)\,dt.$$

Thus,

$$\begin{aligned}
\mathcal{L}\{f'(t)\}(s) &= \int_0^\infty e^{-st} f'(t)\,dt = \lim_{T\to\infty}\int_0^T e^{-st} f'(t)\,dt \\
&= \lim_{T\to\infty}\left[-f(0) + e^{-sT} f(T) + s\int_0^T e^{-st} f(t)\,dt\right] \\
&= s\tilde{f}(s) - f(0) + \lim_{T\to\infty} e^{-sT} f(T),
\end{aligned}$$

provided the limit on the right exists. Since $f(t)$ is $O(e^{\alpha t})$, there exist M and $\overline{T}$ such that for $t > \overline{T}$, $|f(t)| < Me^{\alpha t}$. Thus, for $T > \overline{T}$,

$$e^{-sT}|f(T)| < e^{-sT} M e^{\alpha T} = Me^{(\alpha - s)T},$$

which approaches zero as T approaches infinity (provided $s > \alpha$). Consequently, $\mathcal{L}\{f'(t)\}(s) = s\tilde{f}(s) - f(0)$. ∎

This result is easily extended to second-order derivatives. The extension is stated in the following corollary and is verifed in Exercise 5. For extensions when $f(t)$ is only piecewise continuous, see Exercise 36.

Corollary Suppose $f(t)$ and $f'(t)$ are continuous and $f''(t)$ is piecewise continuous on every finite interval $0 \le t \le T$. If $f(t)$ and $f'(t)$ are $O(e^{\alpha t})$, then $\mathcal{L}\{f''(t)\}$ exists for $s > \alpha$, and

$$\mathcal{L}\{f''(t)\}(s) = s^2 \tilde{f}(s) - sf(0) - f'(0). \tag{10.7b}$$

The following examples use these properties and at the same time indicate how Laplace transforms reduce ordinary differential equations to algebraic problems.

Example 10.6 Solve the initial value problem

$$y'' - 2y' + y = 2te^t, \quad y(0) = y'(0) = 0.$$

Solution When we take Laplace transforms of both sides of the differential equation and use linearity of the operator,

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 2\mathcal{L}\{te^t\}.$$

Properties 10.7a,b and 10.3a yield

$$[s^2\tilde{y}(s) - sy(0) - y'(0)] - 2[s\tilde{y}(s) - y(0)] + \tilde{y}(s) = \frac{2}{(s-1)^2}.$$

We now use the initial conditions $y(0) = y'(0) = 0$,

$$s^2\tilde{y} - 2s\tilde{y} + \tilde{y} = \frac{2}{(s-1)^2},$$

and solve this equation for $\tilde{y}(s)$,

$$\tilde{y}(s) = \frac{2}{(s-1)^4}.$$

The required function $y(t)$ can now be obtained by taking the inverse Laplace transform of $\tilde{y}(s)$,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} && \text{(by linearity)} \\ &= 2e^t\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} && \text{(by 10.3b)} \\ &= 2e^t\left(\frac{t^3}{3!}\right) && \text{(from Table 10.1)} \\ &= \frac{1}{3}t^3e^t. \bullet \end{aligned}$$

Example 10.7 Solve the initial value problem

$$y'' + 4y = 3\cos 2t\, h(t-\pi), \quad y(0) = 1, \quad y'(0) = 0.$$

Solution When we take the Laplace transform of both sides of the differential equation and use the initial conditions,

$$[s^2\tilde{y} - s(1) - 0] + 4\tilde{y} = 3\mathcal{L}\{\cos 2t\, h(t-\pi)\}.$$

We now use property 10.4a,

$$(s^2+4)\tilde{y} = s + 3e^{-\pi s}\mathcal{L}\{\cos 2(t+\pi)\} = s + 3e^{-\pi s}\mathcal{L}\{\cos 2t\} = s + \frac{3e^{-\pi s}}{s^2+4}.$$

When we divide by s^2+4,

$$\tilde{y}(s) = \frac{3se^{-\pi s}}{(s^2+4)^2} + \frac{s}{s^2+4}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t}{4}\sin 2t \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t,$$

property 10.4b gives

$$y(t) = \frac{3}{4}(t-\pi)\sin 2(t-\pi)\,h(t-\pi) + \cos 2t = \frac{3}{4}(t-\pi)\sin 2t\,h(t-\pi) + \cos 2t. \bullet$$

When solving ordinary differential equations by means of Laplace transforms, considerable emphasis is placed on partial fraction decompositions of transform functions $\tilde{y}(s)$, and rightly so, because for ODEs, transform functions are often rational functions of s. Once the transform is decomposed into constituent fractions, and provided the decomposition is not too complicated, inverse transforms of individual terms can be located in tables. Unfortunately, transforms arising from PDEs are seldom rational functions, and there is little point in our giving a detailed discussion of partial fractions.

It is often necessary in applications to find the inverse transform of the product of two functions $\tilde{f}(s)\tilde{g}(s)$ when inverse transforms of $\tilde{f}(s)$ and $\tilde{g}(s)$ are known. *Convolutions* are defined for this purpose. The **convolution** of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = (f*g)(t) = \int_0^t f(u)g(t-u)\,du. \tag{10.8}$$

Some of the properties of convolutions are developed in Exercise 6. The importance of convolutions lies in the following theorem.

Theorem 10.3 If $f(t)$ and $g(t)$ are $O(e^{\alpha t})$ and piecewise continuous on every finite interval $0 \le t \le T$, then

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}, \quad s > \alpha. \tag{10.9a}$$

Proof If $\tilde{f}(s) = \mathcal{L}\{f(t)\}$ and $\tilde{g}(s) = \mathcal{L}\{g(t)\}$, then

$$\tilde{f}(s)\tilde{g}(s) = \int_0^\infty e^{-su}f(u)\,du \int_0^\infty e^{-s\tau}g(\tau)\,d\tau = \int_0^\infty\int_0^\infty e^{-s(u+\tau)}f(u)g(\tau)\,d\tau\,du.$$

Suppose we change variables of integration in the inner integral with respect to τ by setting $t = u + \tau$. Then

$$\tilde{f}(s)\tilde{g}(s) = \int_0^\infty\int_u^\infty e^{-st}f(u)g(t-u)\,dt\,du.$$

Now $g(t)$ is defined only for $t \ge 0$. If we set $g(t) \equiv 0$ for $t < 0$, we may write

$$\tilde{f}(s)\tilde{g}(s) = \lim_{T\to\infty}\int_0^T\int_0^\infty e^{-st}f(u)g(t-u)\,dt\,du.$$

We would like to interchange orders of integration, but to do so requires that the inner integral converge uniformly with respect to u. To verify that this is indeed the case, we note that since $f(t)$ and $g(t)$ are $O(e^{\alpha t})$ and piecewise continuous on every finite interval $0 \le t \le T$, there exists a constant M such that for all $t \ge 0$, $|f(t)| \le Me^{\alpha t}$ and $|g(t)| \le Me^{\alpha t}$. For each $u \ge 0$, we therefore have $|e^{-st}f(u)g(t-u)| < M^2e^{-st}e^{\alpha u}e^{\alpha(t-u)} = M^2e^{-t(s-\alpha)}$. Thus,

$$\left|\int_0^\infty e^{-st}f(u)g(t-u)\,dt\right| < M^2\int_0^\infty e^{-t(s-\alpha)}dt = M^2\left\{\frac{e^{-t(s-\alpha)}}{\alpha-s}\right\}_0^\infty = \frac{M^2}{s-\alpha},$$

provided $s > \alpha$, and the improper integral is uniformly convergent with respect to u. The order of integration in the expression for $\tilde{f}(s)\tilde{g}(s)$ may therefore be interchanged, and we obtain

$$\begin{aligned}\tilde{f}(s)\tilde{g}(s) &= \lim_{T\to\infty}\int_0^\infty e^{-st}\int_0^T f(u)g(t-u)\,du\,dt\\ &= \lim_{T\to\infty}\left[\int_0^T e^{-st}\int_0^T f(u)g(t-u)\,du\,dt + \int_T^\infty e^{-st}\int_0^T f(u)g(t-u)\,du\,dt\right].\end{aligned}$$

Since

$$\begin{aligned}\left|\int_T^\infty e^{-st}\int_0^T f(u)g(t-u)\,du\,dt\right| &< \int_T^\infty\int_0^T M^2e^{-t(s-\alpha)}du\,dt\\ &= M^2T\left\{\frac{e^{-t(s-\alpha)}}{\alpha-s}\right\}_T^\infty] = \frac{M^2Te^{-T(s-\alpha)}}{s-\alpha},\end{aligned}$$

provided $s > \alpha$, it follows that

$$\lim_{T\to\infty}\int_T^\infty e^{-st}\int_0^T f(u)g(t-u)\,du\,dt = 0.$$

Further, due to the fact that $g(t-u) = 0$ for $u > t$, we may write, for $T > t$,

$$\int_0^T e^{-st}\int_0^T f(u)g(t-u)\,du\,dt = \int_0^T e^{-st}\int_0^t f(u)g(t-u)\,du\,dt = \int_0^T e^{-st}f*g\,dt.$$

Thus,

$$\tilde{f}(s)\tilde{g}(s) = \lim_{T\to\infty}\int_0^T e^{-st}f*g\,dt = \mathcal{L}\{f*g\}. \blacksquare$$

More important in practice is the inverse of equation 10.9a.

Corollary If $\mathcal{L}^{-1}\tilde{f}(s)\}(t) = f(t)$ and $\mathcal{L}^{-1}\{\tilde{g}(s)\}(t) = g(t)$, where $f(t)$ and $g(t)$ are $O(e^{\alpha t})$ and piecewise continuous on every finite interval, then

$$\mathcal{L}^{-1}\{\tilde{f}(s)\tilde{g}(s)\}(t) = \int_0^t f(u)g(t-u)\,du. \tag{10.9b}$$

As an example to illustrate this corollary, consider finding $\mathcal{L}^{-1}\{2/[s^2(s^2+4)]\}$. Since $\mathcal{L}^{-1}\{2/(s^2+4)\} = \sin 2t$ and $\mathcal{L}^{-1}\{1/s^2\} = t$, we can state that the inverse transform of $2/[s^2(s^2+4)]$ is

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^2(s^2+4)}\right\} &= \int_0^t u\sin 2(t-u)\,du = \left\{\frac{u}{2}\cos 2(t-u) + \frac{1}{4}\sin 2(t-u)\right\}_0^t \\ &= \frac{t}{2} - \frac{1}{4}\sin 2t.\end{aligned}$$

Convolutions are particularly important in solving ODEs that contain unspecified nonhomogeneities.

Example 10.8 Find the solution of the initial value problem

$$y'' + 2y' - y = f(t), \quad y(0) = A, \quad y'(0) = B$$

for arbitrary constants A and B and an arbitrary function $f(t)$.

Solution When we take Laplace transforms,

$$[s^2\tilde{y} - As - B] + 2[s\tilde{y} - A] - \tilde{y} = \tilde{f}(s),$$

and solve for $\tilde{y}(s)$,

$$\tilde{y}(s) = \frac{\tilde{f}(s)}{s^2+2s-1} + \frac{As+B+2A}{s^2+2s-1}.$$

To find the inverse transform of this function, we first note that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-2}\right\} = \frac{1}{\sqrt{2}}e^{-t}\sinh\sqrt{2}t.$$

Convolution property 10.9b on the first term of $\tilde{y}(s)$ now yields

$$\begin{aligned}y(t) &= \int_0^t f(u)\frac{1}{\sqrt{2}}e^{-(t-u)}\sinh\sqrt{2}(t-u)\,du + \mathcal{L}^{-1}\left\{\frac{A(s+1)+(B+A)}{(s+1)^2-2}\right\} \\ &= \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sinh\sqrt{2}(t-u)\,du + e^{-t}\mathcal{L}^{-1}\left\{\frac{As+(B+A)}{s^2-2}\right\} \\ &= \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sinh\sqrt{2}(t-u)\,du + e^{-t}\left(A\cosh\sqrt{2}t + \frac{B+A}{\sqrt{2}}\sinh\sqrt{2}t\right).\bullet\end{aligned}$$

EXERCISES 10.1

1. (a) Verify shifting property 10.3.

(b) Use 10.3a and Table 10.1 to calculate Laplace transforms for the following:

(i) $f(t) = t^3e^{-5t}$ (ii) $f(t) = e^{-t}\cos 2t + e^{3t}\sin 2t$ (iii) $f(t) = e^{at}\cosh 4t - e^{-at}\sinh 4t$

(c) Use 10.3b and Table 10.1 to calculate inverse Laplace transforms for the following:

(i) $\tilde{f}(s) = \dfrac{1}{s^2-2s+5}$ (ii) $\tilde{f}(s) = \dfrac{1}{\sqrt{s+3}}$ (iii) $\tilde{f}(s) = \dfrac{s}{s^2+4s+1}$

2. (a) Verify shifting property 10.4.
(b) Use 10.4a and Table 10.1 to calculate Laplace transforms for the following:

(i) $f(t) = \begin{cases} 0, & 0 < t < 3 \\ t-2, & t > 3 \end{cases}$ (ii) $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & t > a \end{cases}$

(iii) $f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & t > a \end{cases}$ (iv) $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & a < t < b \\ 0, & t > b \end{cases}$

(c) Use 10.4b and Table 10.1 to calculate inverse Laplace transforms for the following:

$$\text{(i)} \quad \tilde{f}(s) = \frac{e^{-2s}}{s^2} \qquad \text{(ii)} \quad \tilde{f}(s) = \frac{e^{-3s}}{s^2+1} \qquad \text{(iii)} \quad \tilde{f}(s) = \frac{se^{-5s}}{s^2-2}$$

3. Verify property 10.5.

4. (a) Verify equation 10.6.
(b) Find Laplace transforms for the following functions:

(i) $f(t) = t, \quad 0 < t < a, \quad f(t+a) = f(t)$

(ii) $f(t) = \begin{cases} 1, & 0 < t < a \\ -1, & a < t < 2a \end{cases}, \quad f(t+2a) = f(t)$

(iii) $f(t) = |\sin at|$

5. Verify equation 10.7b.

6. Verify the following properties for convolutions:

$$f * g = g * f \tag{10.10a}$$

$$f * (kg) = (kf) * g = k(f * g), \quad k = \text{constant}, \tag{10.10b}$$

$$(f * g) * h = f * (g * h) \tag{10.10c}$$

$$f * (g + h) = f * g + f * h \tag{10.10d}$$

In Exercises 7–10 use convolutions to find the inverse transform for the function.

7. $\tilde{f}(s) = \dfrac{1}{s(s+1)}$

8. $\tilde{f}(s) = \dfrac{1}{(s^2+1)(s^2+4)}$

9. $\tilde{f}(s) = \dfrac{s}{(s+4)(s^2-2)}$

10. $\tilde{f}(s) = \dfrac{s}{(s^2-4)(s^2-9)}$

In Exercises 11–16 find the Laplace transform of the function.

11. $f(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$

12. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2t, & t > 1 \end{cases}$

13. $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}, \quad f(t+2a) = f(t)$

14. $f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & a < t < 2a \end{cases}, \quad f(t+2a) = f(t)$

15. $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & t > a \end{cases}$

16. $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & a < t < a+1 \\ 0, & t > a+1 \end{cases}$

In Exercises 17–26 find the inverse Laplace transform for the function.

17. $\tilde{f}(s) = \dfrac{s}{s^2 - 3s + 2}$

18. $\tilde{f}(s) = \dfrac{4s + 1}{(s^2 + s)(4s^2 - 1)}$

19. $\tilde{f}(s) = \dfrac{e^{-3s}}{s + 5}$

20. $\tilde{f}(s) = \dfrac{e^{-2s}}{s^2 + 3s + 2}$

21. $\tilde{f}(s) = \dfrac{1}{s^3 + 1}$

22. $\tilde{f}(s) = \dfrac{5s - 2}{3s^2 + 4s + 8}$

23. $\tilde{f}(s) = \dfrac{e^{-s}(1 - e^{-s})}{s(s^2 + 1)}$

24. $\tilde{f}(s) = \dfrac{s}{(s + 1)^5}$

25. $\tilde{f}(s) = \dfrac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$

26. $\tilde{f}(s) = \dfrac{s^2}{(s^2 - 4)^2}$

In Exercises 27–33 solve the differential equation.

27. $y'' + 2y' - y = e^t, \quad y(0) = 1, \quad y'(0) = 2$

28. $y'' + y = 2e^{-t}, \quad y(0) = y'(0) = 0$

29. $y'' + 2y' + y = t, \quad y(0) = 0, \quad y'(0) = 1$

30. $y''' - 3y'' + 3y' - y = t^2 e^t, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$

31. $y'' + 9y = \cos 2t, \quad y(0) = 1, \quad y(\pi/2) = -1$

32. $y''' - 3y'' + 3y' - y = t^2 e^t$

33. $y'' - a^2 y = f(t)$

34. Verify that the Laplace transform of a function $f(t)$ that is piecewise continuous on every finite interval $0 \le t \le T$ and is $O(e^{\alpha t})$ exists for $s > \alpha$.

35. (a) Prove that when n is a nonnegative integer, t^n is $O(e^{\epsilon t})$ for every $\epsilon > 0$.
(b) Prove that when $f(t)$ is $O(e^{\alpha t})$, $t^n f(t)$ is $O(e^{(\alpha+\epsilon)t})$ for every $\epsilon > 0$.

36. (a) Let $f(t)$ be $O(e^{\alpha t})$ and be continuous for $t \ge 0$ except for a finite discontinuity at $t = t_0 > 0$; and let $f'(t)$ be piecewise continuous on every finite interval $0 \le t \le T$. Show that

$$\mathcal{L}\{f'(t)\} = s\tilde{f}(s) - f(0) - e^{-st_0}[f(t_0+) - f(t_0-)].$$

(b) What is the result in part (a) if $t_0 = 0$?

37. Let $f(t)$ and $f'(t)$ be $O(e^{\alpha t})$, let $f'(t)$ be piecewise continuous on every finite interval $0 \le t \le T$, and let $f(t)$ have only a finite number of finite discontinuities for $t \ge 0$. Verify the **initial value theorem**,

$$\lim_{s\to\infty} s\tilde{f}(s) = \lim_{t\to 0^+} f(t).$$

Assume the result that $\lim\limits_{s\to\infty} \tilde{f}(s) = 0$ for functions that are piecewise continuous and of exponential order.

§10.2 Laplace Transform Solutions for Problems on Unbounded Domains

In this section we apply Laplace transforms to initial boundary value problems on unbounded domains. Such problems do not require the complex inversion formula of Section 10.3. We begin with a heat conduction problem on a semi-infinite interval.

Example 10.9 Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0, \tag{10.11a}$$

$$U(0,t) = \overline{U}, \quad t > 0, \tag{10.11b}$$

$$U(x,0) = 0, \quad x > 0, \tag{10.11c}$$

for temperature in a semi-infinite rod that is initially at temperature 0°C. For time $t > 0$, its end $x = 0$ is held at constant temperature $\overline{U}$.

Solution When we take Laplace transforms of the PDE and use initial condition 10.11c, we obtain

$$s\tilde{U} = k\mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\}.$$

Since the integration with respect to t in the Laplace transform and the differentiation with respect to x are independent, we interchange the order of operations on the right,

$$s\tilde{U} = k\frac{\partial^2 \tilde{U}}{\partial x^2}.$$

Because only derivatives with respect to x remain, we replace the partial derivative with an ordinary derivative,

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = 0, \quad x > 0. \tag{10.12a}$$

This ordinary differential equation is subject to the transform of condition 10.11b,

$$\tilde{U}(0,s) = \frac{\overline{U}}{s}. \tag{10.12b}$$

For problems on finite domains, we have found it convenient to express general solutions of equations like 10.12a in terms of hyperbolic functions. On infinite and semi-infinite intervals, it is advantageous to use exponential functions,

$$\tilde{U}(x,s) = Ae^{\sqrt{s/k}x} + Be^{-\sqrt{s/k}x}. \tag{10.13}$$

Because $U(x,t)$ must remain bounded as x becomes infinite, so also must $\tilde{U}(x,s)$. We therefore set $A = 0$, in which case condition 10.12b requires $B = \overline{U}/s$. Thus,

$$\tilde{U}(x,s) = \frac{\overline{U}}{s}e^{-\sqrt{s/k}x}. \tag{10.14}$$

The inverse Laplace transform of this function is found in tables

$$U(x,t) = \overline{U}\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s/k}x}}{s}\right\} = \overline{U}\,\text{erfc}\left(\frac{x}{2\sqrt{kt}}\right), \tag{10.15}$$

where erfc(x) is the complementary error function

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du. \tag{10.16}$$

Notice that for any $x > 0$ and any $t > 0$, temperature $U(x, t)$ is positive. This indicates that the abrupt change in temperature at the end $x = 0$ from 0°C to $\overline{U}$ is felt instantaneously at every point in the rod. In other words, energy is "transmitted" infinitely fast along the rod, a property of the heat equation that we mentioned in Section 6.6.•

When $U(0, t)$ is a function of time in this example, say $U(0, t) = f_1(t)$, transform 10.14 is replaced by

$$\tilde{U}(x, s) = \tilde{f}_1(s) e^{-\sqrt{s/k}x}. \tag{10.17}$$

Because $\mathcal{L}^{-1}\{e^{-a\sqrt{s}}\} = [a/(2\sqrt{\pi t^3})]e^{-a^2/(4t)}$, it follows by convolution property 10.9b that

$$\begin{aligned} U(x, t) &= \int_0^t f_1(t - u) \frac{x}{2\sqrt{k\pi u^3}} e^{-x^2/(4ku)}\, du \\ &= \frac{x}{2\sqrt{k\pi}} \int_0^t u^{-3/2} f_1(t - u) e^{-x^2/(4ku)}\, du \end{aligned} \tag{10.18a}$$

or, alternatively, that

$$U(x, t) = \frac{x}{2\sqrt{k\pi}} \int_0^t (t - u)^{-3/2} f_1(u) e^{-x^2/[4k(t-u)]}\, du. \tag{10.18b}$$

In the next example we illustrate how a semi-infinite string falling under gravity reacts to one end being fixed.

Example 10.10 A semi-infinite string is supported from below so that it lies motionless on the x-axis. At time $t = 0$, the support is removed and gravity is permitted to act on the string. If the end $x = 0$ is fixed at the origin, find the displacement of the string.

Solution The initial boundary value problem is

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2} - g, \quad x > 0, \quad t > 0, \tag{10.19a}$$

$$y(0, t) = 0, \quad t > 0, \tag{10.19b}$$

$$y(x, 0) = 0, \quad x > 0, \tag{10.19c}$$

$$y_t(0, t) = 0, \quad x > 0, \tag{10.19d}$$

where $g = 9.81$. When we apply the Laplace transform to the PDE and use the initial conditions,

$$s^2 \tilde{y} = c^2 \frac{d^2 \tilde{y}}{dx^2} - \frac{g}{s}.$$

Thus, $\tilde{y}(x, s)$ must satisfy the ODE

$$\frac{d^2 \tilde{y}}{dx^2} - \frac{s^2}{c^2} \tilde{y} = \frac{g}{c^2 s}, \quad x > 0, \tag{10.20a}$$

subject to the transform of condition 10.19b,

$$\tilde{y}(0, s) = 0. \tag{10.20b}$$

A general solution of ODE 10.20a is

$$\tilde{y}(x, s) = Ae^{sx/c} + Be^{-sx/c} - \frac{g}{s^3}.$$

For this function to remain bounded as $x \to \infty$, we must set $A = 0$, in which case boundary condition 10.20b requires $B = g/s^3$. Hence,

$$\tilde{y}(x, s) = -\frac{g}{s^3}(1 - e^{-sx/c}). \tag{10.21}$$

The inverse transform of this function is

$$y(x, t) = -\frac{gt^2}{2} + \frac{g}{2}\left(t - \frac{x}{c}\right)^2 h\left(t - \frac{x}{c}\right), \tag{10.22}$$

where $h(t - x/c)$ is the Heaviside unit step function. What this says is that a point x in the string falls freely under gravity for $0 < t < x/c$, after which it falls with constant velocity $-gx/c$ [since for $t > x/c$, $y(x, t) = (g/2)(-2xt/c + x^2/c^2)$]. A picture of the string at any given time t_0 is shown in Figure 10.5. It is parabolic for $0 < x < ct_0$ and horizontal for $x > ct_0$. As t_0 increases, the parabolic portion lengthens and the horizontal section drops.●

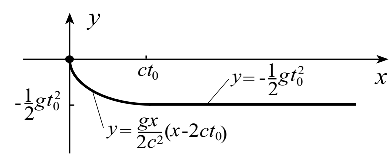


Figure 10.5

EXERCISES 10.2

Part A Heat Conduction

1. (a) Solve the following heat conduction problem when heat flow across the end $x = 0$ of the rod is specified,

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0,$$
$$U_x(0, t) = -\kappa^{-1}Q_0 = \text{constant}, \quad t > 0,$$
$$U(x, 0) = 0, \quad x > 0.$$

(b) Plot the solution on the interval $0 \le x \le 5$ with $k = 10^{-6}$, $\kappa = 10$, and $Q_0 = 1000$ for $t = 10^5$ and $t = 10^6$.

(c) Describe the temperature of the left end of the rod.

2. Show that every solution $U(x, t)$ of the one-dimensional heat conduction equation

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x, t),$$

which at time $t = 0$ has value $U(x,0) = f(x)$, must have a Laplace transform of the form

$$\tilde{U}(x,s) = Ae^{\sqrt{s/k}x} + Be^{-\sqrt{s/k}x} - \sqrt{\frac{k}{s}} \int_0^x \left[\frac{f(u)}{k} + \frac{\tilde{g}(u,s)}{\kappa}\right] \sinh \sqrt{\frac{s}{k}}(x-u)\, du,$$

where A and B are independent of x. In Exercises 3–6 we use this result to solve various heat conduction problems on infinite and semi-infinite intervals.

3. (a) Use the result of Exercise 2 to solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0,$$
$$U(0,t) = f_1(t), \quad t > 0,$$
$$U(x,0) = U_0 = \text{constant}, \quad x > 0.$$

(b) Simplify the solution when $f_1(t) = \overline{U} = \text{constant}$.

4. (a) Use the result of Exercise 2 to solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0,$$
$$\frac{\partial U(0,t)}{\partial x} = -\frac{f_1(t)}{\kappa}, \quad t > 0,$$
$$U(x,0) = U_0 = \text{constant}, \quad x > 0.$$

(b) Simplify the solution when $f_1(t) = Q_0 = \text{constant}$.

5. (a) Use the result of Exercise 2 to solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0,$$
$$-\kappa\frac{\partial U(0,t)}{\partial x} + \mu U(0,t) = \mu f_1(t), \quad t > 0,$$
$$U(x,0) = 0, \quad x > 0.$$

(b) Simplify the solution when $f_1(t) = U_m = \text{constant}$.

6. (a) Use the result of Exercise 2, and the fact that the transform must remain bounded as $x \to \pm\infty$, to show that the transform of the function satisfying the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$
$$U(x,0) = U_0 h(x), \quad -\infty < x < \infty,$$

must be of the form

$$\tilde{U}(x,s) = \begin{cases} Ae^{\sqrt{s/k}x}, & x < 0 \\ Be^{-\sqrt{s/k}x} + \dfrac{U_0}{2s}(2 - e^{-\sqrt{s/k}x}), & x > 0 \end{cases}.$$

(b) By demanding that the expression for $\tilde{U}(x,s)$ and its first derivative with respect to x agree at $x = 0$, show that

$$\tilde{U}(x,s) = \frac{U_0}{2s}\begin{cases} e^{\sqrt{s/k}x}, & x < 0 \\ 2 - e^{-\sqrt{s/k}x}, & x \geq 0 \end{cases}.$$

(c) Find the inverse transform $U(x,t)$.

Part B Vibrations

7. Show that every solution $y(x,t)$ of the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}$$

that also satisfies the initial conditions

$$y(x,0) = f(x), \qquad y_t(x,0) = g(x)$$

must have a Laplace transform of the form

$$\tilde{y}(x,s) = Ae^{sx/c} + Be^{-sx/c} - \frac{1}{cs}\int_0^x \left[sf(u) + g(u) + \frac{\tilde{F}(u,s)}{\rho} \right] \sinh \frac{s}{c}(x-u)\, du,$$

where A and B are independent of x. In Exercises 8–9 we use this result to solve vibration problems on semi-infinite intervals.

8. At time $t = 0$ a semi-infinite taut string lies motionless along the positive x-axis. If its left end is subjected to vertical motion described by $f_1(t)$ for $t > 0$, find its subsequent displacements.

9. Solve Exercise 8 if $f_1(t)$ represents a force on the end $x = 0$ of the string; that is, replace the Dirichlet condition with the Neumann condition $\partial y(0,t)/\partial x = -\tau^{-1} f_1(t)$.

§10.3 The Complex Inversion Integral

Finding the inverse Laplace transform in Section 10.1 was a matter of organization and tables; we used properties 10.3b and 10.4b (and partial fractions) to organize a given transform $\tilde{f}(s)$ into a form for which the inverse transform could be found in tables. In Section 10.2, for PDEs on infinite and semi-infinite intervals, tables and convolutions were once again prominent. For PDEs on finite domains, however, transform functions are so complicated that their inverses can seldom be found in tables. What we need, then, is a direct method for inverting the Laplace transform. In this section we use the theory of functions of a complex variable to derive such a formula. Appendix D gives a review of those aspects of the theory of complex functions necessary for the remainder of this chapter and the next one. The reader could either take the opportunity to review this material now, or, to refer to it when needed.

We first note that the results in equations 10.3–10.9 remain valid when s is complex; the complex derivation may be somewhat different from its real counterpart, but each result is valid when s is complex.

The following theorem shows that Laplace transforms are analytic functions of the complex variable s.

Theorem 10.4 If $f(t)$ is $O(e^{\alpha t})$ and piecewise continuous on every finite interval $0 \le t \le T$, the Laplace transform $\tilde{f}(s) = \tilde{f}(x+yi)$ of $f(t)$ is an analytic function of s in the half-plane $x > \alpha$.

Proof If the real and imaginary parts of $\tilde{f}(s)$ are denoted by $u(x,y)$ and $v(x,y)$,

$$\tilde{f}(s) = u + vi = \int_0^\infty e^{-(x+yi)t} f(t)\,dt,$$

then

$$u(x,y) = \int_0^\infty e^{-xt}\cos yt\,f(t)\,dt, \qquad v(x,y) = \int_0^\infty -e^{-xt}\sin yt\,f(t)\,dt.$$

To verify the analyticity of $\tilde{f}(s)$, we show that $u(x,y)$ and $v(x,y)$ have continuous first partial derivatives that satisfy the Cauchy-Riemann equations when $x > \alpha$ (see equations D.2). Now,

$$\left\{\begin{array}{l} |e^{-xt}\cos yt\,f(t)| \\ |e^{-xt}\sin yt\,f(t)| \end{array}\right\} \le e^{-xt}|f(t)|,$$

and since $f(t)$ is $O(e^{\alpha t})$, there exist constants M and T such that for all $t > T$, $|f(t)| < Me^{\alpha t}$. Consequently, whenever $x \ge \alpha' > \alpha$ and $t > T$,

$$\left\{\begin{array}{l} |e^{-xt}\cos yt\,f(t)| \\ |e^{-xt}\sin yt\,f(t)| \end{array}\right\} < e^{-xt}Me^{\alpha t} \le Me^{(\alpha-\alpha')t},$$

and

$$\begin{aligned}\left\{\begin{array}{l} |u(x,y)| \\ |v(x,y)| \end{array}\right\} &< \int_0^T e^{-xt}|f(t)|\,dt + \int_T^\infty Me^{(\alpha-\alpha')t}dt \\ &\le \int_0^T e^{-\alpha' t}|f(t)|\,dt + M\left\{\frac{e^{(\alpha-\alpha')t}}{\alpha-\alpha'}\right\}_0^\infty\end{aligned}$$

$$= \int_0^T e^{-\alpha' t}|f(t)|\,dt + \frac{M}{\alpha - \alpha'}.$$

Thus, the integrals representing u and v converge absolutely and uniformly with respect to x and y in the half-plane $x \geq \alpha' > \alpha$. Since $f(t)$ is piecewise continuous, u and v are continuous functions for $x \geq \alpha'$. Now,

$$\int_0^\infty \frac{\partial}{\partial x}[e^{-xt}\cos yt f(t)]\,dt = \int_0^\infty -te^{-xt}\cos yt\, f(t)\,dt$$

and

$$\int_0^\infty \frac{\partial}{\partial y}[-e^{-xt}\sin yt f(t)]\,dt = \int_0^\infty -te^{-xt}\cos yt\, f(t)\,dt.$$

Since $tf(t)$ is $O(e^{(\alpha+\epsilon)t})$ for any $\epsilon > 0$ and is piecewise continuous on every finite interval $0 \leq t \leq T$, a similar argument to that above shows that this integral is absolutely and uniformly convergent with respect to x and y for $x \geq \alpha' > \alpha$. Because $\alpha' > \alpha$ is arbitrary, it follows that this integral converges to a continuous function that is equal to both $\partial u/\partial x$ and $\partial v/\partial y$ for $x > \alpha$. We have shown then, that the first of the Cauchy-Riemann equations $\partial u/\partial x = \partial v/\partial y$ is satisfied for $x > \alpha$. In a similar way, we can show that $\partial u/\partial y = -\partial v/\partial x$, and therefore $\tilde{f}(s)$ is analytic for $x > \alpha$. ∎

To obtain the complex inversion integral for $\mathcal{L}^{-1}\{\tilde{f}(s)\}(t)$, we use the extension of Cauchy's integral formula (see equation D.14) contained in the following theorem.

Theorem 10.5 Let $f(z)$ be a complex function analytic in a domain containing the half-plane $x \geq \gamma$ (Figure 10.6), and let $f(z)$ be $O(z^{-k})$, $(k > 0)$ as $|z| \to \infty$ in that half plane*. Then, if z_0 is any complex number with real part greater than γ,

$$f(z_0) = -\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i}\frac{f(z)}{z - z_0}dz. \qquad (10.23)$$

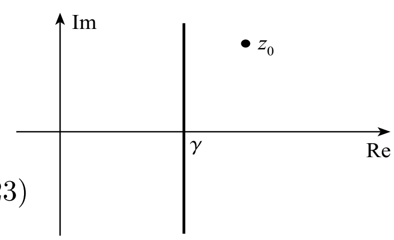


Figure 10.6

When a function $f(t)$ is $O(e^{\alpha t})$, we know that its transform $\tilde{f}(s)$ is analytic for $x > \alpha$ (see Theorem 10.4). It follows from equation 10.23 that when $\tilde{f}(s)$ is $O(s^{-k})$ in a half-plane $x \geq \gamma > \alpha$, we can write $\tilde{f}(s)$ in the form

$$\tilde{f}(s) = -\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i}\frac{\tilde{f}(z)}{z - s}dz$$

for $x > \gamma$. If we formally take inverse transforms of both sides of this equation and interchange the order of integation and $\mathcal{L}^{-1}$, we obtain

$$f(t) = -\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i} -\tilde{f}(z)\mathcal{L}^{-1}\left\{\frac{1}{s - z}\right\}dz = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i} e^{zt}\tilde{f}(z)\,dz.$$

This expression,

* A function $f(z)$ is said to be $O(z^{-k})$ as $|z| \to \infty$ if there exist constants M and r such that $|f(z)z^k| < M$ for $|z| > r$.

$$f(t) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - \beta i}^{\gamma + \beta i} e^{st} \tilde{f}(s)\, ds, \tag{10.24}$$

is called the **complex inversion integral** for the Laplace transform. We could express the contour integral as a complex combination of real improper integrals, but even for the simplest of functions, the integrations involved in the real form are impossible to evaluate (see Exercise 17). Fortunately, in Theorem 10.6 we prove that residues of $e^{st}\tilde{f}(s)$ may be used to evaluate the integral.

Theorem 10.6 Let $\tilde{f}(s)$ be a function for which the inversion integral along a line $x = \gamma$ represents the inverse function $f(t)$, and let $\tilde{f}(s)$ be analytic except for isolated singularities s_n $(n = 1, 2, \ldots)$ in the half-plane $x < \gamma$. Then the series of residues of $e^{st}\tilde{f}(s)$ at $s = s_n$ converges to $f(t)$ for each positive t,

$$f(t) = \text{sum of residues of } e^{st}\tilde{f}(s) \text{ at its singularities,}$$

provided a sequence C_n of contours can be found that satisfies the following properties:

(1) C_n consists of the straight line $x = \gamma$ from $\gamma - \beta_n i$ to $\gamma + \beta_n i$ and some curve Γ_n beginning at $\gamma + \beta_n i$, ending at $\gamma - \beta_n i$, and lying in $x \leq \gamma$,

(2) C_n encloses $s_1, s_2, \ldots, s_n$,

(3) $\lim_{n \to \infty} \beta_n = \infty$,

(4) $\lim_{n \to \infty} \int_{\Gamma_n} e^{st}\tilde{f}(s)\, ds = 0$ (Figure 10.7).

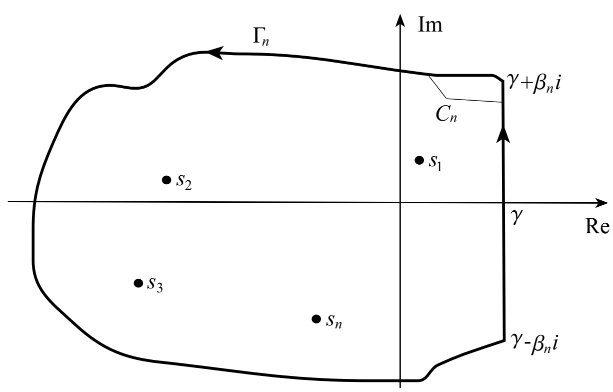


Figure 10.7

Proof Since $e^{st}\tilde{f}(s)$ is analytic in C_n except at $s_1, s_2, \ldots, s_n$, Cauchy's residue theorem states that

$$\begin{aligned}\begin{pmatrix}\text{Sum of residues of}\\ e^{st}\tilde{f}(s) \text{ at } s_1, s_2, \ldots, s_n\end{pmatrix} &= \frac{1}{2\pi i}\oint_{C_n} e^{st}\tilde{f}(s)\, ds \\ &= \frac{1}{2\pi i}\int_{\gamma - \beta_n i}^{\gamma + \beta_n i} e^{st}\tilde{f}(s)\, ds + \frac{1}{2\pi i}\int_{\Gamma_n} e^{st}\tilde{f}(s)\, ds.\end{aligned}$$

When we take limits on n (and use conditions (3) and (4) in the theorem),

$$\begin{pmatrix}\text{Sum of residues of}\\ e^{st}\tilde{f}(s) \text{ at } s_1, s_2, \ldots\end{pmatrix} = \frac{1}{2\pi i} \lim_{n \to \infty} \int_{\gamma - \beta_n i}^{\gamma + \beta_n i} e^{st}\tilde{f}(s)\, ds = f(t). \blacksquare$$

It is not essential, as condition (2) requires, that C_n contain precisely n of the singularities of $\tilde{f}(s)$. In fact, this could be very difficult to accomplish, depending on how the singularities are enumerated. What is essential is that as n increases, the C_n expand to eventually enclose all singularities of $\tilde{f}(s)$.

As a result of this theorem, finding the inverse transform of a function $\tilde{f}(s)$ is now a matter of calculating residues of the function $e^{st}\tilde{f}(s)$ at its singularities. When s_0 is a singularity of $e^{st}\tilde{f}(s)$, the residue at s_0 is defined as the coefficient of $(s-s_0)^{-1}$ in the Laurent expansion of $e^{st}\tilde{f}(s)$ about s_0. It can be found in two ways:

(1) Find the Laurent expansion of $e^{st}\tilde{f}(s)$ about s_0, or at least enough of it to identify the coefficient of $(s-s_0)^{-1}$.

(2) When it is known that s_0 is a pole of order m, the following formula yields the residue of $e^{st}\tilde{f}(s)$ at s_0:

$$\text{Res}[e^{st}\tilde{f}(s), s_0] = \lim_{s\to s_0}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}[(s-s_0)^m e^{st}\tilde{f}(s)]\right\}. \qquad (10.25)$$

Example 10.11 Use Theorem 10.6 to find inverse Laplace transforms when $\tilde{f}(s)$ is equal to (a)$1/s^m$, $m \geq 2$ an integer, (b) $1/(s^2+9)$, (c) $s^2/(s^2+1)^2$. Assume that contours can be found to satisfy Theorem 10.6.

Solution (a) The function $\tilde{f}(s) = 1/s^m$ has a pole of order m at $s=0$. According to equation 10.25, the residue of $e^{st}\tilde{f}(s)$ there is

$$\begin{aligned}\text{Res}\left[\frac{e^{st}}{s^m}, 0\right] &= \lim_{s\to 0}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[\frac{s^m e^{st}}{s^m}\right]\right\} = \frac{1}{(m-1)!}\lim_{s\to 0}\left[\frac{d^{m-1}}{ds^{m-1}}(e^{st})\right]\\ &= \frac{t^{m-1}}{(m-1)!}.\end{aligned}$$

By Theorem 10.6,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^m}\right\} = \frac{t^{m-1}}{(m-1)!}.$$

(b) The function $\tilde{f}(s) = 1/(s^2+9)$ has poles of order 1 at $s = \pm 3i$. The residue of $e^{st}\tilde{f}(s)$ at $3i$ is

$$\text{Res}\left[\frac{e^{st}}{s^2+9}, 3i\right] = \lim_{s\to 3i}\frac{(s-3i)e^{st}}{(s+3i)(s-3i)} = \frac{e^{3it}}{6i} = -\frac{i}{6}e^{3it}.$$

Similarly, $\text{Res}\left[\dfrac{e^{st}}{s^2+9}, -3i\right] = (i/6)e^{-3it}$. By Theorem 10.6, then,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = -\frac{i}{6}e^{3it} + \frac{i}{6}e^{-3it} = \frac{1}{3}\sin 3t.$$

(c) The function $\tilde{f}(s) = s^2/(s^2+1)^2$ has poles of order 2 at $s = \pm i$. The residue of $e^{st}\tilde{f}(s)$ at i is

$$\begin{aligned}\text{Res}\left[\frac{s^2 e^{st}}{(s^2+1)^2}, i\right] &= \lim_{s\to i}\frac{d}{ds}\left[\frac{(s-i)^2 e^{st}s^2}{(s+i)^2(s-i)^2}\right]\\ &= \lim_{s\to i}\left[\frac{(s+i)^2(2se^{st}+ts^2e^{st}) - s^2e^{st}(2)(s+i)}{(s+i)^4}\right] = \frac{1}{4}e^{it}(t-i).\end{aligned}$$

Similarly, $\text{Res}\left[\dfrac{s^2e^{st}}{(s^2+1)^2}, -i\right] = (1/4)e^{-it}(t+i)$. By Theorem 10.6, then,

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+1)^2}\right\} = \frac{1}{4}e^{it}(t-i) + \frac{1}{4}e^{-it}(t+i) = \frac{t}{2}\cos t + \frac{1}{2}\sin t.\bullet$$

More complicated illustrations of Theorem 10.6 are contained in the next example. This example is more typical of problems encountered in Section 10.4, when Laplace transforms are used to solve initial boundary value problems on bounded domains.

Example 10.12 Find inverse transforms for the following functions

$$\text{(a)}\quad \tilde{f}(x,s) = \frac{\sinh\sqrt{s}x}{s\,\sinh\sqrt{s}} \qquad\qquad \text{(b)}\quad \tilde{f}(x,s) = \frac{1}{s^3}(1-\cosh sx) + \frac{\sinh s\sinh sx}{s^3\cosh s}$$

Assume once again that contours can be found to satisfy Theorem 10.6.

Solution (a) The function $\tilde{f}(x,s)$ has isolated singularities at the zeros of $\sinh\sqrt{s}$; that is, when $\sqrt{s} = n\pi i$ or $s = -n^2\pi^2$, $n \geq 0$ an integer. To determine the nature of the singularity at $s = 0$, we find the Laurent expansion of $\tilde{f}(x,s)$ about $s = 0$. We do this with expansions of the hyperbolic functions,

$$\tilde{f}(x,s) = \frac{1}{s}\left[\frac{\sqrt{s}x + \dfrac{1}{3!}(\sqrt{s}x)^3 + \cdots}{\sqrt{s} + \dfrac{1}{3!}(\sqrt{s})^3 + \cdots}\right] = \frac{1}{s}\left[x + \frac{s}{6}(x^3 - x) + \cdots\right].$$

Consequently, $\tilde{f}(x,s)$ has a pole of order 1 at $s = 0$. The following expansion shows that the residue of $e^{st}\tilde{f}(x,s)$ at this pole is x:

$$\begin{aligned} e^{st}\tilde{f}(x,s) &= \left[1 + st + \frac{(st)^2}{2!} + \cdots\right]\left(\frac{1}{s}\right)\left[x + \frac{s}{6}(x^3 - x) + \cdots\right] \\ &= \frac{1}{s}\left[x + \frac{s}{6}(6xt + x^3 - x) + \cdots\right]. \end{aligned}$$

Because the derivative of $\sinh\sqrt{s}$ does not vanish at the remaining singularities $s = -n^2\pi^2$ $(n > 0)$, these are also poles of order 1, and residues of $e^{st}\tilde{f}(x,s)$ at these poles are given by limit 10.25,

$$\begin{aligned} \text{Res}\left[\frac{e^{st}\sinh\sqrt{s}x}{\sinh\sqrt{s}}, -n^2\pi^2\right] &= \lim_{s\to -n^2\pi^2}(s + n^2\pi^2)e^{st}\frac{\sinh\sqrt{s}x}{s\,\sinh\sqrt{s}} \\ &= e^{-n^2\pi^2 t}\frac{\sinh n\pi x i}{-n^2\pi^2}\lim_{s\to -n^2\pi^2}\frac{s + n^2\pi^2}{\sinh\sqrt{s}}. \end{aligned}$$

L'Hôpital's rule can be used to evaluate this limit, which, combined with the facts that $\sinh i\theta = i\sin\theta$ and $\cosh i\theta = \cos\theta$, gives

$$\begin{aligned} -\frac{i}{n^2\pi^2}e^{-n^2\pi^2 t}\sin n\pi x \lim_{s\to -n^2\pi^2}\frac{1}{\dfrac{1}{2\sqrt{s}}\cosh\sqrt{s}} &= -\frac{2i}{n^2\pi^2}e^{-n^2\pi^2 t}\sin n\pi x\frac{n\pi i}{\cosh n\pi i} \\ = \frac{2}{n\pi}e^{-n^2\pi^2 t}\sin n\pi x\frac{1}{\cos n\pi} &= \frac{2(-1)^n}{n\pi}e^{-n^2\pi^2 t}\sin n\pi x. \end{aligned}$$

Thus, the sum of the residues of $e^{st}\tilde{f}(x,s)$ at its singularities is

$$f(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin n\pi x.$$

Transforms of this type arise in heat conduction problems.
(b) This transform has singularities at $s = 0$ and $s = (2n-1)\pi i/2$, n an integer (the zeros of $\cosh s$). The Laurent expansion of $\tilde{f}(x,s)$ about $s = 0$ can be found by expanding the hyperbolic functions in Maclaurin series,

$$\begin{aligned}\tilde{f}(x,s) &= \frac{1}{s^3}\left(1 - 1 - \frac{s^2x^2}{2!} - \frac{s^4x^4}{4!} - \cdots\right) + \frac{1}{s^3}\left(\frac{s + \dfrac{s^3}{3!} + \dfrac{s^5}{5!} + \cdots}{1 + \dfrac{s^2}{2!} + \dfrac{s^4}{4!} + \cdots}\right)\left(sx + \frac{s^3x^3}{3!} + \cdots\right)\\ &= \left(-\frac{x^2}{2s} - \frac{x^4 s}{24} - \cdots\right) + \left(s - \frac{s^3}{3} + \cdots\right)\left(\frac{x}{s^2} + \frac{x^3}{6} + \cdots\right)\\ &= \frac{x}{2s}(2-x) + \frac{xs}{24}(-x^3 + 4x^2 - 8) + \cdots .\end{aligned}$$

Consequently, $\tilde{f}(x,s)$ has a pole of order 1 at $s = 0$. Multiplication of this series by the Maclaurin series for e^{st} gives

$$\begin{aligned}e^{st}\tilde{f}(x,s) &= \left[1 + st + \frac{(st)^2}{2!} + \cdots\right]\left[\frac{x}{2s}(2-x) + \frac{xs}{24}(-x^3 + 4x^2 - 8) + \cdots\right]\\ &= \frac{x}{2s}(2-x) + \frac{xt}{2}(2-x) + \cdots ,\end{aligned}$$

and therefore the residue of $e^{st}\tilde{f}(x,s)$ at $s = 0$ is $x(2-x)/2$. Because the derivative of $\cosh s$ does not vanish at $s = (2n-1)\pi i/2$, these singularities are also poles of order 1, and residues of $e^{st}\tilde{f}(x,s)$ at these poles are given by the limits

$$\begin{aligned}&\lim_{s\to(2n-1)\pi i/2}\left[s - \frac{(2n-1)\pi i}{2}\right]e^{st}\left[\frac{1}{s^3}(1 - \cosh sx) + \frac{\sinh s \sinh sx}{s^3\cosh s}\right]\\ &= \frac{e^{(2n-1)\pi t i/2}}{-(2n-1)^3\pi^3 i/8}\sinh\frac{(2n-1)\pi i}{2}\sinh\frac{(2n-1)\pi x i}{2}\lim_{s\to(2n-1)\pi i/2}\frac{s - (2n-1)\pi i/2}{\cosh s}\\ &= \frac{8e^{(2n-1)\pi t i/2}}{(2n-1)^3\pi^3 i}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{2}\lim_{s\to(2n-1)\pi i/2}\frac{1}{\sinh s}\\ &= \frac{8(-1)^{n+1}e^{(2n-1)\pi t i/2}}{(2n-1)^3\pi^3 i}\sin\frac{(2n-1)\pi x}{2}\frac{1}{\sinh\dfrac{(2n-1)\pi i}{2}}\\ &= -\frac{8e^{(2n-1)\pi t i/2}}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{2}.\end{aligned}$$

The sum of the residues of $e^{st}\tilde{f}(x,s)$ at its singularities is therefore

$$f(x,t) = \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=-\infty}^{\infty}\frac{e^{(2n-1)\pi t i/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2}.$$

To simplify this expression, we separate it into two summations, one over positive n and the other over nonpositive n, and in the latter we set $m = 1 - n$,

$$\begin{aligned}f(x,t) &= \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2}\\ &\quad - \frac{8}{\pi^3}\sum_{n=-\infty}^{0}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2}\\ &= \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2}\\ &\quad - \frac{8}{\pi^3}\sum_{m=1}^{\infty}\frac{e^{[2(1-m)-1]\pi ti/2}}{[2(1-m)-1]^3}\sin\frac{[2(1-m)-1]\pi x}{2}.\end{aligned}$$

If we now replace m by n in the second summation and combine it with the first,

$$\begin{aligned}f(x,t) &= \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2}\\ &\quad - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2}\\ &= \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi ti/2}+e^{-(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2}\\ &= \frac{x}{2}(2-x) - \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cos\frac{(2n-1)\pi t}{2}\sin\frac{(2n-1)\pi x}{2}.\end{aligned}$$

Transforms of this type occur in vibration problems.●

EXERCISES 10.3

In Exercises 1–16 use residues to find the inverse Laplace transform of the given function. Do not verify the existence of contours satisfying the requirements of Theorem 10.6.

1. $\tilde{f}(s) = \dfrac{s}{(s-1)^3}$

2. $\tilde{f}(s) = \dfrac{s}{(s^2+4)^2}$

3. $\tilde{f}(s) = \dfrac{1}{s^2(s+3)}$

4. $\tilde{f}(s) = \dfrac{s^2+2}{(s+1)^2(s-3)^3}$

5. $\tilde{f}(s) = \dfrac{s^2}{(s^2+1)(s^2+4)}$

6. $\tilde{f}(s) = \dfrac{s}{s^2-1}$

7. $\tilde{f}(s) = \dfrac{s^3}{(s^2-4)^3}$

8. $\tilde{f}(s) = \dfrac{1}{(s^2-2s+2)^2}$

9. $\tilde{f}(s) = \dfrac{s-1}{(s^2-2s+2)^2}$

10. $\tilde{f}(s) = \dfrac{s^2}{(s^2-2s+2)^2}$

11. $\tilde{f}(x,s) = \dfrac{1}{s}\left(x - \dfrac{\sinh\sqrt{s}x}{\sinh\sqrt{s}}\right)$

12. $\tilde{f}(x,u,s) = \dfrac{\sinh sx \sinh s(1-u)}{s\,\sinh s}$

13. $\tilde{f}(x,s) = \dfrac{2\sinh sx}{s^3\sinh s}(1-\cosh s) + \dfrac{2}{s^3}(\cosh sx - 1) + \dfrac{x}{s}(1-x)$

14. $\tilde{f}(x,s) = \dfrac{1}{s^3} + \dfrac{\cosh sx}{s^2 \sinh s}$

15. $\tilde{f}(x,s) = \dfrac{\sinh sx}{(4s^2 + \pi^2)\sinh s}$

16. $\tilde{f}(x,s) = \dfrac{\sinh sx}{(s^2 + \pi^2)\sinh s}$

17. We have claimed that to use inversion integral 10.24 directly is usually impossible. Set up the complex combination of real improper integrals for 10.24 when $\tilde{f}(s) = 1/s^2$; that is, express 10.24 in the form

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = I_1 + I_2 i,$$

where I_1 and I_2 are real, improper integrals. Use the line $\gamma = 1$.

§10.4 Applications to Partial Differential Equations on Bounded Domains

Laplace transforms can be used to eliminate the time variable from initial boundary value problems. This reduces the PDE to an ODE or a PDE with one fewer variable. We illustrate with the following examples.

Example 10.13 Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{10.26a}$$

$$U(0,t) = 0, \quad t > 0, \tag{10.26b}$$

$$U(L,t) = 0, \quad t > 0, \tag{10.26c}$$

$$U(x,0) = x, \quad 0 < x < L. \tag{10.26d}$$

Solution When we take Laplace transforms with respect to t on both sides of the PDE and use property 10.7a,

$$s\tilde{U}(x,s) - x = k\frac{\partial^2 \tilde{U}}{\partial x^2}.$$

Thus, $\tilde{U}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = -\frac{x}{k} \tag{10.27a}$$

subject to the transforms of boundary conditions 10.26b,c,

$$\tilde{U}(0,s) = 0, \tag{10.27b}$$

$$\tilde{U}(L,s) = 0. \tag{10.27c}$$

A general solution of the ODE is

$$\tilde{U}(x,s) = C_1 \cosh\sqrt{\frac{s}{k}}x + C_2 \sinh\sqrt{\frac{s}{k}}x + \frac{x}{s},$$

and the boundary conditions require

$$0 = C_1, \qquad 0 = C_1 \cosh\sqrt{\frac{s}{k}}L + C_2 \sinh\sqrt{\frac{s}{k}}L + \frac{L}{s}.$$

From these,

$$\tilde{U}(x,s) = \frac{1}{s}\left(x - \frac{L\sinh\sqrt{s/k}x}{\sinh\sqrt{s/k}L}\right). \tag{10.28}$$

It remains now to find the inverse transform of $\tilde{U}(x,s)$. We do this by calculating residues of $e^{st}\tilde{U}(x,s)$ at the singularities of $\tilde{U}(x,s)$. To discover the nature of the singularity at $s = 0$, we expand $\tilde{U}(x,s)$ in a Laurent series around $s = 0$,

$$\begin{aligned}
\tilde{U}(x,s) &= \frac{1}{s}\left\{x - \frac{L[\sqrt{s/k}x + (\sqrt{s/k}x)^3/3! + \cdots]}{\sqrt{s/k}L + (\sqrt{s/k}L)^3/3! + \cdots}\right\} \\
&= \frac{1}{s}\left[x - \frac{x + sx^3/(6k) + \cdots}{1 + sL^2/(6k) + \cdots}\right] \\
&= \frac{1}{s}\left[\frac{sx(L^2 - x^2)}{6k} + \cdots\right] = \frac{x(L^2 - x^2)}{6k} + \text{terms in } s, s^2, \ldots.
\end{aligned}$$

It follows that $\tilde{U}(x,s)$ has a removable singularity at $s=0$.

The remaining singularities of $\tilde{U}(x,s)$ occur at the zeros of $\sinh\sqrt{s/k}L$; that is, when $\sqrt{s/k}L = n\pi i$ or $s=-n^2\pi^2k/L^2$, n a positive integer. Because the derivative of $\sinh\sqrt{s/k}L$ does not vanish at $s=-n^2\pi^2k/L^2$, this function has zeros of order 1 at $s=-n^2\pi^2k/L^2$. It follows that $\tilde{U}(x,s)$ has poles of order 1 at these singularities, and, according to formula 10.25, the residue of $e^{st}\tilde{U}(x,s)$ at $s=-n^2\pi^2k/L^2$ is

$$\begin{aligned}\text{Res}\left[e^{st}\tilde{U}(x,s), -\frac{n^2\pi^2k}{L^2}\right] &= \lim_{s\to -n^2\pi^2k/L^2}\left(s+\frac{n^2\pi^2k}{L^2}\right)\frac{e^{st}}{s}\left(x-\frac{L\sinh\sqrt{s/k}x}{\sinh\sqrt{s/k}L}\right)\\ &= -\frac{e^{-n^2\pi^2kt/L^2}}{-n^2\pi^2k/L^2}L\sinh\frac{n\pi xi}{L}\lim_{s\to -n^2\pi^2k/L^2}\left(\frac{s+n^2\pi^2k/L^2}{\sinh\sqrt{s/k}L}\right).\end{aligned}$$

L'Hôpital's rule yields

$$\begin{aligned}\text{Res}\left[e^{st}\tilde{U}(x,s), -\frac{n^2\pi^2k}{L^2}\right] &= \frac{iL^3}{n^2\pi^2k}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\lim_{s\to -n^2\pi^2k/L^2}\frac{1}{\frac{L}{2\sqrt{ks}}\cosh\sqrt{s/k}L}\\ &= \frac{2iL^2}{n^2\pi^2k}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\frac{1}{\frac{L}{n\pi ki}\cosh n\pi i}\\ &= \frac{2L}{n\pi}(-1)^{n+1}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.\end{aligned}$$

We sum these residues to find the inverse Laplace transform of $\tilde{U}(x,s)$,

$$U(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.\bullet \tag{10.29}$$

Before proceeding to further problems, some general comments are appropriate:

1. In the above example, the Laplace transform was applied to the time variable to eliminate the time derivative from the PDE and obtain an ODE in $\tilde{U}(x,s)$. The Laplace transform cannot be applied to the space variable x, because the range of x is only $0\le x\le L$. It is the power of finite Fourier transforms to eliminate the space variable, not the Laplace transform.
2. The Laplace transform immediately incorporates the initial condition into the solution, and boundary conditions on $U(x,t)$ become boundary conditions for $\tilde{U}(x,s)$. Contrast this with finite Fourier transforms, which immediately incorporate boundary conditions and use the initial condition on $U(x,t)$ as an initial condition for $\tilde{U}(\lambda_n,t)$.
3. Mathematically, the solution is not complete because the existence of a sequence of contours satisfying the properties of Theorem 10.6 has not been established, but we omit this part of the problem. We could circumvent this difficulty by now verifying that function 10.29 does indeed satisfy initial boundary value problem 10.26.

Problems with arbitrary initial conditions are more difficult to handle. This is illustrated in the next example.

Example 10.14 Solve the vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{10.30a}$$

$$y(0,t) = 0, \quad t > 0, \tag{10.30b}$$

$$y(L,t) = 0, \quad t > 0, \tag{10.30c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{10.30d}$$

$$y_t(x,0) = 0, \quad 0 < x < L, \tag{10.30e}$$

(see Exercise 20 in Section 4.2, with $g(x) \equiv 0$).

Solution When we take Laplace transforms of the PDE with respect to t and use initial conditions 10.30d,e in property 10.7b,

$$s^2\tilde{y} - sf(x) = c^2 \frac{\partial^2 \tilde{y}}{\partial x^2}.$$

Thus, $\tilde{y}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{y}}{dx^2} - \frac{s^2}{c^2}\tilde{y} = -\frac{s}{c^2} f(x) \tag{10.31a}$$

subject to transforms of boundary conditions 10.30b,c,

$$\tilde{y}(0,s) = 0, \quad \tilde{y}(L,s) = 0. \tag{10.31b}$$

Variation of parameters (see Section 4.3) leads to the following form for a general solution of ODE 10.31a

$$\tilde{y}(x,s) = C_1 \cosh\frac{sx}{c} + C_2 \sinh\frac{sx}{c} - \frac{1}{c}\int_0^x f(u)\sinh\frac{s}{c}(x-u)\,du.$$

Boundary conditions 10.31b require

$$0 = C_1, \qquad 0 = C_1 \cosh\frac{sL}{c} + C_2 \sinh\frac{sL}{c} - \frac{1}{c}\int_0^L f(u)\sinh\frac{s}{c}(L-u)\,du,$$

from which

$$\begin{aligned}\tilde{y}(x,s) &= \frac{\sinh\frac{sx}{c}}{c\sinh\frac{sL}{c}}\int_0^L f(u)\sinh\frac{s}{c}(L-u)\,du - \frac{1}{c}\int_0^x f(u)\sinh\frac{s}{c}(x-u)\,du \\ &= \int_0^L f(u)\tilde{p}(x,u,s)\,du - \frac{1}{c}\int_0^x f(u)\sinh\frac{s}{c}(x-u)\,du, \end{aligned} \tag{10.32a}$$

where

$$\tilde{p}(x,u,s) = \frac{\sinh\frac{sx}{c}\sinh\frac{s}{c}(L-u)}{c\sinh\frac{sL}{c}}. \tag{10.32b}$$

To obtain $y(x,t)$ by residues requires the singularities of $\tilde{y}(x,s)$. Provided $f(x)$ is piecewise continuous, integration with respect to u in 10.32a and any differentiation with respect to s can be interchanged, and therefore the second integral in

10.32a has no singularities. Singularities of the first integral are determined by those of $\tilde{p}(x,u,s)$. For the singularity at $s=0$, we note that

$$\begin{aligned}\tilde{p}(x,u,s) &= \frac{1}{c}\sinh\frac{s}{c}(L-u)\left(\frac{\sinh\frac{sx}{c}}{\sinh\frac{sL}{c}}\right)\\
&= \frac{1}{c}\left[\frac{s}{c}(L-u)+\frac{s^3}{3!c^3}(L-u)^3+\cdots\right]\left[\frac{\frac{sx}{c}+\frac{1}{3!}\left(\frac{sx}{c}\right)^3+\cdots}{\frac{sL}{c}+\frac{1}{3!}\left(\frac{sL}{c}\right)^3+\cdots}\right]\\
&= \left[\frac{s}{c^2}(L-u)+\frac{s^3}{6c^4}(L-u)^3+\cdots\right]\left(\frac{x+\frac{x^3s^2}{6c^2}+\cdots}{L+\frac{L^3s^2}{6c^2}+\cdots}\right)\\
&= \frac{s}{c^2}(L-u)\frac{x}{L}+\text{terms in } s^2, s^3, \ldots,\end{aligned}$$

and therefore $\tilde{p}(x,u,s)$ has a removable singularity at $s=0$. The remaining singularities of $\tilde{p}(x,u,s)$ are $s=n\pi ci/L$, n a nonzero integer. Because the derivative of $\sinh(sL/c)$ does not vanish at $s=n\pi ci/L$, these singularities are poles of order 1. According to formula 10.25, the residue of $\tilde{p}(x,u,s)$ at $s=n\pi ci/L$ is

$$\begin{aligned}\text{Res}\left[\tilde{p}(x,u,s),\frac{n\pi ci}{L}\right] &= \lim_{s\to n\pi ci/L}\left(s-\frac{n\pi ci}{L}\right)\tilde{p}(x,u,s)\\
&= \lim_{s\to n\pi ci/L}\left(s-\frac{n\pi ci}{L}\right)\frac{\sinh\frac{sx}{c}\sinh\frac{s}{c}(L-u)}{c\sinh\frac{sL}{c}}\\
&= \sinh\frac{n\pi xi}{L}\sinh\frac{n\pi i(L-u)}{L}\lim_{s\to n\pi ci/L}\frac{s-\frac{n\pi ci}{L}}{c\sinh\frac{sL}{c}}\\
&= -\sin\frac{n\pi x}{L}\sin\frac{n\pi}{L}(L-u)\lim_{s\to n\pi ci/L}\frac{1}{L\cosh\frac{sL}{c}} \qquad \text{(by l'Hôpital's rule)}\\
&= \frac{(-1)^n}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}\frac{1}{\cosh n\pi i}\\
&= \frac{1}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}.\end{aligned}$$

The residue of e^{st} times the first integral in 10.32a at $s=n\pi ci/L$ is now

$$\lim_{s\to n\pi ci/L}\left(s-\frac{n\pi ci}{L}\right)e^{st}\int_0^L f(u)\tilde{p}(x,u,s)\,du.$$

When we interchange the limit on s with the integration with respect to u, the residue becomes

$$\begin{aligned}\int_0^L \lim_{s\to n\pi ci/L}\left[e^{st}\left(s-\frac{n\pi ci}{L}\right)f(u)\tilde{p}(x,u,s)\right]du &= \int_0^L e^{n\pi cti/L}f(u)\frac{1}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}du\\
&= \frac{1}{L}e^{n\pi cti/L}\sin\frac{n\pi x}{L}\int_0^L f(u)\sin\frac{n\pi u}{L}du.\end{aligned}$$

The inverse transform of $\tilde{y}(x,s)$ is the sum of all such residues,

$$y(x,t) = \frac{1}{L}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{n\pi cti/L}\sin\frac{n\pi x}{L}\int_0^L f(u)\sin\frac{n\pi u}{L}du. \tag{10.33}$$

To simplify this summation, we divide it into two parts,

$$\begin{aligned} y(x,t) &= \frac{1}{L}\sum_{n=1}^{\infty} e^{n\pi cti/L}\sin\frac{n\pi x}{L}\int_0^L f(u)\sin\frac{n\pi u}{L}du \\ &\quad + \frac{1}{L}\sum_{n=-\infty}^{-1} e^{n\pi cti/L}\sin\frac{n\pi x}{L}\int_0^L f(u)\sin\frac{n\pi u}{L}du, \end{aligned}$$

and replace n by $-n$ in the second summation,

$$\begin{aligned} y(x,t) &= \frac{1}{L}\sum_{n=1}^{\infty} e^{n\pi cti/L}\sin\frac{n\pi x}{L}\int_0^L f(u)\sin\frac{n\pi u}{L}du \\ &\quad + \frac{1}{L}\sum_{n=1}^{\infty} e^{-n\pi cti/L}\sin\left(\frac{-n\pi x}{L}\right)\int_0^L f(u)\sin\left(\frac{-n\pi u}{L}\right)du \\ &= \frac{1}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}(e^{n\pi cti/L}+e^{-n\pi cti/L})\int_0^L f(u)\sin\frac{n\pi u}{L}du \\ &= \sum_{n=1}^{\infty} a_n\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L} \quad \text{where } a_n = \frac{2}{L}\int_0^L f(u)\sin\frac{n\pi u}{L}du. \end{aligned} \tag{10.34}$$

This is identical to the solution obtained by separation of variables in Exercise 20 in Section 4.2 when $g(x)$ is set equal to zero.●

Examples 10.13 and 10.14 were homogeneous problems. Convolutions can be used to handle problems with unspecified nonhomogeneities.

Example 10.15 Solve Example 10.13 if the end $x = 0$ of the rod has a prescribed temperature $f(t)$ and the initial temperature is zero throughout. Compare the solution with that obtained by variation of constants and by finite Fourier transforms.

Solution The initial boundary value problem is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{10.35a}$$

$$U(0,t) = f(t), \quad t > 0, \tag{10.35b}$$

$$U(L,t) = 0, \quad t > 0, \tag{10.35c}$$

$$U(x,0) = 0, \quad 0 < x < L. \tag{10.35d}$$

When the Laplace transform is applied to the PDE and initial temperature 10.35d is used, the transform $\tilde{U}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = 0, \tag{10.36a}$$

$$\tilde{U}(0,s) = \tilde{f}(s), \tag{10.36b}$$

$$\tilde{U}(L,s) = 0. \tag{10.36c}$$

The solution of this system is

$$\tilde{U}(x,s) = \frac{\tilde{f}(s)\sinh\sqrt{s/k}(L-x)}{\sinh\sqrt{s/k}L}. \tag{10.37}$$

To find the inverse transform of this function, first consider finding the inverse of $\tilde{p}(x,s) = \sinh\sqrt{s/k}(L-x)/\sinh\sqrt{s/k}L$. This function has singularities when $\sqrt{s/k}L = n\pi i$ or $s = -n^2\pi^2 k/L^2$, n a nonnegative integer. Expansion of $\tilde{p}(x,s)$ in a Laurent series around $s = 0$ immediately shows that $s = 0$ is a removable singularity. The remaining singularities are poles of order 1, and the residue of $e^{st}\tilde{p}(x,s)$ at $s = -n^2\pi^2k/L^2$ is

$$\begin{aligned}
\operatorname{Res}\left[e^{st}\tilde{p}(x,s), -\frac{n^2\pi^2 k}{L^2}\right] &= \lim_{s\to -n^2\pi^2k/L^2}\left(s+\frac{n^2\pi^2k}{L^2}\right)e^{st}\frac{\sinh\sqrt{s/k}(L-x)}{\sinh\sqrt{s/k}L} \\
&= e^{-n^2\pi^2kt/L^2}\sinh\frac{n\pi i(L-x)}{L}\lim_{s\to -n^2\pi^2k/L^2}\left(\frac{s+n^2\pi^2k/L^2}{\sinh\sqrt{s/k}L}\right) \\
&= ie^{-n^2\pi^2kt/L^2}\sin\frac{n\pi(L-x)}{L}\lim_{s\to -n^2\pi^2k/L^2}\frac{1}{\dfrac{L}{2\sqrt{ks}}\cosh\sqrt{s/k}L} \\
&= ie^{-n^2\pi^2kt/L^2}(-1)^{n+1}\sin\frac{n\pi x}{L}\frac{2nk\pi i}{L^2\cosh n\pi i} \\
&= \frac{2nk\pi}{L^2}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.
\end{aligned}$$

Convolutions can now be used to invert $\tilde{U}(x,s)$,

$$\begin{aligned}
U(x,t) = \mathcal{L}^{-1}\{\tilde{f}(s)\tilde{p}(x,s)\} &= \int_0^t f(u)p(x,t-u)\,du \\
&= \int_0^t f(u)\left[\frac{2k\pi}{L^2}\sum_{n=1}^{\infty} ne^{-n^2\pi^2k(t-u)/L^2}\sin\frac{n\pi x}{L}\right]du \\
&= \frac{2k\pi}{L^2}\sum_{n=1}^{\infty} c_n(t)\sin\frac{n\pi` x}{L},
\end{aligned} \tag{10.38a}$$

where

$$c_n(t) = n\int_0^t f(u)e^{-n^2\pi^2k(t-u)/L^2}du. \tag{10.38b}$$

With variation of constants (see Section 4.3), the dependent variable is changed to $V(x,t) = U(x,t) - f(t)(1 - x/L)$, resulting in a problem with homogeneous boundary conditions for $V(x,t)$,

$$\begin{aligned}
\frac{\partial V}{\partial t} &= k\frac{\partial^2 V}{\partial x^2} - f'(t)\left(1-\frac{x}{L}\right), \quad 0 < x < L, \quad t > 0, \\
V(0,t) &= 0, \quad t > 0, \\
V(L,t) &= 0, \quad t > 0, \\
V(x,0) &= -f(0)\left(1-\frac{x}{L}\right) = 0, \quad 0 < x < L,
\end{aligned}$$

provided we assume that $f(0)=0$. (The $f(0)\neq 0$ situation is discussed in Exercise 12.) Variation of constants

$$V(x,t)=\sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L}$$

leads to

$$a_n(t)=\frac{-2}{n\pi}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}du,$$

and therefore

$$U(x,t)=f(t)\left(1-\frac{x}{L}\right)-\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}du\right]\sin\frac{n\pi x}{L}. \quad (10.39)$$

That this is identical to solution 10.38 is verified by integrating expression 10.38b by parts,

$$\begin{aligned} c_n(t) &= n\left\{\frac{L^2}{n^2\pi^2k}f(u)e^{-n^2\pi^2k(t-u)/L^2}\right\}_0^t - n\int_0^t \frac{L^2}{n^2\pi^2k}f'(u)e^{-n^2\pi^2k(t-u)/L^2}du \\ &= \frac{L^2}{n\pi^2k}f(t)-\frac{L^2}{n\pi^2k}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}du, \end{aligned}$$

and substituting into equation 10.38a,

$$\begin{aligned} U(x,t) &= \frac{2k\pi}{L^2}\sum_{n=1}^{\infty}\left[\frac{L^2}{nk\pi^2}f(t)-\frac{L^2}{nk\pi^2}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}du\right]\sin\frac{n\pi x}{L} \\ &= f(t)\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi x}{L}-\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}du\right]\sin\frac{n\pi x}{L}. \end{aligned}$$

This is identical to solution 10.39 when we notice that the coefficients in the Fourier sine series of $1-x/L$ are $2/(n\pi)$.

The finite Fourier transform

$$\tilde{f}(\lambda_n)=\int_0^L f(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx$$

applied to problem 10.35 gives the solution in form 10.38.●

When we write solution 10.38 for problem 10.35 in the form

$$U(x,t)=\frac{2k\pi}{L^2}\sum_{n=1}^{\infty}b_ne^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L} \quad \text{where } b_n=n\int_0^t f(u)e^{n^2\pi^2ku/L^2}du, \quad (10.40)$$

we see that the exponentials in the series enhance convergence for large values of t. For instance, if the temperature of the left end is maintained at 100°C for $t>0$, the temperature function reduces to

$$U(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-e^{-n^2\pi^2kt/L^2})\sin\frac{n\pi x}{L}, \quad (10.41)$$

which can also be expressed in the form

$$U(x,t) = 100\left(1 - \frac{x}{L}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}. \tag{10.42}$$

Suppose the rod is 1/5 m in length and is made from stainless steel with thermal diffusivity $k = 3.87 \times 10^{-6}$ m^2/s. Consider finding the temperature at the midpoint $x = 1/10$ of the rod at the four times $t = 2, 5, 30$ and 100 minutes. Series 10.42 gives

$$U(0.1, 120) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-0.1145861n^2}\sin\frac{n\pi}{2} = 0.10°\text{C},$$

$$U(0.1, 300) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-0.28646526n^2}\sin\frac{n\pi}{2} = 3.80°\text{C},$$

$$U(0.1, 1800) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-1.7187915n^2}\sin\frac{n\pi}{2} = 38.6°\text{C},$$

$$U(0.1, 6000) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-5.7293052n^2}\sin\frac{n\pi}{2} = 49.8°\text{C}.$$

To obtain these temperatures, we required only four nonzero terms from the first series, three from the second, one each from the third and fourth. This substantiates our claim that as t increases, fewer and fewer terms in series 10.42 are required for accurate calculations of temperature.

Laplace transforms can be used to give a completely different representation for the temperature in the rod when $f(t) = 100$. To find this representation, we return to expression 10.37 for the Laplace transform $\tilde{U}(x,s)$ of $U(x,t)$ and set $\tilde{f}(s) = 100/s$, the transform of $f(t) = 100$,

$$\begin{aligned}\tilde{U}(x,s) &= \frac{100\sinh\sqrt{s/k}(L-x)}{s\ \sinh\sqrt{s/k}L} = \frac{100}{s}\frac{e^{\sqrt{s/k}(L-x)} - e^{-\sqrt{s/k}(L-x)}}{e^{\sqrt{s/k}L} - e^{-\sqrt{s/k}L}} \\ &= \frac{100}{s}\frac{e^{-\sqrt{s/k}L}[e^{\sqrt{s/k}(L-x)} - e^{-\sqrt{s/k}(L-x)}]}{1 - e^{-2\sqrt{s/k}L}}.\end{aligned}$$

If we regard $1/(1-e^{-2\sqrt{s/k}L})$ as the sum of an infinite geometric series with common ratio $e^{-2\sqrt{s/k}L}$, we may write

$$\begin{aligned}\tilde{U}(x,s) &= \frac{100}{s}[e^{-\sqrt{s/k}x} - e^{-\sqrt{s/k}(2L-x)}]\sum_{n=0}^{\infty}e^{-2n\sqrt{s/k}L} \\ &= 100\sum_{n=0}^{\infty}\left[\frac{e^{-\sqrt{s/k}(2nL+x)}}{s} - \frac{e^{-\sqrt{s/k}[2(n+1)L-x]}}{s}\right].\end{aligned} \tag{10.43}$$

Tables of Laplace transforms indicate that

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \text{erfc}\left(\frac{a}{2\sqrt{t}}\right),$$

where erfc(x) is the complementary error function in equation 10.16. Hence, $U(x,t)$ may be expressed as a series of complementary error functions,

$$U(x,t) = 100\sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2nL+x}{2\sqrt{kt}}\right) - \operatorname{erfc}\left(\frac{2(n+1)L-x}{2\sqrt{kt}}\right)\right]$$
$$= 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)L-x}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{2nL+x}{2\sqrt{kt}}\right)\right], \qquad (10.44)$$

where we have used the fact that $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. The error function erf(x) is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du. \qquad (10.45)$$

This representation of $U(x,t)$ is valuable for small values of t (as opposed to 10.42, which converges rapidly for large t). To see this, consider temperature at the midpoint of the above stainless steel rod at $t = 300$ s,

$$U(0.1,300) = 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)/5-0.1}{2\sqrt{3.87\times 10^{-6}(300)}}\right) - \operatorname{erf}\left(\frac{2n/5+0.1}{2\sqrt{3.87\times 10^{-6}(300)}}\right)\right].$$

For $n > 0$, all terms in this series essentially vanish, and

$$U(0.1,300) = 100[\operatorname{erf}(4.40) - \operatorname{erf}(1.467)] = 3.80°\text{C}.$$

For $t = 1800$,

$$U(0.1,1800) = 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)/5-0.1}{2\sqrt{3.87\times 10^{-6}(1800)}}\right) - \operatorname{erf}\left(\frac{2n/5+0.1}{2\sqrt{3.87\times 10^{-6}(1800)}}\right)\right].$$

Once again, only the $n = 0$ term is required; it yields $U(0.1,1800) = 38.6°$C. Finally, for $t = 6000$,

$$U(0.1,6000) = 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)/5-0.1}{2\sqrt{3.87\times 10^{-6}(6000)}}\right) - \operatorname{erf}\left(\frac{2n/5+0.1}{2\sqrt{3.87\times 10^{-6}(6000)}}\right)\right].$$

In this case, the $n = 0$ and $n = 1$ terms give $U(0.1,6000) = 49.9°$C. For larger values of t, more and more terms are required.

The error function representiation in equation 10.44 once again substantiates our claim in Section 6.6 that heat propagates with infinite speed. Because the error function is an increasing function of its argument, and the argument $(2nL+2L-x)/(2\sqrt{kt})$ of the first error function in 10.44 is greater than the second argument, $(2nL+x)/(2\sqrt{kt})$, it follows that each term in 10.44 is positive, Since this is true for every x in $0 < x < L$ and every $t > 0$, the temperature at every point in the rod for every $t > 0$ is positive. This means that the effect of changing the temperature of the end $x = 0$ of the rod from 0°C to 100°C at time $t = 0$ is instantaneously felt at every point in the rod. The amount of heat transmitted to other parts of the rod may be minute, but nonetheless, heat is transmitted instantaneously to all parts of the rod.

EXERCISES 10.4

Use Laplace transforms to solve all problems in this set of exercises.
Part A Heat Conduction

1. A homogeneous, isotropic rod with insulated sides has temperature $\sin m\pi x/L$ (m an integer) at time $t = 0$. For time $t > 0$, its ends ($x = 0$ and $x = L$) are held at temperature 0°C. Find a formula for temperature $U(x,t)$ in the rod for $0 < x < L$ and $t > 0$.

2. Solve Example 4.2 in Section 4.2 when the initial temperature is $U_0 =$ constant.

3. Repeat Exercise 1 if the initial temperature is 10°C throughout.

4. Solve Exercise 8 in Section 4.3.

5. Solve Exercise 2 in Section 4.2.

6. Solve Example 4.2 in Section 4.2 when the initial temperature is $f(x)$ (in place of x).

7. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at temperature 0°C throughout. For $t > 0$, its left end, $x = 0$, is kept at 0°C and its right end, $x = L$, is kept at constant temperature U_L°C. Find two expressions for temperature in the rod, one in terms of exponentials in time and the other in terms of error functions.

8. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at constant temperature U_0°C throughout. For $t > 0$, its end $x = 0$ is insulated, and heat is added to the end $x = L$ at a constant rate Q W/m^2. Find the temperature in the rod for $0 < x < L$ and $t > 0$.

9. (a) A homogeneous, isotropic rod with insulated sides has, for time $t > 0$, its ends at $x = 0$ and $x = L$ kept at temperature zero. Initially its temperature is Ax, where A is constant. Show that temperature in the rod can be expressed in two ways:

$$U(x,t) = \frac{2AL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L},$$

and $$U(x,t) = A\left\{x - L\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{(2n+1)L+x}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{(2n+1)L-x}{2\sqrt{kt}}\right)\right]\right\}.$$

(b) Which of the two solutions do you expect to converge more rapidly for small t? For large t?

(c) Verify your conjecture in part (b) by calculating the temperature at the midpoint of a stainless steel rod ($k = 3.87 \times 10^{-6}$) of length 1/5 m when $A = 500$ and (i) $t = 30$ s (ii) $t = 5$ min (iii) $t = 100$ min.

10. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at temperature 0°C throughout. For $t > 0$, its left end, $x = 0$, is kept at 0°C and heat is added to the end $x = L$ at a constant rate $Q > 0$ W/m^2. Find two series representations for $U(x,t)$, one in terms of error functions and one in terms of time exponentials.

11. Solve Exercise 13 in Section 7.2.

12. Show that the Laplace transform solution and the eigenfunction expansion solution to the problem in Example 10.15 are identical when $f(0) \neq 0$.

13. A homogeneous, isotropic rod with insulated sides has initial temperature distribution $U_L x/L$, $0 \le x \le L$ (U_L a constant). For time $t > 0$, its ends $x = 0$ and $x = L$ are held at temperatures 0°C and U_L°C, respectively. Find the temperature distribution in the rod for $t > 0$.

14. Repeat Exercise 13 if the initial temperature distribution is $f(x) = ax$, $0 \le x \le L$, where a is a constant. The ends $x = 0$ and $x = L$ are held at constant temperatures U_0°C and U_L°C, respectively, for $t > 0$.

15. Solve Exercise 5 in Section 4.3 in the case that $k \ne L^2/(n^2\pi^2)$ for any positive integer n. (See also Exercise 8 in Section 7.2.)

16. Solve Exercise 16 in Section 4.3 with zero initial temperature.

Part B Vibrations

17. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. If it is given an initial displacement (at time $t = 0$) of $f(x) = kx(L - x)$, (k a constant), and zero initial velocity, find its subsequent displacement.

18. Solve Exercise 19(a) in Section 4.2.

19. Repeat Exercise 17 for zero initial displacement and an unspecified initial velocity $g(x)$.

20. Solve Exercise 37(a) in Section 7.2.

21. Solve Exercise 21 in Section 4.3.

22. Solve Exercise 21(a) in Section 4.2.

For Exercises 23–28 solve Exercises 30–35 in Section 7.2.

29. Repeat Example 10.14 if gravity is taken into account. See also Exercise 41 in Section 7.2.

30. Solve Exercise 28 in Section 7.2.

31. Show that Laplace transforms lead to the solution in part (c) for the problem in Exercise 20 of Section 4.3.

32. (a) Find a series solution for displacements in the bar of Exercise 24 of Section 7.2 if the constant force per unit area F is replaced by an impulse force $F = F_0\delta(t)$. Use the fact that

$$\int_0^\infty f(t)\delta(t)\,dt = f(0+).$$

(b) Show that the displacement of the end $x = L$ is $cF_0/(AE)$ times the square wave function

$$M_{2L/c}(t) = \begin{cases} 1, & 0 < t < 2L/c \\ -1, & 2L/c < t < 4L/c \end{cases}, \qquad M_{2L/c}(t + 4L/c) = M_{2L/c}(t).$$

33. Solve Exercise 42 in Section 7.2.

34. A taut string of length L is initially at rest along the x-axis. For time $t > 0$, its ends are subjected to prescribed displacements

$$y(0,t) = f_1(t), \qquad y(L,t) = f_2(t).$$

Find its displacement for $0 < x < L$ and $t > 0$.

35. (a) Show that the Laplace transform of the displacement function $y(x,t)$ for the vibrations in Exercise 45 of Section 7.2 is

$$\tilde{y}(x,s) = \frac{F_0\omega c \sinh(sx/c)}{s(s^2+\omega^2)[AE\cosh(sL/c) + mcs\sinh(sL/c)]}.$$

(b) Resonance occurs if either of the zeros $s = \pm i\omega$ of $s^2 + \omega^2$ coincides with a zero of

$$h(s) = AE\cosh(sL/c) + mcs\sinh(sL/c).$$

By expressing zeros of $h(s)$ in the form $s = c(\mu + \lambda i)$, show that

$$\tanh 2\mu L = \frac{-2AEmc^2\mu}{A^2E^2 + m^2c^4(\mu^2 + \lambda^2)}$$

and that therefore $\mu = 0$. Verify that resonance occurs if $\omega = c\lambda$ where λ is a root of the equation

$$\tan\lambda L = \frac{AE}{mc^2\lambda}.$$

36. Solve Example 4.4 in Section 4.2, but with an unspecified initial displacement $f(x)$. (Hint: Replace s by icq^2 in the ODE for $\tilde{y}(x,s)$.)

37. (a) The top of the bar in Exercise 20 is attached to a spring with constant k. If $x = 0$ corresponds to the top end of the bar when the spring is unstretched, show that the Laplace transform of the displacement function for cross sections of the bar is

$$\tilde{y}(x,s) = \frac{g}{s^3} - \frac{kgc\cosh[s(L-x)/c]}{s^3[AE\sinh(sL/c) + kc\cosh(sL/c)]}.$$

(b) Verify that $\tilde{y}(x,s)$ has a pole of order 1 at $s = 0$. What is the residue of $e^{st}\tilde{y}(x,s)$ at $s = 0$?
(c) By setting $s = c(\mu + \lambda i)$ to obtain zeros of

$$h(s) = AEs\sinh(sL/c) + kc\cosh(sL/c),$$

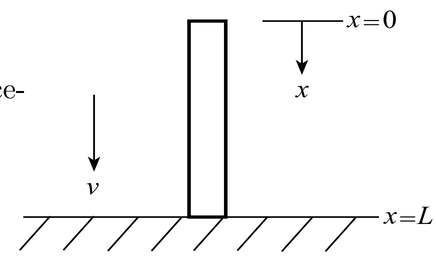
show that μ must be zero and that λ must satisfy

$$\tan\lambda L = \frac{k}{AE\lambda}.$$

(d) Find $y(x,t)$. (See also Exercise 38 in Section 7.2.)

38. (a) An unstrained elastic bar falls vertically under gravity with its axis in the vertical position (figure to the right). When its velocity is $v > 0$, it strikes a solid object and remains in contact with it thereafter. Show that the Laplace transform of displacements $y(x,t)$ of cross sections of the bar is

$$\tilde{y}(x,s) = \left(\frac{v}{s^2} + \frac{g}{s^3}\right)\left[1 - \frac{\cosh(sx/c)}{\cosh(sL/c)}\right].$$



(b) Use residues to find

$$y(x,t) = \frac{g(L^2 - x^2)}{2c^2} + \frac{8Lv}{\pi^2 c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}$$
$$+ \frac{16L^2 g}{\pi^3 c^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}.$$

(c) Verify that the second series in part (b) may be expressed in the form

$$-\frac{g}{4c^2}[K(x+ct)+K(x-ct)],$$

where $K(x)$ is the even, odd-harmonic extension of $L^2 - x^2$, $0 \le x \le L$, to a function of period $4L$. (See Exercise 21 in Section 3.2 for the definition of an even, odd-harmonic function.)

(d) Verify that the first series in part (b) may be expressed in the form

$$\frac{v}{2c}[M_L(x+ct)-M_L(x-ct)],$$

where $M_L(x)$ is the odd, odd-harmonic extension of x, $0 \le x \le L$, to a function of period $4L$. (See Exercise 20 in Section 3.2 for the definition of an odd, odd-harmonic function.)

(e) Find an expression for the force $F(t)$ due to the bar on the cross section at $x = L$. Sketch graphs of $F(t)$ when $v < 2Lg/c$ and $v > 2gL/c$.

39. A bar 1/4 m long is falling as in Exercise 38 when it stikes an object squarely. Use the result of Exercise 38 to find a formula for the length of time of contact of the bar with the object. Use this formula to find the contact time for a steel bar with $\rho = 7.8\times 10^3$ kg/m^3 and $E = 2.1\times 10^{11}$ kg/m^2 when $v = 2$ m/s.

§10.5 Applications to Problems in Polar, Cylindrical, and Spherical Coordinates

Laplace transforms can also be used to solve problems in polar, cylindrical, and spherical coordinates, but calculations are sometimes more complex. We illustrate with the following examples.

Example 10.16 Use Laplace transforms to solve Example 9.1 of Section 9.1.

Solution The initial boundary value problem for $U(r,t)$ is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < a, \quad t > 0, \tag{10.46a}$$

$$\frac{\partial U(a,t)}{\partial r} = 0, \quad t > 0, \tag{10.46b}$$

$$U(r,0) = a^2 - r^2, \quad 0 < r < a. \tag{10.46c}$$

When we take Laplace transforms of the PDE and use initial condition 10.46c,

$$s\tilde{U}(r,s) - (a^2 - r^2) = k\left(\frac{\partial^2 \tilde{U}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{U}}{\partial r}\right);$$

that is, $\tilde{U}(r,s)$ must satisfy the ODE

$$r\frac{d^2\tilde{U}}{dr^2} + \frac{d\tilde{U}}{dr} - \frac{sr}{k}\tilde{U} = \frac{r^3 - a^2 r}{k}, \quad 0 < r < a, \tag{10.47a}$$

subject to the transform of boundary condition 10.46b,

$$\tilde{U}'(a,s) = 0. \tag{10.47b}$$

The change of independent variable $u = \sqrt{s/k}ri$ replaces the homogeneous equation

$$r\frac{d^2\tilde{U}}{dr^2} + \frac{d\tilde{U}}{dr} - \frac{sr}{k}\tilde{U} = 0 \tag{10.48}$$

with

$$u\frac{d^2\tilde{U}}{du^2} + \frac{d\tilde{U}}{du} + u\tilde{U} = 0. \tag{10.49}$$

This is Bessel's differential equation of order zero, with general solution

$$AJ_0(u) + BY_0(u).$$

Thus, a general solution of ODE 10.48 is

$$AJ_0\left(\sqrt{\frac{s}{k}}ri\right) + BY_0\left(\sqrt{\frac{s}{k}}ri\right). \tag{10.50}$$

When the particular solution $-r^2/s + (a^2 s - 4k)/s^2$ of 10.47a is added to this, a general solution of 10.47a is

$$\tilde{U}(r,s) = AJ_0\left(\sqrt{\frac{s}{k}}ri\right) + BY_0\left(\sqrt{\frac{s}{k}}ri\right) - \frac{4k}{s^2} + \frac{a^2 - r^2}{s}. \tag{10.51}$$

Because $U(r,t)$ must remain bounded as r approaches zero, so also must $\tilde{U}(r,s)$. This implies that B must vanish, in which case boundary condition 10.47b requires

$$i\sqrt{\frac{s}{k}}AJ_0'\left(\sqrt{\frac{s}{k}}ai\right) - \frac{2a}{s} = 0.$$

When this equation is solved for A and the result is substituted into 10.51,

$$\tilde{U}(r,s) = \frac{2aJ_0(\sqrt{s/k}ri)}{i\sqrt{s^3/k}J_0'(\sqrt{s/k}ai)} - \frac{4k}{s^2} + \frac{a^2 - r^2}{s}. \tag{10.52}$$

This function has singularities at $s = 0$ and values of s satisfying $J_0'(\sqrt{s/k}ai) = 0$. If we set $\sqrt{s/k}i = \lambda_n$, singularities occur for $s = -k\lambda_n^2$ where $J_0'(\lambda_n a) = 0$. Power series 8.18 in Section 8.3 can be used to expand $\tilde{U}(r,s)$ about $s = 0$,

$$\begin{aligned}
\tilde{U}(r,s) &= \frac{2\sqrt{k}a}{s^{3/2}i}\left[\frac{1 - \dfrac{(\sqrt{s/k}ri)^2}{4} + \dfrac{(\sqrt{s/k}ri)^4}{64} - \cdots}{-\dfrac{(\sqrt{s/k}ai)}{2} + \dfrac{(\sqrt{s/k}ai)^3}{16} - \cdots}\right] - \frac{4k}{s^2} + \frac{a^2 - r^2}{s} \\
&= \frac{2\sqrt{k}a}{s^{3/2}i}\left[-\frac{2}{\sqrt{s/k}ai} - \frac{\sqrt{s/k}i}{a}\left(\frac{a^2}{4} - \frac{r^2}{2}\right) + \cdots\right] - \frac{4k}{s^2} + \frac{a^2 - r^2}{s} \\
&= \frac{a^2}{2s} + \cdots .
\end{aligned}$$

When this result is multiplied by e^{st},

$$e^{st}\tilde{U}(r,s) = \left(1 + st + \frac{s^2t^2}{2} + \cdots\right)\left(\frac{a^2}{2s} + \cdots\right).$$

Thus, the residue of $e^{st}\tilde{U}(r,s)$ at $s = 0$ is $a^2/2$. Because the derivative of J_0' does not vanish at its zeros, the remaining singularities at $s = -k\lambda_n^2$ are poles of order 1, and residues of $e^{st}\tilde{U}(r,s)$ at these poles are

$$\begin{aligned}
&\lim_{s \to -k\lambda_n^2} (s + k\lambda_n^2)e^{st}\left[\frac{2aJ_0(\sqrt{s/k}ri)}{\sqrt{s^3/k}iJ_0'(\sqrt{s/k}ai)} - \frac{4k}{s^2} + \frac{a^2 - r^2}{s}\right] \\
&= \frac{2a}{-k\lambda_n^3}e^{-k\lambda_n^2 t}J_0(\lambda_n r)\lim_{s \to -k\lambda_n^2}\frac{s + k\lambda_n^2}{J_0'(\sqrt{s/k}ai)} \\
&= \frac{-2a}{k\lambda_n^3}e^{-k\lambda_n^2 t}J_0(\lambda_n r)\lim_{s \to -k\lambda_n^2}\frac{1}{\dfrac{ai}{2\sqrt{ks}}J_0''(\sqrt{s/k}ai)} \qquad \text{(by l'Hôpital's rule)} \\
&= \frac{-4}{k\lambda_n^3}e^{-k\lambda_n^2 t}J_0(\lambda_n r)\frac{1}{\dfrac{-1}{k\lambda_n}J_0''(\lambda_n a)} \\
&= \frac{4}{\lambda_n^2 J_0''(\lambda_n a)}e^{-k\lambda_n^2 t}J_0(\lambda_n r).
\end{aligned}$$

But, because $J_0(\lambda_n r)$ satisfies equation 10.48 when $s = -k\lambda_n^2$,

$$r\frac{d^2J_0(\lambda_n r)}{dr^2} + \frac{dJ_0(\lambda_n r)}{dr} + \lambda_n^2 rJ_0(\lambda_n r) = 0$$

or,

$$\lambda_n^2 r J_0''(\lambda_n r) + \lambda_n J_0'(\lambda_n r) + \lambda_n^2 r J_0(\lambda_n r) = 0.$$

When we set $r = a$ in this equation and note that $J_0'(\lambda_n a) = 0$, we obtain

$$J_0''(\lambda_n a) = -J_0(\lambda_n a).$$

Residues of $e^{st}\tilde{U}(r,s)$ at $s = -k\lambda_n^2$ can therefore be expressed as

$$\frac{-4}{\lambda_n^2 J_0(\lambda_n a)} e^{-k\lambda_n^2 t} J_0(\lambda_n r).$$

The sum of the residues at $s = 0$ and $s = -k\lambda_n^2$ yields the temperature function

$$U(r,t) = \frac{a^2}{2} - 4\sum_{n=1}^{\infty} \frac{e^{-k\lambda_n^2 t} J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n a)}. \bullet \tag{10.53}$$

The following vibration problem has a nonhomogeneous boundary condition.

Example 10.17 A circular membrane of radius a is initially at rest on the xy-plane. Find its displacement for time $t > 0$ if its edge is forced to undergo periodic oscillations described by $A \sin \omega t$, A a constant. Assume that resonance does not occur.

Solution The initial boundary value problem for displacements $z(r,t)$ of the membrane is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} \right), \quad 0 < r < a, \quad t > 0, \tag{10.54a}$$

$$z(a,t) = A \sin \omega t, \quad t > 0, \tag{10.54b}$$

$$z(r,0) = 0, \quad 0 < r < a, \tag{10.54c}$$

$$z_t(r,0) = 0, \quad 0 < r < a. \tag{10.54d}$$

When we apply the Laplace transform to the PDE and use the initial conditions,

$$s^2 \tilde{z} = c^2 \left(\frac{d^2 \tilde{z}}{dr^2} + \frac{1}{r}\frac{d\tilde{z}}{dr} \right);$$

that is, $\tilde{z}(r,s)$ must satisfy

$$r\frac{d^2 \tilde{z}}{dr^2} + \frac{d\tilde{z}}{dr} - \frac{s^2 r}{c^2}\tilde{z} = 0, \tag{10.55a}$$

subject to the transform of boundary condition 10.54b,

$$\tilde{z}(a,s) = \frac{A\omega}{s^2 + \omega^2}. \tag{10.55b}$$

The change of independent variable $u = sri/c$ replaces this ODE with

$$u\frac{d^2 \tilde{z}}{du^2} + \frac{d\tilde{z}}{du} + u\tilde{z} = 0, \tag{10.56}$$

Bessel's differential equation of order zero. Since a general solution is $BJ_0(u) + DY_0(u)$, it follows that

$$\tilde{z}(r,s) = BJ_0\left(\frac{sri}{c}\right) + DY_0\left(\frac{sri}{c}\right). \tag{10.57}$$

Because $z(r,t)$ must remain bounded as r approaches zero, so also must $\tilde{z}(r,s)$. This implies that D must vanish, in which case boundary condition 10.55b requires

$$\frac{A\omega}{s^2+\omega^2} = BJ_0\left(\frac{sai}{c}\right).$$

When this is solved for B, we obtain

$$\tilde{z}(r,s) = \frac{A\omega}{s^2+\omega^2}\frac{J_0(sri/c)}{J_0(sai/c)}. \tag{10.58}$$

This function has singularities at $s = \pm\omega i$ and values of s satisfying $J_0(sai/c) = 0$. If we set $si/c = \lambda_n$, singularities occur for $s = -c\lambda_n i$ where $J_0(\lambda_n a) = 0$. (For every positive value of λ_n satisfying this equation, $\lambda_{-n} = -\lambda_n$ is also a solution.) Provided $\omega \neq c\lambda_n$ for any n (the nonresonant case), all singularities are poles of order 1. The residue of $e^{st}\tilde{z}(r,s)$ at $s = \omega i$ is

$$\begin{aligned}
\lim_{s\to\omega i}(s-\omega i)e^{st}\tilde{z}(r,s) &= \lim_{s\to\omega i}(s-\omega i)\frac{A\omega e^{st}}{(s+\omega i)(s-\omega i)}\frac{J_0(sri/c)}{J_0(sai/c)}\\
&= \frac{A\omega e^{\omega ti}}{2\omega i}\frac{J_0(-\omega r/c)}{J_0(-\omega a/c)}\\
&= -\frac{i}{2}Ae^{\omega ti}\frac{J_0(\omega r/c)}{J_0(\omega a/c)}.
\end{aligned}$$

Similarly, the residue at $s = -\omega i$ is

$$\frac{i}{2}Ae^{-\omega ti}\frac{J_0(\omega r/c)}{J_0(\omega a/c)}.$$

Residues at $s = -c\lambda_n i$ are

$$\begin{aligned}
&\lim_{s\to -c\lambda_n i}(s+c\lambda_n i)e^{st}\frac{A\omega}{s^2+\omega^2}\frac{J_0(sri/c)}{J_0(sai/c)}\\
&= \frac{A\omega}{\omega^2-c^2\lambda_n^2}e^{-c\lambda_n ti}J_0(\lambda_n r)\lim_{s\to -c\lambda_n i}\frac{s+c\lambda_n i}{J_0(sai/c)}\\
&= \frac{A\omega}{\omega^2-c^2\lambda_n^2}e^{-c\lambda_n ti}J_0(\lambda_n r)\lim_{s\to -c\lambda_n i}\frac{1}{\frac{ai}{c}J_0'(sai/c)} \qquad \text{(by l'Hôpital's rule)}\\
&= \frac{-A\omega c i e^{-c\lambda_n ti}}{a(\omega^2-c^2\lambda_n^2)}J_0(\lambda_n r)\frac{1}{J_0'(\lambda_n a)} = \frac{A\omega c i e^{-c\lambda_n ti}}{a(\omega^2-c^2\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}.
\end{aligned}$$

Ths sum of the residues at $s = \pm\omega i$ and $s = -c\lambda_n i$ yields the displacement of the membrane

$$\begin{aligned}
z(r,t) &= -\frac{i}{2}Ae^{\omega ti}\frac{J_0(\omega r/c)}{J_0(\omega a/c)} + \frac{i}{2}Ae^{-\omega ti}\frac{J_0(\omega r/c)}{J_0(\omega a/c)} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{A\omega c i e^{-c\lambda_n ti}}{a(\omega^2-c^2\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}\\
&= A\frac{J_0(\omega r/c)}{J_0(\omega a/c)}\left(\frac{e^{\omega ti}-e^{-\omega ti}}{2i}\right) + \sum_{n=1}^{\infty}\frac{A\omega c i e^{-c\lambda_n ti}}{a(\omega^2-c^2\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}
\end{aligned}$$

$$+ \sum_{n=-\infty}^{-1} \frac{A\omega c i e^{-c\lambda_n t i}}{a(\omega^2 - c^2\lambda_n^2)} \frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$$

$$= A\frac{J_0(\omega r/c)}{J_0(\omega a/c)} \sin \omega t + \frac{A\omega c i}{a} \sum_{n=1}^{\infty} \frac{e^{-c\lambda_n t i}}{\omega^2 - c^2\lambda_n^2} \frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$$

$$+ \frac{A\omega c i}{a} \sum_{n=1}^{\infty} \frac{e^{-c\lambda_{-n} t i}}{\omega^2 - c^2\lambda_{-n}^2} \frac{J_0(\lambda_{-n} r)}{J_1(\lambda_{-n} a)}.$$

Since $\lambda_{-n} = -\lambda_n$, and J_0 and J_1 are even and odd functions, respectively, it follows that

$$z(r,t) = A \sin \omega t \frac{J_0(\omega r/c)}{J_0(\omega a/c)} - \frac{A\omega c i}{a} \sum_{n=1}^{\infty} \frac{e^{c\lambda_n t i} - e^{-c\lambda_n t i}}{\omega^2 - c^2\lambda_n^2} \frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}$$

$$= A \sin \omega t \frac{J_0(\omega r/c)}{J_0(\omega a/c)} + \frac{2A\omega c}{a} \sum_{n=1}^{\infty} \frac{\sin c\lambda_n t}{\omega^2 - c^2\lambda_n^2} \frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}. \tag{10.59}$$

The solution of this problem, obtained by finite Fourier transforms in Exercise 22 of Section 9.2, is

$$z(r,t) = -\frac{2AC}{a} \sum_{n=1}^{\infty} \frac{c\lambda_n \sin \omega t - \omega \sin c\lambda_n t}{(\omega^2 - c^2\lambda_n^2) J_1(\lambda_n a)} J_0(\lambda_n r).$$

The Laplace transform solution is preferable; it expresses part of the finite Fourier transform solution in closed form.•

EXERCISES 10.5

Part A Heat Conduction

1. Solve Exercise 1(b) in Section 9.1.

2. Solve Exercise 1(c) in Section 9.1.

3. Laplace transforms do not handle problems in polar coordinates efficiently when initial conditions contain unspecified functions. To illustrate this, find the Laplace transform of the PDE for Exercise 1(a) in Section 9.1. How difficult is it to solve the ODE in $\tilde{U}(r,s)$?

4. Solve Example 9.5 in Section 9.2.

5. (a) An infinitely long cylinder of radius a is initially at temperature 0°C throughout. If the surface $r = a$ has variable temperature $f(t)$ for $t > 0$, find the temperature inside the cylinder.
(b) Simplify the solution when $f(t) = \overline{U}$, a constant. Do you obtain the solution to Exercise 4?

6. Solve Exercise 2(b) in Section 9.2.

7. (a) A cylinder occupying the region $0 \le r \le a$, $0 \le z \le L$, is initially at constant temperature U_0°C throughout. What is the initial boundary value problem for temperature in the cylinder if its surface is held at 0°C for $t > 0$?
(b) If a finite Fourier transform is used to remove the z-variable from the problem in $U(r,z,t)$, what is the initial boundary value problem for $\tilde{U}(r,\mu_m,t)$ (where $\mu_m = m\pi/L$ are eigenvalues associated with this transform)?

(c) Show that when the Laplace transform is applied to the PDE in $\tilde{U}(r,\mu_m,t)$, the transform function $\tilde{\tilde{U}}(r,\mu_m,s)$ must satisfy

$$r\frac{d^2\tilde{\tilde{U}}}{dr^2} + \frac{d\tilde{\tilde{U}}}{dr} - r\left(\frac{s}{k} + \mu_m^2\right)\tilde{\tilde{U}} = -\frac{rU_0\tilde{1}_m}{k}, \quad 0 < r < a,$$
$$\tilde{\tilde{U}}(a,\mu_m,s) = 0,$$

where $\tilde{1}_m = \sqrt{2L}[1 + (-1)^{m+1}]/(m\pi)$ is the finite Fourier transform of the unity function.

(d) Verify that the solution for $\tilde{\tilde{U}}(r,\mu_m,s)$ is

$$\tilde{\tilde{U}}(r,\mu_m,s) = \frac{U_0\tilde{1}_m}{s + k\mu_m^2}\left[1 - \frac{J_0(\sqrt{\mu_m^2 + s/k}ri)}{J_0(\sqrt{\mu_m^2 + s/k}ai)}\right].$$

(e) Prove that $\tilde{\tilde{U}}(r,\mu_m,s)$ has a removable singularity at $s = -k\mu_m^2$ and poles of order 1 at $s = -k(\lambda_n^2 + \mu_m^2)$ where $J_0(\lambda_n a) = 0$. Show that the residues of $e^{st}\tilde{\tilde{U}}(r,\mu_m,s)$ at these poles are

$$\frac{2U_0\tilde{1}_m}{a\lambda_n}e^{-k(\lambda_n^2+\mu_m^2)t}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}.$$

(f) Finally, invert the Laplace transform and the finite Fourier transform to find $U(r,z,t)$.

Vibrations

8. Solve Exercise 23 in Section 9.1.

9. Solve Exercise 24 in Section 9.1.

10. Solve Exercise 23 in Section 9.2 in the nonresonance case.

CHAPTER 11 FOURIER AND HANKEL TRANSFORMS

§11.1 Introduction

In Chapters 3–9 we restricted consideration to problems on bounded spatial domains, but many important problems take place on infinite or semi-infinite domains. For example, suppose a rod of infinite length is initially at temperature $f(x)$, $-\infty < x < \infty$. The initial value problem for temperature $U(x,t)$ in the rod when the sides are insulated is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \tag{11.1a}$$

$$U(x,0) = f(x), \quad -\infty < x < \infty. \tag{11.1b}$$

It may be argued that there is no such thing as an infinite rod. Physically it must be finite, and therefore boundary effects must be taken into account. This can be countered by stating that, for small t, the rod may be so long that boundary effects are negligibly small in that part of the rod under consideration. Consequently, if there is a simple solution to the infinite problem that is an excellent approximation to the Fourier series solution of the bounded problem, then clearly there is an advantage in considering the infinite problem.

In this chapter we illustrate that separation of variables on problems with infinite spatial domains leads to integral representations of the solution called *Fourier integrals*. The Fourier integral replaces the Fourier series representation for finite intervals; it is direct result of the fact that eigenvalues of the separated equation form a continuous, rather than discrete, set. When the solution of an infinite spatial problem is known to be even or odd, the Fourier integral takes on a simplified form called the *Fourier cosine* or *sine integral*. These integrals also arise naturally in problems on semi-infinite intervals ($0 < x < \infty$) when the boundary condition at $x = 0$ is Neumann or Dirichlet. Generalized Fourier integrals arise when the boundary condition at $x = 0$ is of Robin type. Associated with each Fourier integral is an integral transform that provides a convenient alternative to separation of variables. These transforms are as valuable for homogeneous problems as they are for nonhomogeneous problems (unlike finite Fourier transforms, which are not normally used on homogeneous problems.)

We begin by illustrating the continuous nature of "eigenvalues" for infinite spatial problems. Separation of variables $U(x,t) = X(x)T(t)$ on problem 11.1 yields

$$X'' + \alpha X = 0, \qquad T' + k\alpha T = 0, \qquad \alpha = \text{constant}. \tag{11.2}$$

The solution for $T(t)$ is $Ce^{-k\alpha t}$, which clearly indicates that α must be nonnegative. We therefore set $\alpha = \lambda^2$, in which case

$$X(x) = A\cos\lambda x + B\sin\lambda x. \tag{11.3}$$

Alternatively, we could argue that the solution $X(x)$ of $X'' + \alpha X = 0$ must be bounded as $x \to \pm\infty$, and this would again imply that α be nonnegative. Thus, any function of the form

$$e^{-k\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

for arbitrary A, B, and λ satisfies PDE 11.1a. For problems on bounded intervals, boundary conditions determine a discrete set of eigenvalues λ_n and an equation relating A and B. Separated functions are then superposed as infinite series. For infinite intervals, no boundary conditions exist, and hence A, B, and λ are all arbitrary. But suppose for the moment that A and B are functions of λ. It is straightforward to show that when the integral

$$U(x,t) = \int_0^\infty e^{-k\lambda^2 t}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda \tag{11.4}$$

is suitably convergent so that integrations with respect to λ may be interchanged with differentiations with respect to x and t, such functions satisfy 11.1a (see Exercise 2). This integral is a superposition of separated functions over all values of the parameter λ, and it satisfies PDE 11.1a for arbitrary $A(\lambda)$ and $B(\lambda)$. To determine these functions, we demand that $U(x,t)$ as defined by the integral in equation 11.4 satisfy initial condition 11.1b:

$$f(x) = \int_0^\infty [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda, \quad -\infty < x < \infty. \tag{11.5}$$

The solution of problem 11.1 is therefore defined by improper integral 11.4, provided we can find functions $A(\lambda)$ and $B(\lambda)$ satisfying equation 11.5. Equation 11.5 is called the **Fourier integral representation** of $f(x)$; it is the integral analogue of the Fourier series of a periodic function. In Section 11.2 we investigate conditions under which a function has a Fourier integral representation, and we determine formulas for $A(\lambda)$ and $B(\lambda)$.

EXERCISES 11.1

1. Why does the integral superposition in equation 11.4 not extend over the interval $-\infty < \lambda < \infty$?

2. Show that if partial derivatives of the improper integral in 11.4 with respect to x and t may be interchanged with the λ-integation, then $U(x,t)$ satisfies PDE 11.1a.

§11.2 The Fourier Integral Formulas

To state conditions under which the Fourier integral of a function represents the function, we require the concept of absolute integrability.

Definition 11.1 A function $f(x)$ is said to be **absolutely integrable** on the interval $-\infty < x < \infty$ if $\displaystyle\int_{-\infty}^{\infty} |f(x)|\,dx$ converges.

For example, the functions e^{-x^2} and $(x^2+1)^{-1}$ are absolutely integrable on $-\infty < x < \infty$, but x, $\sin x$, and $1/\sqrt{|x|}$ are not.

Corresponding to Theorem 3.2 in Section 3.1 for Fourier series, we have the following result for Fourier integrals.

Theorem 11.1 If $f(x)$ is piecewise continuous on every finite interval and absolutely integrable on $-\infty < x < \infty$, then at every x at which $f(x)$ has a right- and left-derivative,

$$\frac{f(x+)+f(x-)}{2} = \int_0^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda \tag{11.6a}$$

provided $A(\lambda)$ and $B(\lambda)$ are calculated by the formulas

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx, \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx. \tag{11.6b}$$

Equation 11.6 is called the **Fourier integral formula** for the function $f(x)$. It is verified in Appendix B. Since functions that are piecewise smooth must have right- and left-derivatives, we may state the following corollary to Theorem 11.1.

Corollary If $f(x)$ is absolutely integrable on $-\infty < x < \infty$ and piecewise smooth on every finite interval, then $f(x)$ can be expressed in Fourier integral form 11.6.

One of the most important functions that we encounter in this chapter is contained in the following example.

Example 11.1 Find the Fourier integral representation of the Gaussian $f(x) = e^{-ax^2}$, $a > 0$ a constant.

Solution The function and its derivative are continuous, and the well-known result from statistics

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}, \tag{11.7}$$

shows that the function is absolutely integrable (see Exercise '`stat Int`' for verification of this formula). Hence, the function has a Fourier integral representation

$$e^{-ax^2} = \int_0^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda$$

where

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-ax^2}\cos\lambda x\,dx \quad \text{and} \quad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-ax^2}\sin\lambda x\,dx.$$

To evaluate $A(\lambda)$, we note that the presence of the exponential e^{-ax^2} permits differentiation under the integral to obtain

$$\frac{dA}{d\lambda} = \frac{1}{\pi}\int_{-\infty}^{\infty} -xe^{-ax^2}\sin\lambda x\,dx.$$

Integration by parts now gives

$$\frac{dA}{d\lambda} = \frac{1}{\pi}\left\{\frac{e^{-ax^2}}{2a}\sin\lambda x\right\}_{-\infty}^{\infty} - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-ax^2}}{2a}\lambda\cos\lambda x\,dx = -\frac{\lambda}{2a}A(\lambda).$$

In other words, $A(\lambda)$ must satisfy the ODE

$$\frac{dA}{d\lambda} + \frac{\lambda}{2a}A = 0.$$

An initial condition for this differential equation is

$$A(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \frac{1}{\sqrt{a\pi}}.$$

The solution of this problem is

$$A(\lambda) = \frac{1}{\sqrt{a\pi}}e^{-\lambda^2/(4a)}.$$

Because $e^{-ax^2}\sin\lambda x$ is an odd function, we quickly conclude that $B(\lambda) = 0$. We may therefore write that

$$e^{-ax^2} = \int_0^{\infty}\frac{e^{-\lambda^2/(4a)}}{\sqrt{a\pi}}\cos\lambda x\,d\lambda = \frac{1}{\sqrt{a\pi}}\int_0^{\infty} e^{-\lambda^2/(4a)}\cos\lambda x\,d\lambda.$$

An alternative derivation of $A(\lambda)$ using complex contour integrals is described in Exercise 12. Figures 11.1 illustrate convergence of this improper integral to e^{-ax^2} for three values of a. Figure 11.1a is a plot of $e^{-x^2/100}$ and its Fourier integral representation on the interval $0 \le x \le 20$ using an upper limit of integration equal to 1 to approximate the improper integral; the curves are indistinguishable. Figure 11.1b is a plot of e^{-x^2} and its Fourier integral representation for $0 \le x \le 3$ using an upper limit of 10; once again the curves are indistinguishable. Figure 11.1c is a plot of e^{-100x^2} and its Fourier integral representation on the interval $0 \le x \le 0.3$ using an upper limit of 25; it is not an adequate representation. In Figure 11.1d, we have used an upper limit of 50, and the representation is indistinguishable from the Gaussian.●

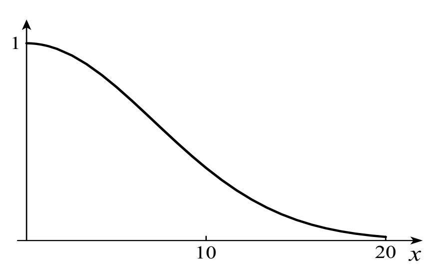


Figure 11.1a

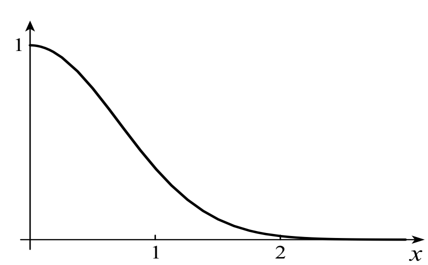


Figure 11.1b

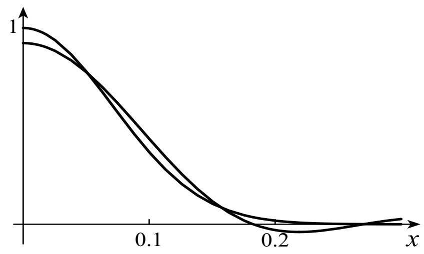


Figure 11.1c

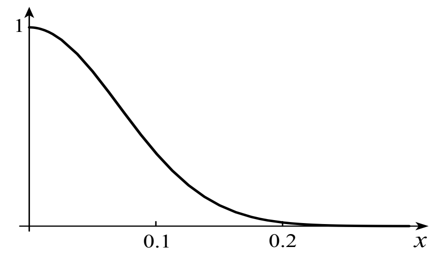


Figure 11.1d

Using a finite value for the upper limit of integration in formula 11.6a, as we did in this example, is equivalent to taking the partial sum of a Fourier series to approximate the series.

When a function $f(x)$ satisfying the conditions of Theorem 11.1 (or its corollary) is even, it is obvious that

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\cos\lambda x\,dx, \qquad B(\lambda) = 0. \tag{11.8b}$$

in which case

$$\frac{f(x+) + f(x-)}{2} = \int_0^\infty A(\lambda)\cos\lambda x\,d\lambda. \tag{11.8a}$$

This result is called the **Fourier cosine integral formula**. The function e^{-kx^2} in Example 11.1 is represented in the form of a Fourier cosine integral.

Example 11.2 Find the Fourier integral representation for the function

$$f(x) = \begin{cases} k(L - |x|)/L, & |x| \le L \\ 0, & |x| > L \end{cases}.$$

Solution Because $f(x)$ is even (Figure 11.2), it has a cosine integral representation, where

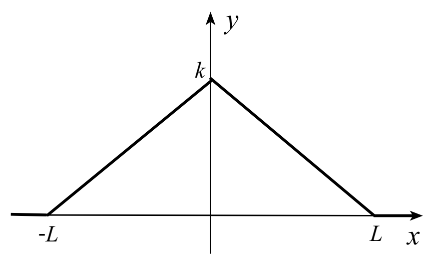


Figure 11.2

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi}\int_0^\infty f(x)\cos\lambda x\,dx = \frac{2}{\pi}\int_0^L \frac{k}{L}(L-x)\cos\lambda x\,dx \\ &= \frac{2k}{\pi L}\left\{\frac{L-x}{\lambda}\sin\lambda x - \frac{1}{\lambda^2}\cos\lambda x\right\}_0^L = \frac{2k}{\pi L\lambda^2}(1-\cos\lambda L). \end{aligned}$$

Since $f(x)$ is continuous, we may write

$$f(x) = \int_0^\infty \frac{2k}{\pi L\lambda^2}(1-\cos\lambda L)\cos\lambda x\,d\lambda = \frac{2k}{\pi L}\int_0^\infty \frac{1-\cos\lambda L}{\lambda^2}\cos\lambda x\,d\lambda.$$

Figure 11.3a shows how the Fourier cosine integral approximates $f(x)$ for $x > 0$ when the infinite interval of integration for the improper integral is replaced by the finite interval $0 \le \lambda \le 5$. The approximation is better in Figure 11.3b when the interval of integration is $0 \le \lambda \le 10$.●

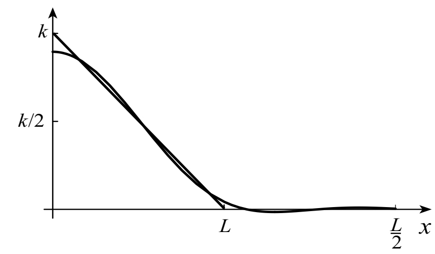


Figure 11.3a

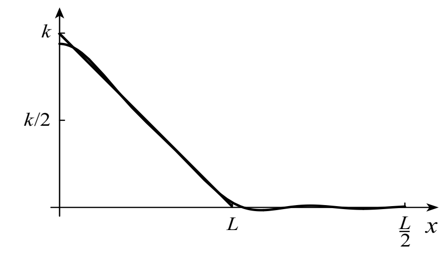


Figure 11.3b

When $f(x)$ is an odd function, coefficient $A(\lambda) = 0$, and $f(x)$ may be represented by the **Fourier sine integral formula**

$$\frac{f(x+) + f(x-)}{2} = \int_0^\infty B(\lambda) \sin \lambda x \, d\lambda, \tag{11.9a}$$

where

$$B(\lambda) = \frac{2}{\pi} \int_0^\infty f(x) \sin \lambda x \, dx. \tag{11.9b}$$

Example 11.3 Find the Fourier integral representation for the function

$$(\text{sgn} x)e^{-|x|} = \begin{cases} e^{-x}, & x > 0 \\ -e^{x}, & x < 0 \end{cases}.$$

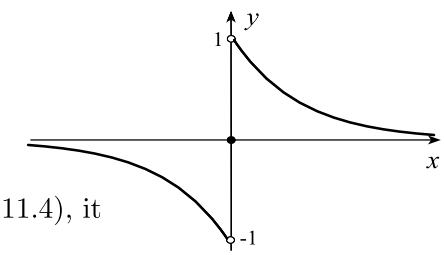


Figure 11.4

Solution Because the function is odd (Figure 11.4), it has a sine integral representation, where

$$B(\lambda) = \frac{2}{\pi} \int_0^\infty e^{-x} \sin \lambda x \, dx = \frac{2}{\pi} \left\{ \frac{-e^{-x}}{1 + \lambda^2} (\sin \lambda x + \lambda \cos \lambda x) \right\}_0^\infty = \frac{2\lambda}{\pi(1 + \lambda^2)}.$$

Hence,

$$(\text{sgn} x)e^{-|x|} = \int_0^\infty \frac{2\lambda}{\pi(1 + \lambda^2)} \sin \lambda x \, d\lambda,$$

provided the function is assigned the value zero at $x = 0$. Figure 11.5a shows how the Fourier sine integral approximates $f(x)$ for $x > 0$ when the infinite interval of integration for the improper integral is replaced by the finite interval $0 \le \lambda \le 10$. The approximation is better in Figure 11.5b when the interval of integration is $0 \le \lambda \le 20$. Convergence of the improper integral is much slower in this example, compared to previous examples, due to the discontinuity of the function at $x = 0$.●

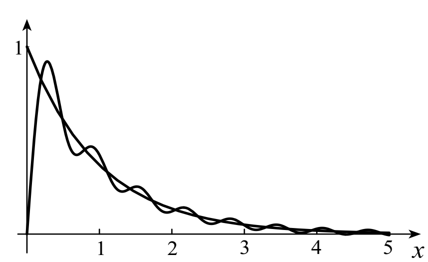


Figure 11.5a

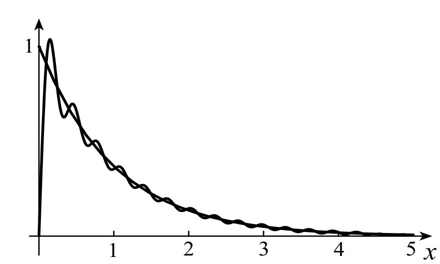


Figure 11.5b

The Fourier sine and cosine integral formulas also provide integral representations for functions that are defined only for $0 < x < \infty$. Indeed, when $f(x)$ is absolutely integrable on $0 < x < \infty$, and $f(x)$ is piecewise smooth on every finite interval $0 \le x \le X$, integrals 11.8 and 11.9 converge to $[f(x+)+f(x-)]/2$ for $x > 0$. At $x = 0$, the Fourier cosine integral converges to $f(0+)$, and the sine integral yields the value zero.

Theorem 11.1 would seem to eliminate many functions that we might wish to represent in the form of a Fourier integral. For instance, it would be quite reasonable to have a sinusoidal initial temperature distribution $f(x)$ in problem 11.1. But such a function is not absolutely integrable on $-\infty < x < \infty$; absolutely integrable functions must necessarily have limit zero as $x \to \pm\infty$. Thus, Fourier integrals cannot presently be used to solve problem 11.1 when $f(x)$ is sinusoidal. *Generalized functions*, the class of functions that contain the Dirac delta function as a special case (see Section 2.1 and Chapter 12) can be used to weaken the condition of absolute integrability. In this chapter, however, we shall maintain this restriction unless otherwise specified and concentrate our attention on how Fourier integrals and Fourier transforms are used to solve problems, rather than attempt to enlarge the class of problems to which the techniques can be applied.

Fourier integral formula 11.6 can be used, in conjunction with separation of variables, to solve problems with spatial domain $-\infty < x < \infty$. In many of these problems, Fourier integral 11.6 reduces to the cosine or sine integral 11.8 or 11.9. Additionally, sine and cosine integrals are useful for problems on the semi-infinite domain $0 < x < \infty$ when the boundary condition at $x = 0$ is Dirichlet or Neumann. We choose not to give illustrations; that is, we shall not show how to use Fourier integrals to solve homogeneous problems on infinite or semi-infinite intervals, and we do so for the following reason. Finite Fourier transforms in Chapter 7 deal with nonhomogeneities in PDEs and boundary conditions for problems on finite domains; they are not used on homogeneous problems where separation of variables and generalized Fourier series suffice. *Fourier transforms*, which will be introduced in Sections 11.3 and 11.5, once again handle nonhomogeneities in a PDE or boundary condition (on infinite and semi-infinite intervals), but they often provide simpler solutions for homogeneous problems. As a result, it is preferable to avoid separation of variables and Fourier integrals and use Fourier transforms on all problems, homogeneous and nonhomogeneous.

EXERCISES 11.2

In Exercises 1–5 find the Fourier integral representation of the function. Draw a graph of the function to which the integral converges. In addition, plot the Fourier integral on the given interval using the suggested values of λ to illustrate its approximation to $f(x)$.

1. $f(x) = e^{-a|x|}$, $a > 0$ constant Use $a = 1$ on $-5 \le x \le 5$ with $0 \le \lambda \le 5$.

2. $f(x) = h(x-a) - h(x-b)$, $b > a$ constants. $h(x-a)$ is the Heaviside unit step function
Use $a = 1$ and $b = 2$ on $0 \le x \le 3$ with $0 \le \lambda \le 20$.

3. $f(x) = \begin{cases} (b/a)(a-|x|), & |x| < a \\ 0, & |x| > a \end{cases}$, $a > 0$, $b > 0$ constants Use $a = 1$ and $b = 1$ on $-2 \le x \le 2$ with $0 \le \lambda \le 10$.

4. $f(x) = \begin{cases} b(a^2 - x^2)/a^2, & |x| < a \\ 0, & |x| > a \end{cases}$, $a > 0$, $b > 0$ constants Use $a = 1$ and $b = 1$ on $-2 \le x \le 2$ with $0 \le \lambda \le 10$.

5. $f(x) = e^{-ax}h(x)$, $a > 0$ constant, where $h(x)$ is the Heaviside unit step function. Use $a = 1$ on $-1 \le x \le 3$ with $0 \le \lambda \le 20$.

6. What is the Fourier cosine integral for the function $f(x) = e^{-kx^2}$ $(k > 0)$, defined only for $x \ge 0$?

In Exercises 7–10 $f(x)$ is defined only for $x \ge 0$. Find its Fourier sine and cosine integral representations. To what does each integral converge at $x = 0$?

7. $f(x) = h(x - a) - h(x - b)$, $b > a > 0$ constants (see Exercise 2)

8. $f(x) = \begin{cases} (b/a)(a - |x - c|), & |x - c| < a \\ 0, & |x - c| > a \end{cases}$, a, b, and c positive constants with $c > a > 0$

9. $f(x) = e^{-ax}\cos bx$, $a > 0$, $b > 0$ constants

10. $f(x) = e^{-ax}\sin bx$, $a > 0$, $b > 0$ constants

11. To evaluate

$$I = \int_{-\infty}^{\infty} e^{-kx^2}\,dx = 2\int_0^{\infty} e^{-kx^2}\,dx,$$

we write

$$\frac{I^2}{4} = \left(\int_0^{\infty} e^{-kx^2}\,dx\right)\left(\int_0^{\infty} e^{-ky^2}\,dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-k(x^2+y^2)}\,dx$$

and transform the double integral to polar coordinates. Show that $I = \sqrt{\pi/k}$.

12. In this exercise we use complex residue theory to evaluate $A(\lambda)$ in Example 11.1.

(a) Transform the complex combination of real integrals

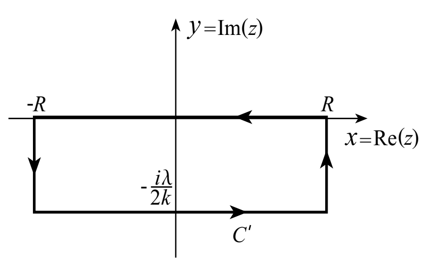
$$I = \int_{-\infty}^{\infty} e^{-ax^2} e^{i\lambda x}\,dx$$

by means of $z = x - i\lambda/(2a)$ into the contour integral

$$I = e^{-\lambda^2/(4a)} \int_C e^{-az^2}\,dz$$

along the line $\mathrm{Im}(z) = -i\lambda/(2a)$.

(b) Use the contour integral $\oint_{C'} e^{-az^2}\,dz$, where C' is the rectangle in the figure below, to find I.



(c) Take real and imaginary parts of I to find $A(\lambda)$ and $B(\lambda)$.

§11.3 The Fourier Transform

In this section we introduce the Fourier transform in order to solve nonhomogeneous initial boundary value problems on the infinite interval $-\infty < x < \infty$. But as suggested in Section 11.2, they are also advantageous for solving homogeneous problems. To obtain the transform, we express Fourier integral 11.6a in complex form, reminiscent of the complex form for Fourier series (see Exercise 27 in Section 3.1),

$$\begin{aligned}
\frac{f(x+)+f(x-)}{2} &= \int_0^\infty \left[A(\lambda)\left(\frac{e^{i\lambda x}+e^{-i\lambda x}}{2}\right)+B(\lambda)\left(\frac{e^{i\lambda x}-e^{-i\lambda x}}{2i}\right)\right] d\lambda \\
&= \int_0^\infty \left[e^{i\lambda x}\left(\frac{A(\lambda)-iB(\lambda)}{2}\right)+e^{-i\lambda x}\left(\frac{A(\lambda)+iB(\lambda)}{2}\right)\right] d\lambda \\
&= \int_0^\infty e^{i\lambda x}\left[\frac{A(\lambda)-iB(\lambda)}{2}\right] d\lambda + \int_0^{-\infty} e^{i\lambda x}\left[\frac{A(-\lambda)+iB(-\lambda)}{2}\right](-d\lambda) \\
&= \int_0^\infty C(\lambda)e^{i\lambda x}d\lambda + \int_{-\infty}^0 C(\lambda)e^{i\lambda x}d\lambda = \int_{-\infty}^\infty C(\lambda)e^{i\lambda x}d\lambda,
\end{aligned}$$

where

$$C(\lambda) = \begin{cases} [A(\lambda)-iB(\lambda)]/2, & \lambda > 0 \\ [A(-\lambda)+iB(-\lambda)]/2, & \lambda < 0 \end{cases}.$$

But using equation 11.6b, we may write, for $\lambda > 0$,

$$C(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\cos\lambda x\,dx - \frac{i}{2\pi}\int_{-\infty}^\infty f(x)\sin\lambda x\,dx = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}dx,$$

and for $\lambda < 0$,

$$C(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\cos(-\lambda x)\,dx + \frac{i}{2\pi}\int_{-\infty}^\infty f(x)\sin(-\lambda x)\,dx = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}dx.$$

If, as has been our custom, we define, or redefine, if necessary, $f(x)$ as the average value of left- and right-hand limits at any point of discontinuity, we have shown that Fourier integral 11.6 may be expressed in the complex form

$$f(x) = \int_{-\infty}^\infty C(\lambda)e^{i\lambda x}d\lambda \quad \text{where} \quad C(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}dx. \tag{11.10a}$$

A somewhat more critical analysis of the improper integrals leading to the first integral in 11.10a indicates that the integral should be taken in the sense of Cauchy's principal value,

$$f(x) = \lim_{R\to\infty}\int_{-R}^R C(\lambda)e^{i\lambda x}dx \tag{11.10b}$$

(see Exercise 32). We shall continue to write the integral in 11.10a for brevity, but if convergence difficulties arise, we shall replace it with 11.10b.

It is clear that by redefining $C(\lambda)$, we could also write

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty C(\lambda)e^{i\lambda x}d\lambda \quad \text{where} \quad C(\lambda) = \int_{-\infty}^\infty f(x)e^{-i\lambda x}dx, \tag{11.11}$$

or,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(\lambda) e^{i\lambda x} d\lambda \quad \text{where} \quad C(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx. \quad (11.12)$$

Any of the these three pairs can be used to define the Fourier transform; we pick the second simply because it involves the factor 2π only in the latter stages of applications. It is customary to use ω in place of λ for Fourier transforms.

Definition 11.2 The **Fourier transform** of a function $f(x)$ is defined as

$$\tilde{f}(\omega) = \mathcal{F}\{f(x)\}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx. \quad (11.13a)$$

The associated inverse transform is

$$f(x) = \mathcal{F}^{-1}\{\tilde{f}(\omega)\}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega x} d\omega. \quad (11.13b)$$

Convergence of the integral is guaranteed when $f(x)$ is piecewise smooth on every finite interval and absolutely integrable on $-\infty < x < \infty$. When these conditions are satisfied, definition 11.13 leads to a continuous Fourier transform function. We state this as the first of many properties of Fourier transforms.

Theorem 11.2 The Fourier transform of a function that is piecewise smooth on every finite interval and absolutely integrable for $-\infty < x < \infty$ is a continuous function.

Absolute integrability of $f(x)$ is a much more demanding condition than being of exponential order for existence of the Laplace transform of a function $f(t)$. This is due to the fact that in the definition of the Laplace transform, the factor e^{-st} suppresses $f(t)$ substantially for large t, but the factor $e^{-i\omega x} = \cos\omega x - i\sin\omega x$ in definition 11.13 has no such effect, the function $f(x)$ must be integrable on its own. This effectively eliminates most of the functions that we worked with in Chapter 10 such as polynomials, sines and cosines, and exponentials; they are not absolutely integrable on the real line.

We have introduced Fourier transform 11.13 in order to solve (initial) boundary value problems on infinite intervals $-\infty < x < \infty$. We shall show how to do this in Section 11.4. Although the finite Fourier transform of Chapter 7 was introduced for similar problems on finite domains, our treatment of the finite and "infinite" transforms are quite different. There are many finite Fourier transforms (each associated with a Sturm-Liouville system); because of this, we made no attempt to discuss general properties of finite Fourier transforms. As a result, when we apply a finite Fourier transform to a PDE, we must work our way through the integrals involved, bringing into play boundary and/or initial conditions at appropriate times. On the other hand, because there are only three Fourier transforms (the above transform and the sine and cosine transforms in Section 11.5), it is possible to develop properties of these transforms that make it unnecessary to return to their integral definitions when solving (initial) boundary value problems. This makes it much simpler to apply Fourier transforms; it is reminiscent of our treatment of Laplace transforms in Chapter 10.

Although we have used the tilde notation, $\tilde{f}$, for both the Laplace transform and the Fourier transform, and we will also use it for Fourier sine and cosine transforms in Section 11.5, context always makes it clear which transform is appropriate. In addition, none of the problems that we consider in this chapter simultaneously use more than one type of transform.

Equations 11.13 should be compared with equations 7.3 in Chapter 7 for the finite Fourier transform. Finite Fourier transforms are associated with Sturm-Liouville systems. When $[\lambda_n, y_n(x)]$ are eigenpairs of Sturm-Liouville system 5.3 in Chapter 5, the finite Fourier transform of a function $f(x)$ is

$$\tilde{f}(\lambda_n) = \int_a^b p(x)f(x)y_n(x)\,dx,$$

and the inverse transform is

$$f(x) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_n)y_n(x).$$

The finite Fourier transform is a sequence of numbers $\{\tilde{f}(\lambda_n)\}$, or a discrete function defined only for integers n; the inverse transform is a superposition over all eigenfunctions. Fourier transform 11.13a defines a continuous function $\tilde{f}(\omega)$, and this is due to the fact that "eigenfunctions" of the differential equation

$$\frac{d^2X}{dx^2} + \omega^2 X = 0, \quad X(x) \text{ bounded},$$

are $A\cos\omega x + B\sin\omega x$, where "eigenvalues" ω are arbitrary. Inverse transform 11.13b is an integral superposition over all ω.

It is straightforward to identify the real and imaginary parts of $\tilde{f}(\omega)$,

$$\begin{aligned}\tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(x)(\cos\omega x - i\sin\omega x)\,dx \\ &= \int_{-\infty}^{\infty} f(x)\cos\omega x\,dx - i\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx.\end{aligned}$$

If $f(x)$ is an even function, then the second of these integrals vanishes, and $f(x)$ is odd, the first vanishes.

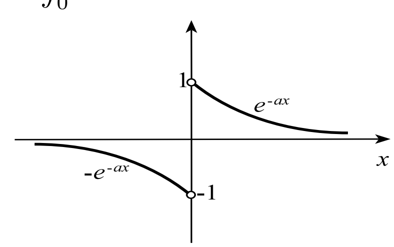
Theorem 11.3 When $f(x)$ is an even function, its Fourier transform is a real function, given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = 2\int_0^{\infty} f(x)\cos\omega x\,dx; \tag{11.14a}$$

and when $f(x)$ is an odd function, its Fourier transform is purely imaginary, given by

$$\tilde{f}(\omega) = -i\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = -2i\int_0^{\infty} f(x)\sin\omega x\,dx. \tag{11.14b}$$

Example 11.4 Find the Fourier transform of the function in Figure 11.6 where $a > 0$ is a constant.



Solution Because the function is continuous and absolutely integrable on the real line, its Fourier transform exists. Furthermore, since the function is odd, its Fourier transform is given by equation 11.14b,

Figure 11.6

$$\tilde{f}(\omega) = -2i \int_0^{\infty} e^{-ax} \sin \omega x \, dx.$$

We could integrate this by parts, but it is easier to use a complex integral,

$$\tilde{f}(\omega) = -2i \,\mathrm{Im}\left[\int_0^{\infty} e^{-ax} e^{i\omega x} dx\right] = -2i \,\mathrm{Im}\left[\int_0^{\infty} e^{(-a+\omega i)x} dx\right]$$
$$= -2i \,\mathrm{Im}\left[\left\{\frac{e^{(-a+\omega i)x}}{-a+\omega i}\right\}_0^{\infty}\right] = -2i \,\mathrm{Im}\left[\frac{1}{a-\omega i}\right] = \frac{-2\omega i}{a^2+\omega^2}. \bullet$$

Because the real and imaginary parts of the Fourier transform are improper integrals (see equation 11.13), they can often be calculated using residues from the theory of complex functions. This is especially so when $f(x)$ is a rational function of x. We illustrate this in the next example.

Example 11.5 Find the Fourier transform of the function $f(x) = \dfrac{1}{a^2+x^2}$, where $a > 0$ is a constant.

Solution Because the function is continuous and absolutely integrable on the real line, its Fourier transform exists. Furthermore, since the function is even, its Fourier transform is given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} \frac{\cos \omega x}{a^2+x^2} dx.$$

We use residues to evaluate this integral. According to equation D.17 in Appendix D,

$$\tilde{f}(\omega) = -2\pi \,\mathrm{Im}\left\{\mathrm{Res}\left[\frac{e^{i\omega z}}{a^2+z^2}, ai\right]\right\},$$

provided $\omega > 0$. For $\omega > 0$ then,

$$\tilde{f}(\omega) = -2\pi \,\mathrm{Im}\left\{\lim_{z \to ai}\left[\frac{(z-ai)e^{i\omega z}}{(z+ai)(z-ai)}\right]\right\} = \frac{\pi}{a} e^{-a\omega}.$$

The integral definition of $\tilde{f}(\omega)$ makes it clear that $\tilde{f}(\omega)$ is an even function of ω. Consequently, when $\omega < 0$, we write that

$$\tilde{f}(\omega) = \tilde{f}(-\omega) = \frac{\pi}{a} e^{-a(-\omega)} = \frac{\pi}{a} e^{a\omega}.$$

Both cases $\omega > 0$ and $\omega < 0$ are contained in

$$\tilde{f}(\omega) = \frac{\pi}{a} e^{-a|\omega|}.$$

This result is also valid for $\omega = 0$ since

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} \frac{1}{a^2+x^2} dx = \frac{\pi}{a}. \bullet$$

In our third example, the function $f(x)$ is defined only over a finite interval so that the improper integral in definition 11.13 reduces to an ordinary integral.

Example 11.6 Find the Fourier transform for the function in Figure 11.7.

Solution Since the function is even, continuous, and absolutely integrable, its Fourier transform is given by equation 11.14a,

$$\tilde{f}(\omega) = 2\int_0^\infty f(x)\cos\omega x\,dx$$
$$= 2\int_0^1 (1-x)\cos\omega x\,dx.$$

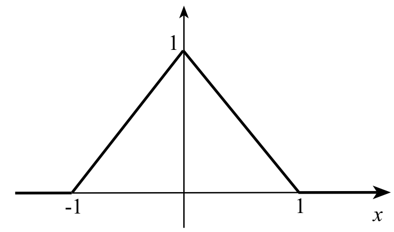


Figure 11.7

Integration by parts gives

$$\tilde{f}(\omega) = 2\left\{\frac{1}{\omega}(1-x)\sin\omega x\right\}_0^1 - 2\int_0^1 -\frac{1}{\omega}\sin\omega x\,dx \quad (\text{provided } \omega \neq 0)$$
$$= \frac{2}{\omega}\left\{-\frac{1}{\omega}\cos\omega x\right\}_0^1 = \frac{2}{\omega^2}(1-\cos\omega) = \frac{4}{\omega^2}\sin^2\frac{\omega}{2}.$$

When $\omega = 0$,

$$\tilde{f}(0) = 2\int_0^\infty f(x)\,dx = 2\int_0^1 (1-x)\,dx = 2\left\{x - \frac{x^2}{2}\right\}_0^1 = 1.$$

Theorem 11.2 promised continuity of the Fourier transform. The only value of ω at which continuity could be questioned is $\omega = 0$. Since $\tilde{f}(0)$ is defined, continuity is established if $\lim_{\omega\to 0}\tilde{f}(\omega) = 1$ also. This is easily verified

$$\lim_{\omega\to 0}\tilde{f}(\omega) = \lim_{\omega\to 0}\left(\frac{4}{\omega^2}\sin^2\frac{\omega}{2}\right) = \lim_{\omega\to 0}\left[\frac{\sin(\omega/2)}{\omega/2}\right]^2 = 1.$$

In other words, we can write that for all ω,

$$\tilde{f}(\omega) = \frac{4}{\omega^2}\sin^2\frac{\omega}{2},$$

knowing that the limit as $\omega \to 0$ gives the value of the transform at $\omega = 0$.●

One of the most important functions in the application of Fourier transforms to heat conduction problems is contained in the following example.

Example 11.7 Find the Fourier transform of e^{-ax^2}.

Solution The function is continuous and absolutely integrable (see Example 11.1). Once again the function is even, and therefore its Fourier transform is given by equation 11.14a,

$$\tilde{f}(\omega) = 2\int_0^\infty e^{-ax^2}\cos\omega x\,dx.$$

This integral was evaluated in Example 11.1 and Exercise 12 of Section 11.2. The result is

$$\tilde{f}(\omega) = \sqrt{\frac{\pi}{a}}e^{-\omega^2/(4a)}.●$$

The Inverse Fourier Transform

When $\tilde{f}$ is the Fourier transform of f, we say that f is the inverse Fourier transform of $\tilde{f}$, and we write $f = \mathcal{F}^{-1}\{\tilde{f}\}$. It can be retrieved from its transform with the following improper integral

$$f(x) = \mathcal{F}^{-1}\{\tilde{f}\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega x}d\omega. \tag{11.15}$$

Should convergence difficulties be associated with this integral, it should be interpreted as its Cauchy's principal value,

$$f(x) = \frac{1}{2\pi}\,⨍_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega x}d\omega = \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R} \tilde{f}(\omega)e^{i\omega x}d\omega. \tag{11.16}$$

Furthermore, be reminded that these equations are valid with the agreement that $f(x)$ is defined, or redefined if necessary, at points of discontinuity as its average of right- and left-hnad limits.

Contour integral 10.24 for the inverse Laplace transform was unusable, but fortunately, it was replaced by residues. The improper integral in equation 11.15 for the inverse Fourier transform is calculationally viable, but there are often more efficient ways to evaluate the inverse transform.

In order that definitions of Fourier transform and inverse Fourier transforms be more symmetric, some authors multiply each of the integrals in equations 11.13 and 11.15 by $1/\sqrt{2\pi}$. Others interchange the exponentials using $e^{i\omega x}$ for the Fourier transform and $e^{-i\omega x}$ for the inverse transform. Whichever convention is adopted, solutions of initial boundary value problems are ultimately identical.

Corresponding to Theorem 11.3 for the Fourier transform, we have the following for the inverse transform.

Theorem 11.4 When $\tilde{f}(\omega)$ is a (real and) even function, its inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\cos\omega x\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} \tilde{f}(\omega)\cos\omega x\,d\omega; \tag{11.17a}$$

and when $\tilde{f}(\omega)$ is a (real and) odd function, its inverse Fourier transform is given by

$$f(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\sin\omega x\,d\omega = \frac{i}{\pi}\int_{0}^{\infty} \tilde{f}(\omega)\sin\omega x\,d\omega. \tag{11.17b}$$

Example 11.8 Find the inverse Fourier transform of $\tilde{f}(\omega) = \dfrac{1}{(a^2+\omega^2)^2}$, where $a > 0$ is a constant.

Solution Because the transform is even, we can use equation 11.17a,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\omega x}{(a^2+\omega^2)^2}d\omega.$$

We use residues to evaluate this integral. According to equation D.17a in Appendix D,

$$\tilde{f}(\omega) = \frac{1}{2\pi}\left\{-2\pi\,\mathrm{Im}\left\{\mathrm{Res}\left[\frac{e^{ixz}}{(a^2+z^2)^2}, ai\right]\right\}\right\},$$

provided $x > 0$. For $x > 0$ then,

$$f(x) = -\operatorname{Im}\left\{\operatorname{Res}\left[\frac{e^{ixz}}{(a^2+z^2)^2}, ai\right]\right\} = -\operatorname{Im}\left\{\lim_{z\to ai}\frac{d}{dz}\left[\frac{(z-ai)^2 e^{ixz}}{(z+ai)^2(z-ai)^2}\right]\right\}$$

$$= -\operatorname{Im}\left\{\lim_{z\to ai}\left[\frac{ix(z+ai)^2e^{ixz} - 2e^{ixz}(z+ai)}{(z+ai)^4}\right]\right\}$$

$$= -\operatorname{Im}\left\{\frac{ix(2ai)e^{-ax} - 2e^{-ax}}{(2ai)^3}\right\} = \frac{1}{4a^3}(1+ax)e^{-ax}.$$

The integral representation of $f(x)$ makes it clear that $f(x)$ is an even function of x. Consequently, when $x < 0$, we write that

$$f(x) = f(-x) = \frac{1}{4a^3}(1-ax)e^{ax}.$$

Both cases $x > 0$ and $x < 0$ are contained in

$$f(x) = \frac{1}{4a^3}(1+a|x|)e^{-a|x|}.$$

In addition, this formula is also valid at $x = 0$ since

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(a^2+\omega^2)^2}d\omega = \frac{1}{4a^3}.\bullet$$

Example 11.9 Find the inverse Fourier transform of $\tilde{f}(\omega) = e^{-a\omega^2}$?

Solution We could evaluate the improper integral in equation 11.15, but it is easier to use the result of Example 11.7,

$$\mathcal{F}\left\{e^{-ax^2}\right\} = \sqrt{\frac{\pi}{a}}e^{-\omega^2/(4a)}.$$

If we replace a with $1/(4a)$, we get

$$\mathcal{F}\left\{e^{-x^2/(4a)}\right\} = \sqrt{4\pi a}e^{-a\omega^2} \quad \text{and therefore} \quad \mathcal{F}^{-1}\left\{e^{-a\omega^2}\right\} = \frac{1}{2\sqrt{a\pi}}e^{-x^2/(4a)}.\bullet$$

Example 11.10 Find the Fourier transform for the function $f(x) = h(x+a) - h(x-a)$ (Figure 11.8a) and illustrate graphically that integral 11.17a converges to $[f(x+) + f(x-)]/2$.

Solution Because the function is even, its Fourier transform is given by equation 11.14a,

$$\tilde{f}(\omega) = 2\int_0^{\infty}[h(x+a) - h(x-a)]\cos\omega x\,dx = 2\int_0^a \cos\omega x\,dx = 2\left\{\frac{\sin\omega x}{\omega}\right\}_0^a = \frac{2}{\omega}\sin a\omega,$$

provided $\omega \neq 0$. But

$$\tilde{f}(0) = 2\int_0^{\infty}[h(x+a) - h(x-a)]\,dx = 2\int_0^a dx = 2a.$$

This is the limit of $\tilde{f}(\omega)$ as $\omega \to 0$, and therefore we write for all ω,

$$\tilde{f}(\omega) = \frac{2}{\omega}\sin a\omega.$$

Since $\tilde{f}(\omega)$ is an even function and $f(x)$ has discontinuities, we use formula 11.17a to write

$$\frac{f(x+)+f(x-)}{2} = \frac{2}{\pi}\int_0^{\infty} \frac{1}{\omega} \sin a\omega \cos \omega x \, d\omega.$$

We have shown approximations to this improper integral with $a = 1$ in Figures 11.8a,b using upper limits of integration equal to 50 and 100, respectively.●

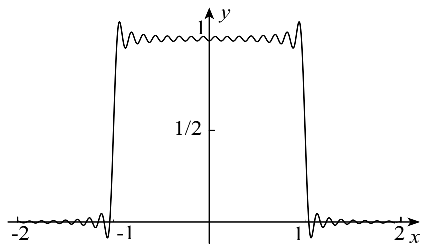


Figure 11.8a

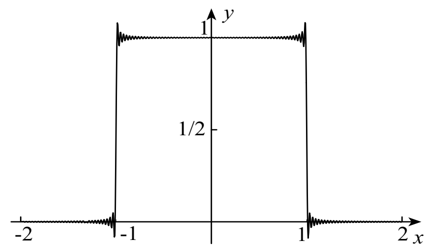


Figure 11.8b

We use Fourier transforms to solve (initial) boundary value problems on infinite domains. We shall show how to do this in Section 11.4. We prepare the way by developing properties of the transform and its inverse that make it unnecessary to return to integral definitions 11.13 and 11.15 each time a Fourier transform and its inverse are required. Many of these properties are sufficiently easy to verify that proofs are relegated to the exercises. First, we note that the Fourier transform and its inverse are linear operators; that is,

$$\mathcal{F}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{F}\{f_1\} + c_2\mathcal{F}\{f_2\}, \tag{11.18a}$$

$$\mathcal{F}^{-1}\{c_1 \tilde{f}_1 + c_2 \tilde{f}_2\} = c_1\mathcal{F}^{-1}\{\tilde{f}_1\} + c_2\mathcal{F}^{-1}\{\tilde{f}_2\}. \tag{11.18b}$$

(See Exercise 1 for verification.)

Shifting Properties of the Fourier Transform

Like shifting property 10.3 for the Laplace transform, we have the following shifting property for the Fourier transform.

Theorem 11.5 When $f(x)$ is piecewise smooth on every finite interval, and $f(x)$ and $e^{-ax}f(x)$ are absolutely integrable on the real line,

$$\mathcal{F}\{e^{-ax}f(x)\}(\omega) = \tilde{f}(\omega - ai), \tag{11.19a}$$

$$\mathcal{F}^{-1}\{\tilde{f}(\omega - ai)\}(x) = e^{-ax}f(x). \tag{11.19b}$$

(See Exercise 3 for a proof).

Example 11.11 Find the Fourier transform of the function $f(x) = e^{-bx}[h(x+a) - h(x-a)]$, where a and b are positive constants.

Solution Example 11.10 derived the transform of $h(x+a) - h(x-a)$,

$$\mathcal{F}\{h(x+a) - h(x-a)\} = \frac{2}{\omega}\sin a\omega.$$

Equation 11.19a now gives

$$\tilde{f}(\omega) = \frac{2}{\omega - bi}\sin a(\omega - bi).●$$

Example 11.12 Find the Fourier transform of the function $f(x) = e^{-ax}h(x)$, where a is a positive constant. Can you use property 11.19a?

Solution We cannot use property 11.19a since $h(x)$ is not absolutely integrable on the real line. We return to definition 11.13,

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-ax}h(x)e^{-i\omega x}\,dx = \int_{0}^{\infty} e^{-(a+\omega i)x}\,dx = \left\{\frac{e^{-(a+\omega i)x}}{-(a+\omega i)}\right\}_0^{\infty} = \frac{1}{a+\omega i}.\bullet$$

We cannot help but notice the similarity of this result to the Laplace transform $1/(s+a)$ for the function e^{-at}. The function e^{-ax} does not have a Fourier transform (since it is not absolutely integrable on the real line), but $e^{-ax}h(x)$ is absolutely integrable. General discussions of this situation can be found in Exercise 33.

The second shifting property is contained in the next theorem. It should be compared to property 10.4 of the Laplace transform.

Theorem 11.6 When $f(x)$ is piecewise smooth on every finite interval and absolutely integrable on the real line,

$$\mathcal{F}\{f(x-a)\}(\omega) = e^{-ia\omega}\mathcal{F}\{f\}(\omega), \tag{11.20a}$$

$$\mathcal{F}^{-1}\{e^{-ia\omega}\tilde{f}\}(x) = \mathcal{F}^{-1}\{\tilde{f}\}(x-a). \tag{11.20b}$$

(See Exercise 2 for verification).

Example 11.13 Find the Fourier transform of the function in Figure 11.9.

Solution This is the function in Figure 11.7 translated 3 units to the right. Hence,

$$\tilde{f}(\omega) = \frac{4e^{-3\omega i}}{\omega^2}\sin^2\frac{\omega}{2}.\bullet$$

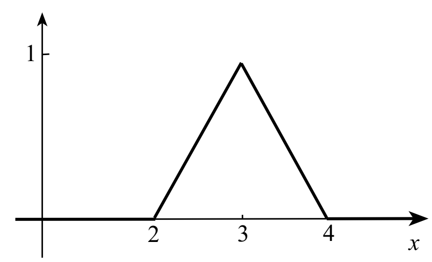


Figure 11.9

Multiplication by x^n

The following theorem indicates that multiplying a function $f(x)$ by x^n results in its Fourier transform being differentiated n times. Compare this to the analogous property 10.5 for the Laplace transform.

Theorem 11.7 If $f(x)$ is piecewise smooth on every finite interval and $f(x)$ and $x^n f(x)$, $n > 0$ an integer, are absolutely integrable on the real line, then

$$\mathcal{F}\{x^n f(x)\}(\omega) = i^n \frac{d^n}{d\omega^n}[\mathcal{F}\{f\}(\omega)], \tag{11.21a}$$

$$\mathcal{F}^{-1}\{\tilde{f}^{(n)}(\omega)\}(x) = (-ix)^n \mathcal{F}^{-1}\{\tilde{f}\}(x). \tag{11.21b}$$

(See Exercise 4 for a proof.)

Example 11.14 Find the Fourier transform of $f(x) = x^2 e^{-ax^2}$.

Solution Since $\mathcal{F}\{e^{-ax^2}\} = \sqrt{\pi/a}e^{-\omega^2/(4a)}$ (see Example 11.7), property 11.21a gives

$$\mathcal{F}\{x^2 e^{-ax^2}\} = i^2 \frac{d^2}{d\omega^2}\left[\sqrt{\frac{\pi}{a}} e^{-\omega^2/(4a)}\right] = \frac{\sqrt{\pi}}{2a^{3/2}}\left(1 - \frac{\omega^2}{2a}\right) e^{-\omega^2/(4a)}. \bullet$$

Transforms of Derivatives

The following theorem and its corollary eliminate much of the work when Fourier transforms are applied to (initial) boundary value problems.

Theorem 11.8 Suppose $f(x)$ is continuous for $-\infty < x < \infty$ and $f'(x)$ is piecewise continuous on every finite interval. If both functions are absolutely integrable on $-\infty < x < \infty$, then

$$\mathcal{F}\{f'(x)\} = i\omega \mathcal{F}\{f(x)\}, \tag{11.22a}$$

$$\mathcal{F}^{-1}\{i\omega \tilde{f}(\omega)\} = \frac{d}{dx}[\mathcal{F}^{-1}\{\tilde{f}(\omega)\}]. \tag{11.22b}$$

Proof When integration by parts is used on the definition of $\mathcal{F}\{f'(x)\}$,

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x) e^{-i\omega x} dx = \left\{f(x) e^{-i\omega x}\right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega) e^{-i\omega x} dx$$
$$= i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = i\omega \mathcal{F}\{f(x)\}. \blacksquare$$

It is straightforward to extend this result to second derivatives (see the corollary below) and higher-order derivatives (see Exercise 5).

Corollary Suppose $f(x)$ and $f'(x)$ are continuous for $-\infty < x < \infty$ and $f''(x)$ is piecewise continuous on every finite interval. If all three functions are absolutely integrable on $-\infty < x < \infty$, then

$$\mathcal{F}\{f''(x)\} = -\omega^2 \mathcal{F}\{f(x)\}, \tag{11.23a}$$

$$\mathcal{F}^{-1}\{-\omega^2 \tilde{f}(\omega)\} = \frac{d^2}{dx^2}[\mathcal{F}^{-1}\{\tilde{f}(\omega)\}]. \tag{11.23b}$$

Convolutions and the Fourier Transform

In applications of Fourier transforms to initial boundary value problems, it is often necessary to find the inverse transform of the product of two functions $\tilde{f}$ and $\tilde{g}$, both of whose inverse transforms are known; that is, we require $\mathcal{F}^{-1}\{\tilde{f}\tilde{g}\}$, knowing that $\mathcal{F}^{-1}\{\tilde{f}\} = f$ and $\mathcal{F}^{-1}\{\tilde{g}\} = g$. In Theorem 11.9, it is shown that

$$\mathcal{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\} = \int_{-\infty}^{\infty} f(v) g(x - v)\, dv.$$

This integral, called the **convolution** of the functions $f(x)$ and $g(x)$, is often given the notation $(f * g)(x)$ or $f(x) * g(x)$:

$$(f * g)(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(v) g(x - v)\, dv. \tag{11.24}$$

Comparison of convolutions 10.8 for Laplace transforms and 11.24 for Fourier transforms indicates that integrands are identical, only limits of integration change.

Theorem 11.9 Suppose that $f(x)$ and $g(x)$ are piecewise smooth on every finite interval and absolutely integrable on $-\infty < x < \infty$. If either $\tilde{f}(\omega)$ or $\tilde{g}(\omega)$ is absolutely integrable on $-\infty < \omega < \infty$, then

$$\mathcal{F}^{-1}\{\tilde{f}\tilde{g}\} = f * g. \tag{11.25}$$

Proof Let us assume that $\tilde{g}(\omega)$ is absolutely integrable. (The proof is similar if $\tilde{f}(\omega)$ is absolutely integrable.) By equation 11.13b,

$$\mathcal{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\tilde{g}(\omega)e^{i\omega x}d\omega,$$

and when we substitute the integal definition of $\tilde{f}(\omega)$,

$$\mathcal{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right]\tilde{g}(\omega)e^{i\omega x}d\omega.$$

The fact that $f(x)$ and $\tilde{g}(\omega)$ are both absolutely integrable permits us to interchange the order of integration and write

$$\mathcal{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\}(x) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(\omega)e^{i\omega(x-v)}d\omega\right]f(v)\,dv = \int_{-\infty}^{\infty} f(v)g(x-v)\,dv. \blacksquare$$

The simplicity of the proof of Theorem 11.9 is a direct result of the assumption that $\tilde{g}(\omega)$ is absolutely integrable. This condition can be weakened, but because functions that we encounter satisfy this condition, we pursue the discussion no further.

By making a change of variable of integration in equation 11.24, it is easily shown that convolutions are symmetric; that is, $f*g = g*f$. Other properties of convolutions are discussed in Exercise 6. An example of convolutions that we encounter in heat conduction problems is finding the inverse transform of $\tilde{f}(\omega)e^{-k\omega^2 t}$ where $\tilde{f}(\omega)$ is the transform of an initial temperature distribution, k is thermal diffusivity, and t is time. According to Example 11.9, $\mathcal{F}^{-1}\{e^{-k\omega^2 t}\} = [1/(2\sqrt{k\pi t})]e^{-x^2/(4kt)}$, and hence convolutions yield

$$\mathcal{F}^{-1}\{\tilde{f}(\omega)e^{-k\omega^2 t}\} = \int_{-\infty}^{\infty} f(v)\frac{1}{2\sqrt{k\pi t}}e^{-(x-v)^2/(4kt)}dv = \frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(v)e^{-(x-v)^2/(4kt)}dv.$$

In this chapter, we use the Dirac delta function to model heat source at a point, and a force applied to a vibrating string at a point. As a result, we need its Fourier transform. With defining relation 2.13,

$$\mathcal{F}\{\delta(x-x_0)\}(\omega) = \int_{-\infty}^{\infty}\delta(x-x_0)e^{-i\omega x}\,dx = e^{-i\omega x_0}. \tag{11.26}$$

EXERCISES 11.3

1. Verify that the Fourier transform and its inverse are linear operators.

2. Verify property 11.20a.

3. Verify property 11.19a.

4. Verify property 11.21a.

5. Verify property 11.23a.

6. Verify the following properties for convolutions:

$$f * g = g * f \tag{11.27a}$$
$$f * (kg) = (kf) * g = k(f * g), \quad k = \text{constant}, \tag{11.27b}$$
$$(f * g) * h = f * (g * h) \tag{11.27c}$$
$$f * (g + h) = f * g + f * h \tag{11.27d}$$

7. Verify the following shifting properties for the Fourier transform $\tilde{f}(\omega) = \mathcal{F}\{f\}(\omega)$:

$$\mathcal{F}\{f(x)\cos ax\}(\omega) = \frac{\tilde{f}(\omega - a) + \tilde{f}(\omega + a)}{2}, \tag{11.28a}$$
$$\mathcal{F}\{f(x)\sin ax\}(\omega) = \frac{\tilde{f}(\omega - a) - \tilde{f}(\omega + a)}{2i}, \tag{11.28b}$$
$$\mathcal{F}^{-1}\{\tilde{f}(\omega)\cos a\omega\}(x) = \frac{f(x - a) + f(x + a)}{2}, \tag{11.28c}$$
$$\mathcal{F}^{-1}\{\tilde{f}(\omega)\sin a\omega\}(x) = \frac{f(x + a) - f(x - a)}{2i}. \tag{11.28d}$$

8. (a) Show that

$$\mathcal{F}\{\mathcal{F}\{f\}\}(\omega) = 2\pi f(-\omega). \tag{11.29}$$

(b) Illustrate the property in part (a) with the function $f(x) = e^{-ax^2}$.

(c) Use equation 11.29 and Example 11.10 to find the Fourier transform of $f(x) = \dfrac{\sin ax}{x}$.

(d) Does the result in part (c) violate Theorem 11.2? Explain.

In Exercises 9–12 use residues to find the Fourier transform of the function.

9. $f(x) = \dfrac{1}{a^4 + x^4}, \quad a > 0$ constant

10. $f(x) = \dfrac{1}{(a^2 + x^2)^2}, \quad a > 0$ constant

11. $f(x) = \dfrac{x}{(a^2 + x^2)^2}, \quad a > 0$ constant

12. $f(x) = \dfrac{1}{1 + x + x^2}$

In Exercises 13–26 find the Fourier transform of the function.

13. $f(x) = e^{-a|x|}, \quad a > 0$ constant

14. $f(x) = xe^{-a|x|}, \quad a > 0$ constant

15. $f(x) = xe^{-ax^2}, \quad a > 0$ constant

16. $f(x) = \dfrac{x}{a^2 + x^2}, \quad a > 0$ constant

17. $f(x) = |x|e^{-a|x|}, \quad a > 0$ constant

18. $f(x) = x^n e^{-ax} h(x), \quad a > 0$ constant, $n \geq 0$ an integer

19. $f(x) = h(x - a) - h(x - b), \quad b > a$ constants

20. $f(x) = x[h(x + a) - h(x - a)], \quad a > 0$ constant

21. $f(x) = \begin{cases} (b/a)(a - |x|), & |x| < a \\ 0, & |x| > a \end{cases}, \quad a > 0,\ b > 0$ constants

22. $f(x) = \begin{cases} b(a^2 - x^2)/a^2, & |x| < a \\ 0, & |x| > a \end{cases}, \quad a > 0,\ b > 0$ constants

23. $f(x) = e^{-kx^2} \cos ax, \quad a > 0,\ k > 0$ constants

24. $f(x) = e^{-kx^2} \sin ax, \quad a > 0,\ k > 0$ constants

25. $f(x) = \cos ax \left[h\left(x + \frac{\pi}{2a}\right) - h\left(x - \frac{\pi}{2a}\right)\right], \quad a > 0$ constant

26. $f(x) = \sin ax \left[h\left(x + \frac{\pi}{a}\right) - h\left(x - \frac{\pi}{a}\right)\right], \quad a > 0$ constant

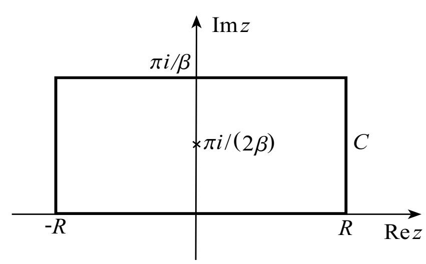
The improper integrals in the next two exercises are required for evaluation of the Fourier transforms in Exercise 29. These Fourier transforms are essential for solving initial boundary value problems in Section 11.4.

27. (a) To evaluate the improper integrals

$$\int_0^\infty \frac{\cosh \alpha x}{\cosh \beta x} \cos \omega x \, dx \qquad \text{and} \qquad \int_0^\infty \frac{\sinh \alpha x}{\cosh \beta x} \sin \omega x \, dx,$$

where $0 < \alpha < \beta$, consider the contour integral $\oint_C \frac{e^{\alpha z}}{\cosh \beta z} e^{i\omega z} \, dz = \oint_C \frac{e^{(\alpha+\omega i)z}}{\cosh \beta z} dz,$ where C is the contour in the figure to the right. Use residues to show that the value of the contour integral is

$$\frac{2\pi}{\beta} e^{\pi(-\omega+\alpha i)/(2\beta)}.$$



(b) Verify that integrals along the vertical line segments approach zero as $R \to \infty$.

(c) Combine integrals along the horizontal line segments and take limits as $R \to \infty$ to show that

$$\int_0^\infty \frac{\cosh(\alpha + \omega i)x}{\cosh \beta x} dx = \frac{\pi e^{\pi(-\omega+\alpha i)/(2\beta)}}{\beta[1 + e^{\pi(-\omega+\alpha i)/\beta}]}.$$

(d) Finally, take real and imaginary parts to obtain

$$\int_0^\infty \frac{\cosh \alpha x}{\cosh \beta x} \cos \omega x \, dx = \frac{\pi \cos \frac{\pi\alpha}{2\beta} \cosh \frac{\pi\omega}{2\beta}}{\beta\left(\cosh \frac{\pi\omega}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}, \quad \int_0^\infty \frac{\sinh \alpha x}{\cosh \beta x} \sin \omega x \, dx = \frac{\pi \sin \frac{\pi\alpha}{2\beta} \sinh \frac{\pi\omega}{2\beta}}{\beta\left(\cosh \frac{\pi\omega}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}.$$

28. (a) To evaluate the improper integrals

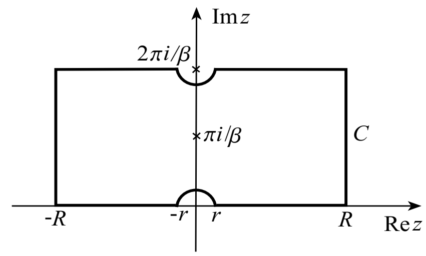
$$\int_0^\infty \frac{\sinh \alpha x}{\sinh \beta x} \cos \omega x \, dx \qquad \text{and} \qquad \int_0^\infty \frac{\cosh \alpha x}{\sinh \beta x} \sin \omega x \, dx,$$

where $0 < \alpha < \beta$, consider the contour integral

$$\oint_C \frac{e^{\alpha z}}{\sinh \beta z} e^{i\omega z}\, dz = \oint_C \frac{e^{(\alpha+\omega i)z}}{\sinh \beta z} dz,$$

where C is the contour in the figure to the right. Use residues to show that the value of the contour integral is

$$-\frac{2\pi i}{\beta} e^{\pi(-\omega+\alpha i)/\beta}.$$



(b) Verify that integrals along the vertical line segments approach zero as $R \to \infty$.

(c) Prove that in the limit as $r \to 0$, the integrals over the semi-circles at $z = 0$ and $z = 2\pi i/\beta$ have values

$$\frac{-\pi i}{\beta}, \qquad \text{and} \qquad -\frac{\pi i e^{2\pi(-\omega+\alpha i)/\beta}}{\beta}, \quad \text{respectively.}$$

(d) Combine integrals along the horizontal line segments and the two semi-circles, and take limits as $R \to \infty$ and $r \to 0$ to show that

$$\int_0^\infty \frac{\sinh{(\alpha+\omega i)x}}{\sinh \beta x} dx = \frac{\pi i[1 - e^{\pi(-\omega+\alpha i)/\beta}]}{2\beta[1 + e^{\pi(-\omega+\alpha i)/\beta}]}.$$

(e) Finally, take real and imaginary parts to obtain

$$\int_0^\infty \frac{\sinh \alpha x}{\sinh \beta x} \cos \omega x\, dx = \frac{\pi \sin \frac{\pi\alpha}{\beta}}{2\beta\left(\cosh \frac{\pi\omega}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}, \quad \int_0^\infty \frac{\cosh \alpha x}{\sinh \beta x} \sin \omega x\, dx = \frac{\pi \sinh \frac{\pi\omega}{\beta}}{2\beta\left(\cosh \frac{\pi\omega}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}.$$

29. Use Exercises 27 and 28 to show that when $0 < \alpha < \beta$,

$$\mathcal{F}\left\{\frac{\cosh \alpha x}{\cosh \beta x}\right\}(\omega) = \frac{2\pi \cos \frac{\pi\alpha}{2\beta} \cosh \frac{\pi\omega}{2\beta}}{\beta\left(\cosh \frac{\pi\omega}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}, \qquad \mathcal{F}\left\{\frac{\sinh \alpha x}{\cosh \beta x}\right\}(\omega) = \frac{-2\pi i \sin \frac{\pi\alpha}{\beta} \sinh \frac{\pi\omega}{2\beta}}{\beta\left(\cosh \frac{\pi\omega}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)},$$

$$\mathcal{F}\left\{\frac{\sinh \alpha x}{\sinh \beta x}\right\}(\omega) = \frac{\pi \sin \frac{\pi\alpha}{\beta}}{\beta\left(\cosh \frac{\pi\omega}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}, \qquad \mathcal{F}\left\{\frac{\cosh \alpha x}{\sinh \beta x}\right\}(\omega) = \frac{\pi \sinh \frac{\pi\omega}{\beta}}{\beta\left(\cosh \frac{\pi\omega}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}.$$

30. (a) Show that

$$\mathcal{F}^{-1}\{f(-\omega)\}(x) = \frac{1}{2\pi}\mathcal{F}\{f\}(x). \tag{11.30}$$

(b) Illustrate the property in part (a) with the function $f(x) = e^{-ax^2}$.

(c) Use equation 11.30 and Exercise 29 to show that when $0 < \alpha < \beta$,

$$\mathcal{F}^{-1}\left\{\frac{\cosh \alpha\omega}{\cosh \beta\omega}\right\}(x) = \frac{\cos \frac{\pi\alpha}{2\beta} \cosh \frac{\pi x}{2\beta}}{\beta\left(\cosh \frac{\pi x}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}, \qquad \mathcal{F}^{-1}\left\{\frac{\sinh \alpha\omega}{\cosh \beta\omega}\right\}(x) = \frac{-i \sin \frac{\pi\alpha}{\beta} \sinh \frac{\pi x}{2\beta}}{\beta\left(\cosh \frac{\pi x}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)},$$

$$\mathcal{F}^{-1}\left\{\frac{\sinh \alpha\omega}{\sinh \beta\omega}\right\}(x) = \frac{\sin \frac{\pi\alpha}{\beta}}{2\beta\left(\cosh \frac{\pi x}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}, \qquad \mathcal{F}^{-1}\left\{\frac{\cosh \alpha\omega}{\sinh \beta\omega}\right\}(x) = \frac{\sinh \frac{\pi x}{\beta}}{2\beta\left(\cosh \frac{\pi x}{\beta} + \cos \frac{\pi\alpha}{\beta}\right)}.$$

31. Verify formally each of the following results, often called **Parseval's relations**:

$$\int_{-\infty}^{\infty} \tilde{f}(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\tilde{g}(x)\,dx \tag{11.31a}$$

$$2\pi \int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_{-\infty}^{\infty} \tilde{f}(\omega)\tilde{g}(-\omega)\,d\omega \tag{11.31b}$$

$$2\pi \int_{-\infty}^{\infty} [f(x)]^2\,dx = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2\,d\omega \tag{11.31c}$$

32. Verify that improper integral 11.10a should be taken in the sense of Cauchy's principal value 11.10b.

The following exercises should be attempted only by readers who are already familiar with the Laplace transform. In these exercises, $\mathcal{L}\{f(x)\}$ denotes the Laplace transform of a function $f(x)$.

33. (a) Show that when $f(x)$ is absolutely integrable on $0 < x < \infty$, and $f(x) = 0$ for $x < 0$,

$$\mathcal{F}\{f\}(\omega) = \mathcal{L}\{f\}(i\omega). \tag{11.32}$$

(b) Use the result in part (a) to calculate Fourier transforms for the functions in Exercise 18, and in Exercise 19 (when $a > 0$).

34. (a) The inverse result of property 11.32 can be stated as follows: Suppose that when ω in the Fourier transform $\tilde{f}(\omega)$ is replaced by $-is$, the function $\tilde{f}(-is)$ has no poles on the imaginary s-axis or in the right half-plane. If $\tilde{f}(-is)$ has an inverse Laplace transform, this is also the inverse Fourier transform of $\tilde{f}(\omega)$,

$$\mathcal{F}^{-1}\{\tilde{f}(\omega)\} = \begin{cases} \mathcal{L}^{-1}\{\tilde{f}(-is)\}, & x > 0 \\ 0, & x < 0. \end{cases} \tag{11.33}$$

Use this result to find inverse Fourier transforms for the following:

(i) $\tilde{f}(\omega) = \dfrac{1}{(8 + i\omega)^3}$ (ii) $\tilde{f}(\omega) = \dfrac{b}{a}\left[\left(\dfrac{1 - e^{-i\omega a}}{\omega^2}\right) - \dfrac{ia}{\omega}\right]$, $a > 0$, $b > 0$ constants

(b) Can the result in part (a) be used to find $\mathcal{F}^{-1}\left\{\dfrac{i}{\omega}e^{-ia\omega}\right\}$?

35. (a) Show that when $f(x)$ is absolutely integrable on $-\infty < x < 0$, and $f(x) = 0$ for $x > 0$,

$$\mathcal{F}\{f(x)\}(\omega) = \mathcal{L}\{f(-x)\}(-i\omega). \tag{11.34}$$

(b) Use the result in part (a) to find Fourier transforms for the following:

(i) $f(x) = \begin{cases} -x(x + L), & -L \le x \le 0 \\ 0, & \text{otherwise} \end{cases}$ (ii) $f(x) = e^{cx}[h(x - a) - h(x - b)]$, $a < b < 0$, $c > 0$

36. (a) Let $f(x)$ be a function that has a Fourier transform. Denote by $f^+(x)$ and $f^-(x)$ the right and left halves respectively, of $f(x)$:

$$f^+(x) = \begin{cases} 0, & x < 0 \\ f(x), & x > 0 \end{cases}; \qquad f^-(x) = \begin{cases} f(x), & x < 0 \\ 0, & x > 0 \end{cases}.$$

Show that

$$\mathcal{F}\{f\}(\omega) = \mathcal{F}\{f^+\}(\omega) + \mathcal{F}\{f^-\}(\omega).$$

(b) Use the result in part (a) in conjunction with equations 11.32 and 11.34 to find Fourier transforms for the following:

(i) $f(x)$ in Exercise 21 (ii) $f(x) = \sin ax[h(x + 2n\pi/a) - h(x - 2n\pi/a)]$, $n > 0$ an integer, $a > 0$

§11.4 Application of the Fourier Transform to Initial Boundary Value Problems

Fourier transform 11.13 not only handles nonhomogeneous PDEs on infinite intervals, but it also provides a valuable alternative to separation of variables and Fourier integral 11.6 for homogeneous problems. We begin with homogeneous, heat conduction problem 11.1. When we apply Fourier transform 11.13 to PDE 11.1a,

$$\int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{-i\omega x} dx = k \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2} e^{-i\omega x} dx.$$

When we interchange the operations of integration with respect to x and differentiation with respect to t on the left, and use property 11.23a for the transform on the right,

$$\frac{d\tilde{U}}{dt} = -k\omega^2 \tilde{U}(\omega, t).$$

A general solution of this ODE in $\tilde{U}(\omega, t)$ is

$$\tilde{U}(\omega, t) = Ce^{-k\omega^2 t}.$$

The Fourier transform of initial condition 11.1b is $\tilde{U}(\omega, 0) = \tilde{f}(\omega)$, and this condition requires $C = \tilde{f}(\omega)$. Thus,

$$\tilde{U}(\omega, t) = \tilde{f}(\omega) e^{-k\omega^2 t},$$

and the inverse Fourier transform now gives

$$U(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-k\omega^2 t} e^{i\omega x} d\omega. \tag{11.35a}$$

For heat conduction problems like this one, a much more useful form of the solution, which expresses $U(x, t)$ as a real integral involving $f(x)$, rather than a complex integal in $\tilde{f}(\omega)$, can be obtained with convolutions. Because the inverse transform of $e^{-k\omega^2 t}$ is $1/(2\sqrt{k\pi t})e^{-x^2/(4kt)}$ (see Example 11.7), convolution property 11.24 yields

$$\begin{aligned} U(x, t) &= \int_{-\infty}^{\infty} f(u) \frac{1}{2\sqrt{k\pi t}} e^{-(x-u)^2/(4kt)} du \\ &= \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} f(u) e^{-(x-u)^2/(4kt)} du. \end{aligned} \tag{11.35b}$$

This form for the solution clearly indicates the dependence of $U(x, t)$ on the initial temperature distribution $f(x)$. It also has another advantage. Because representation 11.35b does not contain the Fourier transform of $f(x)$, it may represent a solution to problem 11.1 even when $f(x)$ has no Fourier transform. Indeed, provided $f(x)$ is piecewise continuous on some bounded interval, and continuous and bounded outside this interval, it can be shown that $U(x, t)$ so defined satisfies problem 11.1. This is illustrated in the first two special cases that follow.

Case 1: $f(x) = U_0$, a constant

In this case, we would expect that $U(x, t) = U_0$ for all x and t. That representation 11.35b gives this result is easily demonstrated by setting $v = (x - u)/(2\sqrt{kt})$ and $dv = -du/(2\sqrt{kt})$,

$$U(x,t) = \frac{U_0}{2\sqrt{k\pi t}} \int_{\infty}^{-\infty} e^{-v^2}(-2\sqrt{kt}\,dv) = \frac{U_0}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv = U_0$$

(see Exercise '`stat Int`' in Section 11.2 for the value of this integral). Thus, integral 11.35b has given the correct solution in spite of the fact that the function $f(x) = U_0$ does not have a Fourier transform.

Case 2: $f(x) = U_0 h(x)$

In this case, we set $v = (x-u)/(2\sqrt{kt})$ and $dv = -du/(2\sqrt{kt})$ in

$$U(x,t) = \frac{U_0}{2\sqrt{k\pi t}} \int_0^{\infty} e^{-(x-u)^2/(4kt)} du$$

to obtain

$$\begin{aligned} U(x,t) &= \frac{U_0}{2\sqrt{k\pi t}} \int_{x/(2\sqrt{kt})}^{-\infty} e^{-v^2}(-2\sqrt{kt}\,dv) = \frac{U_0}{\sqrt{\pi}} \int_{-\infty}^{x/(2\sqrt{kt})} e^{-v^2} dv \\ &= \frac{U_0}{\sqrt{\pi}} \left[\int_{-\infty}^{0} e^{-v^2} dv + \int_0^{x/(2\sqrt{kt})} e^{-v^2} dv \right] \\ &= \frac{U_0}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \text{erf}\left(\frac{x}{2\sqrt{kt}} \right) \right] = \frac{U_0}{2} \left[1 + \text{erf}\left(\frac{x}{2\sqrt{kt}} \right) \right]. \end{aligned}$$

This solution indicates how heat that is concentrated in one-half of a rod diffuses into the other half. It indicates, in particular, that temperature at every point in the left half of the rod ($x < 0$) is positive for every $t > 0$. This substantiates our claim in Section 6.6 that heat propagates with infinite speed. We have plotted this function for $t = 10^6$ and $t = 10^7$, and the initial temperature distribution $U_0 h(x)$, in Figure 11.10 for $k = 10^{-6}$. They show that although heat propagates with infinite speed, the amount is very small so that it takes a very long time for temperature to approach its steady state value $U_0/2$ throughout the rod.

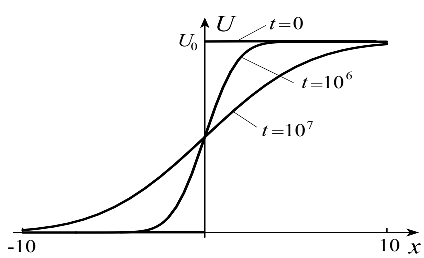


Figure 11.10

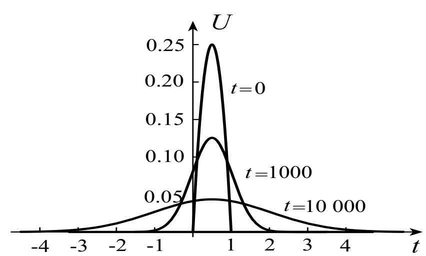


Figure 11.11

Case 3: $f(x) = x(L-x)$, $0 \le x \le L$, and vanishes otherwise

In this case, representation 11.35b gives

$$U(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_0^L u(L-u) e^{-(x-u)^2/(4kt)} du.$$

We have plotted the initial temperature function along with the temperature at $t = 1000$ and $t = 10\,000$ in Figure 11.11 using $k = 114 \times 10^{-6}$ (the thermal diffusivity of copper) and $L = 1$.

In the following example, heat is generated over the interval $-x_0 \le x \le x_0$ at a constant rate.

Example 11.15 Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}[h(x+x_0) - h(x-x_0)], \quad -\infty < x < \infty, \quad t > 0, \quad (11.36a)$$

$$U(x,0) = f(x), \quad -\infty < x < \infty. \quad (11.36b)$$

Solution When we take Fourier transforms of the PDE, and use property 11.23a and Example 11.10,

$$\frac{d\tilde{U}}{dt} = -k\omega^2\tilde{U} + \frac{2k}{\kappa\omega}\sin x_0\omega. \quad (11.37a)$$

The transform $\tilde{U}(\omega,t)$ must satisfy this ODE subject to the transform of initial condition 11.36b,

$$\tilde{U}(\omega,0) = \tilde{f}(\omega). \quad (11.37b)$$

A general solution of the ODE is

$$\tilde{U}(\omega,t) = Ce^{-k\omega^2 t} + \frac{2}{\kappa\omega^3}\sin x_o\omega,$$

and the initial condition requires

$$\tilde{f}(\omega) = C + \frac{2}{\kappa\omega^3}\sin x_0\omega.$$

Thus,

$$\tilde{U}(\omega,t) = \left[\tilde{f}(\omega) - \frac{2}{\kappa\omega^3}\sin x_0\omega\right]e^{-k\omega^2 t} + \frac{2}{\kappa\omega^3}\sin x_0\omega,$$

and $U(x,t)$ is the inverse transform thereof. According to convolution property 11.25, the inverse transform of $\tilde{f}(\omega)e^{-k\omega^2 t}$ can be expressed as

$$\frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/(4kt)}du,$$

and therefore

$$U(x,t) = \frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/(4kt)}du - \frac{1}{\kappa\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^3}(1-e^{-k\omega^2 t})\sin x_0\omega\, e^{i\omega x}d\omega. \bullet$$

Example 11.16 In Exercise 27 of Section 2.3 we introduced the telegraph equation

$$\frac{\partial^2 I}{\partial x^2} = CL\frac{\partial^2 I}{\partial t^2} + (RC+GL)\frac{\partial I}{\partial t} + RG\,I, \quad -\infty < x < \infty, \quad t > 0, \quad (11.38a)$$

where C, L, R, and G represent capacitance, inductance, resistance, and conductance, all per unit length of a transmission line. Function $I(x,t)$ is the current in the line for $t > 0$ and $-\infty < x < \infty$, but it could also be the voltage. There will be two initial conditions accompanying the PDE,

$$I(x,0) = f_1(x), \quad -\infty < x < \infty, \quad (11.38b)$$

$$I_t(x,0) = f_2(x), \quad -\infty < x < \infty. \quad (11.38c)$$

By defining new parameters

$$2\beta = \frac{R}{L} + \frac{G}{C}, \qquad c^2 = \frac{1}{CL}, \qquad k = \frac{RG}{CL},$$

we can express the PDE in a simpler, and already recognized form,

$$c^2 I_{xx} = I_{tt} + 2\beta I_t + kI. \tag{11.38d}$$

This is the PDE for transverse vibrations of a string subjected to a damping force proportional to velocity $(2\beta I_t)$ and a restoring force proportional to displacement (kI) (see equations 2.46 and 2.47 in Section 2.3). Find the current $I(x,t)$ for $t > 0$ and $-\infty < x < \infty$.

Solution When we apply the Fourier transform to the PDE and use property 11.23a,

$$-c^2\omega^2 \tilde{I} = \frac{d^2\tilde{I}}{dt^2} + 2\beta\frac{d\tilde{I}}{dt} + k\,\tilde{I},$$

or,

$$\frac{d^2\tilde{I}}{dt^2} + 2\beta\frac{d\tilde{I}}{dt} + (k + c^2\omega^2)\tilde{I} = 0, \quad t > 0, \tag{11.39a}$$

subject to

$$\tilde{I}(\omega, 0) = \tilde{f}_1(\omega), \tag{11.39b}$$

$$\tilde{I}'(\omega, 0) = \tilde{f}_2(\omega). \tag{11.39c}$$

Solutions of ODE 11.39a depend on values of the parameters β, k, and c, and the value of ω. The auxiliary equation associated with the differential equation is

$$m^2 + 2\beta m + (k + c^2\omega^2) = 0 \qquad \Longrightarrow \qquad m = -\beta \pm \sqrt{\beta^2 - k - c^2\omega^2}.$$

We consider three cases:

Case 1 $\beta^2 - k < 0$

In this case, roots of the auxiliary equation are complex for all values of ω, $m = -\beta \pm \sqrt{k - \beta^2 + c^2\omega^2}\,i$, and a general solution of differential equation 11.39a is

$$\tilde{I}(\omega, t) = e^{-\beta t}(A\cos\sqrt{k - \beta^2 + c^2\omega^2}\,t + B\sin\sqrt{k - \beta^2 + c^2\omega^2}\,t).$$

When initial conditions 11.39b,c are applied, the result is

$$\tilde{I}(\omega, t) = e^{-\beta t}\left[\tilde{f}_1(\omega)\cos\sqrt{k - \beta^2 + c^2\omega^2}\,t + \frac{\tilde{f}_2(\omega) + \beta\tilde{f}_1(\omega)}{\sqrt{k - \beta^2 + c^2\omega^2}}\sin\sqrt{k - \beta^2 + c^2\omega^2}\,t\right].$$

Current in the transmission line is

$$I(x,t) = \frac{e^{-\beta t}}{2\pi}\int_{-\infty}^{\infty}\left[\tilde{f}_1(\omega)\cos\sqrt{k - \beta^2 + c^2\omega^2}\,t + \frac{\tilde{f}_2(\omega) + \beta\tilde{f}_1(\omega)}{\sqrt{k - \beta^2 + c^2\omega^2}}\sin\sqrt{k - \beta^2 + c^2\omega^2}\,t\right]e^{i\omega x}d\omega.$$

Case 2 $\beta^2 - k = 0$

In this case, roots of the auxiliary equation are $m = -\beta \pm c\omega i$, and a general solution of differential equation 11.39a is

$$\tilde{I}(\omega,t) = e^{-\beta t}(A\cos c\omega t + B\sin c\omega t).$$

When initial conditions 11.39b,c are applied, the result is

$$\tilde{I}(\omega,t) = e^{-\beta t}\left[\tilde{f}_1(\omega)\cos c\omega t + \frac{\tilde{f}_2(\omega)+\beta\tilde{f}_1(\omega)}{c\omega}\sin c\omega t\right].$$

Current in the transmission line is

$$I(x,t) = \frac{e^{-\beta t}}{2\pi}\int_{-\infty}^{\infty}\left[\tilde{f}_1(\omega)\cos c\omega t + \frac{\tilde{f}_2(\omega)+\beta\tilde{f}_1(\omega)}{c\omega}\sin c\omega t\right]e^{i\omega x}\,d\omega.$$

This solution can be expressed in closed form. Since $f_1(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\tilde{f}_1(\omega)e^{i\omega x}d\omega$, it follows that

$$\begin{aligned}
\frac{e^{-\beta t}}{2\pi}\int_{-\infty}^{\infty}\tilde{f}_1(\omega)\cos c\omega t e^{i\omega x}d\omega &= \frac{e^{-\beta t}}{2\pi}\int_{-\infty}^{\infty}\tilde{f}_1(\omega)\left(\frac{e^{ic\omega t}+e^{-ic\omega t}}{2}\right)e^{i\omega x}d\omega \\
&= \frac{e^{-\beta t}}{2\pi}\int_{-\infty}^{\infty}\tilde{f}_1(\omega)[e^{i\omega(x+ct)}+e^{i\omega(x-ct)}]d\omega \\
&= \frac{e^{-\beta t}}{2}[f_1(x+ct)+f_1(x-ct)].
\end{aligned}$$

According to Exercise 11.10 in Section 11.3, $\mathcal{F}\{h(x+ct)-h(x-ct)\} = \dfrac{2}{\omega}\sin c\omega t.$ Convolution property 11.25 then gives

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\frac{\tilde{f}_2(\omega)+\beta\tilde{f}_1(\omega)}{\omega}\sin c\omega t\right\} &= \frac{1}{2}\int_{-\infty}^{\infty}[h(x-u+ct)-h(x-u-ct)][f_2(u)+\beta f_1(u)]\,du \\
&= \frac{1}{2}\int_{-\infty}^{\infty}[h(u-x+ct]-h(u-x-ct)][f_2(u)+\beta f_1(u)]\,du \\
&= \frac{1}{2}\int_{x-ct}^{x+ct}[f_2(u)+\beta f_1(u)]\,du.
\end{aligned}$$

Consequently, current can be expressed as

$$I(x,t) = \frac{e^{-\beta t}}{2}[f_1(x+ct)+f_1(x-ct)] + \frac{e^{-\beta t}}{2c}\int_{x-ct}^{x+ct}[f_2(u)+\beta f_1(u)]\,du.$$

Case 3 $\beta^2 - k > 0$

In this case, roots of the auxiliary equation depend on values of ω,

$$m = \begin{cases} -\beta \pm \sqrt{k-\beta^2+c^2\omega^2}\,i, & |c\omega| > \sqrt{\beta^2-k} \\ -\beta \pm \sqrt{\beta^2-k-c^2\omega^2}, & |c\omega| < \sqrt{\beta^2-k} \end{cases}$$

and a general solution of differential equation 11.39a is

$$\tilde{I}(\omega,t) = \begin{cases} e^{-\beta t}(A\cos\sqrt{k-\beta^2+c^2\omega^2}t + B\sin\sqrt{k-\beta^2+c^2\omega^2}t, & |c\omega| > \sqrt{\beta^2-k} \\ e^{-\beta t}(A\cosh\sqrt{\beta^2-k-c^2\omega^2}t + B\sinh\sqrt{\beta^2-k-c^2\omega^2}t, & |c\omega| < \sqrt{\beta^2-k}. \end{cases}$$

When initial conditions 11.39b,c are applied, the result is

$$\tilde{I}(\omega,t)=\begin{cases} e^{-\beta t}\left[\tilde{f}_1(\omega)\cos\sqrt{k-\beta^2+c^2\omega^2}\,t+\dfrac{\tilde{f}_2(\omega)+\beta\tilde{f}_1(\omega)}{\sqrt{k-\beta^2+c^2\omega^2}}\sin\sqrt{k-\beta^2+c^2\omega^2}\,t\right], & |c\omega|>\sqrt{\beta^2-k} \\ e^{-\beta t}\left[\tilde{f}_1(\omega)\cosh\sqrt{\beta^2-k-c^2\omega^2}\,t+\dfrac{\tilde{f}_2(\omega)+\beta\tilde{f}_1(\omega)}{\sqrt{\beta^2-k-c^2\omega^2}}\sinh\sqrt{\beta^2-k-c^2\omega^2}\,t\right], & |c\omega|<\sqrt{\beta^2-k}. \end{cases}$$

Current in the transmission line is

$$\begin{aligned} I(x,t)&=\frac{e^{-\beta t}}{2\pi}\int_{-\infty}^{-c^{-1}\sqrt{\beta^2-k}}\left[\tilde{f}_1(\omega)\cos\sqrt{k-\beta^2+c^2\omega^2}\,t+\frac{\tilde{f}_2(\omega)+\beta\tilde{f}_1(\omega)}{\sqrt{k-\beta^2+c^2\omega^2}}\sin\sqrt{k-\beta^2+c^2\omega^2}\,t\right]e^{i\omega x}d\omega \\ &+\frac{e^{-\beta t}}{2\pi}\int_{-c^{-1}\sqrt{\beta^2-k}}^{c^{-1}\sqrt{\beta^2-k}}\left[\tilde{f}_1(\omega)\cosh\sqrt{\beta^2-k-c^2\omega^2}\,t+\frac{\tilde{f}_2(\omega)+\beta\tilde{f}_1(\omega)}{\sqrt{\beta^2-k-c^2\omega^2}}\sinh\sqrt{\beta^2-k-c^2\omega^2}\,t\right]e^{i\omega x}d\omega \\ &+\frac{e^{-\beta t}}{2\pi}\int_{c^{-1}\sqrt{\beta^2-k}}^{\infty}\left[\tilde{f}_1(\omega)\cos\sqrt{k-\beta^2+c^2\omega^2}\,t+\frac{\tilde{f}_2(\omega)+\beta\tilde{f}_1(\omega)}{\sqrt{k-\beta^2+c^2\omega^2}}\sin\sqrt{k-\beta^2+c^2\omega^2}\,t\right]e^{i\omega x}d\omega.\bullet \end{aligned}$$

EXERCISES 11.4

Part A Heat Conduction

1. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2}+\frac{k}{\kappa}g(x,t),\quad -\infty<x<\infty,\quad t>0,$$
$$U(x,0)=f(x),\quad -\infty<x<\infty.$$

(b) Simplify the solution in part (a) in the case that $g(x,t)\equiv 0$ and

$$\text{(i)}\quad f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases};\qquad\qquad \text{(ii)}\quad f(x)=\begin{cases}0, & |x|<a\\ 1, & |x|>a\end{cases}.$$

Plot the solutions on the interval $-5\le x\le 5$ with $k=10^{-6}$ and $a=1$ for $t=10^5$ and $t=10^6$.

2. Express the solution of the following initial value problem as a real improper integral,

$$\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2}+\alpha\frac{\partial U}{\partial x},\quad -\infty<x<\infty,\quad t>0,$$
$$U(x,0)=f(x),\quad -\infty<x<\infty,$$

where k and α are positive constants. Hint: See Exercise 20 in Section 11.3.

Part B Vibrations

3. Solve the following problem for displacements of an infinite string

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2},\quad -\infty<x<\infty,\quad t>0,$$
$$y(x,0)=f(x),\quad -\infty<x<\infty,$$
$$y_t(x,0)=g(x),\quad -\infty<x<\infty.$$

Determine the d'Alembert form of the solution.

4. Repeat Exercise 3 if a restoring force proportional to displacement acts on all points of the string. Is there a d'Alembert solution?

5. Repeat Exercise 3 if a damping force proportional to velocity acts on all points of the string. Is there a d'Alembert solution?

Part C Potential, Steady-state Heat Conduction, Static Deflection of Membranes

6. Solve the Dirichlet boundary value problem for steady-state temperature in the infinite strip $0 < y < L'$, $-\infty < x < \infty$, when boundary temperatures are $f(x)$ along $y = 0$ and $g(x)$ along $y = L'$.

7. Repeat Exercise 6 if the boundary condition along $y = 0$ is Neumann $\partial U(x, 0)/\partial y = f(x)$.

8. (a) Use the Fourier transform to show that the solution to Laplace's equation for the upper half-plane subject to the condition that $V(x, 0) = f(x)$ is

$$V(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(x - u)^2 + y^2}\, du.$$

This is called **Poisson's integral formula for the half-plane**.

(b) What is the solution when $f(x) = h(x)$?

§11.5 The Fourier Sine and Cosine Transforms

Fourier transforms are also associated with the Fourier cosine and sine integrals 11.8 and 11.9. They can be regarded as special cases of the Fourier transform when the function is either even or odd, or as transforms for functions defined only on the semi-infinite interval $0 < x < \infty$. We take the latter approach in the following definition.

Definition 11.3 The **Fourier cosine transform** of a function $f(x)$ defined for $0 < x < \infty$ is denoted by $\tilde{f} = \mathcal{F}_c\{f\}$ with values given by

$$\tilde{f}(\omega) = \mathcal{F}_c\{f\}(\omega) = \int_0^\infty f(x)\cos\omega x\,dx, \tag{11.40a}$$

and inverse transform

$$f(x) = \mathcal{F}_c^{-1}\{\tilde{f}\}(x) = \frac{2}{\pi}\int_0^\infty \tilde{f}(\omega)\cos\omega x\,d\omega. \tag{11.40b}$$

The **Fourier sine transform** of a function $f(x)$ defined for $0 < x < \infty$ is denoted by $\tilde{f} = \mathcal{F}_s\{f\}$ with values given by

$$\tilde{f}(\omega) = \mathcal{F}_s\{f\}(\omega) = \int_0^\infty f(x)\sin\omega x\,dx, \tag{11.41a}$$

and inverse transform

$$f(x) = \mathcal{F}_s^{-1}\{\tilde{f}\}(x) = \frac{2}{\pi}\int_0^\infty \tilde{f}(\omega)\sin\omega x\,d\omega. \tag{11.41b}$$

According to equations 11.8 and 11.9, the transforms are the coefficients in the Fourier cosine and sine integrals of $f(x)$, and the inverse transforms are the integral formulas. The transforms exist when $f(x)$ is piecewise smooth on every finite interval $0 \le x \le X$ and absolutely integrable on $0 < x < \infty$.

Example 11.17 Find the Fourier cosine and sine transforms of the function $f(x) = e^{-ax}$ $(a > 0)$, defined for $x \ge 0$?

Solution The Fourier cosine and sine transforms are given by the integrals

$$\mathcal{F}_c\{e^{-ax}\}(\omega) = \int_0^\infty e^{-ax}\cos\omega x\,dx \quad \text{and} \quad \mathcal{F}_s\{e^{-ax}\}(\omega) = \int_0^\infty e^{-ax}\sin\omega x\,dx.$$

We can evaluate both of these by considering the improper integral

$$\int_0^\infty e^{-ax}e^{i\omega x}\,dx = \int_0^\infty e^{(-a+\omega i)x}\,dx = \left\{\frac{e^{(-a+\omega i)x}}{-a+\omega i}\right\}_0^\infty = \frac{1}{a-\omega i} = \frac{a+\omega i}{a^2+\omega^2}.$$

Real and imaginary parts of this equation give

$$\mathcal{F}_c\{e^{-ax}\}(\omega) = \frac{a}{\omega^2+a^2} \quad \text{and} \quad \mathcal{F}_s\{e^{-ax}\}(\omega) = \frac{\omega}{a^2+\omega^2}.$$

With these transforms, we may write the function e^{-ax}, for $x > 0$, in either of the forms

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{a}{\omega^2+a^2}\cos\omega x\,d\omega = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}d\omega$$

or

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{\omega}{\omega^2+a^2}\sin\omega x\,d\omega.\bullet$$

When $f(x)$ is an even, rational function with a Fourier cosine transform, the transform can be calculated with residues.

Example 11.18 Find the Fourier cosine transform of the even, rational function $f(x) = \dfrac{1}{a^2+x^2}$, where $a > 0$ is a constant.

Solution The Fourier cosine transform is

$$\tilde{f}(\omega) = \int_0^\infty \frac{\cos\omega x}{a^2+x^2}dx.$$

If we express this in the form

$$\tilde{f}(\omega) = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos\omega x}{a^2+x^2}dx,$$

then we could use residues to evaluate the integral. This was done in Example 11.5,

$$\tilde{f}(\omega) = \frac{\pi}{2a}e^{-a\omega}.\bullet$$

When $f(x)$ is an odd, rational function with a Fourier sine transform, the transform can be calculated with residues.

Example 11.19 Find the Fourier sine transform of the odd, rational function $f(x) = \dfrac{x}{(a^2+x^2)^2}$, where $a > 0$ is a constant.

Solution The Fourier sine transform is

$$\tilde{f}(\omega) = \int_0^\infty \frac{x\sin\omega x}{(a^2+x^2)^2}dx.$$

If we express this in the form

$$\tilde{f}(\omega) = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin\omega x}{(a^2+x^2)^2}dx,$$

then we can use residues to evaluate the integral. According to equation D.16b in Appendix D,

$$\begin{aligned}
\tilde{f}(\omega) &= \pi\,\mathrm{Re}\left\{\mathrm{Res}\left[\frac{ze^{i\omega z}}{(a^2+z^2)^2}, ai\right]\right\} = \pi\,\mathrm{Re}\left\{\lim_{z\to ai}\left\{\frac{d}{dz}\left[\frac{(z-ai)^2ze^{i\omega z}}{(z+ai)^2(z-ai)^2}\right]\right\}\right\}\\
&= \pi\,\mathrm{Re}\left\{\lim_{z\to ai}\left[\frac{(z+ai)^2(e^{i\omega z}+i\omega ze^{i\omega z}) - 2ze^{i\omega z}(z+ai)}{(z+ai)^4}\right]\right\}\\
&= \pi\,\mathrm{Re}\left\{\frac{(2ai)[e^{i\omega(ai)}+i\omega(ai)e^{i\omega(ai)}] - 2(ai)e^{i\omega(ai)}}{(2ai)^3}\right\}\\
&= \frac{\pi\omega}{4a}e^{-a\omega}.\bullet
\end{aligned}$$

The Fourier sine and cosine transforms are linear operators (Exercise 1). They have properties similar to those for the Fourier transform. In particular, corresponding to shifting property 11.20, we have

$$\mathcal{F}_s\{f(x-a)h(x-a)\}(\omega) = (\cos a\omega)\mathcal{F}_s\{f\}(\omega) + (\sin a\omega)\mathcal{F}_c\{f\}(\omega), \quad (11.42a)$$
$$\mathcal{F}_c\{f(x-a)h(x-a)\}(\omega) = (\cos a\omega)\mathcal{F}_c\{f\}(\omega) - (\sin a\omega)\mathcal{F}_s\{f\}(\omega). \quad (11.42b)$$

The presence of $h(x-a)$ is due to the fact that $f(x)$ need not be defined for $x < 0$ and therefore $f(x-a)$ may not be defined for $x < a$. (See Exercise 3 for verification of these.)

Results corresponding to 11.22a and 11.23a for the sine and cosine transforms are

$$\mathcal{F}_c\{f'\}(\omega) = \omega\mathcal{F}_s\{f\}(\omega) - f(0+), \quad (11.43a)$$
$$\mathcal{F}_c\{f''\}(\omega) = -\omega^2\mathcal{F}_c\{f\}(\omega) - f'(0+), \quad (11.43b)$$
$$\mathcal{F}_s\{f'\}(\omega) = -\omega\mathcal{F}_c\{f\}(\omega), \quad (11.43c)$$
$$\mathcal{F}_s\{f''\}(\omega) = -\omega^2\mathcal{F}_s\{f\}(\omega) + \omega f(0+). \quad (11.43d)$$

(See Exercise 5 for verification.) The limits in 11.43a,b,d allow for the possibility of $f(x)$ being undefined at $x = 0$ (but its right-hand limit must exist). The fact that the sine transform of the first derivative of a function involves the cosine transform of the function implies that the transform is not useful in differential equations involving both first and second derivatives. The same is true for the cosine transform.

The following convolution properties for sine and cosine transforms are verified in Exercise 7. When $f = \mathcal{F}_c^{-1}\{\tilde{f}\}$ and $g = \mathcal{F}_c^{-1}\{\tilde{g}\}$,

$$\mathcal{F}_c^{-1}\{\tilde{f}\tilde{g}\}(x) = \frac{1}{2}\int_0^\infty f(v)[g(x-v) + g(x+v)]\,dv, \quad (11.44a)$$
$$= \frac{1}{2}\int_0^\infty g(v)[f(x-v) + f(x+v)]\,dv, \quad (11.44b)$$

provided $f(x)$ and $g(x)$ are extended as even functions for $x < 0$.

When $f = \mathcal{F}_s^{-1}\{\tilde{f}\}$ and $g = \mathcal{F}_c^{-1}\{\tilde{g}\}$ (note that $\tilde{g}$ is a Fourier cosine transform),

$$\mathcal{F}_s^{-1}\{\tilde{f}\tilde{g}\}(x) = \frac{1}{2}\int_0^\infty f(v)[g(x-v) - g(x+v)]\,dv, \quad (11.44c)$$
$$= \frac{1}{2}\int_0^\infty g(v)[f(x+v) + f(x-v)]\,dv, \quad (11.44d)$$

provided $f(x)$ and $g(x)$, respectively, are extended as odd and even functions for $x < 0$. There are other convolution results, but these prove the most useful in applications.

EXERCISES 11.5

1. Verify that the Fourier sine and cosine transforms and their inverses are linear operators.

2. What are the results corresponding to equation 11.21a for $\mathcal{F}_s$ and $\mathcal{F}_c$ when $n = 1$ and $n = 2$?

3. Verify equations 11.42.

4. (a) Prove that when $f(x)$ is an even function with a Fourier transform,

$$\mathcal{F}\{f\} = 2\mathcal{F}_c\{f\}. \tag{11.45a}$$

(b) Prove that when $f(x)$ is an odd function with a Fourier transform,

$$\mathcal{F}\{f\} = -2i\mathcal{F}_s\{f\}. \tag{11.45b}$$

5. Verify equations 11.43.

6. Verify the following results for Fourier sine and cosine transforms, corresponding to equation 11.29,

$$\mathcal{F}_c\{\mathcal{F}_c\{f\}\}(\omega) = \frac{\pi}{2}f(\omega), \tag{11.46a}$$

$$\mathcal{F}_s\{\mathcal{F}_s\{f\}\}(\omega) = \frac{\pi}{2}f(\omega). \tag{11.46b}$$

7. (a) Verify convolution properties 11.44a,b for the Fourier cosine transform.
(b) Verify convolution properties 11.44c,d for the Fourier sine transform.

In Exercises 8–12 find the Fourier sine and cosine transforms of the function.

8. $f(x) = \delta(x - x_0), \quad x_0 > 0$

9. $f(x) = e^{-ax^2}, \quad a > 0$ constant

10. $f(x) = xe^{-ax}, \quad a > 0$ constant

11. $f(x) = h(x - a) - h(x - b), \quad b > a > 0$ constants

12. $f(x) = \begin{cases} (b/a)(a - |x - c|), & |x - c| < a \\ 0, & |x - c| > a \end{cases}, \quad a, b$, and c all positive constants with $c > a$

13. Find the Fourier sine transform of the function $f(x) = xe^{-ax^2}$.

In Exercises 14–15 use residues to find the Fourier sine transform of the function.

14. $f(x) = \dfrac{x}{a^2 + x^2}, \quad a > 0$ constant

15. $f(x) = \dfrac{1}{x(a^2 + x^2)}, \quad a > 0$ constant

In Exercises 16–17 use residues to find the Fourier cosine transform of the function.

16. $f(x) = \dfrac{x^2}{a^4 + x^4}, \quad a > 0$ constant

17. $f(x) = \dfrac{x^2}{(a^2 + x^2)^2}, \quad a > 0$ constant

18. Use Exercises 27 and 28 in Section 11.3 to show that when $0 < \alpha < \beta$,

$$\mathcal{F}_s\left\{\frac{\sinh\alpha x}{\cosh\beta x}\right\}(\omega) = \frac{\pi\sin\frac{\pi\alpha}{2\beta}\sinh\frac{\pi\omega}{2\beta}}{\beta\left(\cosh\frac{\pi\omega}{\beta} + \cos\frac{\pi\alpha}{\beta}\right)}, \qquad \mathcal{F}_s\left\{\frac{\cosh\alpha x}{\sinh\beta x}\right\}(\omega) = \frac{\pi\sinh\frac{\pi\omega}{\beta}}{2\beta\left(\cosh\frac{\pi\omega}{\beta} + \cos\frac{\pi\alpha}{\beta}\right)},$$

$$\mathcal{F}_c\left\{\frac{\cosh\alpha x}{\cosh\beta x}\right\}(\omega) = \frac{\pi\cos\frac{\pi\alpha}{2\beta}\cosh\frac{\pi\omega}{2\beta}}{\beta\left(\cosh\frac{\pi\omega}{\beta} + \cos\frac{\pi\alpha}{\beta}\right)}, \qquad \mathcal{F}_c\left\{\frac{\sinh\alpha x}{\sinh\beta x}\right\}(\omega) = \frac{\pi\sin\frac{\pi\alpha}{\beta}}{2\beta\left(\cosh\frac{\pi\omega}{\beta} + \cos\frac{\pi\alpha}{\beta}\right)}.$$

19. (a) Show that

$$\mathcal{F}_c^{-1}\{f\}(x) = \frac{2}{\pi}\mathcal{F}_c\{f\}(x), \tag{11.47a}$$

$$\mathcal{F}_s^{-1}\{f\}(x) = \frac{2}{\pi}\mathcal{F}_s\{f\}(x). \tag{11.47b}$$

(b) Use the results in part (a) and Exercise 18 to verify that, when $0 < \alpha < \beta$,

$$\mathcal{F}_s^{-1}\left\{\frac{\sinh \alpha\omega}{\cosh \beta\omega}\right\}(x) = \frac{2\sin\frac{\pi\alpha}{2\beta}\sinh\frac{\pi x}{2\beta}}{\beta\left(\cosh\frac{\pi x}{\beta} + \cos\frac{\pi\alpha}{\beta}\right)}, \qquad \mathcal{F}_s^{-1}\left\{\frac{\cosh \alpha\omega}{\sinh \beta\omega}\right\}(x) = \frac{\sinh\frac{\pi x}{\beta}}{\beta\left(\cosh\frac{\pi x}{\beta} + \cos\frac{\pi\alpha}{\beta}\right)},$$

$$\mathcal{F}_c^{-1}\left\{\frac{\cosh \alpha\omega}{\cosh \beta\omega}\right\}(x) = \frac{2\cos\frac{\pi\alpha}{2\beta}\cosh\frac{\pi x}{2\beta}}{\beta\left(\cosh\frac{\pi x}{\beta} + \cos\frac{\pi\alpha}{\beta}\right)}, \qquad \mathcal{F}_c^{-1}\left\{\frac{\sinh \alpha\omega}{\sinh \beta\omega}\right\}(x) = \frac{\sin\frac{\pi\alpha}{\beta}}{\beta\left(\cosh\frac{\pi x}{\beta} + \cos\frac{\pi\alpha}{\beta}\right)}.$$

20. The **error function**, $\operatorname{erf}(x)$, is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du.$$

Because this function is increasing for $x > 0$ and $\lim_{x\to\infty}\operatorname{erf}(x) = 1$, it does not have Fourier transforms. The **complementary error function**, $\operatorname{erfc}(x)$, defined by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du,$$

does have Fourier transforms. Use properties 11.43 and Exercise 9 to derive the following results:

(a) $\mathcal{F}_s\{\operatorname{erfc}(ax)\} = \dfrac{1 - e^{-\omega^2/(4a^2)}}{\omega}$, $a > 0$ constant

(b) $\mathcal{F}_c\left\{ax\operatorname{erfc}(ax) - \dfrac{1}{\sqrt{\pi}}e^{-a^2x^2}\right\} = \dfrac{a}{\omega^2}[-1 + e^{-\omega^2/(4a^2)}]$, $a > 0$ constant

The following exercise should be attempted only by readers who are already familiar with the Laplace transform. In this exercise, $\mathcal{L}\{f(x)\}$ denotes the Laplace transform of a function $f(x)$.

21. (a) Show that when $f(x)$ is absolutely integrable on $0 < x < \infty$, and $f(x) = 0$ for $x < 0$,

$$\mathcal{F}_s\{f(x)\}(\omega) = -\operatorname{Im}[\mathcal{L}\{f(x)\}(i\omega), \tag{11.48a}$$
$$\mathcal{F}_c\{f(x)\}(\omega) = \operatorname{Re}[\mathcal{L}\{f(x)\}(i\omega). \tag{11.48b}$$

(b) Use the results in part (a) to calculate Fourier sine and cosine transforms for the functions in Exercises 11 and 12.

22. When the boundary condition at $x = 0$ for an initial boundary value problem on the semi-infinite interval $x > 0$ is of Robin type, separation of variables leads to the system

$$X'' + \omega^2 X = 0, \quad x > 0$$
$$-lX'(0) + hX(0) = 0,$$
$$X(x) \text{ bounded as } x \to \infty.$$

An eigenfunction of this system is

$$X_\omega(x) = \frac{1}{\sqrt{1 + [h/(\omega l)]^2}}\left(\cos\omega x + \frac{h}{\omega l}\sin\omega x\right) \tag{11.49}$$

for arbitrary ω, which we take as positive. Associated therewith is a generalized Fourier integral formula that states that a function $f(x)$ satisfying the conditions of Theorem 11.1 can be represented in the form

$$\frac{f(x+) + f(x-)}{2} = \frac{2}{\pi}\int_0^\infty G(\omega)X_\omega(x)\,d\omega, \tag{11.50a}$$

where

$$G(\omega) = \int_0^\infty f(x)X_\omega(x)\,dx. \tag{11.50b}$$

From this formula we define a generalized Fourier transform,

$$\tilde{f}(\omega) = \mathcal{G}\{f(x)\} = \int_0^\infty f(x)X_\omega(x)\,dx, \tag{11.51a}$$

and an inverse transform

$$f(x) = \mathcal{G}^{-1}\{\tilde{f}(\omega)\} = \frac{2}{\pi}\int_0^\infty \tilde{f}(\omega)X_\omega(x)\,d\omega. \tag{11.51b}$$

Find transforms of the following functions:
(a) $f(x) = e^{-ax}$, $a > 0$ constant
(b) $f(x) = h(x-a) - h(x-b)$, $b > a > 0$ constants

§11.6 Application of Fourier Sine and Cosine Transforms to Initial Boundary Value Problems

Fourier sine and cosine transforms are used to solve initial boundary value problems associated with second order partial differential equations on the semi-infinite interval $x > 0$. Because property 11.43d for the Fourier sine transform utilizes the value of the function at $x = 0$, the sine transform is applied to problems with a Dirichlet boundary condition at $x = 0$. Similarly, property 11.43b indicates that the cosine transform should be used when the boundary condition at $x = 0$ is of Neumann type.

Example 11.20 Solve the vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad x > 0, \quad t > 0, \tag{11.52a}$$

$$y(0,t) = f_1(t), \quad t > 0, \tag{11.52b}$$

$$y(x,0) = f(x), \quad x > 0, \tag{11.52c}$$

$$y_t(x,0) = g(x), \quad x > 0, \tag{11.52d}$$

for displacement of a semi-infinite string with prescribed motion at its left end $x = 0$.

Solution Because the boundary condition at $x = 0$ is Dirichlet, we apply the Fourier sine transform to the PDE and use property 11.43d for the transform of $\partial^2 y/\partial x^2$,

$$\frac{d^2\tilde{y}}{dt^2} = -\omega^2 c^2 \tilde{y}(\omega,t) + \omega c^2 f_1(t).$$

Thus, the Fourier sine transform $\tilde{y}(\omega,t)$ of $y(x,t)$ must satisfy the ODE

$$\frac{d^2\tilde{y}}{dt^2} + \omega^2 c^2 \tilde{y} = \omega c^2 f_1(t)$$

subject to transforms of initial conditions 11.52c,d,

$$\tilde{y}(\omega,0) = \tilde{f}(\omega), \quad \tilde{y}'(\omega,0) = \tilde{g}(\omega).$$

Variation of parameters leads to the following general solution of the ODE

$$\tilde{y}(\omega,t) = A\cos c\omega t + B\sin c\omega t + c\int_0^t f_1(u)\sin c\omega(t-u)\,du.$$

The initial conditions require A and B to satisfy

$$\tilde{f}(\omega) = A, \qquad \tilde{g}(\omega) = c\omega B.$$

Hence,

$$\tilde{y}(\omega,t) = \tilde{f}(\omega)\cos c\omega t + \frac{\tilde{g}(\omega)}{c\omega}\sin c\omega t + c\int_0^t f_1(u)\sin c\omega(t-u)\,du, \tag{11.53}$$

and $y(x,t)$ is the inverse transform of this function

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \tilde{y}(\omega,t)\sin\omega x\,d\omega. \tag{11.54}$$

The first term in this integral is

$$\frac{2}{\pi}\int_0^\infty \tilde{f}(\omega)\cos c\omega t\sin\omega x\,d\omega = \frac{2}{\pi}\int_0^\infty \frac{1}{2}\tilde{f}(\omega)[\sin\omega(x-ct)+\sin\omega(x+ct)]\,d\omega$$
$$= \frac{1}{2}[f(x-ct)+f(x+ct)],$$

provided $f(x)$ is extended as an odd function.

According to Exercise 11 in Section 11.5, the Fourier cosine transform of $h(x+ct)-h(x-ct)$ is $(\sin c\omega t)/\omega$. Consequently, convolution identity 11.44d implies that the inverse sine transform of $[\tilde{g}(\omega)/(c\omega)]\sin c\omega t$ is

$$\frac{1}{2c}\int_0^\infty [h(v)-h(v-ct)][g(x+v)+g(x-v)]\,dv = \frac{1}{2c}\left[\int_0^{ct} g(x+v)\,dv + \int_0^{ct} g(x-v)\,dv\right],$$

provided $g(x)$ is extended as an odd function for $x<0$. When we set $u=x+v$ and $u=x-v$, respectively, in these integrals, the result is

$$\frac{1}{2c}\left[\int_x^{x+ct} g(u)\,du + \int_x^{x-ct} g(u)(-du)\right] = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du.$$

The inverse transform of the integral term in $\tilde{y}(\omega,t)$ can also be expressed in closed form if we set $u=c(t-v)$,

$$c\int_0^t f_1(v)\sin c\omega(t-v)\,dv = c\int_{ct}^0 f_1\left(t-\frac{u}{c}\right)\sin\omega u\left(-\frac{du}{c}\right) = \int_0^{ct} f_1\left(t-\frac{u}{c}\right)\sin\omega u\,du.$$

But this is the Fourier sine transform of the function

$$\begin{cases} f_1\left(t-\dfrac{x}{c}\right), & x<ct \\ 0, & x>ct \end{cases}$$

or

$$\begin{cases} 0, & t<x/c \\ f_1\left(t-\dfrac{x}{c}\right), & t>x/c \end{cases} = f_1\left(t-\frac{x}{c}\right)h\left(t-\frac{x}{c}\right).$$

The solution is therefore

$$y(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du + f_1\left(t-\frac{x}{c}\right)h\left(t-\frac{x}{c}\right). \quad (11.55)$$

The first two terms are the d'Alembert part of the solution. The last term is due to the nonhomogeneity at the end $x=0$; it can be interpreted physically, and this is most easily done when $f(x)=g(x)=0$. In this case, the complete solution is

$$y(x,t) = f_1\left(t-\frac{x}{c}\right)h\left(t-\frac{x}{c}\right).$$

A point x on the string remains at rest until time $t=x/c$, when it begins to execute the same motion as the end $x=0$. The time x/c taken by the disturbance to reach x is called **retarded time**. The disturbance $f_1(t)$ at $x=0$ travels down the string with velocity c.

The solution of the original problem is a superposition of the d'Alembert displacement and the displacement due to the end effect at $x=0$.●

Example 11.21 The temperature of a semi-infinite rod at time $t = 0$ is $f(x)$, $x \geq 0$. For time $t > 0$, heat is added to the rod uniformly over the end $x = 0$ at a variable rate $f_1(t)$ W/m^2. The initial boundary value problem for temperature $U(x, t)$ in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0, \tag{11.56a}$$

$$U_x(0, t) = -\kappa^{-1} f_1(t), \quad t > 0, \tag{11.56b}$$

$$U(x, 0) = f(x), \quad x > 0. \tag{11.56c}$$

Find $U(x, t)$.

Solution Because the boundary condition at $x = 0$ is Neumann, we apply the Fourier cosine transform to the PDE and use property 11.43b,

$$\frac{d\tilde{U}}{dt} = -k\omega^2 \tilde{U}(\omega, t) + \frac{k}{\kappa} f_1(t).$$

Thus, the Fourier cosine transform $\tilde{U}(\omega, t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\omega^2 \tilde{U} = \frac{k}{\kappa} f_1(t)$$

subject to the transform of initial condition 11.56c,

$$\tilde{U}(\omega, 0) = \tilde{f}(\omega).$$

A general solution of the ODE is

$$\tilde{U}(\omega, t) = Ce^{-k\omega^2 t} + \frac{k}{\kappa} \int_0^t e^{-k\omega^2(t-u)} f_1(u)\, du,$$

and the initial condition requires $\tilde{f}(\omega) = C$. Consequently,

$$\tilde{U}(\omega, t) = \tilde{f}(\omega) e^{-k\omega^2 t} + \frac{k}{\kappa} \int_0^t e^{-k\omega^2(t-u)} f_1(u)\, du, \tag{11.57}$$

and the required temperature is the inverse cosine transform of this function. According to Exercise 9 in Section 11.5, the Fourier cosine transform of e^{-ax^2} is $\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}} e^{-\omega^2/(4a)}$, or, conversely, the inverse Fourier cosine transform of $e^{-k\omega^2 t}$ is $\dfrac{1}{\sqrt{k\pi t}} e^{-x^2/(4kt)}$. Convolution property 11.44a therefore gives the inverse cosine transform of $\tilde{f}(\omega) e^{-k\omega^2 t}$ as

$$\frac{1}{2}\int_0^\infty f(v) \frac{1}{\sqrt{k\pi t}} [e^{-(x-v)^2/(4kt)} + e^{-(x+v)^2/(4kt)}]\, dv$$
$$= \frac{1}{2\sqrt{k\pi t}} \int_0^\infty f(v) [e^{-(x-v)^2/(4kt)} + e^{-(x+v)^2/(4kt)}]\, dv.$$

Furthermore, the inverse cosine transform of $e^{-k\omega^2(t-u)}$ is $\dfrac{1}{\sqrt{k\pi(t-u)}} e^{-x^2/[4k(t-u)]}$, and therefore the inverse transform of the integral term can be expressed in the form

$$\mathcal{F}_c^{-1}\left\{\int_0^t e^{-k\omega^2(t-u)} f_1(u)\, du\right\} = \int_0^t \frac{f_1(u)}{\sqrt{k\pi(t-u)}} e^{-x^2/[4k(t-u)]}\, du.$$

Thus, the temperature function is

$$U(x,t) = \frac{1}{2\sqrt{k\pi t}}\int_0^\infty f(v)[e^{-(x-v)^2/(4kt)} + e^{-(x+v)^2/(4kt)}]\,dv + \frac{\sqrt{k}}{\kappa\sqrt{\pi}}\int_0^t \frac{f_1(u)}{\sqrt{t-u}}e^{-x^2/[4k(t-u)]}\,du.\bullet \tag{11.58}$$

Example 11.22 Solve the following potential problem in the quarter plane $x > 0$, $y > 0$,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad x > 0, \quad y > 0, \tag{11.59a}$$

$$V(0,y) = g(y), \quad y > 0, \tag{11.59b}$$

$$V_y(x,0) = f(x), \quad x > 0. \tag{11.59c}$$

Solution Superposition can be used to express $V(x,y)$ as the sum of functions $V_1(x,y)$ and $V_2(x,y)$ satisfying

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0, \quad x > 0, \quad y > 0, \qquad \frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} = 0, \quad x > 0, \quad y > 0,$$

$$V_1(0,y) = g(y), \quad y > 0, \qquad V_2(0,y) = 0, \quad y > 0,$$

$$\frac{\partial V_1(x,0)}{\partial y} = 0, \quad x > 0, \qquad \frac{\partial V_2(x,0)}{\partial y} = f(x), \quad x > 0.$$

To find $V_1(x,y)$ we apply Fourier cosine transform 11.40a (with respect to y) to its PDE and use property 11.43b,

$$\frac{d^2\tilde{V}_1}{dx^2} - \omega^2\tilde{V}_1(x,\omega) = 0, \quad x > 0.$$

This transform function $\tilde{V}_1(x,\omega)$ is also subject to

$$\tilde{V}_1(0,\omega) = \tilde{g}(\omega).$$

A general solution of the ODE is

$$\tilde{V}_1(x,\omega) = Ae^{\omega x} + Be^{-\omega x}.$$

For $\tilde{V}_1(x,\omega)$ to remain bounded as $x \to \infty$, A must be zero, and the boundary condition then implies that $B = \tilde{g}(\omega)$. Hence,

$$\tilde{V}_1(x,\omega) = \tilde{g}(\omega)e^{-\omega x}.$$

To invert this transform, first recall from to Example 11.17 that

$$\mathcal{F}_c\left\{e^{-ay}\right\}(\omega) = \frac{a}{a^2+\omega^2} \qquad \text{when } a > 0.$$

With Exercise 19 in Section 11.5, we can say that

$$\mathcal{F}_c^{-1}\{e^{-a\omega}\}(y) = \frac{2}{\pi}\frac{a}{a^2+y^2}.$$

Convolution property 11.44c, now gives

$$V_1(x,y) = \frac{1}{2}\int_0^\infty g(v)\left(\frac{2}{\pi}\right)\left[\frac{x}{(y-v)^2+x^2} + \frac{x}{(y+v)^2+x^2}\right]dv$$

$$= \frac{x}{\pi}\int_0^\infty g(v)\left[\frac{1}{x^2+(y-v)^2} + \frac{1}{x^2+(y+v)^2}\right]dv.$$

Taking Fourier sine transforms with respect to x in order to find $V_1(x,y)$ leads to a nonhomogeneous ODE in $\tilde{V}_1(\omega,y)$ that is more difficult to solve.

To find $V_2(x,y)$ we apply the Fourier sine transform with respect to x to its PDE and use property 11.43d,

$$\frac{d^2\tilde{V}_2}{dy^2} - \omega^2\tilde{V}_2(\omega,y) = 0.$$

The transform must also satisfy

$$\frac{d\tilde{V}_2(\omega,0)}{dy} = \tilde{f}(\omega).$$

A general solution of the ODE is

$$\tilde{V}_2(\omega,y) = Ae^{\omega y} + Be^{-\omega y}.$$

For $\tilde{V}_2(\omega,y)$ to remain bounded as $y \to \infty$, A must be zero, and the boundary condition on $\tilde{V}_2$ then implies that $B = -\tilde{f}(\omega)/\omega$. Hence,

$$\tilde{V}_2(\omega,y) = -\frac{\tilde{f}(\omega)}{\omega}e^{-\omega y}$$

and

$$V_2(x,y) = \frac{2}{\pi}\int_0^\infty -\frac{\tilde{f}(\omega)}{\omega}e^{-\omega y}\sin\omega x\,d\omega.$$

The final solution is

$$V(x,y) = \frac{x}{\pi}\int_0^\infty g(v)\left[\frac{1}{x^2+(y-v)^2} + \frac{1}{x^2+(y+v)^2}\right]dv + \frac{2}{\pi}\int_0^\infty -\frac{\tilde{f}(\omega)}{\omega}e^{-\omega y}\sin\omega x\,d\omega.\bullet$$

Time-dependent heat and vibration problems on infinite or semi-infinite intervals require Fourier transforms. The boundary value problem in Example 11.22 also requires Fourier transforms since both x and y are on semi-infinite intervals. When solving Laplace's (or Poisson's) equation in the xy-plane where one of x or y is of finite extent, it may not be advantageous to introduce Fourier transforms; separation of variables or finite Fourier transforms may be preferable. We illustrate in the following example.

Example 11.23 A thin plate has edges along $y = 0$, $y = L'$, and $x = 0$ for $0 \le y \le L'$. The other edge is so far to the right that its effect may be considered negligible. Assuming no heat flow in the z-direction, find the steady-state temperature inside the plate (for $x > 0$, $0 < y < L'$) if sides $y = 0$ and $y = L'$ are held at constant temperature U_0°C, and side $x = 0$ has temperature $f(y)$, $0 \le y \le L'$.

Solution The boundary value problem for steady-state temperature $U(x,y)$ is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad x > 0, \quad 0 < y < L', \tag{11.60a}$$

$$U(x,0) = U(x,L') = U_0, \quad x > 0, \tag{11.60b}$$

$$U(0,y) = f(y), \quad 0 < y < L'. \tag{11.60c}$$

The finite Fourier transform associated with y is

$$\widetilde{f}(\lambda_n) = \int_0^{L'} f(y)Y_n(y)\,dy,$$

where $\lambda_n^2 = n^2\pi^2/L'^2$ and $Y_n(y) = \sqrt{2/L'}\sin\lambda_n y$ are eigenpairs of the Sturm-Liouville system

$$Y'' + \lambda^2 Y = 0, \quad 0 < y < L', \qquad Y(0) = Y(L') = 0.$$

When we apply the transform to the PDE, and use integration by parts,

$$\begin{aligned}\frac{\partial^2 \widetilde{U}}{\partial x^2} &= -\int_0^{L'} \frac{\partial^2 U}{\partial y^2} Y_n(y)\,dy = -\left\{\frac{\partial U}{\partial y} Y_n\right\}_0^{L'} + \int_0^{L'} \frac{\partial U}{\partial y} Y_n'\,dy \\ &= \left\{UY_n'\right\}_0^{L'} - \int_0^{L'} UY_n''\,dy = U_0Y_n'(L') - U_0Y_n'(0) - \int_0^{L'} U\left(-\lambda_n^2 Y_n\right)\,dy \\ &= U_0[Y_n'(L') - Y_n'(0)] + \lambda_n^2\widetilde{U}.\end{aligned}$$

Thus, $\widetilde{U}(x, \lambda_n)$ must satisfy the ODE

$$\frac{d^2\widetilde{U}}{dx^2} - \lambda_n^2\widetilde{U} = U_0[Y_n'(L') - Y_n'(0)],$$

subject to

$$\widetilde{U}(0, \lambda_n) = \widetilde{f}(\lambda_n).$$

A general solution of the differential equation is

$$\widetilde{U}(x, \lambda_n) = Ae^{\lambda_n x} + Be^{-\lambda_n x} - \frac{U_0[Y_n'(L') - Y_n'(0)]}{\lambda_n^2}.$$

For this to remain bounded as $x \to \infty$, we must set $A = 0$, in which case the boundary condition requires

$$\widetilde{f}(\lambda_n) = B - \frac{U_0[Y_n'(L') - Y_n'(0)]}{\lambda_n^2}.$$

Thus,

$$\widetilde{U}(x, \lambda_n) = \widetilde{f}(\lambda_n)e^{-\lambda_n x} - \frac{U_0[Y_n'(L') - Y_n'(0)]}{\lambda_n^2}(1 - e^{-\lambda_n x}).$$

The inverse finite Fourier transform now gives

$$\begin{aligned}U(x, y) &= \sum_{n=1}^{\infty} \widetilde{U}(x, \lambda_n)Y_n(y) \\ &= \sum_{n=1}^{\infty}\left\{\widetilde{f}(\lambda_n)e^{-\lambda_n x} - \frac{U_0[Y_n'(L') - Y_n'(0)]}{\lambda_n^2}(1 - e^{-\lambda_n x})\right\}\sqrt{\frac{2}{L'}}\sin\lambda_n y \\ &= \sqrt{\frac{2}{L'}}\sum_{n=1}^{\infty}\widetilde{f}(\lambda_n)e^{-n\pi x/L'}\sin\frac{n\pi y}{L'} + \frac{2U_0}{\pi}\sum_{n=1}^{\infty}\frac{[1 + (-1)^{n+1}]}{n}(1 - e^{-n\pi x/L'})\sin\frac{n\pi y}{L'}.\end{aligned}$$

Since

$$\widetilde{1} = \int_0^{L'}\sqrt{\frac{2}{L'}}\sin\frac{n\pi y}{L'}dy = \frac{\sqrt{2L'}[1 + (-1)^{n+1}]}{n\pi},$$

it follows that

$$U(x,y) = U_0 + \sqrt{\frac{2}{L'}} \sum_{n=1}^{\infty} \left\{ \widetilde{f}(\lambda_n) - \frac{\sqrt{2L'}U_0[1+(-1)^{n+1}]}{n\pi} \right\} e^{-n\pi x/L'} \sin\frac{n\pi y}{L'}. \bullet$$

EXERCISES 11.6

Part A Heat Conduction

1. Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0, \\ U(0,t) &= \overline{U} = \text{constant}, \quad t > 0, \\ U(x,0) &= 0, \quad x > 0. \end{aligned}$$

(Hint: See Exercise 20 in Section 11.5 when inverting the transform.) Is the solution the same as that in Example 10.9?

(b) Plot the solution on the interval $0 \le x \le 5$ with $k = 10^{-6}$ and $\overline{U} = 1$ for $t = 10^5$ and $t = 10^6$.

(c) Comment on the possibility of using the transformation $W = U - \overline{U}$ to remove the nonhomogeneity from the boundary condition.

2. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0, \\ U_x(0,t) &= -\kappa^{-1}Q_0 = \text{constant}, \quad t > 0, \\ U(x,0) &= 0, \quad x > 0. \end{aligned}$$

(Hint: See Exercise 20 in Section 11.5 when inverting the transform.) Plot the solution on the interval $0 \le x \le 5$ with $k = 10^{-6}$, $\kappa = 10$, and $Q_0 = 1000$ for $t = 10^5$ and $t = 10^6$.

(b) Describe the temperature of the left end of the rod.

3. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \quad x > 0, \quad t > 0, \\ U(0,t) &= f_1(t), \quad t > 0, \\ U(x,0) &= f(x), \quad x > 0. \end{aligned}$$

(b) Simplify the solution in part (a) when $g(x,t) \equiv 0$, $f_1(t) \equiv 0$, and $f(x) = U_0 = \text{constant}$.

(c) Simplify the solution in part (a) when $g(x,t) \equiv 0$, $f(x) \equiv 0$, and $f_1(t) = \overline{U} = \text{constant}$. Is it the solution of Exercise 1?

4. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \quad x > 0, \quad t > 0,$$
$$U_x(0,t) = -\kappa^{-1}f_1(t), \quad t > 0,$$
$$U(x,0) = f(x), \quad x > 0.$$

(b) Simplify the solution in part (a) when $g(x,t) \equiv 0$, $f_1(t) \equiv 0$, and $f(x) = U_0 =$ constant.
(c) Simplify the solution in part (a) when $g(x,t) \equiv 0$, $f(x) \equiv 0$, and $f_1(t) = Q_0 =$ constant. Is it the solution of Exercise 2?

5. Use the Fourier transform of Exercise 22 in Section 11.5 to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0,$$
$$-\kappa\frac{\partial U(0,t)}{\partial x} + \mu U(0,t) = \mu U_m = \text{constant}, \quad t > 0,$$
$$U(x,0) = 0, \quad x > 0.$$

6. Use the Fourier transform of Exercise 22 in Section 11.5 to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \quad x > 0, \quad t > 0,$$
$$-\kappa\frac{\partial U(0,t)}{\partial x} + \mu U(0,t) = \mu f_1(t), \quad t > 0,$$
$$U(x,0) = f(x), \quad x > 0.$$

Part B Vibrations

7. Solve the vibration problem of Example 11.20 if a unit force acts at the point $x = x_0$ on the string for all $t > 0$.

8. Repeat Example 11.20 if the Dirichlet boundary condition at $x = 0$ is replaced by the Neumann condition

$$y_x(0,t) = -\tau^{-1}f_1(t).$$

Constant τ is the tension in the string. This boundary condition describes the situation where the end $x = 0$ of the string, taken as massless, moves vertically with tension and an external force $f_1(t)$ acting on the end.

Part C Potential, Steady-state Heat Conduction, Static Deflection of Membranes

9. Find the electrostatic potential in the source-free region $0 < x < L$, $y > 0$ when potential along $y = 0$ is zero and potentials along $x = 0$ and $x = L$ are $f_1(y)$ and $f_2(y)$, respectively.

10. Solve the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad x > 0, \quad 0 < y < L,$$
$$V(0,y) = 0, \quad 0 < y < L,$$
$$V(x,0) = f(x), \quad x > 0,$$
$$V(x,L) = g(x), \quad x > 0.$$

11. Solve the boundary value problem in Exercise 10 if the boundary condition along the x-axis is Neumann $V_y(x, 0) = f(x)$.

12. Solve the boundary value problem in Exercise 10 if the boundary condition along $y = L$ is Neumann $V_y(x, L) = g(x)$.

13. Solve the boundary value problem in Exercise 10 if the boundary condition along $x = 0$ is homogeneous Neumann $V_x(0, y) = 0$.

14. Solve the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad x > 0, \quad y > 0,$$
$$V(0, y) = g(y), \quad y > 0,$$
$$V(x, 0) = f(x), \quad x > 0.$$

15. (a) Solve the boundary value problem for potential in the semi-infinite strip $0 < y < L'$, $x > 0$ when:
(i) potential on $y = 0$ and $y = L'$ is zero and that on $x = 0$ is $f(y)$ (simplify the solution when $f(y)$ is constant),
(ii) potential on $x = 0$ and $y = 0$ is zero and that on $y = L'$ is $g(x)$,
(iii) potential on $x = 0$ and $y = L'$ is zero and that on $y = 0$ is $g(x)$,
(iv) potentials on $x = 0$, $y = 0$, and $y = L'$ are $f(y)$, $g_1(x)$, and $g_2(x)$, respectively. (Hint: Superpose solutions of the types in (i), (ii), and (iii).)
(b) Try to solve the problem in (iv) by using:
(i) a Fourier sine transform on x (ii) a finite Fourier transform on y.

16. A thin plate has edges along $y = 0$, $y = L'$, and $x = 0$ for $0 \le y \le L'$. The other edge is so far to the right that its effect may be considered negligible. Assuming no heat flow in the z-direction, find the steady-state temperature inside the plate (for $x > 0$, $0 < y < L'$) if side $y = 0$ is held at temperature 0°C, side $y = L'$ is insulated, and, along $x = 0$:
(a) temperature is held at a constant U_0°C.
(b) heat is added to the plate at a constant rate $Q_0 > 0$ W/m^2 over the interval $0 < y < L'/2$ and extracted at the same rate for $L'/2 < y < L'$.
(c) heat is transferred to a medium at constant temperature U_m according Newton's law of cooling.

17. What are the solutions to Exercise 16 if edge $y = 0$ is insulated instead of held at temperature 0°C.

18. Does the function

$$U(x, y) = \begin{cases} -Q_0 x/\kappa, & 0 < y < L'/2 \\ Q_0 x/\kappa, & L'/2 < y < L' \end{cases}$$

satisfy the PDE and the boundary conditions on $x = 0$, $y = 0$, and $y = L'$ in Exercise 16(b)? Why is this not the solution?

19. (a) A thin plate has edges along $y = 0$, $y = L'$, and $x = 0$ for $0 \le y \le L'$. The other edge is so far to the right that its effect may be considered negligible. Assuming no heat flow in the z-direction, find the steady-state temperature inside the plate (for $x > 0$, $0 < y < L'$) if side $x = 0$ is held at temperature $f(y)$, side $y = 0$ is held at temperature zero, and along side $y = L'$ heat is transferred according to Newton's law of cooling to a medium at constant temperature U_m.

(b) Simplify the solution in part (a) when $U_m = 0$ and $f(y) = U_0$, a constant.

20. Repeat Exercise 19 when side $y = 0$ is insulated.

21. (a) A uniform charge distribution of density σ coulombs per cubic metre occupies the region bounded by the planes $x = 0$, $y = 0$, and $x = L$ $(y \geq 0)$. If the planes $x = 0$ and $y = 0$ are kept at zero potential and $x = L$ is maintained at a constant potential V_L, find the potential between the planes using:
(i) a finite Fourier transform.
(ii) a transformation to remove the constant nonhomogeneities σ and V_L.
(b) Can we apply a Fourier sine transform with respect to y?

22. If the charge distribution in Exercise 21 is a function of y, $\sigma(y) = e^{-y}$, find the potential between the plates.

23. Solve Exercise 22 when $V_L = 0$, using
(a) a finite Fourier transform (b) the Fourier sine transform

24. (a) Show that the Fourier sine transform with respect to x of the solution of the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad x > 0, \quad y > 0,$$
$$V(0, y) = 0, \quad y > 0,$$
$$V(x, 0) = f(x), \quad x > 0,$$

is $\tilde{V}(\omega, y) = \tilde{f}(\omega)e^{-\omega y}$.
(b) Use Example 11.17 and the result of Exercise 8b in Section 11.3 to show that

$$\mathcal{F}_C\left\{\frac{y}{x^2 + y^2}\right\} = \frac{\pi}{2}e^{-\omega y}, \quad y > 0.$$

(c) Now use convolution property 11.25 to show that

$$V(x, y) = \frac{y}{\pi}\int_0^\infty f(u)\left[\frac{1}{(x-u)^2 + y^2} - \frac{1}{(x+u)^2 + y^2}\right] du.$$

(d) Simplify the solution in part (c) when $f(x) = 1$.
(e) What is the solution when the boundary condition along $x = 0$ is $V(0, y) = g(y)$?

25. Simplify the solution to part (a)(i) of Exercise 15 when $f(y) = 1$.

§11.7 Hankel Transforms

Fourier transforms have been used to remove Cartesian coordinates from initial boundary value problems on infinite intervals; Fourier sine and cosine transforms are applicable to Cartesian coordinates on semi-infinite intervals. For problems in polar and cylindrical coordinates wherein the radial coordinate has range $r \geq 0$, the Hankel transform is prominent. It is based on Bessel's differential equation

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda^2 r - \frac{\nu^2}{r}\right) R = 0, \quad r > 0, \quad \nu \geq 0. \tag{11.61}$$

We have already seen that solutions of this differential equation that are bounded near $r = 0$ are multiples of

$$R(r) = J_\nu(\lambda r). \tag{11.62}$$

In order to associate a transform with $J_\nu(\lambda r)$, we must be aware of the behaviour of Bessel functions for large r. It is shown in the theory of asymptotics that $J_\nu(r)$ may be approximated for large r by

$$J_\nu(r) \approx \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi}{4} - \frac{\nu\pi}{2}\right), \tag{11.63}$$

the approximation being better the larger the value of r. This means that for larger r, $J_\nu(r)$ is oscillatory with an amplitude that decays at the same rate as $1/\sqrt{r}$.

Corresponding to the corollary of Theorem 11.1 in Section 11.2, we have the following **Hankel integral formula**.

Theorem 11.10 If $\sqrt{r}f(r)$ is absolutely integrable on $0 < r < \infty$, and $f(r)$ is piecewise smooth on every finite interval, then for $0 < r < \infty$,

$$\frac{f(r+) + f(r-)}{2} = \int_0^\infty \lambda A(\lambda) J_\nu(\lambda r)\, d\lambda \quad \text{where} \quad A(\lambda) = \int_0^\infty r f(r) J_\nu(\lambda r)\, dr. \tag{11.64}$$

In view of the asymptotic behaviour of $J_\nu(r)$ in expression 11.63, it is clear that absolute integrability of $\sqrt{r}f(r)$ guarantees convergence of the integral for $A(\lambda)$. Associated with this integral formula is the **Hankel transform** $\tilde{f}_\nu(\lambda)$ of a function $f(r)$,

$$\tilde{f}_\nu(\lambda) = \int_0^\infty r f(r) J_\nu(\lambda r)\, dr, \tag{11.65a}$$

and its inverse

$$f(r) = \int_0^\infty \lambda \tilde{f}_\nu(\lambda) J_\nu(\lambda r)\, d\lambda, \tag{11.65b}$$

where it is understood in 11.65b that $f(r)$ is defined as the average of left- and right-limits at points of discontinuity. We place a subscript ν on $\tilde{f}_\nu(\lambda)$ to remind ourselves that the Hankel transform is dependent on the choice of ν in differential equation 11.61; changing ν changes the transform.

Example 11.24 Find the Hankel transform of $f(r) = \begin{cases} r^\nu, & 0 < r < a \\ 0, & r > a \end{cases}$.

Solution By definition 11.65a,

$$\tilde{f}_\nu(\lambda) = \int_0^a r^{\nu+1} J_\nu(\lambda r)\, dr.$$

If we set $u = \lambda r$, then

$$\begin{aligned}\tilde{f}_\nu(\lambda) &= \int_0^{\lambda a} \left(\frac{u}{\lambda}\right)^{\nu+1} J_\nu(u) \frac{du}{\lambda} = \frac{1}{\lambda^{\nu+2}} \int_0^{\lambda a} u^{\nu+1} J_\nu(u)\, du \\ &= \frac{1}{\lambda^{\nu+2}} \int_0^{\lambda a} \frac{d}{du}[u^{\nu+1} J_{\nu+1}(u)]\, du \quad \text{(see equation 8.42 in Section 8.3)} \\ &= \frac{1}{\lambda} a^{\nu+1} J_{\nu+1}(\lambda a).\end{aligned}$$

The inverse Hankel transform then gives

$$\int_0^\infty \lambda \left[\frac{1}{\lambda} a^{\nu+1} J_{\nu+1}(\lambda a)\right] J_\nu(\lambda r)\, d\lambda = \begin{cases} r^\nu, & 0 < r < a \\ a^\nu/2, & r = a \\ 0, & r > a, \end{cases}$$

and from this we obtain the following useful integration formula

$$\int_0^\infty J_{\nu+1}(\lambda a) J_\nu(\lambda r)\, d\lambda = \begin{cases} \dfrac{1}{a}\left(\dfrac{r}{a}\right)^\nu, & 0 < r < a \\ 1/(2a), & r = a \\ 0, & r > a.\bullet \end{cases}$$

Example 11.25 Use the Hankel transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad r > 0, \quad t > 0, \tag{11.66a}$$

$$U(r, 0) = f(r), \quad r > 0. \tag{11.66b}$$

Solution Because the Bessel function $J_0(r)$ results when separation of variables is performed on the PDE, we apply the Hankel transform associated with $J_0(r)$, namely

$$\tilde{f}(\lambda) = \int_0^\infty r f(r) J_0(\lambda r)\, dr,$$

where we have suppressed the zero subscript on $\tilde{f}(\lambda)$. Application of this transform to the PDE gives

$$\begin{aligned}\frac{d\tilde{U}}{dt} &= k \int_0^\infty r\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) J_0(\lambda r)\, dr \\ &= k\left\{r\frac{\partial U}{\partial r} J_0(\lambda r)\right\}_0^\infty - k\int_0^\infty \frac{\partial U}{\partial r}\left\{\frac{d}{dr}[rJ_0(\lambda r)] - J_0(\lambda r)\right\} dr \\ &= -k\int_0^\infty \frac{\partial U}{\partial r} r \frac{d}{dr}[J_0(\lambda r)]\, dr \quad \left(\text{provided } \lim_{r\to\infty} \sqrt{r}\frac{\partial U}{\partial r} = 0\right) \\ &= -k\left\{Ur\frac{d}{dr}[J_0(\lambda r)]\right\}_0^\infty + k\int_0^\infty U \frac{d}{dr}\left[r\frac{dJ_0(\lambda r)}{dr}\right] dr \\ &= k\int_0^\infty U[-\lambda^2 r J_0(\lambda r)]\, dr \quad \left(\text{provided } \lim_{r\to\infty} \sqrt{r}U = 0\right) \\ &= -k\lambda^2 \tilde{U}.\end{aligned}$$

Thus, $\tilde{U}(\lambda, t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\lambda^2\tilde{U} = 0 \tag{11.67a}$$

subject to the transform of condition 11.66b,

$$\tilde{U}(\lambda, 0) = \tilde{f}(\lambda) = \int_0^\infty r f(r) J_0(\lambda r)\, dr. \tag{11.67b}$$

The solution of this problem is

$$\tilde{U}(\lambda, t) = \tilde{f}(\lambda) e^{-k\lambda^2 t}, \tag{11.68}$$

and therefore

$$U(r, t) = \int_0^\infty \lambda \tilde{f}(\lambda) e^{-k\lambda^2 t} J_0(\lambda r)\, d\lambda. \bullet \tag{11.69}$$

EXERCISES 11.7

Part A Heat Conduction

1. Heat is generated at a constant rate g W/m^3 inside the cylinder $0 < r < a$ for time $t > 0$. If the temperature of space is zero at time $t = 0$, find the temperature at all points for $t > 0$.

2. An infinite wedge is bounded by the straight edges $\theta = 0$ and $\theta = \alpha$ $(0 < \alpha < 2\pi)$. At time $t = 0$, its temperature is zero throughout, and for $t > 0$, its edges $\theta = 0$ and $\theta = \alpha$ are held at constant temperature $\overline{U}$. Find the temperature in the wedge for $t > 0$. Hint: Apply a finite Fourier transform with respect to θ and a Hankel transform with respect to r. You will need the result that

$$\int_0^\infty \frac{J_\nu(x)}{x} dx = \frac{1}{\nu}.$$

Part B Vibrations

3. (a) A very large membrane is given an initial displacement that is only a function $f(r)$ of distance from some fixed point but has no initial velocity. Find an integral representation for its subsequent displacement.

(b) Use Exercise 14 in Section 8.3 to show that when $f(r) = A/\sqrt{1 + (r/a)^2}$, where a and A are positive constants, the solution can be expressed in the form

$$z(r, t) = aA \int_0^\infty e^{-a\lambda} \cos c\lambda t J_0(\lambda r)\, d\lambda.$$

(c) Use Exercise 14 in Section 8.3 once again to simplify the solution to

$$z(r, t) = \frac{aA\sqrt{\sqrt{(r^2 + a^2 - c^2t^2)^2 + 4a^2c^2t^2} + r^2 + a^2 - c^2t^2}}{\sqrt{2}\sqrt{(r^2 + a^2 - c^2t^2)^2 + 4a^2c^2t^2}}.$$

4. Repeat part (a) of Exercise 3 when $f(r)$ is the initial velocity of the membrane and it has no initial displacement.

Part C Potential, Steady-state Heat Conduction, Static Deflections of Membranes

5. A disc $0 \le r < a$ in the xy-plane emits heat into the region $z > 0$ at a constant rate Q W/m^2. If the remainder $(r > a)$ of the plane is insulated, the steady-state temperature in $z > 0$ must satisfy

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0, \quad r > 0, \quad z > 0,$$

$$\frac{\partial U(r,0)}{\partial z} = \begin{cases} -Q/\kappa, & 0 \le r < a \\ 0, & r > a \end{cases}.$$

Find $U(r, z)$.

6. Repeat Exercise 5 if the disc is held at constant temperature $\overline{U}$ and the remainder of the xy-plane is held at temperature zero.

7. Repeat Exercise 5 if the disc is held at constant temperature $\overline{U}$ and the remainder of the xy-plane is insulated.

8. The disc $0 \le r < a$ in the xy-plane is kept at constant temperature $\overline{U}$. Find steady-state temperature in space.

CHAPTER 12 GREEN'S FUNCTIONS FOR ORDINARY DIFFERENTIAL EQUATIONS

§12.1 Generalized Functions

Without justification, we have used the delta function of Section 2.1 to model mathematical idealizations called *point* entities — point charges, point masses, point heat sources, and point forces, to name a few. It is the purpose of this chapter to justify the use of the delta function in ordinary differential equations and to use them to develop Green's functions. In Chapter 13, we define multi-variable delta functions and use them to develop Green's functions for partial differential equations.

First we illustrate once again how delta functions simplify the solution of ODEs containing point "entities". Suppose a 1-N force is applied to the midpoint of a taut string (of negligible mass) as shown in Figure 12.1. We have a point force of one newton acting at the midpoint of the string. The boundary value problem that describes static deflections of the string is

$$-\tau \frac{d^2y}{dx^2} = F(x), \quad 0 < x < L, \tag{12.1a}$$

$$y(0) = 0 = y(L), \tag{12.1b}$$

where τ is the constant tension in the string and $F(x)$ is the force per unit x-length on the string due to the applied force. Although it might seem to be a simple procedure to integrate the differential equation twice and apply the boundary conditions (for determination of constants of integration), integration of $F(x)$ presents a problem. In fact, even representation of $F(x)$ is problematic. It might seem natural to represent $F(x)$ as follows

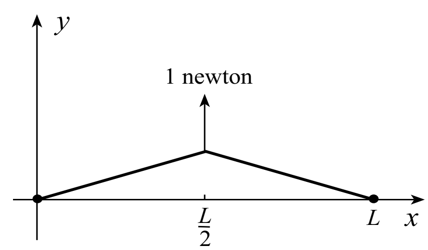


Figure 12.1

$$F(x) = \begin{cases} 0, & 0 < x < L/2 \\ 1, & x = L/2 \\ 0, & L/2 < x < L. \end{cases} \tag{12.2}$$

Antidifferentiation of differential equation 12.1a with this representation gives

$$y(x) = \begin{cases} Ax + B, & 0 < x < L/2 \\ Cx + D, & L/2 < x < L. \end{cases}$$

(Recall from elementary calculus that we antidifferentiate only over an interval, not at a point; hence the absence of an antiderivative "at" $x = L/2$.) If we now apply boundary conditions 12.1b and demand that $y(x)$ be continuous at $x = L/2$, we obtain

$$y(x) = \begin{cases} Ax, & 0 \le x \le L/2 \\ -A(x - L), & L/2 < x \le L. \end{cases}$$

But how do we calculate A? Certainly the size of the force (1 N here) and the tension τ in the string must determine A, but there seems to be no way to use this information. The problem must be representation 12.2 for a point force concentrated at $x = L/2$. Perhaps what we should do is distribute this force along the string, solve the problem, and then take a limit as the distributed force approaches a concentrated force. There is a multitude of ways that $F(x)$ might be defined, but clearly each must satisfy the condition

$$\int_0^L F(x)\,dx = 1. \tag{12.3}$$

Two possibilities, which are symmetric, are shown in Figure 12.2.

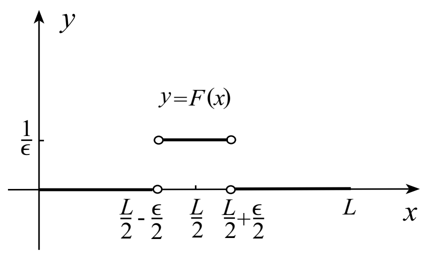


Figure 12.2a

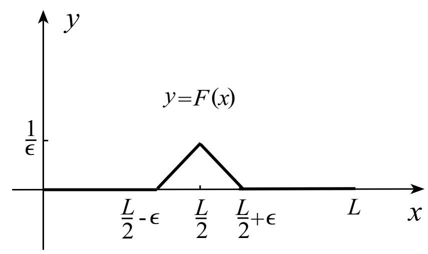


Figure 12.2b

Suppose we use the distribution in Figure 12.2a. Differential equation 12.1a becomes

$$\frac{d^2y}{dx^2} = -\frac{1}{\tau}\begin{cases} 0, & 0 < x < (L-\epsilon)/2 \\ 1/\epsilon, & (L-\epsilon)/2 < x < (L+\epsilon)/2 \\ 0, & (L+\epsilon)/2 < x < L, \end{cases}$$

and integration gives

$$y(x) = -\frac{1}{\tau}\begin{cases} Ax + B, & 0 < x < (L-\epsilon)/2 \\ x^2/(2\epsilon) + Cx + D, & (L-\epsilon)/2 < x < (L+\epsilon)/2 \\ Ex + F, & (L+\epsilon)/2 < x < L. \end{cases}$$

If we apply boundary conditions 12.1b and demand that $y(x)$ and $y'(x)$ be continuous at $x = (L-\epsilon)/2$ and $x = (L+\epsilon)/2$, we find that

$$y(x) = \frac{1}{\tau}\begin{cases} \dfrac{x}{2}, & 0 \le x \le (L-\epsilon)/2 \\ -\dfrac{x^2}{2\epsilon} + \dfrac{Lx}{2\epsilon} - \dfrac{1}{8\epsilon}(L-\epsilon)^2, & (L-\epsilon)/2 \le x \le (L+\epsilon)/2 \\ \dfrac{L-x}{2}, & (L+\epsilon)/2 \le x \le L. \end{cases} \tag{12.4}$$

A graph of this function is shown in Figure 12.3. To obtain the solution of problem 12.1 for a concentrated force, we now let ϵ approach zero. Geometrically, the parabolic section becomes smaller and smaller in width, and in the limit the two straight-line sections meet at $x = L/2$ (Figure 12.4). This implies that the displacement at $L/2$ is $L/(4\tau)$ and the displacement function for the unit point force in Figure 12.1 is that in Figure 12.4, defined algebraically by

$$y(x) = \begin{cases} x/(2\tau), & 0 \le x \le L/2 \\ (L-x)/(2\tau), & L/2 \le x \le L. \end{cases} \tag{12.5}$$

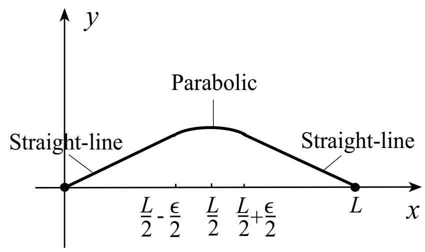


Figure 12.3

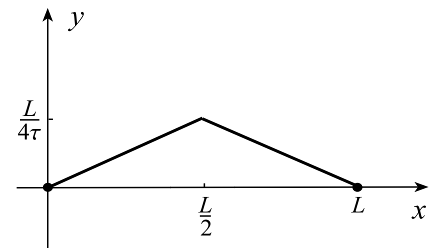


Figure 12.4

In Exercise 7, displacement $y(x)$ for the distributed load in Figure 12.2b is calculated. Although it is different from 12.4, its limit as ϵ approaches zero is once again given by equation 12.5.

Now suppose that we use $\delta(x - L/2)$ to model the unit force at $x = L/2$, so that problem 12.1 reads

$$-\tau \frac{d^2y}{dx^2} = \delta(x - L/2), \quad 0 < x < L, \tag{12.6a}$$

$$y(0) = 0 = y(L). \tag{12.6b}$$

If we integrate the differential equation twice using equation 2.16 and Exercise 8 in Section 2.1, we obtain

$$y(x) = -\frac{1}{\tau}(x - L/2)h(x - L/2) + Cx + D,$$

where $h(x - L/2)$ is the Heaviside unit step function. The boundary conditions require

$$0 = y(0) = D, \quad 0 = y(L) = -\frac{L}{2\tau} + CL + D \qquad \Longrightarrow \qquad C = \frac{1}{2\tau}, \quad D = 0.$$

The deflection of the string is therefore

$$y(x) = -\frac{1}{\tau}(x - L/2)h(x - L/2) + \frac{x}{2\tau} = \begin{cases} x/(2\tau), & 0 \le x \le L/2 \\ (L - x)/(2\tau), & L/2 \le x \le L, \end{cases}$$

solution 12.5. (Because $h(x - L/2)$ is undefined at $x = L/2$, we implicitly assume that limits are taken as $x \to L/2$.) These calculations have demonstrated the simplicity of the delta function representation of a point force as opposed to a distributed force and limits. Exercises 9, 10, 14, and 15 provide further illustrations.

When we solve linear, second-order differential equations

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = f(x),$$

where $P(x)$, $Q(x)$, and $R(x)$ are continuous and $f(x)$ is piecewise continuous, the solution should be continuous and have a continuous first derivative. In fact, for the distributed load in Figure 12.2a we actually imposed these conditions at $x = (L \pm \epsilon)/2$ to obtain displacement 12.4. But notice that limit function 12.5, shown in Figure 12.4, has a discontinuity in $y'(x)$ at $x = L/2$. In other words, when *point* sources influence second-order differential equations, we cannot expect solutions to have continuous first derivatives.

We now continue our discussion of delta functions as representations for concentrated sources. Based on the above example (where the unit force was distributed over an interval on the x-axis), it might seem reasonable to define $\delta(x-c)$ as the limit as $\epsilon \to 0$ of the unit pulse function $P_\epsilon(x,c)$ in Figure 12.5; that is, define

$$\delta(x-c) = \lim_{\epsilon \to 0} P_\epsilon(x,c). \tag{12.7}$$

Because the area under $P_\epsilon(x,c)$ is unity for any $\epsilon > 0$, this definition appears to preserve the "unit" character of the source. But, from the classical point of view of a function as a mapping from domain to range, definition 12.7 is unacceptable. It maps all values $x \neq c$ onto zero, and the value of $\delta(x-c)$ at $x = c$ is somehow "infinite". What we are saying is that $\delta(x-c)$ cannot be defined in a pointwise sense; functions that represent point sources require a completely new approach.

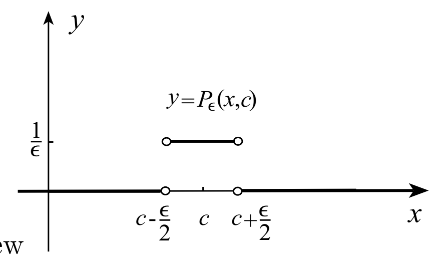


Figure 12.5

To introduce this approach, recall that when $y_n(x)$ are normalized eigenfunctions of a Sturm-Liouville system on an interval $a \le x \le b$, and $f(x)$ is suitably behaved, the finite Fourier transform of $f(x)$ is

$$\tilde{f}(\lambda_n) = \int_a^b p(x) f(x) y_n(x)\, dx \tag{12.8}$$

where $p(x)$ is the weight function of the Sturm-Liouville system. The emphasis here is that given a function $f(x)$, each eigenfunction $y_n(x)$ associates a number $\tilde{f}(\lambda_n)$ with $f(x)$, and the sequence of numbers $\{\tilde{f}(\lambda_n)\}$ is called the finite Fourier transform of $f(x)$. Suppose we switch the emphasis to one of the eigenfunctions, say $y_n(x)$, for a fixed integer n. This eigenfunction associates a number $\tilde{f}(\lambda_n)$ given by equation 12.8 with any given function $f(x)$. Various words are used to describe this association, including mapping, operator, and functional. We think of $y_n(x)$ as a mapping, or operator, that maps, or operates on, functions $f(x)$ to yield a number $\tilde{f}(\lambda_n)$ defined by integral 12.8,

$$f(x) \xrightarrow{y_n(x)} \tilde{f}(\lambda_n) = \int_a^b p(x) f(x) y_n(x)\, dx.$$

By this definition, each eigenfunction $y_n(x)$ associates with a function $f(x)$ its n^{th} Fourier coefficient $\tilde{f}(\lambda_n)$,

$$f(x) \xrightarrow{y_n(x)} \tilde{f}(\lambda_n).$$

The word functional is used to describe a function whose domain is a set of functions. Here $y_n(x)$, the functional, associates a number $\tilde{f}(\lambda_n)$ with any given function $f(x)$.

But we could do this with any function $g(x)$, continuous on $a \le x \le b$, not just eigenfunctions of Sturm-Liouville systems. We can regard $g(x)$ as an operator, mapping, or functional, that associates with any function $f(x)$ a number defined by the following definite integral

$$f(x) \xrightarrow{g(x)} \int_a^b g(x)f(x)\,dx.$$

It is this view of an ordinary function as a mapping, operator, or functional that we adopt to define $\delta(x-c)$. The "generalized" function $\delta(x-c)$, called the (**Dirac**) **delta function**, is the functional that maps a function $f(x)$, continuous at $x=c$, onto its value at $x=c$,

$$f(x) \xrightarrow{\delta(x-c)} f(c). \tag{12.9}$$

For example,

$$x^2+2x-3 \xrightarrow{\delta(x-2)} 5$$

and

$$(x+1)^2\cos x \xrightarrow{\delta(x)} 1.$$

In order that the delta functional have an integral representation, we write

$$f(x) \xrightarrow{\delta(x-c)} f(c) = \int_{-\infty}^{\infty} f(x)\delta(x-c)\,dx. \tag{12.10}$$

Because $\delta(x-c)$ cannot be regarded pointwise, the multiplication in this integral, and the integral itself, are symbolic. When we encounter an integral such as that in equation 12.10, we interpret it as the action of the functional $\delta(x-c)$ operating on $f(x)$ and immediately write $f(c)$. For example,

$$\int_{-\infty}^{\infty}\left(x^2+\frac{2}{x-1}\right)\delta(x)\,dx = -2$$

and

$$\int_{-\infty}^{\infty}\delta(x+2)\,dx = 1$$

(since the left side of the latter integral is interpreted as the delta function $\delta(x+2)$ operating on the function $f(x)\equiv 1$).

Many authors regard the integral in equation 12.10 as the defining relation for the delta function $\delta(x-c)$. In actual fact, equation 12.9 is the defining relation of $\delta(x-c)$ as a mapping, operator, or functional, and the integral in equation 12.10 is a symbolic representation. We have manipulated many ODEs and PDEs containing delta functions, but these equations were never regarded pointwise. They were always integrated and symbolic representation 12.10 was used to simplify any integral involving a delta function.

Because $\delta(x-c)$ picks out the value of a function at $x=c$, we also write

$$\int_a^b f(x)\delta(x-c)\,dx = f(c) \tag{12.11a}$$

whenever $a<c<b$; that is, the limits on the integral need not be $\pm\infty$. Furthermore, if $x=c$ is not between a and b, we set

$$\int_a^b f(x)\delta(x-c)\,dx = 0. \tag{12.11b}$$

For instance,

$$\int_{-2}^{6} \sqrt{x+5}\,\delta(x)\,dx = \sqrt{5}$$

and

$$\int_2^3 (x^2+2x-4)\delta(x+1)\,dx = 0.$$

The above discussion of delta functions from a mapping, an operator, or a functional perspective still does not justify their use as mathematical representations of point entities. Our discussion of problem 12.1 and exercises 9, 10, 14, and 15 suggest that this should be the case, but they are not mathematical verifications.

EXERCISES 12.1

In Exercises 1–6 evaluate the integral.

1. $\displaystyle\int_{-\infty}^{\infty} (x^2-2x+4)\delta(x-1)\,dx$ **2.** $\displaystyle\int_{-8}^{3} \sin(3x+1)\delta(x)\,dx$

3. $\displaystyle\int_{-4}^{20} (e^x+x^2)\delta(x+3)\,dx$ **4.** $\displaystyle\int_{3}^{\infty} (x^2+1/x)\delta(x)\,dx$

5. $\displaystyle\int_{-\infty}^{\infty} (2x^2+x^3+4)\delta(x-4)\,dx$ **6.** $\displaystyle\int_{-\infty}^{\infty} (1+4x-\cos x)\delta(x+10)\,dx$

7. Solve problem 12.1 when $F(x)$ is defined as in Figure 12.2b, and sketch the displacement function. Show that the displacement of Figure 12.4 is obtained in the limit as $\epsilon \to 0^+$.

8. Define your own distributed force function $F(x)$ (subject to condition 12.3) and solve problem 12.1, taking limits as $F(x)$ approaches a point force. Do you obtain the result in Figure 12.4?

9. (a) Calculate the displacement of a taut string (of negligible mass and length L) when two unit point masses are attached at distances $L/3$ from each end. Use distribution functions like that in Figure 12.2a for each mass.
(b) Show that the solution in part (a) is obtained if delta functions are used to represent the point masses.

10. (a) A beam of length L and negligible weight is subjected to a unit downward force at its midpoint. If the left end of the beam ($x=0$) is fixed horizontally and the right end ($x=L$) is free, use a distributed force like that of Figure 12.2a and limits as $\epsilon \to 0^+$ to find the static deflection of the beam. Sketch the graph of the displacement function. Are $y'(x)$, $y''(x)$, and $y'''(x)$ continuous?
(b) Show that the solution in part (a) is obtained if a delta function is used to represent the point force.

11. Find deflections of the beam in Exercise 10 if the point force is placed at the end $x=L$ by:
(a) distributing the force and taking limits as the length over which distribution takes place approaches zero;
(b) representing the force by $-\delta(x-x_0)$ and taking the limit of the solution as $x_0 \to L$.

12. The displacement of a mass M from its equilibrium position at the end of a spring with constant k is described by the differential equation

$$M\frac{d^2y}{dt^2} + ky = F(t)$$

when viscous damping is negligible. In this exercise we determine the displacement $y(t)$ due to an instantaneous unit force $F(t)$ applied at time T,

$$F(t) = \begin{cases} 0, & 0 < t < T \\ 1, & t = T \\ 0, & t > T. \end{cases}$$

We do this by distributing the unit impulse in two ways and also using a delta function.

(a) First, distribute $F(t)$ over a time interval of length ϵ around T according to

$$F_1(t) = \begin{cases} 0, & 0 < t < T - \epsilon/2 \\ 1/\epsilon, & T - \epsilon/2 < t < T + \epsilon/2 \\ 0, & t > T + \epsilon/2. \end{cases}$$

(Notice that the units of $F_1(t)$ are units of force per unit of time, so that the total area "under" the $F_1(t)$ curve is unity.) Solve the differential equation with $F(t)$ replaced by $F_1(t)$ subject to the initial conditions $y(0) = y'(0) = 0$. Find and sketch the limit function as $\epsilon \to 0^+$.

(b) Repeat part (a) with $F(t)$ distributed over the time interval $T < t < T + \epsilon$ according to

$$F_2(t) = \begin{cases} 0, & 0 < t < T \\ 1/\epsilon, & T < t < T + \epsilon \\ 0, & t > T + \epsilon. \end{cases}$$

(c) Use variation of parameters to solve the initial value problem when $F(t)$ is replaced by $\delta(t - T)$.

13. Show that the same function as that in Exercise 12 is obtained if we assume that $y(t) = 0$ for $t < T$ and that for $t \geq T$, $y(t)$ satisfies

$$M\frac{d^2y}{dt^2} + ky = 0, \quad t > T,$$

$$y(T) = 0, \quad y'(T) = \frac{1}{M}.$$

Distributing point forces for multidimensional boundary value problems is more complex. The remaining exercises give examples.

14. A square membrane stretched tightly over the region $0 \leq x, y \leq L$ has edges fixed on the xy-plane. Distribute a unit load at the midpoint of the membrane according to

$$F(x,y) = \begin{cases} -1/\epsilon^2, & (L-\epsilon)/2 < x, y < (L+\epsilon)/2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the static deflection of the membrane due to this load by using the finite Fourier transform associated with the x-variable, or an eigenfunction expansion

$$z(x,y) = \sum_{n=1}^{\infty} a_n(y)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}.$$

(b) Take the limit of the function $z(x, y)$ in part (a) as $\epsilon \to 0^+$ to find the static deflection of the membrane under a unit concentrated load at its centre.

(c) Show that the same solution as in part (b) is obtained if the unit load is represented by the product of delta functions $-\delta(x - L/2)\delta(y - L/2)$.

(d) Is the result in part (b) defined at $(L/2, L/2)$?

15. Repeat parts (a) and (b) of Exercise 14 for a circular membrane of radius R. Distribute the unit load at the midpoint of the membrane according to

$$F(r) = \begin{cases} -1/(\pi\epsilon^2), & 0 \le r < \epsilon \\ 0, & \text{otherwise.} \end{cases}$$

(c) Show that the same solution as in part (b) is obtained if the unit load is represented by the delta function $-\delta(r)/(2\pi r)$. In Section 13.1, we shall see why this is the delta function at $r = 0$.

§12.2 Introductory Example

In this section we use a simple example to illustrate Green's functions. The boundary value problem

$$-\tau\frac{d^2y}{dx^2} = F(x), \tag{12.12a}$$
$$y(0) = y(L) = 0 \tag{12.12b}$$

describes static deflections of a taut string with tension τ, length L, and fixed end points, due to a load $F(x)$ (Figure 12.6). We can solve this problem by using variation of parameters on the general solution $Ax + B$ of the associated homogeneous equation (see Section 4.3). Derivatives of $A(x)$ and $B(x)$ must satisfy

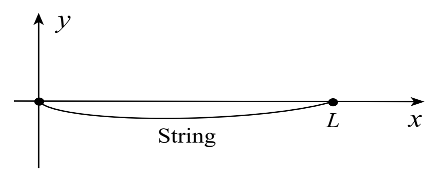


Figure 12.6

$$A'x + B' = 0,$$
$$A' = -\frac{F(x)}{\tau}.$$

Solutions of these equations may be expressed as definite integrals

$$A(x) = \int_0^x -\tau^{-1}F(X)\,dX + C, \qquad B(x) = \int_0^x \tau^{-1}XF(X)\,dX + D,$$

and hence

$$\begin{aligned} y(x) &= x\left[\int_0^x -\tau^{-1}F(X)\,dX + C\right] + \int_0^x \tau^{-1}XF(X)\,dX + D \\ &= \frac{1}{\tau}\int_0^x (X-x)F(X)\,dX + Cx + D. \end{aligned}$$

Boundary conditions 12.12b require the constants C and D to satisfy

$$\begin{aligned} 0 &= y(0) = D, \\ 0 &= y(L) = \frac{1}{\tau}\int_0^L (X-L)F(X)\,dX + CL + D, \end{aligned}$$

and therefore

$$\begin{aligned} y(x) &= \frac{1}{\tau}\int_0^x (X-x)F(X)\,dX + \frac{x}{L\tau}\int_0^L (L-X)F(X)\,dX \\ &= \frac{1}{\tau}\int_0^x \left[(X-x) + \frac{x(L-X)}{L}\right]F(X)\,dX + \frac{x}{L\tau}\int_x^L (L-X)F(X)\,dX \\ &= \frac{1}{L\tau}\int_0^x X(L-x)F(X)\,dX + \frac{x}{L\tau}\int_x^L (L-X)F(X)\,dX \end{aligned}$$

or

$$y(x) = \int_0^L g(x;X)F(X)\,dX, \tag{12.13a}$$

where

$$g(x;X) = \begin{cases} \dfrac{X(L-x)}{L\tau}, & 0 \le X \le x \\ \dfrac{x(L-X)}{L\tau}, & x \le X \le L. \end{cases} \tag{12.13b}$$

The solution of problem 12.12 has therefore been expressed in integral form — the integral of the nonhomogeneity $F(x)$ multiplied by the function $g(x;X)$. The function $g(x;X)$ is called the Green's function for boundary value problem 12.12. It does not depend on $F(x)$; it depends only on the differential operator and the boundary conditions. Once $g(x;X)$ is known, the solution for any $F(x)$ can be represented in the form of a definite integral involving $g(x;X)$ and $F(x)$, and this integral representation clearly displays how the solution depends on $F(x)$. In addition, we shall see that when the boundary conditions are nonhomogeneous, representation of the solution in terms of the Green's function also indicates the nature of the dependence on these nonhomogeneities. Finally, it should be clear that formulation of the solution as a definite integral is a distinct advantage in numerical analysis.

The representation of $g(x;X)$ in equation 12.13b regards X as the independent variable and x as a parameter. By interchanging the two expressions, we obtain a representation wherein X is the parameter and x is the independent variable,

$$g(x;X) = \begin{cases} \dfrac{x(L-X)}{L\tau}, & 0 \le x \le X \\ \dfrac{X(L-x)}{L\tau}, & X \le x \le L. \end{cases} \tag{12.13c}$$

With representation 12.13c, it is straightforward to illustrate three properties of this Green's function that are shared by all Green's functions of one variabale. First,

$$g(x;X) \text{ is continuous for all } x \text{ (including } x = X). \tag{12.14a}$$

Second, the derivative of $g(x;X)$ with respect to x is continuous for all $x \ne X$, and

$$\lim_{x\to X^+} \frac{dg}{dx} - \lim_{x\to X^-} \frac{dg}{dx} = \left(\frac{-X}{L\tau}\right) - \left(\frac{L-X}{L\tau}\right) = -\frac{1}{\tau}. \tag{12.14b}$$

This jump is the reciprocal of the coefficient of d^2y/dx^2 in differential equation 12.12a. Finally, it is straightforward to check that at every $x \ne X$,

$$\begin{gathered} g(x;X) \text{ satisfies the homogeneous version of the} \\ \text{differential equation from which it was derived.} \end{gathered} \tag{12.14c}$$

As we said, properties 12.14a–c are shared by all Greens functions associated with ordinary differential equations. In fact, we shall use them to characterize Green's functions in Section 12.3.

EXERCISES 12.2

1. Consider the boundary value problem

$$\frac{d^2y}{dx^2} + y = F(x), \quad 0 < x < L,$$
$$y(0) = 0 = y'(L).$$

(a) Use variation of parameters to show that the solution can be expressed in the form

$$y(x) = \int_0^L g(x;X)F(X)\,dX,$$

where $g(x;X)$ is the Green's function of the problem defined by

$$g(x;X) = \frac{-1}{\cos L}\begin{cases} \sin X \cos{(L-x)}, & 0 \le X \le x \\ \sin x \cos{(L-X)}, & x \le X \le L \end{cases}.$$

(b) Show that $g(x;X)$ satisfies properties 12.14a–c.

2. Show that if $F(x)$ is set equal to $\delta(x - L/2)$ in equation 12.13a, solution 12.5 of problem 12.1 is obtained.

§12.3 Green's Functions

In this section we associate Green's functions with linear, second-order ordinary differential equations

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = f(x), \quad \alpha < x < \beta. \tag{12.15}$$

Functions $P(x)$, $Q(x)$, and $R(x)$ are assumed continuous for $\alpha \le x \le \beta$, but no assumption is yet made on the behaviour of $f(x)$. Provided $P(x)$ does not vanish on the interval $\alpha \le x \le \beta$, multiplication of ODE 12.15 by $P^{-1}e^{\int (Q/P)\,dx}$ gives

$$\frac{d}{dx}\left[e^{\int (Q/P)\,dx}\frac{dy}{dx}\right] + \frac{R}{P}e^{\int (Q/P)\,dx}y = \frac{1}{P}e^{\int (Q/P)\,dx}f(x).$$

When we set $r(x) = e^{\int (Q/P)\,dx}$, $q(x) = -RP^{-1}e^{\int (Q/P)\,dx}$, and
$F(x) = P^{-1}e^{\int (Q/P)\,dx}$, the equation takes on a form reminiscent of that in Sturm-Liouville theory,

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = F(x), \quad \alpha < x < \beta. \tag{12.16a}$$

In other words, every linear, second-order differential equation for which $P(x) \ne 0$ can be expressed in form 12.16a, where $r(x) > 0$. This is called the **self-adjoint** form of the differential equation. We shall often find it convenient to denote the self-adjoint differential operator on the left side of equation 12.16a by L, in which case the differential equation is expressed more compactly as

$$Ly = F(x), \quad \alpha < x < \beta. \tag{12.16b}$$

To obtain a unique solution of equation 12.16, it is necessary to specify two boundary conditions. The most general boundary conditions that we consider are of the form

$$B_1y = -l_1y'(\alpha) + h_1y(\alpha) - l_3y'(\beta) + h_3y(\beta) = m_1, \tag{12.17a}$$
$$B_2y = l_2y'(\beta) + h_2(\beta) + l_4y'(\alpha) + h_4y(\alpha) = m_2, \tag{12.17b}$$

where l_1, l_2, l_3, l_4, h_1, h_2, h_3, h_4, m_1, and m_2 are given constants. For the most part, we consider conditions of the form

$$B_1y = -l_1y'(\alpha) + h_1y(\alpha) = m_1, \tag{12.18a}$$
$$B_2y = l_2y'(\beta) + h_2y(\beta) = m_2, \tag{12.18b}$$

They are called **unmixed** boundary conditions because one condition is at $x = \alpha$ and the other is at $x = \beta$. On occasion, however, we shall consider **periodic** conditions

$$y(\alpha) = y(\beta), \tag{12.19a}$$
$$y'(\alpha) = y'(\beta). \tag{12.19b}$$

They arise only when $r(\alpha) = r(\beta)$, and they are always homogeneous. We have seen both types of conditions many times throughout Chapters 2–11.

For the moment, we concentrate only on the self-adjoint operator L in equation 12.16, not on the differential equation or the boundary conditions. When $u(x)$ and $v(x)$ are continuously differentiable functions on $\alpha \le x \le \beta$ with piecewise continuous second derivatives, it is straightforward to show that

$$uLv - vLu = \frac{d}{dx}J(u,v) = \frac{d}{dx}[r(uv' - vu')]. \tag{12.20}$$

This equation is known as **Lagrange's identity**. The quantity

$$J(u,v) = r(uv' - vu') \tag{12.21}$$

is called the **conjunct** of u and v. Lagrange's identity is valid at every point except discontinuities of the second derivatives of u and v. Because such discontinuities must be finite, equation 12.20 may be integrated between any two values of x in the interval $\alpha \le x \le \beta$,

$$\int_{x_1}^{x_2} (uLv - vLu)\,dx = \Big\{J(u,v)\Big\}_{x_1}^{x_2}. \tag{12.22}$$

This result is called **Green's formula on the interval** $x_1 \le x \le x_2$. When $x_1 = \alpha$ and $x_2 = \beta$, we obtain Green's formula on $\alpha \le x \le \beta$,

$$\int_{\alpha}^{\beta} (uLv - vLu)\,dx = \Big\{J(u,v)\Big\}_{\alpha}^{\beta}. \tag{12.23}$$

Identities 12.20–12.23 were based on the operator L in differential equation 12.16, but not on the differential equation itself; that is, $F(x)$ was not introduced. Nor were boundary conditions used in the derivation. In other words, identities 12.20–12.23 are properties of the operator L.

Suppose now that $u(x)$ and $v(x)$ satisfy the homogeneous version of differential equation 12.16. Lagrange's identity makes it clear that the conjunct of u and v is constant. This result is sufficiently important that we state it in the form of a theorem.

Theorem 12.1 If $u(x)$ and $v(x)$ satisfy the homogeneous differential equation $Ly = 0$, then $J(u,v)$ is a constant (independent of x).

The constant value vanishes only if $u(x)$ and $v(x)$ are linearly dependent.

With these preliminaries out of the way, we are prepared to define Green's functions for boundary value problems of the form

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = F(x), \quad \alpha < x < \beta, \tag{12.24a}$$

$$B_1 y = m_1, \tag{12.24b}$$

$$B_2 y = m_2, \tag{12.24c}$$

where $r(x)$ is continuously differentiable and does not vanish for $\alpha \le x \le \beta$ and $q(x)$ is continuous therein. (If the boundary conditions are periodic, they are also homogeneous, $m_1 = m_2 = 0$.) When $F(x)$ is a piecewise continuous function, a solution of 12.24 is called **classical** if it is continuously differentiable, has a piecewise continuous second derivative, satisfies the boundary conditions 12.24b,c, and is such that Ly and $F(x)$ are identical at every point of continuity of $F(x)$. We mention this fact because Green's functions do not turn out to be classical solutions. The Green's function $g(x;X)$ associated with problem 12.24, if it exists, is defined as the solution of

$$Lg = \delta(x - X), \tag{12.25a}$$
$$B_1 g = 0, \tag{12.25b}$$
$$B_2 g = 0. \tag{12.25c}$$

It is the solution of the same problem, with two changes. The source function $F(x)$ is replaced by a concentrated unit source, and the boundary conditions are made homogeneous. Because $\delta(x - X)$ is not piecewise continuous, Green's function cannot be called a classical solution of problem 12.25. It turns out that $g(x;X)$ is an ordinary function (as opposed to a generalized function). This is established in Schwartz's theory of distributions, wherein it is also shown that solutions of differential equation 12.25a have the following properties analogous to those in equations 12.14,

(1) $g(x;X)$ is continuous for $\alpha \le x \le \beta$; (12.26a)

(2) $dg(x;X)/dx$ is continuous except for a discontinuity at $x = X$ of magnitude $1/r(X)$; that is

$$\lim_{x \to X^+} \frac{dg}{dx} - \lim_{x \to X^-} \frac{dg}{dx} = \frac{1}{r(X)}; \tag{12.26b}$$

(3) for all $x \neq X$,

$$Lg = 0. \tag{12.26c}$$

These properties, along with boundary conditions 12.25b,c, completely characterize Green's functions; in fact, we now use them to derive formulas for Green's functions.

Condition 12.26c implies that $g(x;X)$ must be of the form

$$g(x;X) = \begin{cases} Eu(x) + Bv(x), & \alpha \le x < X \\ Du(x) + Gv(x), & X < x \le \beta, \end{cases} \tag{12.27}$$

where $u(x)$ and $v(x)$ are continuously differentiable, linearly independent solutions of $Ly = 0$. Inclusion of $x = \alpha$ and $x = \beta$ is a result of continuity condition 12.26a. Continuity at $x = X$ requires

$$Eu(X) + Bv(X) = Du(X) + Gv(X),$$

and condition 12.26b for the jump in dg/dx at $x = X$ implies that

$$Du'(X) + Gv'(X) - Eu'(X) - Bv'(X) = \frac{1}{r(X)}.$$

When these equations are solved for B and D in terms of E and G, and substituted into formula 12.27, the result is

$$g(x;X) = \begin{cases} Eu(x) + Gv(x) - \dfrac{u(X)v(x)}{J(u,v)}, & \alpha \le x \le X \\ Eu(x) + Gv(x) - \dfrac{v(X)u(x)}{J(u,v)}, & X \le x \le \beta. \end{cases}$$

The Heaviside unit step function can be used to combine these two expressions into one,

$$g(x;X) = Eu(x) + Gv(x) - \frac{1}{J(u,v)}[u(X)v(x)h(X-x) + v(X)u(x)h(x-X)].$$

We prefer a slightly different form obtained by using the fact that $h(x-X) = 1 - h(X-x)$,

$$\begin{aligned} g(x;X) &= Eu(x) + Gv(x) - \frac{1}{J(u,v)}\{u(X)v(x)[1-h(x-X)] + v(X)u(x)[1-h(X-x)]\} \\ &= \left[E - \frac{v(X)}{J(u,v)}\right]u(x) + \left[G - \frac{u(X)}{J(u,v)}\right]v(x) \\ &\quad + \frac{1}{J(u,v)}[u(x)v(X)h(X-x) + u(X)v(x)h(x-X)] \\ &= Au(x) + Cv(x) + \frac{1}{J(u,v)}[u(x)v(X)h(X-x) + u(X)v(x)h(x-X)]. \quad (12.28) \end{aligned}$$

We understand that terms involving the Heaviside unit step function are regarded in the limit sense $(x \to X)$ at $x = X$.

The remaining unknowns A and C are evaluated using boundary conditions 12.25b,c. They require

$$0 = B_1 g = AB_1 u + CB_1 v + B_1 a, \qquad (12.29a)$$

$$0 = B_2 g = AB_2 u + CB_2 v + B_2 a, \qquad (12.29b)$$

where $a = J^{-1}[u(x)v(X)h(X-x) + u(X)v(x)h(x-X)]$. These algebraic equations for A and C have a unique solution provided

$$\begin{vmatrix} B_1 u & B_1 v \\ B_2 u & B_2 v \end{vmatrix} \neq 0. \qquad (12.30)$$

Thus, when condition 12.30 is satisfied, $g(x;X)$ is defined by 12.28, where A and C are chosen so that $g(x;X)$ satisfies 12.25b,c.

We briefly examine here the significance of a vanishing determinant and deal with it more fully in Section 12.5. A vanishing determinant is equivalent to the existence of a constant k, which might be zero, such that

$$B_1 u = kB_1 v \quad \text{and} \quad B_2 u = kB_2 v \qquad (12.31a)$$

or,

$$B_1(u - kv) = 0 \quad \text{and} \quad B_2(u - kv) = 0. \qquad (12.31b)$$

Since $u(x)$ and $v(x)$ are linearly independent, we can say that the determinant vanishes if and only if there is a nontrivial solution $u - kv$ of the homogeneous boundary value problem

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = 0, \quad \alpha < x < \beta, \qquad (12.32a)$$

$$B_1 y = 0, \qquad (12.32b)$$

$$B_2 y = 0. \qquad (12.32c)$$

We summarize these results in the following theorem.

Theorem 12.2 When homogeneous system 12.32 has only the trivial solution, the Green's function for problem 12.24 is uniquely given by

$$g(x;X) = Au(x) + Cv(x) + \frac{1}{J(u,v)}[u(x)v(X)h(X-x) + u(X)v(x)h(x-X)], \tag{12.33}$$

where $u(x)$ and $v(x)$ are linearly independent solutions of 12.32a, and A and C are chosen so that $g(x;X)$ satisfies 12.25b,c.

When the boundary conditions are unmixed, determination of $g(x;X)$ can be simplified further.

Corollary When homogeneous system 12.32 has only the trivial solution and boundary conditions are unmixed, the Green's function for problem 12.24 is uniquely given by

$$g(x;X) = \frac{1}{J(u,v)}[u(x)v(X)h(X-x) + u(X)v(x)h(x-X)], \tag{12.34}$$

where $u(x)$ and $v(x)$ are linearly independent solutions of 12.32a satisfying $B_1u = 0$ and $B_2v = 0$.

Proof This function satisfies 12.25a since it satisfies characterizing properties 12.26. In addition, when $x < X$, $g(x;X)$ reduces to $J^{-1}u(x)v(X)$, which, as a function of x, satisfies $B_1g = 0$. Similarly, because $B_2v = 0$, we must have $B_2g = 0$. ∎

Once again, we point out that due to the Heaviside functions, expressions for $g(x;X)$ in 12.33 and 12.34 are not defined at $x = X$. However, the requirement that $g(x;X)$ be continuous at at $x = X$ (equation 12.26a) implies that $g(x;X)$ can be calculated at $x = X$ by either of the limits $\lim_{x\to X^+} g(x;X) = \lim_{x\to X^-} g(x;X)$, and we implicitly understand this when we write 12.33 and 12.34.

Notice that for unmixed boundary conditions, $g(x;X)$ is symmetric in x and X. That this is also true for periodic boundary conditions is verified in Theorem 12.5 of this section.

Example 12.1 Use formula 12.34 to find the Green's function for problem 12.12.

Solution Solutions of $y'' = 0$ satisfying $y(0) = 0$ and $y(L) = 0$, respectively, are $u(x) = x$ and $v(x) = L - x$. With $J(u,v) = r(uv' - vu') = -\tau[x(-1) - (1)(L-x)] = L\tau$, formula 12.34 gives

$$g(x;X) = \frac{1}{L\tau}[x(L-X)h(X-x) + X(L-x)h(x-X)],$$

and this is expression 12.13c.●

Example 12.2 Find the Green's function for the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \quad \alpha < x < \beta$$
$$y(\alpha) = m_1, \quad y'(\beta) = m_2.$$

Solution Solutions of $y'' + 4y = 0$ are $y(x) = A\cos 2x + B\sin 2x$. For a solution to satisfy $y(\alpha) = 0$, we must have

$$0 = A\cos 2\alpha + B\sin 2\alpha \qquad \Longrightarrow \qquad B = -A\cot 2\alpha.$$

Thus,

$$\begin{aligned} y(x) &= A\cos 2x - A\cot 2\alpha \sin 2x = \frac{A}{\sin 2\alpha}(\cos 2x \sin 2\alpha - \sin 2x \cos 2\alpha) \\ &= \frac{-A}{\sin 2\alpha}\sin 2(x-\alpha). \end{aligned}$$

We therefore take $u(x) = \sin 2(x-\alpha)$. Similarly, a function satisfying $y'(\beta) = 0$ is $v(x) = \cos 2(\beta - x)$. With

$$\begin{aligned} J(u,v) &= uv' - vu' = 2\sin 2(x-\alpha)\sin 2(\beta - x) - 2\cos 2(x-\alpha)\cos 2(\beta - x) \\ &= -2\cos 2(\beta - \alpha), \end{aligned}$$

formula 12.34 gives

$$\begin{aligned} g(x;X) = \frac{1}{-2\cos 2(\beta-\alpha)}&[\sin 2(x-\alpha)\cos 2(\beta - X)h(X-x) \\ &+ \sin 2(X-\alpha)\cos 2(\beta - x)h(x-X)].\bullet \end{aligned}$$

Example 12.3 Find the Green's function for

$$\begin{aligned} \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = F(x), &\quad 0 < x < \pi/2, \\ y'(0) = 5, &\quad y(\pi/2) = 2. \end{aligned}$$

Solution Solutions of $y'' + 2y' + 10y = 0$ are always of the form $e^{-x}(A\sin 3x + B\cos 3x)$. Solutions that satisfy $y'(0) = 0$ and $y(\pi/2) = 0$, respectively, are $u(x) = e^{-x}(\sin 3x + 3\cos 3x)$ and $v(x) = e^{-x}\cos 3x$. To find the conjunct of u and v, we express the differential equation in self-adjoint form by multiplying by e^{2x},

$$e^{2x}\frac{d^2y}{dx^2} + 2e^{2x}\frac{dy}{dx} + 10e^{2x}y = e^{2x}F(x)$$

or

$$\frac{d}{dx}\left(e^{2x}\frac{dy}{dx}\right) + 10e^{2x}y = e^{2x}F(x).$$

With $r(x)$ identified as e^{2x},

$$\begin{aligned} J(u,v) = e^{2x}&[e^{-x}(\sin 3x + 3\cos 3x)(-e^{-x}\cos 3x - 3e^{-x}\sin 3x) \\ &- e^{-x}\cos 3x(-10e^{-x}\sin 3x)] \\ = -3&, \end{aligned}$$

and therefore

$$\begin{aligned} g(x;X) = -\frac{1}{3}&[e^{-(x+X)}\cos 3X(\sin 3x + 3\cos 3x)h(X-x) \\ &+ e^{-(x+X)}\cos 3x(\sin 3X + 3\cos 3X)h(x-X)].\bullet \end{aligned}$$

Example 12.4 Find the Green's function for the problem

$$\frac{d^2y}{dx^2} + y = F(x), \quad 0 < x < 1,$$
$$y(0) - y(1) = 0, \quad y'(0) - y'(1) = 0.$$

Solution Since $u(x) = \sin x$ and $v(x) = \cos x$ are solutions of $y'' + y = 0$, we may take (according to formula 12.33)

$$g(x;X) = A\sin x + C\cos x + \frac{1}{J(\sin x, \cos x)}[\sin x \cos X\, h(X - x) + \sin X \cos x\, h(x - X)],$$

where $J(\sin x, \cos x) = \sin x(-\sin x) - \cos x(\cos x) = -1$. The boundary conditions must also be satisfied by $g(x;X)$, and therefore

$$C - A\sin 1 - C\cos 1 + \sin X\cos 1 = 0,$$
$$A - \cos X - A\cos 1 + C\sin 1 - \sin X\sin 1 = 0.$$

These can be solved for A and C,

$$A = \frac{\cos X - \cos(1 + X)}{2(1 - \cos 1)}, \qquad C = \frac{\sin X + \sin(1 - X)}{2(1 - \cos 1)},$$

and

$$g(x;X) = \frac{1}{2(1-\cos 1)}\{\sin x[\cos X - \cos(1+X)] + \cos x[\sin X + \sin(1 - X)]\} - \sin x\cos X\, h(X - x) - \sin X\cos x\, h(x - X).\bullet$$

In order to show how Green's functions yield integral representations of boundary value problems (Section 12.4), we must extend Green's formula 12.23 to encompass delta functions.

Theorem 12.3 Let L be the differential operator in problem 12.24a. When $v(x;X)$ is a solution of $Lv = \delta(x - X)$ and $u(x)$ is continuously differentiable with a piecewise continuous second derivative,

$$\int_\alpha^\beta (uLv - vLu)\,dx = \Big\{r(uv' - vu')\Big\}_\alpha^\beta. \tag{12.35}$$

Proof Suppose $u(x)$ has a discontinuity in its second derivative at a point $\overline{X} < X$. (Similar discussions can be made if $u(x)$ has more than one such point or if $\overline{X} > X$.) Then

$$\int_\alpha^\beta (uLv - vLu)\,dx = \int_\alpha^{\overline{X}} (uLv - vLu)\,dx + \int_{\overline{X}}^{X-\epsilon} (uLv - vLu)\,dx + \int_{X-\epsilon}^{X+\epsilon} (uLv - vLu)\,dx + \int_{X+\epsilon}^{\beta} (uLv - vLu)\,dx,$$

where $\epsilon > 0$ is some small number. Green's formula 12.22 can be applied to the first, second, and fourth of these integrals since $Lv = 0$ therein (see condition 12.26c),

$$\int_\alpha^\beta (uLv - vLu)\,dx = \left\{r(uv' - vu')\right\}_\alpha^{\overline{X}} + \left\{r(uv' - vu')\right\}_{\overline{X}}^{X-\epsilon} + \int_{X-\epsilon}^{X+\epsilon} [u\delta(x - X) - vLu]\,dx + \left\{r(uv' - vu')\right\}_{X+\epsilon}^{\beta}.$$

Because r, u, u', v, and v' are all continuous at $\overline{X}$, terms in $\overline{X}$ vanish, and the expression on the right reduces to

$$\begin{aligned}\int_\alpha^\beta (uLv - vLu)\,dx = \left\{r(uv' - vu')\right\}_\alpha^\beta &+ r(X-\epsilon)[u(X-\epsilon)v'(X-\epsilon; X)\\ &- v(X-\epsilon; X)u'(X-\epsilon)] - r(X+\epsilon)[u(X+\epsilon)v'(X+\epsilon; X)\\ &- v(X+\epsilon; X)u'(X+\epsilon)] + u(X) - \int_{X-\epsilon}^{X+\epsilon} vLu\,dx.\end{aligned}$$

We now take limits as $\epsilon \to 0^+$. Since v, u, u', and u'' are continuous on $X - \epsilon \le x \le X + \epsilon$, the final integral vanishes in the limit, and the remaining terms give

$$\begin{aligned}\int_\alpha^\beta (uLv - vLu)\,dx &= \left\{r(uv' - vu')\right\}_\alpha^\beta + r(X)[u(X)v'(X-; X) - v(X; X)u'(X)]\\ &\quad - r(X)[u(X)v'(X+; X) - v(X; X)u'(X)] + u(X)\\ &= \left\{r(uv' - vu')\right\}_\alpha^\beta + r(X)u(X)[v'(X-; X) - v'(X+; X)] + u(X)\\ &= \left\{r(uv' - vu')\right\}_\alpha^\beta\end{aligned}$$

(because v satisfies condition 12.26b). ∎

A similar proof leads to the following extension of Green's formula.

Theorem 12.4 Let L be the differential operator of problem 12.24. When u and v satisfy $Lu = \delta(x - X)$ and $Lv = \delta(x - Y)$,

$$\int_\alpha^\beta (uLv - vLu)\,dx = \left\{r(uv' - vu')\right\}_\alpha^\beta. \tag{12.36}$$

In this case, the integral of $uLv - vLu$ over the interval $\alpha \le x \le \beta$ is subdivided into five integrals over the intervals $\alpha \le x \le X - \epsilon$, $X - \epsilon \le x \le X + \epsilon$, $X + \epsilon \le x \le Y - \epsilon$, $Y - \epsilon \le x \le Y + \epsilon$, $Y + \epsilon \le x \le \beta$ (for $X < Y$), and Green's formula 12.22 is applied to the first, third, and fifth. Details are given in Exercise 24.

Formula 12.34 indicates that Green's functions for problems with unmixed boundary conditions are symmetric. That this is true for periodic boundary conditions as well is proved in the next theorem.

Theorem 12.5 When boundary conditions in problem 12.24 are unmixed or periodic, Green's function $g(x; X)$ is symmetric,

$$g(x; X) = g(X; x). \tag{12.37}$$

Proof When we set $u = g(x;X)$ and $v = g(x;Y)$ in version 12.36 of Green's formula, the result is

$$\int_{\alpha}^{\beta} [g(x;X)Lg(x;Y) - g(x;Y)Lg(x;X)]\,dx$$
$$= \left\{ r(x)\left[g(x;X)\frac{dg(x;Y)}{dx} - g(x;Y)\frac{dg(x;X)}{dx}\right]\right\}_{\alpha}^{\beta}.$$

It is straightforward to show that when $g(x;X)$ satisfies unmixed boundary conditions 12.18 or periodic conditions 12.19, the right side of this equation must vanish, and therefore

$$0 = \int_{\alpha}^{\beta} [g(x;X)\delta(x-Y) - g(x;Y)\delta(x-X)]\,dx = g(Y;X) - g(X;Y). \blacksquare$$

It is interesting to interpret this symmetry physically, say, in string problem 12.12 of Section 12.2. Green's function $g(x;X)$ for this problem is the deflection of the string due to a unit force at position X. Symmetry of $g(x;X)$ means that the deflection at x due to a unit force at X is identical to the deflection at X due to a unit force at x. This is often referred to as **Maxwell's reciprocity** and is illustrated in Figure 12.7.

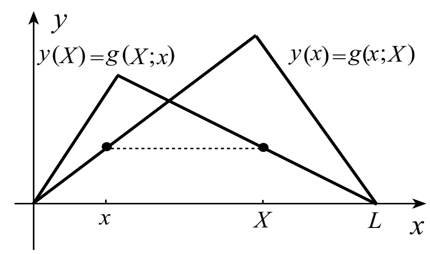


Figure 12.7

EXERCISES 12.3

In Exercises 1–5 write the differential equation in self-adjoint form.

1. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 3y = F(x)$

2. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = F(x)$

3. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - (x+1)y = F(x)$

4. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - (x+1)y = F(x)$

5. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = F(x)$

In Exercises 6–16 find the Green's function for the boundary value problem.

6. $\dfrac{d^2y}{dx^2} = F(x), \quad 0 < x < L, \quad y(0) = 0, \quad y'(L) = 0$

7. $\dfrac{d^2y}{dx^2} + y = F(x), \quad 0 < x < L, \quad y(0) = 0, \quad y'(L) = 0$

8. $\dfrac{d^2y}{dx^2} + k^2y = F(x), \quad 0 < x < \pi, \quad y(0) = 0, \quad y(\pi) = 0$ ($k > 0$ is a constant, but not an integer)

9. $\dfrac{d^2y}{dx^2} = F(x), \quad 0 < x < L, \quad y(0) = y'(0), \quad y'(L) = 0$

10. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = F(x), \quad 0 < x < L, \quad y(0) = 0, \quad y'(L) = 0$

11. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = F(x), \quad 0 < x < \pi/2, \quad y'(0) = 0, \quad y(\pi/2) = 0$

12. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 6y = F(x), \quad 1 < x < 2, \quad y(1) = y'(2), \quad y'(1) = 0$

13. $\dfrac{d^2y}{dx^2} + k^2y = F(x), \quad 0 < x < L, \quad y(0) = 0, \quad y(L) = 0, \quad (k > 0$ a constant). Would you place any restrictions on k?

14. $\dfrac{d^2y}{dx^2} + k^2y = F(x), \quad 0 < x < L, \quad y(0) = 0, \quad y'(L) = 0, \quad (k > 0$ a constant). Would you place any restrictions on k?

15. $\dfrac{d^2y}{dx^2} + k^2y = F(x), \quad \alpha < x < \beta, \quad y(\alpha) = y(\beta), \quad y'(\alpha) = y'(\beta) \quad (k > 0$ a constant). Would you

place any restrictions on k?

16. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = F(x), \quad 0 < \alpha < x < \beta, \quad y(\alpha) = 0, \quad y(\beta) = 0$

17. Find the Green's function for the boundary value problem

$$\frac{d^2y}{dx^2} = F(x), \quad 0 < x < L,$$
$$y(0) = 0, \quad l_2y'(L) + h_2y(L) = 0,$$

using: (a) equation 12.33; (b) equation 12.34. Verify that results are identical.

18. Find the Green's function for the boundary value problem

$$\frac{d^2y}{dx^2} = F(x), \quad 0 < x < L,$$
$$-l_1y'(0) + h_1y(0) = 0, \quad l_2y'(L) + h_2y(L) = 0,$$

using: (a) equation 12.33; (b) equation 12.34. Verify that results are identical.

19. The boundary value problem for static deflection of a beam subjected to a distributed force $F(x)$ is

$$EI\frac{d^4y}{dx^4} = F(x), \quad 0 < x < L,$$

Boundary conditions at $x = 0$ and $x = L$,

where E and I are constants. The Green's function $g(x; X)$ for this fourth-order problem satisfies

$$EI\frac{d^4y}{dx^4} = \delta(x - X),$$

Homogeneous boundary conditions at $x = 0$ and $x = L$.

Thus, it is the solution of the problem due to a unit concentrated force at X (with homogeneous boundary conditions). Solutions of the differential equation are characterized by the following properties:

(i) $g(x;X)$, $dg(x;X)/dx$, and $d^2g(x;X)/dx^2$ are continuous for $0 \le x \le L$ except for a removable discontinuity at $x = X$;

(ii) $d^3g(x;X)/dx^3$ is continuous except for a discontinuity at $x = X$ of magnitude $(EI)^{-1}$; that is,

$$\lim_{x \to X^+} \frac{d^3g}{dx^3} - \lim_{x \to X^-} \frac{d^3g}{dx^3} = \frac{1}{EI};$$

(iii) for any $x \ne X$,

$$EI\frac{d^4g(x;X)}{dx^4} = 0.$$

Use the characterization in (i), (ii), and (iii) to show that $g(x;X)$ can be expressed in the form

$$g(x;X) = \frac{1}{6EI}(x-X)^3h(x-X) + Ax^3 + Bx^2 + Cx + D,$$

where A, B, C, and D are constants. (The constants are evaluated using the homogeneous boundary conditions.)

In Exercises 20–23 use the result of Exercise 19 to find the Green's function for static deflections of a beam of length L ($0 \le x \le L$), where the boundary conditions are as given.

20. $y(0) = y'(0) = 0 = y''(L) = y'''(L)$ (cantilevered)

21. $y(0) = y''(0) = 0 = y(L) = y''(L)$ (simply supported at both ends)

22. $y(0) = y'(0) = 0 = y(L) = y'(L)$ (clamped at both ends)

23. $y(0) = y'(0) = 0 = y(L) = y''(L)$ (clamped at one end, simply supported at the other)

24. Prove Theorem 12.4.

25. When the boundary conditions in problem 12.24 are unmixed, it is sometimes advantageous to represent the Green's function of the problem in terms of orthonormal eigenfunctions of the corresponding Sturm-Liouville system,

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) - q(x)]y = 0, \quad \alpha < x < \beta,$$
$$B_1 y = 0, \quad B_2 y = 0.$$

(Notice that the weight function $p(x)$ is unspecified, but normally there is only one choice of $p(x)$ for which the differential equation gives rise to standard functions.) Show that when $y_n(x)$ are normalized eigenfunctions corresponding to eigenvalues λ_n, Green's function $g(x;X)$ can be expressed in the form

$$g(x;X) = \sum_{n=1}^{\infty} \frac{y_n(X)y_n(x)}{-\lambda_n},$$

provided that $\lambda = 0$ is not an eigenvalue of the SL-system for any weight function. (Hint: Use Green's formula 12.35 with $u = y_n(x)$ and $v = g(x;X)$.)

26. Find an eigenfunction expansion (Exercise 25) for the Green's function of the boundary value problem

$$\frac{d^2y}{dx^2} = F(x), \quad 0 < x < L,$$
$$y(0) = 0, \quad y(L) = 0.$$

§12.4 Solutions of Boundary Value Problems Using Green's Functions

In this section we show how to solve a boundary value problem once the Green's function for the problem is known. First, we consider problems with homogeneous boundary conditions and subsequently, problems with nonhomogeneous boundary conditions.

Problems with Homogeneous Boundary Conditions

Representations of solutions to problems with homogeneous boundary conditions are provided by the following theorem.

Theorem 12.6 When the Green's function $g(x;X)$ for the boundary value problem

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = F(x), \quad \alpha < x < \beta, \tag{12.38a}$$

$$B_1 y = 0, \tag{12.38b}$$

$$B_2 y = 0, \tag{12.38c}$$

exists, the solution of the boundary value problem is

$$y(x) = \int_\alpha^\beta g(x;X)F(X)\,dX. \tag{12.39}$$

Proof The Green's function for problem 12.38 satisfies equations 12.25. If we substitute 12.39 into 12.38a and reverse orders of integration with respect to X and differentiations with respect to x,

$$\begin{aligned} Ly = L\int_\alpha^\beta g(x;X)F(X)\,dX &= \int_\alpha^\beta [Lg(x;X)]F(X)\,dX \\ &= \int_\alpha^\beta \delta(x-X)F(X)\,dX \qquad \text{(by 12.25a)} \\ &= F(x). \end{aligned}$$

Furthermore, because $g(x;X)$ satisfies 12.25b,c, $y(x)$ must satisfy 12.38b,c. ∎

As a result of this theorem, once we know the Green's function for a boundary value problem, the solution for any source function $F(x)$ can be obtained by integration. Think of the integral as a superposition. Because the Green's function is the solution of problem 12.38 due to a unit point source at X, we interpret $g(x;X)F(X)\,dX$ as the effect due to that part $F(X)\,dX$ of the source over the interval dX of the x-axis, and the integral adds over all sources from $x=\alpha$ to $x=\beta$. Were the source composed of both a distributed portion $F(x)$ and n concentrated parts of magnitudes F_j at points x_j, the solution of problem 12.38 would be

$$\begin{aligned} y(x) &= \int_\alpha^\beta g(x;X)\left[F(X) + \sum_{j=1}^n F_j\delta(X-x_j)\right]dX \\ &= \int_\alpha^\beta g(x;X)F(X)\,dX + \sum_{j=1}^n F_j g(x;x_j). \end{aligned} \tag{12.40}$$

We give two illustrative examples.

Example 12.5 A taut string of length L has its ends fixed at $x = 0$ and $x = L$ on the x-axis. A concentrated mass of M kg is attached to the string at $x = L/3$ (Figure 12.8). Find the deflections in the string if gravity on the string itself is also taken into account.

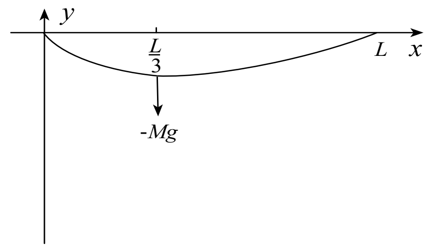


Figure 12.8

Solution The boundary value problem for deflections of the string is

$$-\tau\frac{d^2y}{dx^2} = -9.81 - 9.81M\delta(x - L/3),$$
$$y(0) = 0 = y(L).$$

According to equation 12.13c and Example 12.1, the Green's function for this problem is

$$g(x;X) = \frac{1}{L\tau}[x(L-X)h(X-x) + X(L-x)h(x-X)].$$

The solution is therefore defined by integral 12.39,

$$\begin{aligned}
y(x) &= \int_0^L g(x;X)[-9.81 - 9.81M\delta(X - L/3)]\,dX \\
&= -9.81\int_0^L g(x;X)\,dX - 9.81Mg(x;L/3) \\
&= \frac{-9.81}{L\tau}\int_0^x X(L-x)\,dX - \frac{9.81}{L\tau}\int_x^L x(L-X)\,dX \\
&\quad - \frac{9.81M}{L\tau}\left[x\left(L-\frac{L}{3}\right)h\left(\frac{L}{3}-x\right) + \left(\frac{L}{3}\right)(L-x)h\left(x-\frac{L}{3}\right)\right] \\
&= \frac{-9.81}{L\tau}(L-x)\left(\frac{x^2}{2}\right) - \frac{9.81}{L\tau}\frac{x(L-x)^2}{2} \\
&\quad - \frac{9.81M}{L\tau}\left[\frac{2Lx}{3}h\left(\frac{L}{3}-x\right) + \left(\frac{L}{3}\right)(L-x)h\left(x-\frac{L}{3}\right)\right] \\
&= \frac{-9.81x(L-x)}{2\tau} - \frac{9.81M}{3\tau}\begin{cases} 2x, & 0 \le x \le L/3 \\ L-x, & L/3 \le x \le L \end{cases}.
\end{aligned}$$

This is superposition of the displacement due to gravity (the first term) and that due to the concentrated load (the second term) (Figure 12.9). The parts of the string are parabolic in shape, but because a large value of M was used in the plot, they appear almost straight.●

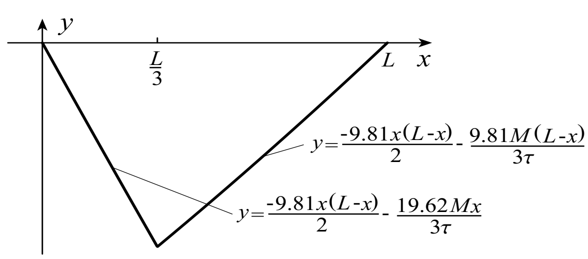


Figure 12.9

Example 12.6 Solve the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \quad 0 < x < 3,$$
$$y(0) = 0 = y'(3)$$

when (a) $F(x) = 2x$ and (b) $F(x) = h(x-1) - h(x-2)$.

Solution The Green's function for this problem can be obtained from Example 12.2 by setting $\alpha = 0$ and $\beta = 3$,

$$g(x;X) = \frac{-1}{2\cos 6}[\sin 2x \cos(6-2X)h(X-x) + \sin 2X \cos(6-2x)h(x-X)].$$

With source function $F(x)$, the solution of the boundary value problem is

$$y(x) = \int_0^3 g(x;X)F(X)\,dX.$$

(a) When $F(x) = 2x$,

$$\begin{aligned}
y(x) &= \int_0^3 2Xg(x;X)\,dX \\
&= \frac{-1}{2\cos 6}\int_0^x 2X \sin 2X \cos(6-2x)\,dX \\
&\quad - \frac{1}{2\cos 6}\int_x^3 2X \sin 2x \cos(6-2X)\,dX \\
&= \frac{-\cos(6-2x)}{\cos 6}\left\{\frac{-X\cos 2X}{2} + \frac{\sin 2X}{4}\right\}_0^x \\
&\quad - \frac{\sin 2x}{\cos 6}\left\{\frac{-X\sin(6-2X)}{2} + \frac{\cos(6-2X)}{4}\right\}_x^3 \\
&= \frac{x}{2} - \frac{\sin 2x}{4\cos 6}.
\end{aligned}$$

(This solution could also be derived very simply by finding the general solution of $y'' + 4y = 2x$ and using boundary conditions to evaluate arbitrary constants.)
(b) For $F(x) = h(x-1) - h(x-2)$, the solution is

$$y(x) = \int_0^3 [h(X-1) - h(X-2)]g(x;X)\,dX = \int_1^2 g(x;X)\,dX.$$

When $x \le 1$,

$$\begin{aligned}
y(x) &= \int_1^2 \frac{-1}{2\cos 6}\sin 2x \cos(6-2X)\,dX \\
&= \frac{\sin 2x}{2\cos 6}\left\{\frac{\sin(6-2X)}{2}\right\}_1^2 = \frac{\sin 2x(\sin 2 - \sin 4)}{4\cos 6};
\end{aligned}$$

when $1 < x < 2$,

$$\begin{aligned} y(x) &= \int_1^x \frac{-1}{2\cos 6} \sin 2X \cos(6-2x)\,dX + \int_x^2 \frac{-1}{2\cos 6} \sin 2x \cos(6-2X)\,dX \\ &= \frac{\cos(6-2x)}{2\cos 6} \left\{\frac{\cos 2X}{2}\right\}_1^x + \frac{\sin 2x}{2\cos 6} \left\{\frac{\sin(6-2X)}{2}\right\}_x^2 \\ &= \frac{1}{4} + \frac{1}{4\cos 6}[\sin 2x \sin 2 - \cos(6-2x)\cos 2]; \end{aligned}$$

and when $2 \le x < 3$,

$$\begin{aligned} y(x) &= \int_1^2 \frac{-1}{2\cos 6} \sin 2X \cos(6-2x)\,dX = \frac{\cos(6-2x)}{2\cos 6} \left\{\frac{\cos 2X}{2}\right\}_1^2 \\ &= \frac{\cos(6-2x)(\cos 4 - \cos 2)}{4\cos 6}. \end{aligned}$$

This solution is not so easily produced using methods from elementary differential equations. It requires integration of the differential equation on three separate intervals and matching of the solution and its first derivative at $x = 1$ and $x = 2$.●

Problems with Nonhomogeneous Boundary Conditions

Suppose now that boundary conditions 12.38b,c are not homogeneous, in which case the problem becomes

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = F(x), \quad \alpha < x < \beta, \tag{12.41a}$$

$$B_1 y = m_1, \tag{12.41b}$$

$$B_2 y = m_2. \tag{12.41c}$$

Only unmixed boundary conditions and periodic conditions, which are always homogeneous, are considered. For unmixed boundary conditions, problem 12.41 takes the form

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = F(x), \quad \alpha < x < \beta, \tag{12.42a}$$

$$B_1 y = -l_1 y'(\alpha) + h_1 y(\alpha) = m_1, \tag{12.42b}$$

$$B_2 y = l_2 y'(\beta) + h_2 y(\beta) = m_2. \tag{12.42c}$$

There are two ways to solve this problem; one is to use superposition, and the other is to use Green's formula. Both methods use Green's function for the associated problem with homogeneous boundary conditions,

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = F(x), \quad \alpha < x < \beta, \tag{12.43a}$$

$$B_1 y = 0, \tag{12.43b}$$

$$B_2 y = 0. \tag{12.43c}$$

In the superposition method, we note that

$$y_1(x) = \int_\alpha^\beta g(x;X)F(X)\,dX,$$

where $g(x;X)$ is the associated Green's function, is a solution of problem 12.43. A solution of problem 12.42 will therefore be $y = y_1 + y_2$ if y_2 satisfies

$$Ly = 0, \quad \alpha < x < \beta, \tag{12.44a}$$

$$B_1 y = m_1, \tag{12.44b}$$

$$B_2 y = m_2. \tag{12.44c}$$

In ODEs it is quite often straightforward to obtain $y_2(x)$ — apply boundary conditions 12.44b,c to a general solution of differential equation 12.44a. We illustrate with the following example.

Example 12.7 Solve the boundary value problem

$$-\tau\frac{d^2y}{dx^2} = F(x), \quad 0 < x < L,$$
$$y(0) = m_1, \quad y(L) = m_2.$$

Solution In Section 12.2 we derived the solution

$$y_1(x) = \int_0^L g(x;X)F(X)\,dX = \frac{L-x}{L\tau}\int_0^x XF(X)\,dX + \frac{x}{L\tau}\int_x^L (L-X)F(X)\,dX$$

for the associated problem with homogeneous boundary conditions. To this we must add the solution of

$$\frac{d^2y}{dx^2} = 0, \quad y(0) = m_1, \quad y(L) = m_2.$$

Since every solution of this differential equation must be of the form $y_2(x) = Ax+B$, to satisfy the boundary conditions we require

$$m_1 = B, \qquad m_2 = AL + B.$$

Thus, $y_2(x) = (m_2 - m_1)x/L + m_1$, and

$$y(x) = y_1(x) + y_2(x) = (m_2 - m_1)\frac{x}{L} + m_1 + \frac{L-x}{L\tau}\int_0^x XF(X)\,dX$$
$$+ \frac{x}{L\tau}\int_x^L (L-X)F(X)\,dX.\bullet$$

This superposition method works well for ODEs but fails to generalize to PDEs; it is not usually possible to produce general solutions of homogeneous PDEs and apply nonhomogeneous boundary conditions to determine arbitrary functions. An alternative approach, which does generalize to PDEs, is to use Green's formula 12.35. This method also illustrates how the solution depends on the nonhomogeneities in the boundary conditions.

Theorem 12.7 When the Green's function $g(x;X)$ for boundary value problem 12.42 exists, the solution of the boundary value problem is

$$y(x) = \int_\alpha^\beta g(x;X)F(X)\,dX - \frac{m_1}{l_1}r(\alpha)g(x;\alpha) - \frac{m_2}{l_2}r(\beta)g(x;\beta). \tag{12.45a}$$

When $l_1 = l_2 = 0$ (and we set $h_1 = h_2 = 1$), the solution is

$$y(x) = \int_\alpha^\beta g(x;X)F(X)\,dX + m_2 r(\beta)\frac{\partial g(x;\beta)}{\partial X} - m_1 r(\alpha)\frac{\partial g(x;\alpha)}{\partial X}. \tag{12.45b}$$

Proof If $y(x)$ is the required solution of problem 12.42 and $v(x)$ is the Green's function $g(x;X)$ for the problem, Green's formula 12.35 becomes

$$\int_\alpha^\beta yLg(x;X)\,dx - \int_\alpha^\beta g(x;X)Ly\,dx = \left\{r(x)\left[y(x)\frac{\partial g(x;X)}{\partial x} - g(x;X)y'(x)\right]\right\}_\alpha^\beta.$$

Because $Ly = F(x)$ and $Lg(x;X) = \delta(x - X)$, we may write

$$\int_\alpha^\beta y(x)\delta(x - X)\,dx - \int_\alpha^\beta g(x;X)F(x)\,dx = \left\{r(x)\left[y(x)\frac{\partial g(x;X)}{\partial x} - g(x;X)y'(x)\right]\right\}_\alpha^\beta$$

or

$$\begin{aligned} y(X) - \int_\alpha^\beta g(x;X)F(x)\,dx = {}& r(\beta)\left[y(\beta)\frac{\partial g(\beta;X)}{\partial x} - g(\beta;X)y'(\beta)\right] \\ & - r(\alpha)\left[y(\alpha)\frac{\partial g(\alpha;X)}{\partial x} - g(\alpha;X)y'(\alpha)\right]. \end{aligned} \tag{12.46}$$

If we now substitute from the boundary conditions

$$B_1 y = -l_1 y'(\alpha) + h_1 y(\alpha) = m_1, \tag{12.47a}$$

$$B_2 y = l_2 y'(\beta) + h_2 y(\beta) = m_2, \tag{12.47b}$$

$$\begin{aligned} y(X) - \int_\alpha^\beta g(x;X)F(x)\,dx &= r(\beta)\left[y(\beta)\frac{\partial g(\beta;X)}{\partial x} - g(\beta;X)\left(\frac{m_2}{l_2} - \frac{h_2}{l_2}y(\beta)\right)\right] \\ &\quad - r(\alpha)\left[y(\alpha)\frac{\partial g(\alpha;X)}{\partial x} - g(\alpha;X)\left(-\frac{m_1}{l_1} + \frac{h_1}{l_1}y(\alpha)\right)\right] \\ &= r(\beta)\left[-\frac{m_2}{l_2}g(\beta;X) + \frac{y(\beta)}{l_2}\left(l_2\frac{\partial g(\beta;X)}{\partial x} + h_2 g(\beta;X)\right)\right] \\ &\quad - r(\alpha)\left[\frac{m_1}{l_1}g(\alpha;X) - \frac{y(\alpha)}{l_1}\left(-l_1\frac{\partial g(\alpha;X)}{\partial x} + h_1 g(\alpha;X)\right)\right]. \end{aligned}$$

But $g(x;X)$ must satisfy homogeneous versions of boundary conditions 12.47; that is,

$$-l_1\frac{\partial g(\alpha;X)}{\partial x} + h_1 g(\alpha;X) = 0, \tag{12.48a}$$

$$l_2\frac{\partial g(\beta;X)}{\partial x} + h_2 g(\beta;X) = 0. \tag{12.48b}$$

Consequently

$$y(X) = \int_\alpha^\beta g(x;X)F(x)\,dx - \frac{m_1}{l_1}r(\alpha)g(\alpha;X) - \frac{m_2}{l_2}r(\beta)g(\beta;X).$$

When we interchange x and X and use the fact that $g(x;X)$ is symmetric, we obtain solution 12.45a. When $l_1 = l_2 = 0$, we interchange x and X in solution 12.46 to obtain 12.45b. ∎

Solutions 12.45a and 12.45b clearly indicate the dependence of $y(x)$ on all three nonhomogeneities in problem 12.42. The integral term accounts for the nonhomogeneity $F(x)$ in the PDE, and the remaining terms contain contributions due to nonhomogeneities in the boundary conditions. With $F(x)$ piecewise continuous, the integral term in 12.45 is continuous in x. Furthermore, because $g(x;X)$ is continuous and $\partial g(x;X)/\partial x$ has a discontinuity only when $x = X$, it follows that the additional terms in 12.45 due to the nonhomogeneities in the boundary conditions are also continuous. In other words, the representation of the solution to a boundary value problem in terms of its Green's function is always a continuous function.

Example 12.8 Use formula 12.45b to solve the boundary value problem of Example 12.7.

Solution The Green's function for this problem is

$$g(x;X) = \frac{1}{L\tau}[x(L-X)h(X-x) + X(L-x)h(x-X)].$$

In Example 12.7 we used the direct method to find the particular solution satisfying the homogeneous differential equation and nonhomogeneous boundary conditions. Alternatively, according to equation 12.45b,

$$\begin{aligned}
y(x) &= \int_0^L g(x;X)F(X)\,dX - \tau m_2 \frac{\partial g(x;L)}{\partial X} + \tau m_1 \frac{\partial g(x;0)}{\partial X} \\
&= \int_0^L g(x;X)F(X)\,dX - \frac{\tau m_2}{L\tau}[-xh(X-x) + (L-x)h(x-X)]_{|X=L} \\
&\quad + \frac{\tau m_1}{L\tau}[-xh(X-x) + (L-x)h(x-X)]_{|X=0} \\
&= \int_0^L g(x;X)F(X)\,dX + \frac{m_2}{L}x + \frac{m_1}{L}(L-x) \\
&= \int_0^L g(x;X)F(X)\,dX + (m_2 - m_1)\frac{x}{L} + m_1 \\
&= (m_2 - m_1)\frac{x}{L} + m_1 + \frac{L-x}{L\tau}\int_0^x XF(X)\,dX \\
&\qquad + \frac{x}{L\tau}\int_x^L (L-X)F(X)\,dX. \bullet
\end{aligned}$$

Example 12.9 Solve the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \quad \alpha < x < \beta,$$
$$y(\alpha) = m_1, \quad y'(\beta) = m_2.$$

Solution According to Example 12.2, the Green's function for this problem is

$$\begin{aligned}
g(x;X) = \frac{-1}{2\cos 2(\beta-\alpha)}&[\sin 2(x-\alpha)\cos 2(\beta - X)h(X-x) \\
&+ \sin 2(X-\alpha)\cos 2(\beta - x)h(x-X)].
\end{aligned}$$

To account for the nonhomogeneities m_1 and m_2 in the boundary conditions, we use the term in 12.45a containing m_2 and the term in 12.45b containing m_1,

$$\begin{aligned} y(x) &= \int_{\alpha}^{\beta} g(x;X)F(X)\,dX - m_1\frac{\partial g(x;\alpha)}{\partial X} - m_2 g(x;\beta) \\ &= \int_{\alpha}^{\beta} g(x;X)F(X)\,dX \\ &\quad + \frac{m_1}{2\cos 2(\beta-\alpha)}[2\sin 2(x-\alpha)\sin 2(\beta-\alpha)h(\alpha-x) + 2\cos 2(\beta-x)h(x-\alpha)] \\ &\quad + \frac{m_2}{2\cos 2(\beta-\alpha)}[\sin 2(x-\alpha)h(\beta-x) + \sin 2(\beta-\alpha)\cos 2(\beta-x)h(x-\beta)] \\ &= \int_{\alpha}^{\beta} g(x;X)F(X)\,dX + \frac{2m_1\cos 2(\beta-x) + m_2\sin 2(x-\alpha)}{2\cos 2(\beta-\alpha)}.\bullet \end{aligned}$$

EXERCISES 12.4

Do the General Results first.

Part A Heat Conduction

1. What is the Green's function for the boundary value problem for steady-state temperature in a rod from $x = 0$ to $x = L$ with constant thermal conductivity k and zero end temperatures?

2. Solve the boundary value problem

$$-\frac{d}{dx}\left(\kappa\frac{dU}{dx}\right) = F(x), \quad \alpha < x < \beta,$$
$$U(\alpha) = 0 = U(\beta)$$

for steady-state temperature in a rod from $x = \alpha$ to $x = \beta$ with variable thermal conductivity $\kappa(x)$ and heat generation $F(x)$. Intepret the Green's function physically.

3. Two rods of lengths L_1 and L_2 and constant thermal conductivities κ_1 and κ_2 are joined end to end (the left end of L_1 at $x = 0$ and the right end of L_2 at $x = L_1 + L_2$). If the ends at $x = 0$ and $x = L_1 + L_2$ are kept at temperature zero, what is the Green's function for steady-state temperature in the rods?

Part B Vibrations

In Exercises 4–9 the function $F(x)$ describes the applied force on a massless string with constant tension τ stretched between two fixed points $x = 0$ and $x = L$. Find and sketch a graph of the displacement $y(x)$ in the string.

4. $F(x) = -k$, where $k > 0$ is a constant

5. $F(x) = \begin{cases} -kx, & 0 < x \le L/2 \\ k(x-L), & L/2 \le x < L \end{cases}$, $k > 0$ a constant

6. $F(x) = \begin{cases} 0, & 0 < x < L/4 \\ -k, & L/4 < x < 3L/4 \\ 0, & 3L/4 < x < L \end{cases}$, $k > 0$ a constant

7. $F(x)$ is due to two concentrated loads $-\overline{k}$ (where $\overline{k} > 0$) placed at $x = L/4$ and $x = 3L/4$.

8. $F(x)$ is due to the combination of the constant force $-k$ in Exercise 4 and the concentrated loads $-\overline{k}$ in Exercise 7.

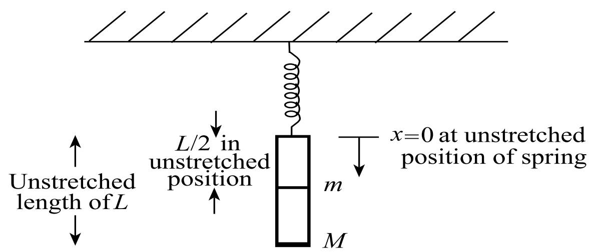
9. $F(x) = \begin{cases} -k, & 0 < x < L/4 \\ 0, & L/4 < x < 3L/4 \\ -k, & 3L/4 < x < L \end{cases}$, $k > 0$ a constant

10. Solve Exercise 15 in Section 2.3.

11. Solve Exercise 10 if a thin ring of mass m is attached halfway along the length of the bar.

12. Solve Exercise 10 if a mass M is attached to the lower end of the bar.

13. The bar in Exercise 12 is hung from a spring with constant k, and a thin ring of mass m is attached halfway along the length of the bar. Find displacements of its cross sections in the coordinate system shown in the figure below.



In Exercises 14–19 the function $F(x)$ describes the applied force on a beam of length L ($0 \le x \le L$), and the conditions represent boundary conditions at the ends of the beam. Use the Green's functions from Exercises 20–23 in Section 12.3 to find the static deflection of the beam. Sketch the deflection curve in Exercises 14–17.

14. $F(x)$ is due to a concentrated load of magnitude unity at $x = L/2$, and the weight of the beam is negligible,

$$y(0) = y'(0) = 0 = y''(L) = y'''(L).$$

(See also Exercise 10 in Section 12.1.)

15. $F(x)$ is due to the load of Exercise 14 placed at $x = L$. (See also Exercise 11 in Section 12.1.)

16. $F(x)$ is due only to the weight per unit x-length w of a uniform beam,

$$y(0) = y''(0) = 0 = y(L) = y''(L).$$

17. $F(x)$ is due to a uniform weight per unit x-length w of a uniform beam and a concentrated load of magnitude k at $x = L/2$,

$$y(0) = y'(0) = 0 = y(L) = y'(L).$$

18. $F(x) = \begin{cases} -w, & 0 < x < L/4 \\ -(w+W), & L/4 < x < 3L/4 \\ -w, & 3L/4 < x < L \end{cases}$, w and W constants,

$$y(0) = y'(0) = 0 = y''(L) = y'''(L).$$

19. $F(x)$ is due to a uniform weight per unit x-length W on $0 < x < L/2$ and a concentrated load of magnitude k at $x = L/4$. The weight of the beam is negligible,

$$y(0) = y'(0) = 0 = y(L) = y''(L).$$

General Results

In Exercises 20–25 find an integral representation for the solution of the boundary value problem.

20. $\dfrac{d^2y}{dx^2} = F(x), \quad 1 < x < 2, \quad y'(1) = m_1,\ y(2) = m_2$. What is the solution when $F(x) = xe^x$?

21. $\dfrac{d^2y}{dx^2} + y = F(x), \quad 0 < x < 1, \quad y(0) = m_1,\ y'(1) = m_2$. What is the solution when $F(x) = \cos x$?

22. $\dfrac{d^2y}{dx^2} + k^2y = F(x), \quad \alpha < x < \beta, \quad y(\alpha) = 0,\ y'(\beta) = 1, \quad k > 0$ a constant. Is there a restriction on the value of k? What is the solution when $F(x) = 1$?

23. $\dfrac{d^2y}{dx^2} + k^2y = F(x), \quad \alpha < x < \beta, \quad y(\alpha) = y(\beta),\ y'(\alpha) = y'(\beta), \quad k > 0$ a constant. (See Exercise 15 in Section 12.3 for the Green's function.) What is the solution when $F(x) = x$?

24. $(x+1)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = F(x), \quad 0 < x < 1, \quad y(0) = 0,\ y(1) = 0$. What is the solution when $F(x) = x$?

25. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 8y = F(x), \quad 0 < x < \pi, \quad y(0) = y(\pi),\ y'(0) = 0$. What is the solution when $F(x) = e^{2x}$?

§12.5 Modified Green's Functions

In Section 12.3, we saw that when homogeneous problem 12.32 has only the trivial solution, Green's function for the operator L and boundary conditions 12.32b,c exists. If we associate the Sturm-Liouville system

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda p(x) - q(x)]y = 0, \tag{12.49a}$$

$$B_1 y = 0, \tag{12.49b}$$

$$B_2 y = 0, \tag{12.49c}$$

with problem 12.32 (in the case of unmixed boundary conditions), it is equivalent to say that Green's function for the operator L and boundary conditions 12.32b,c exists if $\lambda = 0$ is not an eigenvalue for SL-system 12.49. The weight function $p(x)$ is chosen appropriate to solutions of the SL-system when $\lambda \neq 0$. For example, if the differential equation is $y'' = 0$, then the appropriate choice for $p(x)$ is unity since this yields the differential equation $y'' + \lambda^2 y = 0$ of the SL-system in Section 5.2. On the other hand, if the differential equation is $(d/dx)(xdy/dx) - (\nu^2/x)y = 0$, the appropriate choice for $p(x)$ is x since this yields the differential equation $(d/dx)(xdy/dx) + (\lambda^2 x - \nu^2/x)y = 0$ of the SL-system in Section 8.4.

To illustrate that Green's function for the operator L and boundary conditions 12.32b,c does not exist when homogeneous problem 12.32 has nontrivial solutions (or $\lambda = 0$ is an eigenvalue of 12.49), consider the boundary value problem

$$-\kappa\frac{d^2U}{dx^2} = F(x), \quad 0 < x < L,$$
$$U'(0) = 0, \quad U'(L) = 0$$

for steady-state heat conduction in a rod with insulated sides and ends. The associated homogeneous problem has nontrivial solution $U =$ constant. Notice that if we integrate the differential equation from $x = 0$ to $x = L$,

$$\int_0^L F(x)\,dx = \int_0^L -\kappa\frac{d^2U}{dx^2}dx = \left\{-\kappa\frac{dU}{dx}\right\}_0^L = 0.$$

Thus, if there is to be a solution to this problem, $F(x)$ cannot be specified arbitrarily; it must satisfy the condition

$$\int_0^L F(x)\,dx = 0. \tag{12.50}$$

Physically this means that with insulated sides and ends, the only way a steady-state condition can prevail is if the total heat generation is zero.

Since the delta function $\delta(x - X)$ does not satisfy this condition (a unit point source at $x = X$), there can be no solution to

$$-\kappa\frac{d^2g}{dx^2} = \delta(x - X),$$
$$g'(0; X) = 0, \quad g'(L; X) = 0$$

for the associated Green's function $g(x; X)$.

The condition equivalent to 12.50 in problems with more general differential equations and nonhomogeneous boundary conditions is contained in the following theorem.

Theorem 12.8 When a homogeneous boundary value problem

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = 0, \quad \alpha < x < \beta, \tag{12.51a}$$

$$B_1 y = -l_1 y'(\alpha) + h_1 y(\alpha) = 0, \tag{12.51b}$$

$$B_1 y = l_2 y'(\beta) + h_2 y(\beta) = 0, \tag{12.51c}$$

has nontrivial solutions $w(x)$, the nonhomogeneous problem

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] - q(x)y = F(x), \tag{12.52a}$$

$$B_1 y = m_1, \tag{12.52b}$$

$$B_1 y = m_2, \tag{12.52c}$$

has a solution if and only if

$$\int_\alpha^\beta F(x)w(x)\,dx = \frac{m_1}{l_1}r(\alpha)w(\alpha) + \frac{m_2}{l_2}r(\beta)w(\beta), \tag{12.53}$$

for every solution $w(x)$ of the corresponding homogeneous problem. When $l_1 = 0$ (and $h_1 = 1$), the term $(m_1/l_1)r(\alpha)w(\alpha)$ is replaced by $m_1 r(\alpha)w'(\alpha)$; and when $l_2 = 0$ (and $h_2 = 1$), the term $(m_2/l_2)r(\beta)w(\beta)$ is replaced by $-m_2 r(\beta)w'(\beta)$.

It is easy to establish the necessity of condition 12.53. If $y(x)$ is a solution of problem 12.52, then

$$\begin{aligned}
\int_\alpha^\beta F(x)w(x)\,dx &= \int_\alpha^\beta (Ly)w(x)\,dx \\
&= \int_\alpha^\beta y(Lw)\,dx + \Big\{r(wy' - yw')\Big\}_\alpha^\beta \quad \text{(using Green's formula 12.23)} \\
&= r(\beta)[w(\beta)y'(\beta) - y(\beta)w'(\beta)] - r(\alpha)[w(\alpha)y'(\alpha) - y(\alpha)w'(\alpha)] \\
&= r(\beta)\left\{\frac{w(\beta)}{l_2}[m_2 - h_2 y(\beta)] - y(\beta)w'(\beta)\right\} \\
&\quad - r(\alpha)\left\{\frac{w(\alpha)}{l_1}[-m_1 + h_1 y(\alpha)] - y(\alpha)w'(\alpha)\right\} \\
&= \frac{m_2}{l_2}r(\beta)w(\beta) - \frac{r(\beta)y(\beta)}{l_2}[l_2 w'(\beta) + h_2 w(\beta)] \\
&\quad + \frac{m_1}{l_1}r(\alpha)w(\alpha) - \frac{r(\alpha)y(\alpha)}{l_1}[-l_1 w'(\alpha) + h_1 w(\alpha)] \\
&= \frac{m_1}{l_1}r(\alpha)w(\alpha) + \frac{m_2}{l_2}r(\beta)w(\beta).
\end{aligned}$$

When problem 12.51 has nontrivial solutions and consistency condition 12.53 is satisfied, the solution of 12.52 is not unique. If $y(x)$ is a solution, so also is $y(x) + Cw(x)$ for arbitrary C and $w(x)$ a solution of problem 12.51.

To solve problem 12.52 when condition 12.53 is satisfied, we introduce *modified* Green's functions. We do so because there can be no "ordinary" Green's function satisfying

$$Lg = \delta(x - X),$$
$$B_1 g = 0, \quad B_2 g = 0,$$

since consistency condition 12.53 is not satisfied. Two situations arise, depending on whether problem 12.51 has one or two linearly independent solutions. We consider first the case in which problem 12.51 has only one nontrivial solution $w(x)$ that is unique to a multiplicative constant and may therefore be taken as normalized,

$$\int_\alpha^\beta p(x)[w(x)]^2\, dx = 1, \tag{12.54}$$

where $p(x)$ is the weight function for the associated Sturm-Liouville system 12.49. A **modified Green's function** associated with problem 12.52 is defined as a solution $\overline{g}(x;X)$ of

$$L\overline{g} = \delta(x - X) - w(x)w(X), \tag{12.55a}$$
$$B_1\overline{g} = 0, \tag{12.55b}$$
$$B_2\overline{g} = 0. \tag{12.55c}$$

Because the right side of equation 12.55a satisfies consistency condition 12.53, Theorem 12.8 guarantees a solution $\overline{g}(x;X)$. But because the solution is not unique, $\overline{g}(x;X)$ may or may not be symmetric, depending on the method used in its construction. It is important to note, however, that because the differential equation for the ordinary Green's function is modified only by the term $w(x)w(X)$, the modified Green's function satisfies the same continuity properties as the ordinary Green's function. Indeed, we shall use these properties to find $\overline{g}(x;X)$.

Example 12.10 Find modified Green's functions for the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \quad 0 < x < \pi,$$
$$y(0) = 0 = y(\pi).$$

Solution Solutions of the homogeneous differential equation $y'' + 4y = 0$ are of the form $y = A\cos 2x + B\sin 2x$. Since the function $\sin 2x$ satisfies both boundary conditions, the Green's function for this problem does not exist. With weight function $p(x) = 1$, we find that $\|\sin 2x\|^2 = \pi/2$. We define a modified Green's function $\overline{g}(x;X)$ as the solution of

$$\frac{d^2\overline{g}}{dx^2} + 4\overline{g} = \delta(x - X) - \frac{2}{\pi}\sin 2x \sin 2X,$$
$$\overline{g}(0;X) = 0 = \overline{g}(\pi;X).$$

Because $\overline{g}(x;X)$ must satisfy property 12.26c, and a particular solution of $\overline{g}'' + 4\overline{g} = -(2/\pi)\sin 2x \sin 2X$ is $(2\pi)^{-1}x\sin 2X\cos 2x$, we take

$$\overline{g}(x;X) = \frac{x}{2\pi}\sin 2X\cos 2x + \begin{cases} A\sin 2x + B\cos 2x, & 0 \le x < X \\ C\sin 2x + D\cos 2x, & X < x \le \pi. \end{cases}$$

To determine A, B, C, and D, we apply the boundary conditions $\overline{g}(0;X) = 0 = \overline{g}(\pi;X)$,

$$B = 0, \qquad \frac{1}{2}\sin 2X + D = 0,$$

and continuity conditions 12.26a,b at $x = X$,

$$A\sin 2X + B\cos 2X = C\sin 2X + D\cos 2X,$$

$$(2C\cos 2X - 2D\sin 2X) - (2A\cos 2X - 2B\sin 2X) = 1.$$

These four equations require

$$B = 0, \qquad D = -\frac{1}{2}\sin 2X, \qquad C = A + \frac{1}{2}\cos 2X,$$

where $A = A(X)$ is an arbitrary function of X. A modified Green's function is therefore

$$\begin{aligned}\overline{g}(x;X) &= \frac{x}{2\pi}\sin 2X\cos 2x + \begin{cases} A\sin 2x, & 0 \le x \le X \\ \left(A + \dfrac{1}{2}\cos 2X\right)\sin 2x - \dfrac{1}{2}\sin 2X\cos 2x, & X \le x \le \pi \end{cases} \\ &= \frac{x}{2\pi}\sin 2X\cos 2x + \begin{cases} A\sin 2x, & 0 \le x \le X \\ A\sin 2x + \dfrac{1}{2}\sin 2(x-X), & X \le x \le \pi \end{cases} \\ &= \frac{x}{2\pi}\sin 2X\cos 2x + A\sin 2x + \frac{1}{2}\sin 2(x-X)h(x-X).\end{aligned}$$

Notice that the arbitrariness in $\overline{g}(x;X)$ is a constant $A(X)$ times $w(x)$, the solution of the homogeneous problem.•

Modified Green's functions can be used to solve problem 12.52, which has a solution provided m_1, m_2, and $F(x)$ satisfy condition 12.53.

Theorem 12.9 When homogeneous problem 12.51 has only one solution $w(x)$ (unique to a multiplicative constant), and consistency condition 12.53 is satisfied by m_1, m_2, and $F(x)$, the solution of nonhomogeneous problem 12.52 is given by

$$y(x) = \int_\alpha^\beta \overline{g}(X;x)F(X)\,dX + Cw(x) - \frac{m_1}{l_1}r(\alpha)\overline{g}(\alpha,x) - \frac{m^2}{l_2}r(\beta)\overline{g}(\beta;x), \quad (12.56a)$$

or, when $l_1 = l_2 = 0$,

$$y(x) = \int_\alpha^\beta \overline{g}(X;x)F(X)\,dX + Cw(x) + m_2 r(\beta)\frac{\partial\overline{g}(\beta;x)}{\partial X} - m_1 r(\alpha)\frac{\partial\overline{g}(\alpha;x)}{\partial X}, \quad (12.56b)$$

where $\overline{g}(x;X)$ is a symmetric modified Green's function satisfying equation 12.55.

Proof We let $u = y(x)$ be the solution of problem 12.52 and $v = \overline{g}(x;X)$ in Green's formula 12.35,

$$\int_\alpha^\beta (yL\overline{g} - \overline{g}Ly)\,dx = \left\{r\left(y\frac{\partial\overline{g}}{\partial x} - \overline{g}\frac{dy}{dx}\right)\right\}_\alpha^\beta.$$

If we substitute from the differential equations for y and $\overline{g}$,

$$\int_\alpha^\beta \{y[\delta(x-X) - w(x)w(X)] - \overline{g}\, F(x)\}\, dx = \left\{ r \left[y\frac{\partial \overline{g}}{\partial x} - \overline{g} y'(x) \right] \right\}_\alpha^\beta,$$

or,

$$y(X) = \int_\alpha^\beta y(x)w(x)w(X)\, dx + \int_\alpha^\beta F(x)\overline{g}(x;X)\, dx + \left\{ r \left[y(x)\frac{\partial \overline{g}(x;X)}{\partial x} - \overline{g}(x;X)y'(x) \right] \right\}_\alpha^\beta.$$

Suppose now that the boundary conditions are unmixed as in equations 12.47. If $l_1 = l_2 = 0$ (and $h_1 = h_2 = 1$),

$$y(X) = Cw(X) + \int_\alpha^\beta \overline{g}(x;X)F(x)\, dx + r(\beta)m_2\frac{\partial \overline{g}(\beta;X)}{\partial x} - r(\alpha)m_1\frac{\partial \overline{g}(\alpha;X)}{\partial x}.$$

Interchanging x and X gives solution 12.56b. When $l_1 l_2 \neq 0$,

$$\begin{aligned}
y(X) - Cw(X) - \int_\alpha^\beta \overline{g}(x;X)F(x)\, dx &= r(\beta)\left[y(\beta)\frac{\partial \overline{g}(\beta;X)}{\partial x} - \overline{g}(\beta;X)y'(\beta) \right] \\
&\quad - r(\alpha)\left[y(\alpha)\frac{\partial \overline{g}(\alpha;X)}{\partial x} - \overline{g}(\alpha;X)y'(\alpha) \right] \\
&= r(\beta)\left[y(\beta)\frac{\partial \overline{g}(\beta;X)}{\partial x} - \overline{g}(\beta;X)\left(\frac{m_2}{l_2} - \frac{h_2}{l_2}y(\beta) \right) \right] \\
&\quad - r(\alpha)\left[y(\alpha)\frac{\partial \overline{g}(\alpha;X)}{\partial x} - \overline{g}(\alpha;X)\left(-\frac{m_1}{l_1} + \frac{h_1}{l_1}y(\alpha) \right) \right] \\
&= r(\beta)\left\{ -\frac{m_2}{l_2}\overline{g}(\beta;X) + \frac{y(\beta)}{l_2}\left[l_2\frac{\partial \overline{g}(\beta;X)}{\partial x} + h_2\overline{g}(\beta;X) \right] \right\} \\
&\quad - r(\alpha)\left\{ \frac{m_1}{l_1}\overline{g}(\alpha;X) + \frac{y(\alpha)}{l_1}\left[-l_1\frac{\partial \overline{g}(\alpha;X)}{\partial x} + h_1\overline{g}(\alpha;X) \right] \right\}.
\end{aligned}$$

But because $\overline{g}(x;X)$ satisfies

$$-l_1\frac{\partial \overline{g}(\alpha;X)}{\partial x} + h_1\overline{g}(\alpha;X) = 0 \quad \text{and} \quad l_2\frac{\partial \overline{g}(\beta;X)}{\partial x} + h_2\overline{g}(\beta;X) = 0,$$

it follows that

$$y(X) = Cw(X) + \int_\alpha^\beta \overline{g}(x;X)F(x)\, dx - \frac{m_1}{l_1}r(\alpha)\overline{g}(\alpha;X) - \frac{m_2}{l_2}r(\beta)\overline{g}(\beta;X).$$

When we interchange x and X,

$$y(x) = Cw(x) + \int_\alpha^\beta \overline{g}(X;x)F(X)\, dX - \frac{m_1}{l_1}r(\alpha)\overline{g}(\alpha;x) - \frac{m_2}{l_2}r(\beta)\overline{g}(\beta;x). \quad (12.57)$$

If $\overline{g}(x;X)$ is symmetric, this becomes solution 12.56a. ∎

Exercise 9 describes a technique for calculating symmetric modified Green's functions from nonsymmetric ones. The alternative is to use equation 12.57 with a nonsymmetric modified Green's function.

Example 12.11 Solve the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \quad 0 < x < \pi,$$
$$y(0) = 0 = y(\pi).$$

Solution According to Example 12.10, a modified Green's function is

$$\overline{g}(x;X) = \frac{x}{2\pi}\sin 2X\cos 2x + A\sin 2x + \frac{1}{2}\sin 2(x-X)h(x-X).$$

Because $\overline{g}(x;X)$ is not symmetric, we use equation 12.57 to express the solution of the boundary value problem in the form

$$\begin{aligned}
y(x) &= \int_0^\pi \overline{g}(X;x)F(X)\,dX + C\sin 2x \quad (C \text{ a constant})\\
&= \int_0^\pi \left[\frac{X}{2\pi}\sin 2x\cos 2X + A(x)\sin 2X + \frac{1}{2}\sin 2(X-x)h(X-x)\right]F(X)\,dX + C\sin 2x\\
&= C\sin 2x + \frac{\sin 2x}{2\pi}\int_0^\pi X\cos 2X F(X)\,dX + A(x)\int_0^\pi F(X)\sin 2X\,dX\\
&\quad + \frac{1}{2}\int_x^\pi \sin 2(X-x)F(X)\,dX.
\end{aligned}$$

Since the first integral is a constant, the second term may be grouped with $C\sin 2x$. Furthermore, the second integral vanishes because of consistency condition 12.53. Thus, the final solution is

$$y(x) = C\sin 2x + \frac{1}{2}\int_x^\pi \sin 2(X-x)F(X)\,dX.\bullet$$

We have considered the situation in which the homogeneous problem 12.51 corresponding to 12.52 has a single nontrivial solution (unique to a multiplicative constant). The remaining possibility is that all solutions of differential equation 12.51a satisfy boundary conditions 12.51b,c. In such a case, we can always find two orthonormal solutions $v(x)$ and $w(x)$ of $Ly = 0$. If $\psi(x)$ and $\phi(x)$ are linearly independent solutions, two orthonormal solutions are

$$v(x) = \frac{\psi(x)}{\sqrt{\int_\alpha^\beta p(x)[\psi(x)]^2\,dx}}, \qquad w(x) = \frac{\phi(x) - v(x)\int_\alpha^\beta p(x)\phi(x)v(x)\,dx}{\sqrt{\int_\alpha^\beta p(x)\left[\phi(x) - v(x)\int_\alpha^\beta p(x)\phi(x)v(x)\,dx\right]^2 dx}},$$

($\psi(x)$ is normalized to form $v(x)$. For $w(x)$, the component of $\phi(x)$ in the "direction" of $v(x)$ is removed, and the result is then normalized.) We define a modified Green's function $\overline{g}(x;X)$ associated with problem 12.52 as a solution of

$$L\overline{g} = \delta(x-X) - w(x)w(X) - v(x)x(X), \tag{12.58a}$$

$$B_1\overline{g} = 0, \tag{12.58b}$$

$$B_2\overline{g} = 0. \tag{12.58c}$$

Because the right side of differential equation 12.58a satisfies consistency condition 12.53, $\overline{g}(x;X)$ must indeed exist. Green's identity once again gives the solution of problem 12.52 as

$$\begin{aligned}
y(x) = \int_\alpha^\beta \overline{g}(X;x)F(X)\,dX + Cw(x) + Dv(x) - \frac{m_1}{l_1}r(\alpha)\overline{g}(\alpha;x)&\\
- \frac{m^2}{l_2}r(\beta)\overline{g}(\beta;x),&
\end{aligned} \tag{12.59a}$$

or, when $l_1 = l_2 = 0$,

$$y(x) = \int_\alpha^\beta \overline{g}(X;x)F(X)\,dX + Cw(x) + Dv(x) + m_2 r(\beta)\frac{\partial \overline{g}(\beta;x)}{\partial X} - m_1 r(\alpha)\frac{\partial \overline{g}(\alpha;x)}{\partial X}, \tag{12.59b}$$

where C and D are arbitrary constants. In the event that $\overline{g}(x;X)$ is symmetric, $\overline{g}(X;x)$ can be replaced by $\overline{g}(x;X)$.

Example 12.12 Solve the boundary value problem

$$\frac{d^2y}{dx^2} + y = F(x), \quad 0 < x < 2\pi,$$
$$y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$$

Solution The homogeneous problem has nontrivial solutions $\sin x$ and $\cos x$. Because these functions are orthogonal, a modified Green's function for this problem is defined by

$$\frac{d^2\overline{g}}{dx^2} + \overline{g} = \delta(x-X) - \frac{1}{\pi}(\sin x \sin X + \cos x \cos X),$$
$$\overline{g}(0;X) = \overline{g}(2\pi, X), \quad \frac{\partial \overline{g}(0;X)}{\partial x} = \frac{\partial \overline{g}(2\pi;X)}{\partial x}.$$

A solution of the differential equation is

$$\overline{g}(x;X) = \frac{x}{2\pi}\sin(X-x) + \begin{cases} A\sin x + B\cos x, & 0 \le x < X \\ C\sin x + D\cos x, & X < x \le 2\pi. \end{cases}$$

To determine A, B, C, and D, we first apply the boundary conditions

$$B = \sin X + D,$$
$$\frac{\sin X}{2\pi} + A = \frac{\sin X}{2\pi} - \cos X + C,$$

and then continuity conditions 12.26a,b at $x = X$,

$$A\sin X + B\cos X = C\sin X + D\cos X,$$

$$C\cos X - D\sin X - A\cos X + B\sin X = 1.$$

These four conditions require $A = C - \cos X$ and $B = D + \sin X$, where $C = C(X)$ and $D = D(X)$ are arbitrary functions of X. A modified Green's function is therefore

$$\begin{aligned} \overline{g}(x;X) &= \frac{x}{2\pi}\sin(X-x) + C\sin x + D\cos x + \begin{cases} \sin X\cos x - \cos X \sin x, & 0 \le x \le X \\ 0, & X \le x \le 2\pi \end{cases} \\ &= C\sin x + D\cos x + \sin(X-x)\begin{cases} x/(2\pi) + 1, & 0 \le x e X \\ x/(2\pi), & X \le x \le 2\pi \end{cases} \\ &= C\sin x + D\cos x + \sin(X-x)\left[\frac{x}{2\pi} + h(X-x)\right]. \end{aligned}$$

According to formula 12.59, the solution of the boundary value problem is

$$\begin{aligned} y(x) &= \int_0^{2\pi} \overline{g}(X;x)F(X)\,dX + E\sin x + G\cos x \\ &= \int_0^{2\pi} \left\{C\sin X + D\cos X + \sin(x-X)\left[\frac{X}{2\pi} + h(x-X)\right]\right\} F(X)\,dX \\ &\quad + E\sin x + G\cos x \\ &= E\sin x + G\cos x + \int_0^{2\pi} \sin(x-X)\left[\frac{X}{2\pi} + h(x-X)\right] F(X)\,dX, \end{aligned}$$

since $F(x)$ must satisfy the consistency conditions

$$\int_0^{2\pi} F(x)\sin x\,dx = 0 = \int_0^{2\pi} F(x)\cos x\,dx.\bullet$$

EXERCISES 12.5

1. Solve the boundary value problem

$$-\kappa\frac{d^2U}{dx^2} = F(x), \quad 0 < x < L,$$
$$U'(0) = 0 = U'(L)$$

when $F(x)$ satisfies consistency condition 12.50. Calculate the solution in closed form when $F(x) = \cos(\pi x/L)$.

2. Verify that the result in Example 12.12 gives the correct solution when $F(x) = \sin 2x$.

3. (a) Simplify the solution to Example 12.11 when $F(x) = \cos 2x$.
(b) Use equation 12.56b to find the solution when the boundary conditions are nonhomogeneous,

$$y(0) = m_1, \qquad y(\pi) = m_2.$$

What condition must be imposed on m_1 and m_2?

4. Solve the boundary value problem

$$\frac{d^2y}{dx^2} + k^2y = F(x), \quad 0 < x < L, \qquad (k > 0 \text{ a constant}),$$
$$y(0) = 0 = y(L).$$

5. (a) Use the result of Exercise 4 to solve

$$\frac{d^2y}{dx^2} + \frac{9\pi^2}{L^2}y = F(x), \quad 0 < x < L,$$
$$y(0) = m_1, \quad y(L) = m_2.$$

(b) Simplify the solution when $F(x) = x$. What is the consistency condition?

6. Solve the boundary value problem

$$\frac{d^2y}{dx^2} + k^2 y = F(x), \quad 0 < x < L, \qquad (k > 0 \text{ a constant}),$$
$$y(0) = 0 = y'(L).$$

7. (a) Use the result of Exercise 6 to solve

$$\frac{d^2y}{dx^2} + \frac{25\pi^2}{4L^2} y = F(x), \quad 0 < x < L,$$
$$y(0) = m_1, \quad y'(L) = m_2.$$

(b) Simplify the solution when $F(x) = x^2$. What is the consistency condition?

8. A modified Green's function for boundary value problem 12.52, when the corresponding homogeneous problem has only one solution $w(x)$ (unique to a multiplicative constant), is defined by boundary value problem 12.55. In this exercise we show that modified Green's functions can be defined in other ways. The homogeneous boundary value problem associated with the heat conduction problem

$$-\kappa \frac{d^2U}{dx^2} = F(x), \quad 0 < x < L,$$
$$U'(0) = m_1, \quad U'(L) = m_2,$$

has nontrivial solutions $y =$ constant.

(a) Show that when a function $\overline{\overline{g}}(x; X)$ satisfies

$$-\kappa \frac{d^2\overline{\overline{g}}}{dx^2} = \delta(x - X),$$
$$\overline{\overline{g}}'(0; X) = \frac{1}{2\kappa}, \quad \overline{\overline{g}}'(L; X) = -\frac{1}{2\kappa},$$

consistency condition 12.53 for nonhomogeneous problems is satisfied.

(b) Use Green's formula 12.35 to show that $U(x)$ can be exprssed in the form

$$U(x) = \int_0^L \overline{\overline{g}}(X; x) F(X)\, dX + \kappa[m_2 \overline{\overline{g}}(L; x) - m_1 \overline{\overline{g}}(0; x)] + C,$$

where C is an arbitrary constant. Find $\overline{\overline{g}}(x; X)$ and simplify the solution.

(c) Use the result in part (b) to find the solution to the boundary value problem of Exercise 1 when $F(x) = \cos(\pi x/L)$.

9. (a) Show that there is only one modified Green's function $\overline{g}_s(x; X)$ satisfying equations 12.55 that is orthogonal to $w(x)$ and that this function is given by

$$\overline{g}_s(x; X) = \overline{g}(x; X) - w(x) \int_\alpha^\beta \overline{g}(\zeta; X) w(\zeta)\, d\zeta,$$

where $\overline{g}(x; X)$ is any modified Green's function whatsoever.

(b) Use Green's identity 12.36 with $u = \overline{g}_s(x; X)$ and $v = \overline{g}_s(x; Y)$ to show that $\overline{g}_s(x; X)$ is symmetric. Are there any other symmetric modified Green's functions?

10. Use Exercise 9 to find symmetric modified Green's functions for the problem in Exercise 1.

11. Use Exercise 9 to find symmetric modified Green's functions for the problem in Example 12.10.

12. (a) Show that there is only one modified Green's function $\overline{g}_s(x; X)$ satisfying equations 12.58 that is orthogonal to $w(x)$ and $v(x)$ and that this function is given by

$$\overline{g}_s(x; X) = \overline{g}(x; X) - w(x)\int_\alpha^\beta \overline{g}(\zeta; X)w(\zeta)\,d\zeta - v(x)\int_\alpha^\beta \overline{g}(\zeta; X)v(\zeta)\,d\zeta,$$

where $\overline{g}(x; X)$ is any modified Green's function whatsoever.

(b) Use Green's identity 12.36 with $u = \overline{g}_s(x; X)$ and $v = \overline{g}_s(x; Y)$ to show that $\overline{g}_s(x; X)$ is symmetric. Are there any other symmetric modified Green's functions?

13. Use Exercise 12 to find symmetric modified Green's functions for the problem in Example 12.12.

§12.6 Green's Functions for Initial Value Problems

When the conditions that accompany differential equation 12.24a are of the form

$$y(\alpha) = 0, \qquad y'(\alpha) = 0, \tag{12.60}$$

they are called **initial conditions**, and the problem is known as an **initial value problem** rather than a boundary value problem. Because this situation arises most frequently when the independent variable is time t, we rewrite the initial value problem in the form

$$Ly = \frac{d}{dt}\left[r(t)\frac{dy}{dt}\right] - q(t)y = F(t), \quad t > t_0, \tag{12.61a}$$

$$y(t_0) = m_1, \tag{12.61b}$$

$$y'(t_0) = m_2. \tag{12.61c}$$

Initial time t_0 is usually chosen as $t_0 = 0$, but for the sake of generality, we maintain arbitrary t_0.

Were we to follow the lead of boundary value problems, it would be natural to define the Green's function $g(t;T)$ for this problem as the function satisfying the same differential equation with $F(t)$ replaced by a delta function, and the corresponding homogeneous initial conditions

$$Ly = \frac{d}{dt}\left[r(t)\frac{dy}{dt}\right] - q(t)y = \delta(t-T), \quad t > t_0, \tag{12.62a}$$

$$g(t_0;T) = 0, \quad \frac{dg(t_0;T)}{dt} = 0. \tag{12.62b}$$

Unfortunately, this would lead to improper integral representations of solutions of problem 12.61, together with associated convergence problems. Instead, we define the Green's function $g(t;T)$ as what is called a **causal fundamental solution** of 12.61; it is the solution of

$$g(t;T) = 0, \quad t_0 < t < T, \tag{12.63a}$$

$$Lg = \delta(t-T). \tag{12.63b}$$

Physically, $g(t;T)$ is the reaction of the system described by equations 12.61 to a unit impulse at time T. Naturally, for time $t < T$, the system must be identically equal to zero (hence the requirement 12.63a).

Provided $r(t)$ does not vanish for $t \geq t_0$, the solution of problem 12.63 exists and is unique. Furthermore, corresponding to properties 12.26, which characterize the Green's function for boundary value problem 12.24, the following conditions characterize the Green's function for initial value problem 12.61:

$$g(t;T) = 0, \quad t_0 < t < T, \tag{12.64a}$$

$$Lg = \frac{d}{dt}\left[r(t)\frac{dg}{dt}\right] - q(t)g = 0, \quad t > T, \tag{12.64b}$$

$$g(T+;T) = 0, \tag{12.64c}$$

$$\frac{dg(T+;T)}{dt} = \frac{1}{r(T)}. \tag{12.64d}$$

When $u(t)$ and $v(t)$ are linearly independent solutions of differential equation 12.64b, the function

$$g(t;T) = \frac{1}{J(u,v)}[u(T)v(t) - v(T)u(t)]h(t-T) \tag{12.65}$$

clearly satisfies problem 12.64 and must therefore be the Green's function for problem 12.61. This formula replaces 12.34 for boundary value problems, but notice that the condition that the associated homogeneous system has only the trivial solution is absent for initial value problems (it is always satisfied).

Example 12.13 What is the Green's function for the initial value problem

$$M\frac{d^2y}{dt^2} + ky = F(t), \quad t > 0,$$
$$y(0) = m_1, \quad y'(0) = m_2$$

for displacements of a mass M on the end of a spring with constant k?

Solution Since $\sin\sqrt{k/M}t$ and $\cos\sqrt{k/M}t$ are solutions of $My'' + ky = 0$, the Green's function, according to formula 12.65, is

$$\begin{aligned} g(t;T) &= \frac{1}{J(\sin\sqrt{k/M}t, \cos\sqrt{k/M}t)}\left(\sin\sqrt{\frac{k}{M}}T\cos\sqrt{\frac{k}{M}}t - \cos\sqrt{\frac{k}{M}}T\sin\sqrt{\frac{k}{M}}t\right)h(t-T) \\ &= \frac{-1}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}(T-t)h(t-T) = \frac{1}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}(t-T)h(t-T). \bullet \end{aligned}$$

The solution of an initial value problem can be expressed in terms of its Green's function. In particular, the solution of problem 12.61 is

$$y(t) = \int_{t_0}^{t} g(t;T)F(T)\,dT + r(t_0)\left[m_2 g(t;t_0) - m_1\frac{\partial g(t;t_0)}{\partial T}\right]. \tag{12.66}$$

The integral term, which accounts for the nonhomogeneity in the differential equation, is interpreted as the superposition of incremental results. Because the Green's function $g(t;T)$ is the result at time t due to a unit impulse $\delta(t-T)$ at time T, $g(t;T)F(T)\,dT$ is the result at time t due to an incremental "force" $F(T)\,dT$ over dT. The integral then adds over all contributions, beginning at time t_0, to give the final result at time t. The last two terms in 12.66 account for nonhomogeneities in initial conditions 12.61b,c.

Example 12.14 What is the solution of the problem in Example 12.13?

Solution According to formula 12.66, the solution is

$$\begin{aligned} y(t) &= \int_0^t \frac{1}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}(t-T)h(t-T)F(T)\,dT \\ &\quad + M\left(\frac{m_2}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}t + \frac{m_1}{\sqrt{kM}}\sqrt{\frac{k}{M}}\cos\sqrt{\frac{k}{M}}t\right) \\ &= \frac{1}{\sqrt{kM}}\int_0^t \sin\sqrt{\frac{k}{M}}(t-T)F(T)\,dT + \sqrt{\frac{M}{k}}m_2\sin\sqrt{\frac{k}{M}}t + m_1\cos\sqrt{\frac{k}{M}}t. \bullet \end{aligned}$$

EXERCISES 12.6

1. A particle of mass M moves along the x-axis under the action of a force that is an explicit function $F(t)$ $(t \geq 0)$ of time t only. Find an integral representation for its position as a function of time t if at time $t = 0$, it is moving with velocity v_0 at position x_0.

2. A mass M is suspended from a spring (with constant k). Vertical oscillations are initiated at time $t = 0$ by displacing M from its equilibrium position and giving it an initial speed. If motion takes place in a medium that causes a damping force proportional to velocity, and an external force $F(t)$ $(t \geq 0)$ acts on M, find an integral representation for the position of M as a function of time t.

3. (a) Show that the solution of problem 12.61 can be expressed in the form

$$y(t) = \frac{1}{J(u,v)}\left[\int_{t_0}^{t} [u(T)v(t) - v(T)u(t)]F(T)\,dT + r(t_0)[m_1 v'(t_0) - m_2 v(t_0)]u(t) + r(t_0)[m_2 u(t_0) - m_1 u'(t_0)]v(t)\right],$$

where $u(t)$ and $v(t)$ are any two linearly independent solutions of $Ly = 0$.

(b) Use the result in part (a) to show that $y(t)$ can also be written in the form

$$y(t) = \frac{1}{r(t_0)}\int_{t_0}^{t} [u(T)v(t) - v(T)u(t)]F(T)\,dT + m_1 u(t) + m_2 v(t),$$

where $u(t)$ and $v(t)$ are solutions of $Ly = 0$ satisfying

$$u(t_0) = 1, \quad u'(t_0) = 0, \quad v(t_0) = 0, \quad v'(t_0) = 1.$$

4. Use Exercise 3(b) to obtain the solution for Example 12.14.

5. Use Exercise 3(b) to solve Exercise 2.

CHAPTER 13 GREEN'S FUNCTIONS FOR PARTIAL DIFFERENTIAL EQUATIONS

§13.1 Multi-dimensional Delta Functions and Green's Identities

In this chapter we develop Green's functions for boundary value problems (and initial boundary value problems) associated with partial differential equations. Solutions to such problems can then be represented in terms of integrals of source functions and Green's functions. We begin by discussing multi-dimensional delta functions and Green's identities.

Two- and three-dimensional delta functions, like $\delta(x-c)$, are defined from a functional point of view. We discuss two-dimensional functions, but three-dimensional results are analogous. The generalized function $\delta(x-a, y-b)$ maps a function $f(x,y)$ continuous at (a,b) onto its value at (a,b), and once again, we represent the mapping symbolically as an integral, in this case, a double integral,

$$f(x,y) \xrightarrow{\delta(x-a,y-b)} f(a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-a, y-b) f(x,y)\, dA. \qquad (13.1)$$

Because successive applications of delta functions lead to the same result,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-a)\delta(x-b)\, f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} \delta(x-a) f(x,b)\, dx = f(a,b), \qquad (13.2)$$

it follows that

$$\delta(x-a, y-b) = \delta(x-a)\delta(y-b). \qquad (13.3)$$

In other words, the two-dimensional delta function in Cartesian coordinates is the product of two one-dimensional delta functions. Corresponding to property 12.11 in Section 12.1, we take

$$\iint_R \delta(x-a, y-b) f(x,y)\, dA = \begin{cases} f(a,b), & (a,b) \text{ in } R \\ 0, & (a,b) \text{ not in } R. \end{cases} \qquad (13.4)$$

Delta functions in curvilinear coordinates are defined analogously to those in Cartesian coordinates, but their expressions in terms of products of one-dimensional delta functions are complicated by formulas for area and volume elements in curvilinear coordinates. To illustrate, suppose that a point with Cartesian coordinates (x_0, y_0) has polar coordinates (r_0, θ_0). The delta function $\delta(r-r_0, \theta-\theta_0)$ in polar coordinates is that generalized function that assigns to a function $f(r,\theta)$, continuous at (r_0, θ_0), its value at (r_0, θ_0),

$$\iint_{R^2} \delta(r-r_0, \theta-\theta_0) f(r,\theta)\, dA = f(r_0, \theta_0), \qquad (13.5a)$$

where R^2 refers to the xy-plane. But because the area element in polar coordinates is $dA = r\, dr\, d\theta$, equation 13.5 is expressible in the form

$$\int_{-\pi}^{\pi}\int_0^{\infty} \delta(r-r_0, \theta-\theta_0) f(r,\theta) r\, dr\, d\theta = f(r_0, \theta_0). \qquad (13.5b)$$

Since

$$\int_{-\pi}^{\pi}\int_{0}^{\infty} \delta(r-r_0)\delta(\theta-\theta_0)f(r,\theta)\,dr\,d\theta = f(r_0,\theta_0), \tag{13.6}$$

it follows that $r\delta(r-r_0,\theta-\theta_0) = \delta(r-r_0)\delta(\theta-\theta_0)$, or

$$\delta(r-r_0,\theta-\theta_0) = \frac{1}{r}\delta(r-r_0)\delta(\theta-\theta_0). \tag{13.7}$$

Since the delta function $\delta(x-x_0)\delta(y-y_0)$ and that in equation 13.7 pick out the value of a function at the same point, we may write

$$\delta(x-x_0)\delta(y-y_0) = \frac{1}{r}\delta(r-r_0)\delta(\theta-\theta_0). \tag{13.8a}$$

Similarly, transformation laws from delta functions in Cartesian coordinates to those in cylindrical and spherical coordinates are

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = \frac{1}{r}\delta(r-r_0)\delta(\theta-\theta_0)\delta(z-z_0), \tag{13.8b}$$

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = \frac{1}{r^2\sin\phi}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0). \tag{13.8c}$$

Many curvilinear coordinates system, and in particular the above three, have **singular points** — points at which transformations between them and Cartesian coordinates fail to be one-to-one. In polar coordinates, the origin is singular, and in cylindrical and spherical coordinates, the z-axis is singular. Transformation laws 13.8 are not valid at singular points. To understand this, we first note that when the functional on the right side of equation 13.8a operates on a function $f(r,\theta)$, it produces $f(r_0,\theta_0)$, the value of the function at (r_0,θ_0). But if $r_0 = 0$, the value of the function $f(r,\theta)$ does not depend on the value of θ; its value is completely dictated by setting $r = 0$. This means that when $r_0 = 0$, the delta function $\delta(\theta-\theta_0)$ on the right side of 13.8a is redundant. To see how to remove this delta function, notice that if we write $F(0)$ for the value of $f(0,\theta)$, then

$$\int_0^{\infty} \delta(r)f(r,\theta)\,dr = F(0).$$

Integration of this result with respect to θ gives

$$\int_{-\pi}^{\pi}\int_0^{\infty} \delta(r)f(r,\theta)\,dr\,d\theta = \int_{-\pi}^{\pi} F(0)\,d\theta$$

or,

$$\int_{-\pi}^{\pi}\int_0^{\infty} \frac{\delta(r)}{r}f(r,\theta)r\,dr\,d\theta = 2\pi F(0).$$

Thus,

$$\int_{-\pi}^{\pi}\int_0^{\infty} \frac{\delta(r)}{2\pi r}f(r,\theta)r\,dr\,d\theta = F(0).$$

But this equation implies that $\delta(r)/(2\pi r)$ must be the delta function at the origin; that is,

$$\delta(x)\delta(y) = \frac{\delta(r)}{2\pi r}. \tag{13.9}$$

A similar discussion in cylindrical coordinates shows that

$$\delta(x)\delta(y)\delta(z-z_0) = \frac{\delta(r)\delta(z-z_0)}{2\pi r}. \tag{13.10}$$

In spherical coordinates, we obtain

$$\delta(x)\delta(y)\delta(z-z_0) = \begin{cases} \dfrac{\delta(r-r_0)\delta(\phi)}{2\pi r^2 \sin\phi}, & z_0 > 0 \\ \dfrac{\delta(r-r_0)\delta(\phi+\pi)}{2\pi r^2 \sin\phi}, & z_0 < 0 \end{cases} \tag{13.11a}$$

$$\delta(x)\delta(y)\delta(z) = \frac{\delta(r)}{4\pi r^2}. \tag{13.11b}$$

Boundary value problems are associated with elliptic PDEs. We consider only two types in this chapter, those associated with the Helmholtz and Poisson equations. The two-dimensional Helmholtz equation is

$$\nabla^2 u + k^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.12}$$

where A is some open region of the xy-plane (with a piecewise smooth boundary), and Poisson's equation is

$$\nabla^2 u = F(x,y), \quad (x,y) \text{ in } A. \tag{13.13}$$

Green's (second) identity for both of these operators states that for functions $u(x,y)$ and $v(x,y)$ that have continuous first partial derivatives and piecewise continuous second partial derivatives in A,

$$\iint_A (u\nabla^2 v - v\nabla^2 u)\, dA = \oint_{\beta(A)} (u\nabla v - v\nabla u)\cdot \hat{\mathbf{n}}\, ds, \tag{13.14a}$$

where $\hat{\mathbf{n}}$ is the unit outward normal vector to the boundary $\beta(A)$ of A (see Appendix C). This identity is also valid when $u(x,y)$ and/or $v(x,y)$ satisfy $\nabla^2 u + k^2 u = \delta(x-X, y-Y)$ or $\nabla^2 u = \delta(x-X, y-Y)$. These extensions parallel those in Theorems 12.3 and 12.4 in Section 12.3.

The three-dimensional version of Green's identity is

$$\iiint_V (u\nabla^2 v - v\nabla^2 u)\, dV = \oiint_{\beta(V)} (u\nabla v - v\nabla u)\cdot \hat{\mathbf{n}}\, dS, \tag{13.14b}$$

where V is a volume in space with piecewise smooth boundary $\beta(V)$. It is also valid when $u(x,y,z)$ and/or $v(x,y,z)$ satisfy $\nabla^2 u + k^2 u = \delta(x-X, y-Y, z-Z)$ or $\nabla^2 u = \delta(x-X, y-Y, z-Z)$.

§13.2 Green's Functions for Dirichlet Boundary Value Problems

Dirichlet problems for the two-dimensional Helmholtz equation take the form

$$Lu = \nabla^2 u + k^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.15a}$$

$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A). \tag{13.15b}$$

For $k = 0$, we have the special case of Poisson's equation. When $F(x,y)$ has continuous first derivatives and piecewise continuous second derivatives in A, as does $K(x,y)$ on $\beta(A)$, this problem has a unique solution. The special case in which A is a rectangle was discussed in Section 6.7 (see problem 6.70). In practical situations when $F(x,y)$ and $K(x,y)$ may not satisfy these conditions, verification of uniqueness is much more difficult, as is finding the solution by previous methods. Green's functions provide an excellent alternative.

We define the Green's function $G(x,y;X,Y)$ for problem 13.15 as the solution of

$$LG = \nabla^2 G + k^2 G = \delta(x-X, y-Y), \quad (x,y) \text{ in } A, \tag{13.16a}$$

$$G(x,y;X,Y) = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.16b}$$

It is the solution of problem 13.15 due to a unit source at the point (X,Y) when boundary conditions are homogeneous. In Section 13.3, we shall prove that the solution of boundary value problem 13.15 can be expressed in the form

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \oint_{\beta(A)} K(X,Y)\frac{\partial G(x,y;X,Y)}{\partial N}ds, \tag{13.17}$$

where $\partial G/\partial N$ is the outward normal derivative of G with respect to the (X,Y) variables along $\beta(A)$. The solution is expressed in terms of integrals of the associated Green's function and source and boundary terms $F(x,y)$ and $K(x,y)$. We shall also interpret these integrals physically. In this section, we concentrate on methods for finding Green's functions.

For boundary value problems associated with ODEs, we derived general formulas (equations 12.33 and 12.34 in Section 12.3) for Green's functions. This was possible because boundaries for ODEs consist of two points. For PDEs, boundaries consist of curves for two-dimensional problems and surfaces for three-dimensional problems. As a result, it is impossible to find formulas for Green's functions associated with multivariable boundary value problems. What we can do is develop general techniques useful in large classes of problems. In this section, we illustrate four of these techniques for finding the Green's function for Dirichlet problem 13.15 in the case of Poisson's equation. These techniques may also be appropriate for boundary value problems with Neumann or Robin conditions or mixed problems (problems with different types of boundary conditions on different parts of the boundary). Before doing so, however, notice that if we substitute $u = G(x,y;X,Y)$ and $v = G(x,y;R,S)$ into Green's identity 13.14a,

$$\iint_A [G(x,y;R,S)\nabla^2 G(x,y;X,Y) - G(x,y;X,Y)\nabla^2 G(x,y;R,S)]\,dA = 0$$

(since $G(x,y;R,S)$ and $G(x,y;X,Y)$ satisfy boundary condition13.16b). But because G is a solution of PDE 13.16a, we may write

$$0 = \iint_A \{G(x, y; R, S)[\delta(x - X, y - Y) - k^2 G(x, y; X, Y)] \\ - G(x, y; X, Y)[\delta(x - R, y - S) - k^2 G(x, y; R, S)]\}\, dA \\ = G(X, Y; R, S) - G(R, S; X, Y).$$

In other words, the Green's function is symmetric under the interchange of first and second variables with third and fourth,

$$G(x, y; X, Y) = G(X, Y; x, y). \tag{13.18}$$

This result is also valid when boundary condition 13.15b is replaced by either a Neumann or a Robin condition.

Full Eigenfunction Expansion

In this method, the Green's function is expanded in terms of orthonormal eigenfunctions of the associated eigenvalue problem

$$Lu + \lambda^2 u = 0, \quad (x, y) \text{ in } A, \tag{13.19a}$$

$$u(x, y) = 0, \quad (x, y) \text{ on } \beta(A). \tag{13.19b}$$

We illustrate with the following example.

Example 13.1 Find the Green's function associated with the Dirichlet problem for the two-dimensional Laplacian on a rectangle A: $0 \le x \le L$, $0 \le y \le L'$.

Solution Separation of variables on

$$\nabla^2 u + \lambda^2 u = 0, \quad (x, y) \text{ in } A, \tag{13.20a}$$

$$u(x, y) = 0, \quad (x, y) \text{ on } \beta(A), \tag{13.20b}$$

leads to normalized eigenfunctions

$$u_{mn}(x, y) = \frac{2}{\sqrt{LL'}} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'},$$

corresponding to eigenvalues $\lambda_{mn}^2 = (n\pi/L)^2 + (m\pi/L')^2$ (see Section '`The Multi-dimensional Eigenvalue Problem`'). The eigenfunction expansion of $G(x, y; X, Y)$ in terms of these eigenfunctions is

$$G(x, y; X, Y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} u_{mn}(x, y), \tag{13.21}$$

and this representation satisfies the boundary condition that G vanish on the edges of the rectangle. To calculate the coefficients c_{mn}, we substitute this representation into the PDE $\nabla^2 G = \delta(x - X, y - Y)$ for G and expand the delta function in terms of the $u_{mn}(x, y)$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\left(-\frac{n^2\pi^2}{L^2}-\frac{m^2\pi^2}{L'^2}\right)u_{mn}(x,y)$$
$$= \delta(x-X,y-Y)$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\int_0^L\int_0^{L'} \delta(x-X,y-Y)u_{mn}(x,y)\,dy\,dx\right]u_{mn}(x,y)$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(X,Y)u_{mn}(x,y).$$

Consequently, $c_{mn} = u_{mn}(X,Y)/(-\lambda_{mn}^2)$, and

$$G(x,y;X,Y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{u_{mn}(X,Y)}{-\lambda_{mn}^2}u_{mn}(x,y)$$
$$= \frac{-4}{LL'}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{L'}\right)^2}\sin\frac{n\pi X}{L}\sin\frac{m\pi Y}{L'}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}. \tag{13.22}$$

In Exercise 1 it is shown that this full eigenfunction expansion can also be obtained using Green's identity 13.14a. This avoids the interchange of the Laplacian and summation operations and the eigenfunction expansion of $\delta(x-X,y-Y)$.•

A general formula for full eigenfunction expansions can be found in Exercise 2, but such expansions are of limited calculational utility. First, they are possible only when the eigenvalue problem can be separated, and this requires that the boundary of A consist of coordinate curves (or coordinate surfaces, in three-dimensional problems), Second, in the case in which the full eigenfunction expansion is available, a partial eigenfunction expansion that converges more rapidly is also available.

Partial Eigenfunction Expansion

Like the full eigenfunction expanison, this method requires that region A be bounded by coordinate curves (or coordinate surfaces, in three-dimensional problems). It differs in that separation is considered on the homogeneous problem

$$Lu = 0, \quad (x,y) \text{ in } A, \tag{13.23a}$$
$$u(x,y) = 0, \quad (x,y) \text{ on } \beta(A), \tag{13.23b}$$

and is carried out until one variable remains. An eigenfunction expansion for the Green's function is then found in terms of normalized eigenfunctions already determined, with coefficients that are functions of the remaining variable. We illustrate once again with the problem in Example 13.1.

Example 13.2 Find a partial eigenfunction representation for the Green's function in Example 13.1.

Solution Separation of variables on

$$\nabla^2 u = 0, \quad (x,y) \text{ in } A, \tag{13.24a}$$
$$u(x,y) = 0, \quad (x,y) \text{ on } \beta(A), \tag{13.24b}$$

leads to normalized eigenfunctions $f_n(x) = \sqrt{\dfrac{2}{L}} \sin \dfrac{n\pi x}{L}$. We expand $G(x, y; X, Y)$ in terms of these,

$$G(x, y; X, Y) = \sum_{n=1}^{\infty} a_n(y) f_n(x). \tag{13.25}$$

In actual fact, coefficients $a_n(y)$ must also be functions of X and Y, but we shall understand this dependence implicitly rather than express it explicitly. To determine the $a_n(y)$, we substitute this expression into the PDE $\nabla^2 G = \delta(x - X, y - Y)$ for G and expand the delta function in terms of the $f_n(x)$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{-n^2\pi^2}{L^2} a_n f_n(x) + \sum_{n=1}^{\infty} \frac{d^2 a_n}{dy^2} f_n(x) &= \delta(x - X, y - Y)\\ &= \sum_{n=1}^{\infty} \left[\int_0^L \delta(x - X, y - Y) f_n(x)\, dx\right] f_n(x)\\ &= \sum_{n=1}^{\infty} f_n(X)\delta(y - Y) f_n(x).\end{aligned}$$

This equation and the boundary conditions $G(x, 0; X, Y) = 0 = G(x, L'; X, Y)$ require the $a_n(y)$ to satisfy

$$\frac{d^2 a_n}{dy^2} - \frac{n^2\pi^2}{L^2} a_n = \delta(y - Y) f_n(X), \quad 0 < y < L',$$
$$a_n(0) = 0, \quad a_n(L') = 0.$$

We can solve this boundary value problem most easily by using our theory of Green's functions for ODEs. Since a solution of the homogeneous equation that satisfies the first boundary condition is $\sinh{(n\pi y/L)}$, and one that satisfies the second is $\sinh{[n\pi(L' - y)/L]}$, equation 12.34 in Section 12.3 gives

$$a_n(y) = \frac{1}{J}\left[\sinh \frac{n\pi y}{L} \sinh \frac{n\pi(L' - Y)}{L} h(Y - y) + \sinh \frac{n\pi Y}{L} \sinh \frac{n\pi(L' - y)}{L} h(y - Y)\right],$$

where J is the conjunct of $\sinh{(n\pi y/L)}$ and $\sinh{[n\pi(L' - y)/L]}$,

$$\begin{aligned}J &= \frac{1}{f_n(X)}\left[\sinh \frac{n\pi y}{L}\left(\frac{-n\pi}{L}\right) \cosh \frac{n\pi(L' - y)}{L} - \left(\frac{n\pi}{L}\right) \cosh \frac{n\pi y}{L} \sinh \frac{n\pi(L' - y)}{L}\right]\\ &= -\frac{n\pi \sinh{(n\pi L'/L)}}{\sqrt{2L} \sin{(n\pi X/L)}}.\end{aligned}$$

Thus, an alternative to the double-series, full eigenfunction expansion is the single-series, partial eigenfunction expansion

$$G(x, y; X, Y) = \sum_{n=1}^{\infty} \frac{-\sqrt{2}L \sin \dfrac{n\pi X}{L}}{n\pi \sinh \dfrac{n\pi L'}{L}} \left[\sinh \frac{n\pi y}{L} \sinh \frac{n\pi(L' - Y)}{L} h(Y - y)\right.$$

$$+ \sinh \frac{n\pi Y}{L} \sinh \frac{n\pi(L'-y)}{L} h(y-Y)\Bigg] \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

$$= \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi X}{L} \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L} \sinh\frac{n\pi(L'-Y)}{L}}{n\pi \sinh\frac{n\pi L'}{L}}, & 0 \le y \le Y \\ \displaystyle\sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi X}{L} \sin\frac{n\pi x}{L} \sinh\frac{n\pi Y}{L} \sinh\frac{n\pi(L'-y)}{L}}{n\pi \sinh\frac{n\pi L'}{L}}, & Y \le y \le L'. \end{cases} \tag{13.26}$$

It is clear that we could find an equivalent solution by expanding G in a Fourier sine series in y. The result would be

$$G(x,y;X,Y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi Y}{L'} \sin\frac{n\pi y}{L'} \sinh\frac{n\pi x}{L'} \sinh\frac{n\pi(L-X)}{L'}}{n\pi \sinh\frac{n\pi L}{L'}}, & 0 \le x \le X \\ \displaystyle\sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi Y}{L'} \sin\frac{n\pi y}{L'} \sinh\frac{n\pi X}{L'} \sinh\frac{n\pi(L-x)}{L'}}{n\pi \sinh\frac{n\pi L}{L'}}, & X \le x \le L. \end{cases} \tag{13.27}$$

A natural question to ask is: In which problems, should each of these expressions for $G(x,y;X,Y)$ be used? Since each is a Fourier series, rates of convergence of the series will depend on the relative magnitudes of coefficients. The coefficient of $\sin(n\pi x/L)$ in representation 13.26 for $y > Y$ is

$$\frac{-2\sin\frac{n\pi X}{L} \sinh\frac{n\pi Y}{L} \sinh\frac{n\pi(L'-y)}{L}}{n\pi \sinh\frac{n\pi L'}{L}},$$

and for large n we may drop the negative exponentials in the hyperbolic functions and approximate this quantity with

$$-\frac{e^{n\pi Y/L} e^{n\pi(L'-y)/L}}{n\pi e^{n\pi L'/L}} \sin\frac{n\pi X}{L} = \frac{-1}{n\pi} e^{n\pi(Y-y)/L} \sin\frac{n\pi X}{L}.$$

Similarly, when $y < Y$, the coefficient can, for large n, be approximated by

$$\frac{-1}{n\pi} e^{n\pi(y-Y)/L} \sin\frac{n\pi X}{L}.$$

Corresponding coefficients in representation 13.27 are approximated for large n by

$$\frac{-1}{n\pi} e^{n\pi|X-x|/L'} \sin\frac{n\pi Y}{L'}.$$

It follows that to calculate $G(x,y;X,Y)$ at a value of x that is substantially different from X, it would be wise to use representation 13.27, and, conversely, when y is markedly different from Y, representation 13.26 would provide faster convergence.

In addition, when boundary integrals arise for the solution of Dirichlet problem 13.15 (and this occurs for nonhomogeneous boundary conditions 13.15b), it is advantageous to use representation 13.26 for integrations along $y = 0$ and $y = L'$, but use representation 13.27 along $x = 0$ and $x = L$.

Splitting Technique

Sometimes it is convenient to split G into two parts, $G = U + g$, where U contains the singular part of G due to the delta function in PDE 13.16a and g guarantees that G satisfies the boundary conditions associated with L. This splitting technique permits consideration of the singular nature of the Green's function without the annoyance of boundary conditions. (The technique could have been used for ODEs, but it was unnecessary because formulas 12.33 and 12.34 in Section 12.3 were presented for Green's functions.) To be more specific, for the Green's function satisfying problem 13.16, we set $G = U + g$, where $U(x, y; X, Y)$ satisfies the PDE

$$LU = \delta(x - X, y - Y) \tag{13.28}$$

and g satisfies the boundary value problem

$$Lg = 0, \quad (x, y) \text{ in } A, \tag{13.29a}$$

$$g = -U, \quad (x, y) \text{ on } \beta(A). \tag{13.29b}$$

Because $U(x, y; X, Y)$ is not required to satisfy boundary conditions, it is often called the **free-space Green's function for the operator** L. Free-space Green's functions for the Laplace and Helmholtz operators in two and three dimensions are listed in Table 13.1. Each is singular at the source point (X, Y).

	∇^2 Laplacian	$\nabla^2 + k^2$ Helmholtz
xy plane	$\frac{1}{2\pi} \ln \sqrt{(x - X)^2 + (y - Y)^2}$	$\frac{1}{4} Y_0[k\sqrt{(x - X)^2 + (y - Y)^2}]$
xyz space	$\frac{-1}{4\pi\sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}}$	$-\frac{e^{ik\sqrt{(x-X)^2+(y-Y)^2+(z-Z)^2}}}{4\pi\sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}}$, $-\frac{e^{-ik\sqrt{(x-X)^2+(y-Y)^2+(z-Z)^2}}}{4\pi\sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}}$

Table 13.1

We justify the first entry here; the other three are discussed in the exercises. The two-space Green's function $G(x, y; X, Y)$ for the Laplacian is the solution of

$$\nabla^2 G = \delta(x - X, y - Y).$$

It is the effect at point (x, y) due to a unit source at (X, Y). Because the function should be symmetric about the source point, we switch to polar coordinates centred at (X, Y), and search for a function $G(r; 0)$ satisfying

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} = \frac{\delta(r)}{2\pi r}, \tag{13.30}$$

where we have used equation 13.9 for the delta function at the origin. Multiplication by r leads to

$$\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \frac{\delta(r)}{2\pi}.$$

Integration with respect to r from $r = 0$ to an arbitrary value of r gives

$$r\frac{dG}{dr} = \frac{1}{2\pi} \quad \Longrightarrow \quad \frac{dG}{dr} = \frac{1}{2\pi r} \quad \Longrightarrow \quad G(r;0) = \frac{1}{2\pi}\ln r + C.$$

We take $C = 0$. This shows that the effect at a point due to a unit source is $1/(2\pi)$ times the logarithm of the distance from point to source. It follows that the effect at point (x, y) due to a source at (X, Y) is

$$G(x, y; X, Y) = \frac{1}{2\pi}\ln\sqrt{(x - X)^2 + (y - Y)^2}.$$

A similar derivation gives the free-space Green's function for the three-dimensional Laplacian (Exercise 25). Unfortunately, the same technique does not work for the Helmholtz operator. In Exercise 26, we provide an alternative derivation for free-space Green's functions associated with the Laplacian and this technique does extend to Helmholtz operators (Exercises 28 and 29).

We now return to the splitting technique by illustrating it in the following example.

Example 13.3 Find the Green's function for the Dirichlet problem associated with Laplace's equation on a circle $0 \le r \le a$.

Solution The Green's function associated with the Dirichlet problem for the Laplacian on a circle centred at the origin with radius a satisfies

$$\nabla^2 G = \frac{\delta(r - R)\delta(\theta - \Theta)}{r}, \quad 0 < r < a, \quad -\pi < \theta \le \pi, \quad \text{(13.31a)}$$

$$G(a, \theta; R, \Theta) = 0, \quad -\pi < \theta \le \pi. \quad \text{(13.31b)}$$

The free-space Green's function for the two-dimensional Laplacian with singularity at (R, Θ) is

$$\begin{aligned} U(r, \theta; R, \Theta) &= \frac{1}{2\pi}\ln\sqrt{(r\cos\theta - R\cos\Theta)^2 + (r\sin\theta - R\sin\Theta)^2} \\ &= \frac{1}{4\pi}\ln\left[r^2 + R^2 - 2rR\cos(\theta - \Theta)\right] \end{aligned}$$

(see Table 13.1). When we split G into $G = U + g$, function g must satisfy

$$\nabla^2 g = 0, \quad 0 < r < a, \quad -\pi < \theta \le \pi, \quad \text{(13.32a)}$$

$$g(a, \theta; R, \Theta) = -\frac{1}{4\pi}\ln\left[a^2 + R^2 - 2aR\cos(\theta - \Theta)\right], \quad -\pi < \theta \le \pi. \quad \text{(13.32b)}$$

Separation of variables on the PDE, together with boundedness at $r = 0$, leads to a solution of the form

$$g(r, \theta; R, \Theta) = \frac{A_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(A_n r^n \frac{\cos n\theta}{\sqrt{\pi}} + B_n r^n \frac{\sin n\theta}{\sqrt{\pi}}\right)$$

(see equation 6.31a in Section 6.3). Boundary condition 13.32b requires

$$\frac{A_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(A_n a^n \frac{\cos n\theta}{\sqrt{\pi}} + B_n a^n \frac{\sin n\theta}{\sqrt{\pi}}\right) = \frac{-1}{4\pi}\ln\left[a^2 + R^2 - 2aR\cos(\theta-\Theta)\right]$$
$$= \frac{-1}{4\pi}\ln a^2 - \frac{1}{4\pi}\ln\left[1 + \left(\frac{R}{a}\right)^2 - 2\left(\frac{R}{a}\right)\cos(\theta-\Theta)\right].$$

With the result

$$\sum_{n=1}^{\infty}\frac{\alpha^n\cos n\phi}{n} = -\frac{1}{2}\ln(1+\alpha^2-2\alpha\cos\phi), \quad (|\alpha|<1), \tag{13.33}$$

we may write

$$\frac{A_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(A_n a^n \frac{\cos n\theta}{\sqrt{\pi}} + B_n a^n \frac{\sin n\theta}{\sqrt{\pi}}\right)$$
$$= \frac{-1}{4\pi}\ln a^2 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(R/a)^n}{n}\cos n(\theta-\Theta)$$
$$= \frac{-1}{4\pi}\ln a^2 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(R/a)^n}{n}(\cos n\theta\cos n\Theta + \sin n\theta\sin n\Theta).$$

Comparison of coefficients requires

$$\frac{A_0}{\sqrt{2\pi}} = \frac{-1}{4\pi}\ln a^2, \qquad \frac{A_n a^n}{\sqrt{\pi}} = \frac{(R/a)^n}{2\pi n}\cos n\Theta, \qquad \frac{B_n a^n}{\sqrt{\pi}} = \frac{(R/a)^n}{2\pi n}\sin n\Theta,$$

and therefore

$$g(r,\theta;R,\Theta) = \frac{-1}{2\pi}\ln a + \sum_{n=1}^{\infty} r^n\left[\frac{(R/a)^n}{2\pi n a^n}\cos n\theta\cos n\Theta + \frac{(R/a)^n}{2\pi n a^n}\sin n\theta\sin n\Theta\right]$$
$$= \frac{-1}{2\pi}\ln a + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(rR/a^2)^n}{n}\cos n(\theta-\Theta).$$

But identity 13.33 permits evaluation of this series in closed form,

$$g(r,\theta;R,\Theta) = \frac{-1}{2\pi}\ln a - \frac{1}{4\pi}\ln\left[1 + \left(\frac{rR}{a^2}\right)^2 - 2\left(\frac{rR}{a^2}\right)\cos(\theta-\Theta)\right]$$
$$= \frac{1}{2\pi}\ln a - \frac{1}{4\pi}\ln\left[a^4 + R^2r^2 - 2a^2Rr\cos(\theta-\Theta)\right].$$

Finally,

$$G(r,\theta;R,\Theta) = U + g = \frac{1}{4\pi}\ln\left[r^2 + R^2 - 2Rr\cos(\theta-\Theta)\right] + \frac{1}{2\pi}\ln a$$
$$- \frac{1}{4\pi}\ln\left[a^4 + R^2r^2 - 2a^2Rr\cos(\theta-\Theta)\right]$$
$$= \frac{1}{4\pi}\ln\left[a^2\frac{r^2 + R^2 - 2Rr\cos(\theta-\Theta)}{a^4 + R^2r^2 - 2a^2Rr\cos(\theta-\Theta)}\right]. \tag{13.34}$$

This result is also obtained with a partial eigenfunction expansion in Exercise 13.•

The splitting technique points out a distinct difference between Green's functions for one-dimensional problems and those for multidimensional problems. The Green's function $g(x;X)$ for a one-dimensional boundary value problem (associated with a second-order ODE) is a continuous function of x (or can be made so) with a jump discontinuity in its first derivative. Green's functions for multidimensional boundary value problems can always be represented as the sum of a free-space Green's function U and a regular part g, and, according to Table 13.1, free-space Green's functions are always singular at the source point. Thus, multivariable Green's functions always have discontinuities at source points.

Method of Images

The method of images is simply physical reasoning and intelligent guesswork in arriving at the function g in the splitting technique, and as such it works only on Laplace's equation with very simple geometries. When the Green's function G for a domain A is split into $U+g$, the free-space Green's function U can be regarded as the potential due to a unit point source interior to A. This source, by itself, induces a nonzero potential on $\beta(A)$. What is needed is a source distribution exterior to A whose potential g will cancel the effect of U on $\beta(A)$. (The fact that this distribution is exterior to A guarantees that $G=U+g$ satisfies $\nabla^2 G=\delta$ interior to A.)

We illustrate with the following three-dimensional problem.

Example 13.4 Find the Green's function associated with the three-dimensional Dirichlet problem for Laplace's equation in a sphere of radius a.

Solution The Green's function satisfies

$$\nabla^2 G = \frac{\delta(r-R)\delta(\theta-\Theta)\delta(\phi-\Phi)}{r^2\sin\phi}, \quad 0<r<a, \quad 0<\phi<\pi, \quad -\pi<\theta\le\pi, \tag{13.35a}$$

$$G(a,\phi,\theta;R,\Phi,\Theta)=0, \quad 0<\phi<\pi, \quad -\pi<\theta\le\pi. \tag{13.35b}$$

According to Table 13.1, the free-space Green's function with source point (X,Y,Z) is $-1/[4\pi\sqrt{(x-X)^2+(y-Y)^2+(z-Z)^2}]$. When (R,Φ,Θ) are the spherical coordinates of (X,Y,Z), this function becomes

$$U(r,\phi,\theta;R,\Phi,\Theta)=\frac{-1}{4\pi\sqrt{r^2+R^2-2Rr[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}}.$$

What the method of images suggests is finding a source distribution exterior to the sphere, the potential g for which is such that $G=U+g$ vanishes on $r=a$. We might first consider whether a single source of magnitude q at a point $(R^*,\Phi^*,\Theta^*)(R^*>a)$ might suffice. Symmetry would suggest that such a source could eliminate U on $r=a$, which is symmetric around the line through the origin, and (R,Φ,Θ)

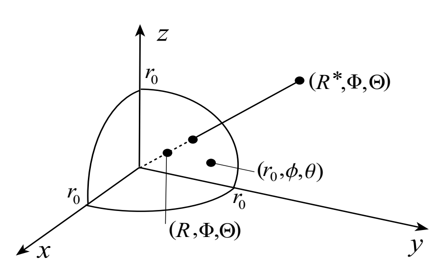


Figure 13.1

(Figure 13.1) only if (R^*, Φ^*, Θ^*) were to lie on the line also. We assume, therefore, that $\Theta^* = \Theta$ and $\Phi^* = \Phi$, in which case the condition that $G = U + g$ vanish on $r = a$ is

$$0 = \frac{-1}{4\pi\sqrt{a^2 + R^2 - 2aR[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}} + \frac{-q}{4\pi\sqrt{a^2 + {R^*}^2 - 2aR^*[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}}$$

or,

$$-q\sqrt{a^2 + R^2 - 2aR[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]} = \sqrt{a^2 + {R^*}^2 - 2aR^*[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}.$$

Since this condition must be valid for all ϕ and θ, we set $\phi = 0$ and $\phi = \pi$,

$$-q\sqrt{a^2 + R^2 - 2aR\cos\Phi} = \sqrt{a^2 + {R^*}^2 - 2aR^*\cos\Phi},$$
$$-q\sqrt{a^2 + R^2 + 2aR\cos\Phi} = \sqrt{a^2 + {R^*}^2 + 2aR^*\cos\Phi}.$$

These two equations imply that $R^* = a^2/R$ and $q = -a/R$, and with these, $U + g$ vanishes identically on $r = a$. Thus, the Green's function for the Laplacian inside a sphere of radius a is

$$\begin{aligned} G(r, \phi, \theta; R, \Phi, \Theta) &= \frac{-1}{4\pi\sqrt{r^2 + R^2 - 2Rr[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}} \\ &\quad + \frac{a}{4\pi R\sqrt{r^2 + \left(\frac{a^2}{R}\right)^2 - 2r\left(\frac{a^2}{R}\right)[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}} \\ &= \frac{-1}{4\pi\sqrt{r^2 + R^2 - 2Rr[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}} \\ &\quad + \frac{a}{4\pi\sqrt{R^2r^2 + a^4 - 2a^2Rr[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}}. \end{aligned} \tag{13.36}$$

EXERCISES 13.2

1. Show that coefficients c_{mn} in representation 13.21 can be obtained by substituting $v = u_{mn}(x, y)$ and $u = G(x, y; X, Y)$ in Green's identity 13.14a.

2. Show that when $u_n(x, y)$ are orthonormal eigenfunctions of the eigenvalue problem

$$\nabla^2 u + \lambda^2 u = 0, \quad (x, y) \text{ in } A, \tag{13.37a}$$
$$u(x, y) = 0, \quad (x, y) \text{ on } \beta(A), \tag{13.37b}$$

associated with the Dirichlet problem

$$\nabla^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.38a}$$
$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A), \tag{13.38b}$$

the full eigenfunction expansion for the Green's function is

$$G(x,y;X,Y) = \sum_{n=1}^{\infty} \frac{u_n(X,Y)u_n(x,y)}{-\lambda_n^2}. \tag{13.39}$$

(This expansion should be compared with that in Exercise 25 of Section 12.3 for the Green's function of an ODE.)

In Exercises 3–8 use Exercise 2 (and its extension to three dimensions) to find full eigenfunction expansions for the Green's function associated with the Dirichlet problem for Poisson's equation on the given domain.

3. $0 \le r < a$, $-\pi < \theta \le \pi$

4. $0 \le r < a$, $0 < \theta < \pi$

5. $0 \le r < a$, $0 < \theta < L$

6. $0 < x < L$, $0 < y < L'$, $0 < z < L''$

7. $0 \le r < a$, $-\pi < \theta \le \pi$, $0 < z < L$

8. $0 \le r < a$, $0 \le \phi \le \pi$, $-\pi < \theta \le \pi$

9. Use the method of images and the result of Example 13.4 to find the Green's function for the Dirichlet problem associated with Poisson's equation in a hemisphere of radius a.

10. Use a "modified" method of images to find the Green's function for the Dirichlet problem associated with the two-dimensional Laplacian on a circle of radius a. Assume that g consists of a potential due to an exterior, negative unit point source plus a constant potential.

11. Use the result of Exercise 10 and the method of images to find the Green's function for the Dirichlet problem associated with Poisson's equation on a semicircle $0 < r < a$, $0 < \theta < \pi$. How does it compare with the representation in Exercise 4.

12. Use the method of images to find the Green's function for the Dirichlet problem for the Laplacian on the rectangle $0 < x < L$, $0 < y < L'$.

13. In this exercise we use a partial eigenfunction expansion to find Green's function 13.34 for problem 13.31.

(a) Show that the partial eigenfunction expansion for $G(r,\theta;R,\Theta)$ is

$$G(r,\theta;R,\Theta) = \frac{A_0(r)}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left[A_n(r)\frac{\cos n\theta}{\sqrt{\pi}} + B_n(r)\frac{\sin n\theta}{\sqrt{\pi}}\right].$$

(b) Substitute the expansion in part (a) into PDE 13.31a, and expand $\delta(r-R)\delta(\theta-\Theta)/r$ in a Fourier series to obtain the following boundary value problems for the coefficients:

$$\frac{d}{dr}\left(r\frac{dA_0}{dr}\right) = \frac{\delta(r-R)}{\sqrt{2\pi}}, \quad A_0(a) = 0;$$
$$\frac{d}{dr}\left(r\frac{dA_n}{dr}\right) - \frac{n^2}{r}A_n = \delta(r-R)\frac{\cos n\Theta}{\sqrt{\pi}}, \quad A_n(a) = 0;$$
$$\frac{d}{dr}\left(r\frac{dB_n}{dr}\right) - \frac{n^2}{r}B_n = \delta(r-R)\frac{\sin n\Theta}{\sqrt{\pi}}, \quad B_n(a) = 0.$$

(c) The systems in part (b) are "singular" in the sense that there is only one boundary condition and the coefficient r in the derivative term vanishes at $r = 0$. As a result, equations 12.33 and 12.34 in Section 12.3 cannot be used to find A_n and B_n. Instead, use properties 12.26a–c from Section 12.3 and the one boundary condition to show that

$$A_0(r) = \begin{cases} \dfrac{\ln(R/a)}{\sqrt{2\pi}}, & 0 \le r \le R \\ \dfrac{\ln(r/a)}{\sqrt{2\pi}}, & R < r \le a, \end{cases}$$

$$A_n(r) = \begin{cases} \dfrac{\cos n\Theta}{2\sqrt{\pi}n}\left[\left(\dfrac{rR}{a^2}\right)^n - \left(\dfrac{r}{R}\right)^n\right], & 0 \le r \le R \\ \dfrac{\cos n\Theta}{2\sqrt{\pi}n}\left[\left(\dfrac{rR}{a^2}\right)^n - \left(\dfrac{R}{r}\right)^n\right], & R < r \le a, \end{cases}$$

$$B_n(r) = \begin{cases} \dfrac{\sin n\Theta}{2\sqrt{\pi}n}\left[\left(\dfrac{rR}{a^2}\right)^n - \left(\dfrac{r}{R}\right)^n\right], & 0 \le r \le R \\ \dfrac{\sin n\Theta}{2\sqrt{\pi}n}\left[\left(\dfrac{rR}{a^2}\right)^n - \left(\dfrac{R}{r}\right)^n\right], & R < r \le a. \end{cases}$$

(d) Find $G(r,\theta;R,\Theta)$ and use identity 13.33 to reduce the function to the form in equation 13.34.

14. Use the technique of Exercise 13 to find a partial eigenfunction expansion for the Green's function of the Dirichlet problem for the Laplacian on the semicircle $0 < r < a$, $0 < \theta < \pi$. Show that it can be expressed in the form of Exercise 11.

15. Use the technique of Exercise 13 to find the partial eigenfunction expansion for the Green's function of Exercise 5.

16. Find a partial eigenfunction expansion for the Green's function of Exercise 6 using eigenfunctions in x and y.

17. Show that when $u_n(x,y)$ are orthonormal eigenfunctions of eigenvalue problem 13.19, the full eigenfunction expansion for the Green's function of the boundary value problem

$$\nabla^2 u + k^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.40a}$$

$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A), \tag{13.40b}$$

is

$$G(x,y;X,Y) = \sum_{n=1}^{\infty} \frac{u_n(X,Y)u_n(x,y)}{k^2 - \lambda_n^2}, \tag{13.41}$$

provided $k \ne \lambda_n$ for any n. (The exceptional case is discussed in Exercise 8 of Section 13.3.)

In Exercises 18–24 use Exercise 17 to state Green's functions for problem 13.40 on the given domain. (See Example 13.1 and Exercises 3–8 for eigenpairs.)

18. $0 < x < L$, $0 < y < L'$

19. $0 \le r < a$, $-\pi < \theta \le \pi$

20. $0 \le r < a$, $0 < \theta < \pi$

21. $0 < r < a$, $0 < \theta < L$

22. $0 < x < L$, $0 < y < L'$, $0 < z < L''$

23. $0 \le r < a$, $-\pi < \theta \le \pi$, $0 < z < L$

24. $0 \le r < a$, $0 \le \phi \le \pi$, $-\pi < \theta \le \pi$

25. Derive the free-space Green's function for the 3-dimensional Laplacian by taking the source at the origin and using spherical coordinates centred there.

26. In this exercise we give a derivation of free-space Green's functions for the two-dimensional Laplacian that can be used to find free-space Green's functions for Helmholtz operators.

(a) Show that $G(r;0) = C\ln r + D$ is a general solution of the homogeneous version of equation 13.30.

(b) By substituting $v = G(r;0)$ and $u = 1$ in Green's second identity 13.14a where A is a circle of radius ϵ centred at the source $r = 0$, show that

$$\epsilon\frac{\partial G(\epsilon;0)}{\partial r} = \frac{1}{2\pi}.$$

(c) Reason that $G(r;0)$ must satisfy

$$r\frac{\partial G(r;0)}{\partial r} = \frac{1}{2\pi} \qquad \text{and} \qquad \lim_{r\to 0} r\frac{\partial G(r;0)}{\partial r} = \frac{1}{2\pi}.$$

(d) Use the results of parts (a) and (c), to find $G(r;0)$.

27. Use the technique of Exercise 26 to derive the free-space Green's function in Table 13.1 for the three-dimensional Laplacian.

28. Use the technique of Exercise 26 to derive the free-space Green's functions in Table 13.1 for the three-dimensional Helmholtz operator $\nabla^2 + k^2$. Hint: Set $G(r;0) = H(r)/r$ in the homogeneous differential equation for $G(r;0)$.

29. Use the technique of Exercise 26 to derive the free-space Green's function in Table 13.1 for the two-dimensional Helmholtz operator $\nabla^2 + k^2$.

§13.3 Solutions of Dirichlet Boundary Value Problems on Finite Regions

In this section we use Green's functions to solve Dirichlet boundary value problems associated with Poisson's equation on finite regions. Results for the Helmholtz equation are discussed in the exercises.

The Dirichlet boundary value problem for Poisson's equation in two dimensions is

$$\nabla^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.42a}$$
$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A), \tag{13.42b}$$

where A is a region with finite area. The following theorem verifies representation 13.17 as the solution of this problem.

Theorem 13.1 When $G(x,y;X,Y)$ is the Green's function for Dirichlet problem 13.42, the solution is

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \oint_{\beta(A)} K(X,Y)\frac{\partial G(x,y;X,Y)}{\partial N} ds, \tag{13.43}$$

where $\partial G/\partial N$ is the outward normal derivative of G with respect to the (X,Y) variables.

Proof If in Green's identity 13.14a, we let $v = G(x,y;X,Y)$ and $u = u(x,y)$ be the solution of problem 13.42,

$$\iint_A (u\nabla^2 G - G\nabla^2 u)\,dA = \oint_{\beta(A)} (u\nabla G - G\nabla u)\cdot \hat{\mathbf{n}}\,ds.$$

Because $\nabla^2 u = F$ and $\nabla^2 G = \delta(x-X, y-Y)$ in A, and $u = K$ and $G = 0$ on $\beta(A)$,

$$\iint_A [u(x,y)\delta(x-X,y-Y) - G(x,y;X,Y)F(x,y)]\,dA$$
$$= \oint_{\beta(A)} K(x,y)\nabla G(x,y;X,Y)\cdot\hat{\mathbf{n}}\,ds$$

or,

$$u(X,Y) = \iint_A G(x,y;X,Y)F(x,y)\,dy\,dx + \oint_{\beta(A)} K(x,y)\frac{\partial G(x,y;X,Y)}{\partial n} ds.$$

When we interchange (x,y) and (X,Y),

$$u(x,y) = \iint_A G(X,Y;x,y)F(X,Y)\,dY\,dX + \oint_{\beta(A)} K(X,Y)\frac{\partial G(X,Y;x,y)}{\partial N} ds$$
$$= \iint_A G(x,y;X,Y)F(X,Y)\,dY\,dX + \oint_{\beta(A)} K(X,Y)\frac{\partial G(x,y;X,Y)}{\partial N} ds$$

(because $G(x,y;X,Y)$ is symmetric). ∎

It can be helpful to interpret the integral terms in soluion 13.43 physically. From an electrostatic point of view, problem 13.42 defines potential in a region A due to an area charge density determined by $F(x,y)$ and a boundary potential

$K(x, y)$. (In actual fact, we are considering any cross section of a z-symmetric three-dimensional problem.) The area integral in solution 13.43 represents that part of the potential due to the interior charge, and the line integral is the boundary potential contribution. The Green's function $G(x, y; X, Y)$ is the potential at (x, y) due to a unit charge at (X, Y) when the boundary potential on $\beta(A)$ vanishes (which would be the case, say, for a grounded metallic surface). The double integral superposes over all elemental contributions $G(x, y; X, Y)F(X, Y)\,dY\,dX$ of internal charge.

From a heat conduction point of view, problem 13.42 describes steady-state temperature in a region A due to internal heat generation determined by $F(x, y)$ and boundary temperature $K(x, y)$. The area integral in solution 13.43 represents that part of the temperature due to internal sources. The Green's function is the temperature at (x, y) due to a unit source at (X, Y) when the boundary temperature is made to vanish. The line integral represents the effect of imposed boundary temperatures.

Finally, problem 13.42 also describes static deflections of a membrane stretched tautly over A. The double integral represents the effect due to applied forces (contained in $F(x, y)$), and the line integral determines the effect of boundary displacements.

We noted in Section 13.2 that $G(x, y; X, Y)$ is not continuous; it has a singularity when $(x, y) = (X, Y)$. The discontinuity cannot be too severe, however, since existence of the area integral in formula 13.43 (which integrates over the singularity) is guaranteed by Theorem 13.1. To illustrate this point, suppose A is the circle $r < a$ and $K = 0$ on $\beta(A)$. According to equation 13.43, the solution to problem 13.42 at any point (r, θ) in this case is

$$u(r, \theta) = \iint_A G(r, \theta; R, \Theta)F(R, \Theta)\,dA.$$

For simplicity, we consider the origin, in which case

$$u(0, \theta) = \iint_A G(0, \theta; R, \Theta)F(R, \Theta)\,dA.$$

Using equation 13.34 for $G(r, \theta; R, \Theta)$,

$$u(0, \theta) = \iint_A \frac{1}{2\pi} \ln\left(\frac{R}{a}\right) F(R, \Theta)\,dA,$$

and indeed we can see that $\ln(R/a)$ is singular at $R = 0$. However, the area element $dA = R\,dR\,d\Theta$ effectively removes this singularity, and

$$u(0, \theta) = \int_{-\pi}^{\pi}\int_0^a \frac{1}{2\pi} \ln\left(\frac{R}{a}\right) F(R, \Theta)R\,dR\,d\Theta$$

must converge. In particular, if $F(R, \Theta) \equiv 1$, integration by parts gives

$$u(0, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^a R\ln\left(\frac{R}{a}\right) dR\,d\Theta = -\frac{a^2}{4}.$$

The three-dimensional counterpart of Theorem 13.1 is the following.

Theorem 13.2 When $G(x, y, z; X, Y, Z)$ is the Green's function for the Dirichlet problem

$$\nabla^2 u = F(x, y, z), \quad (x, y, z) \text{ in } V, \tag{13.44a}$$

$$u(x, y, z) = K(x, y, z), \quad (x, y, z) \text{ on } \beta(V), \tag{13.44b}$$

the solution of the problem is

$$u(x, y, z) = \iiint_V G(x, y, z; X, Y, Z) F(X, Y, Z)\, dV + \iint_{\beta(V)} K(X, Y, Z) \frac{\partial G(x, y, z; X, Y, Z)}{\partial N} dS. \tag{13.45}$$

We now consider some examples.

Example 13.5 Find an integral representation for the solution of the Dirichlet boundary value problem on a circle

$$\nabla^2 u = F(r, \theta), \quad 0 < r < a, \quad -\pi < \theta \le \pi, \tag{13.46a}$$

$$u(a, \theta) = K(\theta), \quad -\pi < \theta \le \pi. \tag{13.46b}$$

Solution According to formula 13.43, the solution can be represented in the form

$$u(r, \theta) = \iint_A G(r, \theta; R, \Theta) F(R, \Theta)\, dA + \oint_{\beta(A)} K(\Theta) \frac{\partial G(r, \theta; a, \Theta)}{\partial R} ds,$$

where $G(r, \theta; R, \Theta)$ is the Green's function in equation 13.34

$$G(r, \theta; R, \Theta) = \frac{1}{4\pi} \ln \left[a^2 \frac{r^2 + R^2 - 2rR\cos(\theta - \Theta)}{a^4 + r^2R^2 - 2a^2rR\cos(\theta - \Theta)} \right].$$

Now,

$$\begin{aligned} \frac{\partial G(r, \theta; a, \Theta)}{\partial R} &= \frac{1}{4\pi} \left[\frac{2R - 2r\cos(\theta - \Theta)}{r^2 + R^2 - 2rR\cos(\theta - \Theta)} - \frac{2r^2R - 2a^2r\cos(\theta - \Theta)}{a^4 + r^2R^2 - 2a^2rR\cos(\theta - \Theta)} \right]_{|R=a} \\ &= \frac{1}{4\pi} \left[\frac{2a - 2r\cos(\theta - \Theta)}{r^2 + a^2 - 2ra\cos(\theta - \Theta)} - \frac{2r^2a - 2a^2r\cos(\theta - \Theta)}{a^4 + r^2a^2 - 2a^3r\cos(\theta - \Theta)} \right] \\ &= \frac{1}{2\pi a} \frac{a^2 - r^2}{r^2 + a^2 - 2ar\cos(\theta - \Theta)}. \end{aligned}$$

Thus,

$$\begin{aligned} u(r, \theta) &= \iint_A G(r, \theta; R, \Theta) F(R, \Theta)\, dA + \int_{-\pi}^{\pi} K(\Theta) \frac{a^2 - r^2}{2\pi a[r^2 + a^2 - 2ar\cos(\theta - \Theta)]} a\, d\Theta \\ &= \iint_A G(r, \theta; R, \Theta) F(R, \Theta)\, dA + \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{K(\Theta)}{r^2 + a^2 - 2ar\cos(\theta - \Theta)} d\Theta. \end{aligned} \tag{13.47}$$

When $F(r, \theta) \equiv 0$, the solution of Laplace's equation is

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{K(\Theta)}{r^2 + a^2 - 2ar\cos(\theta - \Theta)} d\Theta,$$

Poisson's integral formula for a circle (see equation 6.34 in Section 6.3).●

Example 13.6 Find an integral representation for the solution of the following Dirichlet problem on a rectangle:

$$\nabla^2 u = F(x,y), \quad 0 < x < L, \quad 0 < y < L', \tag{13.48a}$$

$$u(x,0) = f(x), \quad 0 < x < L, \tag{13.48b}$$

$$u(L,y) = 0, \quad 0 < y < L', \tag{13.48c}$$

$$u(x,L') = 0, \quad 0 < x < L, \tag{13.48d}$$

$$u(0,y) = g(y), \quad 0 < y < L'. \tag{13.48e}$$

Solution The solution can be represented in the form

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \int_0^L -f(X)\frac{\partial G(x,y;X,0)}{\partial Y}dX + \int_{L'}^0 -g(Y)\frac{\partial G(x,y;0,Y)}{\partial X}(-dY),$$

where G is given by either of formulas 13.26 or 13.27. For the first line integral, we use formula 13.26 in the form

$$G(x,y;X,Y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{-2\sin\dfrac{n\pi X}{L}\sin\dfrac{n\pi x}{L}\sinh\dfrac{n\pi Y}{L}\sinh\dfrac{n\pi(L'-y)}{L}}{n\pi\sinh\dfrac{n\pi L'}{L}}, & 0 \le Y \le y \\ \displaystyle\sum_{n=1}^{\infty} \frac{-2\sin\dfrac{n\pi X}{L}\sin\dfrac{n\pi x}{L}\sinh\dfrac{n\pi y}{L}\sinh\dfrac{n\pi(L'-Y)}{L}}{n\pi\sinh\dfrac{n\pi L'}{L}}, & y \le Y \le L' \end{cases}$$

to calculate

$$\frac{\partial G(x,y;X,0)}{\partial Y} = \sum_{n=1}^{\infty} \frac{-2\sin\dfrac{n\pi X}{L}\sin\dfrac{n\pi x}{L}\left(\dfrac{n\pi}{L}\right)\sinh\dfrac{n\pi(L'-y)}{L}}{n\pi\sinh\dfrac{n\pi L'}{L}}.$$

A similar calculation using formula 13.27 gives

$$\frac{\partial G(x,y;0,Y)}{\partial X} = \sum_{n=1}^{\infty} \frac{-2\sin\dfrac{n\pi Y}{L'}\sin\dfrac{n\pi y}{L'}\left(\dfrac{n\pi}{L'}\right)\sinh\dfrac{n\pi(L-x)}{L'}}{n\pi\sinh\dfrac{n\pi L}{L'}},$$

and therefore

$$\begin{aligned} u(x,y) = &\iint_A G(x,y;X,Y)F(X,Y)\,dA \\ &- \int_0^L f(X)\left[\sum_{n=1}^{\infty} \frac{-2\sin\dfrac{n\pi X}{L}\sin\dfrac{n\pi x}{L}\left(\dfrac{n\pi}{L}\right)\sinh\dfrac{n\pi(L'-y)}{L}}{n\pi\sinh\dfrac{n\pi L'}{L}}\right]dX \\ &- \int_0^{L'} g(Y)\left[\sum_{n=1}^{\infty} \frac{-2\sin\dfrac{n\pi Y}{L'}\sin\dfrac{n\pi y}{L'}\left(\dfrac{n\pi}{L'}\right)\sinh\dfrac{n\pi(L-x)}{L'}}{n\pi\sinh\dfrac{n\pi L}{L'}}\right]dY \end{aligned}$$

$$= \int_0^{L'} \int_0^L G(x,y;X,Y)F(X,Y)\,dX\,dY$$

$$+ \frac{2}{L}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{L}\sinh\frac{n\pi(L'-y)}{L}}{\sinh\frac{n\pi L'}{L}} \int_0^L f(X)\sin\frac{n\pi X}{L}dX$$

$$+ \frac{2}{L'}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi y}{L'}\sinh\frac{n\pi(L-x)}{L'}}{\sinh\frac{n\pi L}{L'}} \int_0^{L'} g(Y)\sin\frac{n\pi Y}{L'}dY. \bullet \qquad (13.49)$$

Special cases of this problem that lead to solutions found in previous chapters are contained in Exercises 1 and 2.

Example 13.7 Find an integral representation for the solution of the Dirichlet problem in a sphere:

$$\nabla^2 u = F(r,\phi,\theta), \quad 0 < r < a, \quad 0 < \phi < \pi, \quad -\pi < \theta \le \pi, \quad (13.50a)$$

$$u(a,\phi,\theta) = K(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi. \quad (13.50b)$$

Solution The solution can be expressed in the form

$$u(r,\phi,\theta) = \iiint_V G(r,\phi,\theta;R,\Phi,\Theta)F(R,\Phi,\Theta)\,dV + \iint_{\beta(V)} K(\Phi,\Theta)\frac{\partial G(r,\phi,\theta;a,\Phi,\Theta)}{\partial R}dS,$$

where the Green's function is contained in equation 13.36. Since

$$\begin{aligned}
\frac{\partial G(r,\phi,\theta;a,\Phi,\Theta)}{\partial R} &= \frac{1}{4\pi}\left\{\frac{a - r[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta)]}{[r^2+a^2-2ra(\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta))]^{3/2}}\right\} \\
&\quad - \frac{a}{4\pi}\left\{\frac{ar^2 - a^2r[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta)]}{[a^2r^2+a^4-2a^3r(\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta))]^{3/2}}\right\} \\
&= \frac{1}{4\pi}\left\{\frac{a - r[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta)]}{[r^2+a^2-2ra(\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta))]^{3/2}}\right\} \\
&\quad - \frac{r}{4\pi a}\left\{\frac{r - a[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta)]}{[r^2+a^2-2ra(\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta))]^{3/2}}\right\} \\
&= \frac{a^2-r^2}{4\pi a\{r^2+a^2-2ra[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]\}^{3/2}},
\end{aligned}$$

we find that

$$\begin{aligned}
u(r,\phi,\theta) &= \iiint_V G(r,\phi,\theta;R,\Phi,\Theta)F(R,\Phi,\Theta)\,dV \\
&\quad + \int_{-\pi}^{\pi}\int_0^{\pi} \frac{(a^2-r^2)K(\Phi,\Theta)a^2\sin\Phi}{4\pi a\{r^2+a^2-2ra[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]\}^{3/2}}d\Phi\,d\Theta \\
&= \iiint_V G(r,\phi,\theta;R,\Phi,\Theta)F(R,\Phi,\Theta)\,dV \\
&\quad + \frac{a^3-r^2a}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi} \frac{K(\Phi,\Theta)\sin\Phi}{\{r^2+a^2-2ra[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]\}^{3/2}}d\Phi\,d\Theta.
\end{aligned}$$

(13.51)

When $F(r,\phi,\theta)=0$, this is Poisson's integral formula 9.41 for a sphere, but the simplicity of the present derivation is unmistakable.●

EXERCISES 13.3

1. Use the result of Example 13.6 to solve Exercise 31 in Section 4.2.

2. Use the result of Example 13.6 to solve Exercise 48 in Section 7.2.

3. Find an integral representation for the solution of the following Dirichlet boundary value problem on a rectangle.

$$\begin{aligned}\nabla^2 u &= F(x,y), \quad 0<x<L, \quad 0<y<L',\\ u(x,0) &= 0, \quad 0<x<L,\\ u(L,y) &= g(y), \quad 0<y<L',\\ u(x,L') &= f(x), \quad 0<x<L,\\ u(0,y) &= 0, \quad 0<y<L'.\end{aligned}$$

4. Find an integral representation for the solution of the Dirichlet problem on a semicircle:

$$\begin{aligned}\nabla^2 u &= F(r,\theta), \quad 0<r<a, \quad 0<\theta<\pi,\\ u(a,\theta) &= f(\theta), \quad 0<\theta<\pi,\\ u(r,0) &= g_1(r), \quad 0<r<a,\\ u(r,\pi) &= g_2(r), \quad 0<r<a.\end{aligned}$$

(See Exercise 11 in Section 13.2 for the Green's function.)

In the remaining exercises, we discuss Dirichlet problems associated with the Helmholtz equation,

$$(\nabla^2+k^2)u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.52a}$$

$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A), \tag{13.52b}$$

where $k>0$ is a constant.

5. Verify that representation 13.43 is the solution of problem 13.52 when there is a Green's function $G(x,y;X,Y)$ satisfying

$$(\nabla^2+k^2)G = \delta(x-X,y-Y), \quad (x,y) \text{ in } A, \tag{13.53a}$$

$$G(x,y;X,Y) = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.53b}$$

6. What is the result corresponding to that in Exercise 5 for three-dimensional problems?

The homogeneous Dirichlet problem for the Laplacian

$$\begin{aligned}\nabla^2 u &= 0, \quad (x,y) \text{ in } A,\\ u(x,y) &= 0, \quad (x,y) \text{ on } \beta(A),\end{aligned}$$

has only the trivial solution. The homogeneous Dirichlet problem

$$(\nabla^2 + k^2)u = 0, \quad (x, y) \text{ in } A,$$
$$u(x, y) = 0, \quad (x, y) \text{ on } \beta(A),$$

on the other hand, may have nontrivial solutions depending on the value of k. In this case, it is necessary to introduce modified Green's functions. We illustrate this in Exercise 7 and discuss it in general in Exercise 8.

7. (a) Show that when A is the square $0 < x, y < L$, the function $w(x, y) = \dfrac{2}{L} \sin \dfrac{\pi x}{L} \sin \dfrac{\pi y}{L}$ is a (nontrivial) solution of

$$\nabla^2 u + \frac{2\pi^2}{L^2} u = 0, \quad (x, y) \text{ in } A,$$
$$u(x, y) = 0, \quad (x, y) \text{ on } \beta(A).$$

(b) Prove that when the problem

$$\nabla^2 u + \frac{2\pi^2}{L^2} u = F(x, y), \quad (x, y) \text{ in } A,$$
$$u(x, y) = 0, \quad (x, y) \text{ on } \beta(A),$$

has a solution $u(x, y)$, then $F(x, y)$ must satisfy the condition

$$\int_0^L \int_0^L F(x, y) w(x, y)\, dy\, dx = 0.$$

(The converse is also valid; that is, when $F(x, y)$ satsifies this condition, the nonhomogeneous problem has a solution $u(x, y)$. It is not unique; $u(x, y) + Cw(x, y)$ is also a solution for any constant C.)

(c) Because the delta function does not satisfy the condition in part (b), there can be no Green's function satisfying

$$\nabla^2 G + \frac{2\pi^2}{L^2} G = \delta(x - X, y - Y), \quad (x, y) \text{ in } A,$$
$$G(x, y; X, Y) = 0, \quad (x, y) \text{ on } \beta(A).$$

We therefore introduce a modified Green's function $\overline{G}(x, y; X, Y)$ satisfying

$$\nabla^2 u + \frac{2\pi^2}{L^2} \overline{G} = \delta(x - X, y - Y) - w(x, y) w(X, Y), \quad (x, y) \text{ in } A,$$
$$\overline{G}(x, y; X, Y) = 0, \quad (x, y) \text{ on } \beta(A).$$

Show that the right side of the PDE for $\overline{G}$ satisfies the condition in part (b).

(d) Find a partial eigenfunction expansion for $\overline{G}$ in terms of the normalized eigenfunctions $\sqrt{2/L} \sin(n\pi x/L)$.

(e) Find an integral representation for the solution of the boundary value problem in part (b) in terms of $F(x, y)$ and $\overline{G}(x, y; X, Y)$.

8. (a) Show that when the homogeneous problem

$$(\nabla^2 + k^2)u = 0, \quad (x,y) \text{ in } A, \tag{13.54a}$$
$$u(x,y) = 0, \quad (x,y) \text{ on } \beta(A), \tag{13.54b}$$

has nontrivial solutions $w(x,y)$, nonhomogeneous problem 13.52 has a solution only if $F(x,y)$ and $K(x,y)$ satisfy the condition that for every such solution $w(x,y)$

$$\iint_A F(x,y)w(x,y)\,dA = -\oint_{\beta(A)} K(x,y)\frac{\partial w(x,y)}{\partial n}ds, \tag{13.55}$$

where $\partial w/\partial n$ is the derivative of w in the outwardly normal direction to $\beta(A)$. (The converse result is also valid; that is, when condition 13.55 is satisfied, problem 13.52 has a solution that is unique to an additive term $Cw(x,y)$, C an arbitrary constant.)

(b) Show that the solution of problem 13.52 can be expressed in the form

$$u(x,y) = \iint_A \overline{G}(X,Y;x,y)F(X,Y)\,dA + \oint_{\beta(A)} K(X,Y)\frac{\partial \overline{G}(X,Y;x,y)}{\partial N}ds + Cw(x,y), \tag{13.56}$$

where $\overline{G}(x,y;X,Y)$ is a modified Green's function satisfying

$$(\nabla^2 + k^2)\overline{G} = \delta(x-X, y-Y) - w(x,y)w(X,Y), \quad (x,y) \text{ in } A, \tag{13.57a}$$
$$\overline{G}(x,y;X,Y) = 0, \quad (x,y) \text{ on } \beta(A), \tag{13.57b}$$

and $w(x,y)$ is a normalized solution of problem 13.54 (with unit weight function).

§13.4 Solutions of Neumann Boundary Value Problems on Finite Regions

The Neumann problem for Poisson's equation

$$\nabla^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.58a}$$

$$\frac{\partial u}{\partial n} = K(x,y), \quad (x,y) \text{ on } \beta(A), \tag{13.58b}$$

is more difficult to handle than the Dirichlet problem because the corresponding homogeneous problem,

$$\nabla^2 u = 0, \quad (x,y) \text{ in } A, \tag{13.59a}$$

$$\frac{\partial u}{\partial n} = 0, \quad (x,y) \text{ on } \beta(A), \tag{13.59b}$$

always has nontrivial solutions $u =$ constant. As a result, problem 13.58 does not have a unique solution; if $u(x,y)$ is a solution, so also is $u(x,y) +$ constant. We already know that for there to be a solution of problem 13.58, $F(x,y)$ and $K(x,y)$ must satisfy the consistency condition

$$\iint_A F(x,y)\, dA = \oint_{\beta(A)} K(x,y)\, ds. \tag{13.60}$$

When problem 13.58 is a steady-state heat conduction problem, condition 13.60 implies that heat generation within A must be compensated by heat crossing its boundary. It would therefore be futile to define the Green's function for problem 13.58 as the solution of

$$\nabla^2 u = \delta(x-X)\delta(y-Y)), \quad (x,y) \text{ in } A, \tag{13.61a}$$

$$\frac{\partial u}{\partial n} = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.61b}$$

Consistency condition 13.60 is not satisfied. We take the lead of Section 12.5 and introduce modified Green's functions. They are functions that satisfy

$$\nabla^2 N = \delta(x-X, y-Y) - \frac{1}{\text{area}(A)}, \quad (x,y) \text{ in } A, \tag{13.62a}$$

$$\frac{\partial N}{\partial n} = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.62b}$$

To see how the area term in this PDE is a direct analogue of the situation for ODEs, see Exercise 1. Solutions to problem 13.62 do exist since condition 13.60 is satisfied,

$$\iint_A \left[\delta(x-X, y-Y) - \frac{1}{\text{area}(A)}\right] dA = 1 - 1 = 0.$$

Some solutions are symmetric with respect to an interchange of (x,y) and (X,Y), others are not. According to the following theorem, symmetric ones are preferable, but not essential.

Theorem 13.3 When consistency condition 13.60 is satisfied, the solution of Neumann problem 13.58 is

$$u(x,y) = \iint_A N(x,y;X,Y)F(X,Y)\, dA - \oint_{\beta(A)} N(x,y;X,Y)K(X,Y)\, ds + C, \tag{13.63}$$

where C is an arbitrary constant and $N(x, y; X, Y)$ is a symmetric modified Green's function satisfying 13.62.

Proof In Green's identity 13.14a on A, we let $v = N(x, y; X, Y)$ and $u = u(x, y)$, the solution of problem 13.58,

$$\iint_A (u\nabla^2 N - N\nabla^2 u)\, dA = \oint_{\beta(A)} (u\nabla N - N\nabla u) \cdot \hat{\mathbf{n}}\, ds.$$

Because $\nabla^2 u = F$ in A, $\nabla^2 N = \delta(x - X, y - X) - 1/\text{area}(A)$, and $\partial u/\partial n = K$ and $\partial N/\partial n = 0$ on $\beta(A)$,

$$\iint_A \left\{ u(x, y) \left[\delta(x - X, y - Y) - \frac{1}{\text{area}A} \right] - N(x, y; X, Y)F(x, y) \right\} dA$$
$$= \oint_{\beta(A)} -N(x, y; X, Y)K(x, y)\, ds$$

or,

$$u(X, Y) = \iint_A N(x, y; X, Y)F(x, y)\, dA - \oint_{\beta(A)} N(x, y; X, Y)K(x, y)\, ds + \frac{C_1}{\text{area}(A)},$$

where $C_1 = \iint_A u(x, y)\, dA$. When we interchange (x, y) and (X, Y),

$$u(x, y) = \iint_A N(X, Y; x, y)F(X, Y)\, dA - \oint_{\beta(A)} N(X, Y; x, y)K(X, Y)\, ds + C \quad (13.64)$$
$$= \iint_A N(x, y; X, Y)F(X, Y)\, dA - \oint_{\beta(A)} N(x, y; X, Y)K(X, Y)\, ds + C,$$

where we have replaced $C_1/\text{area}(A)$ by C, since $u(x, y)$ is unique to an additive constant. ∎

If the modified Green's function $N(x, y; X, Y)$ is not symmetric, equation 13.64 must be used for the solution in place of 13.63.

Example 13.8 Use a modified Green's function to solve the following Neumann boundary value problem on a rectangle.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L',$$
$$\frac{\partial V(0, y)}{\partial x} = 0, \quad 0 < y < L',$$
$$\frac{\partial V(L, y)}{\partial x} = 0, \quad 0 < y < L',$$
$$\frac{\partial V(x, 0)}{\partial y} = 0, \quad 0 < x < L,$$
$$\frac{\partial V(x, L')}{\partial y} = f(x), \quad 0 < x < L.$$

Solution Modified Green's functions $N(x, y; X, Y)$ for this problem must satisfy

$$\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} = \delta(x - X, y - Y) - \frac{1}{LL'}, \quad 0 < x < L, \quad 0 < y < L',$$
$$N_x(0, y) = 0, \quad 0 < y < L',$$
$$N_x(L, y) = 0, \quad 0 < y < L',$$
$$N_y(x, 0) = 0, \quad 0 < x < L,$$
$$N_y(x, L') = 0, \quad 0 < x < L.$$

Substitution of a partial eigenfunction expansion,

$$N(x, y; X, Y) = \sum_{n=0}^{\infty} a_n(y) f_n(x) = \frac{a_0(y)}{\sqrt{L}} + \sum_{n=1}^{\infty} a_n(y) \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L},$$

into the PDE gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} -\frac{n^2\pi^2}{L^2} a_n f_n(x) + \sum_{n=0}^{\infty} \frac{d^2 a_n}{dy^2} f_n(x) \\
&\quad = \delta(x - X, y - Y) - \frac{1}{LL'} \\
&\quad = \sum_{n=0}^{\infty} \left\{ \int_0^L \left[\delta(x - X)\delta(y - Y) - \frac{1}{LL'} \right] f_n(x)\, dx \right\} f_n(x) \\
&\quad = \frac{1}{\sqrt{L}} \left[\delta(y - Y) - \frac{1}{L'} \right] f_0(x) + \sum_{n=1}^{\infty} f_n(X)\delta(y - Y) f_n(x).
\end{aligned}$$

This equation, along with the boundary conditions $N_y(x, 0) = 0 = N_y(x, L')$, requires coefficients $a_n(y)$ to satisfy

$$\frac{d^2 a_0}{dy^2} = \frac{1}{\sqrt{L}} \left[\delta(y - Y) - \frac{1}{L'} \right], \quad 0 < y < L',$$
$$a_0'(0) = a_0'(L') = 0,$$

and for $n > 0$,

$$\frac{d^2 a_n}{dy^2} - \frac{n^2\pi^2}{L^2} a_n = f_n(X)\delta(y - Y), \quad 0 < y < L',$$
$$a_n'(0) = a_n'(L') = 0.$$

Because $-y^2/(2\sqrt{L}L')$ is a solution of $d^2 a_0/dy^2 = -1/(\sqrt{L}L')$, we take

$$a_0(y) = \begin{cases} Ay + B - \dfrac{y^2}{2\sqrt{L}L'}, & 0 \le y < Y \\ Dy + C - \dfrac{y^2}{2\sqrt{L}L'}, & Y < y \le L'. \end{cases}$$

Boundary conditions $a_0'(0) = a_0'(L') = 0$, and continuity conditions 12.26a,b from Section 12.3 require

$$A = 0, \quad D - \frac{1}{\sqrt{L}} = 0, \quad AY + B = DY + C, \quad D - A = \frac{1}{\sqrt{L}}.$$

These yield $A = 0$, $D = 1/\sqrt{L}$, and $B = Y/\sqrt{L} + C$, where C is arbitrary, and hence

$$a_0(y) = \begin{cases} \dfrac{Y}{\sqrt{L}} + C - \dfrac{y^2}{2\sqrt{L}L'}, & 0 \le y \le Y \\ \dfrac{y}{\sqrt{L}} + C - \dfrac{y^2}{2\sqrt{L}L'}, & Y \le y \le L'. \end{cases}$$

Since a general solution of the differential equation $d^2a_n/dy^2 - (n^2\pi^2/L^2)a_n = 0$ is $A\cosh(n\pi y/L) + B\sinh(n\pi y/L)$, we take

$$a_n(y) = \begin{cases} A\cosh\dfrac{n\pi y}{L} + B\sinh\dfrac{n\pi y}{L}, & 0 \le y < Y \\ C\cosh\dfrac{n\pi y}{L} + D\sinh\dfrac{n\pi y}{L}, & Y < y \le L. \end{cases}$$

The boundary conditions require

$$\frac{n\pi}{L}B = 0, \qquad C\sinh\frac{n\pi L'}{L} + D\cosh\frac{n\pi L'}{L} = 0,$$

and continuity conditions 12.26a,b necessitate

$$A\cosh\frac{n\pi Y}{L} + B\sinh\frac{n\pi Y}{L} = C\cosh\frac{n\pi Y}{L} + D\sinh\frac{n\pi Y}{L},$$

$$\left(C\sinh\frac{n\pi Y}{L} + D\cosh\frac{n\pi Y}{L}\right) - \left(A\sinh\frac{n\pi Y}{L} + B\cosh\frac{n\pi Y}{L}\right) = \frac{L}{n\pi}f_n(X).$$

These four equations can be solved for

$$A = \frac{-L\cosh\dfrac{n\pi(L'-Y)}{L}f_n(X)}{n\pi\sinh n\pi L'/L)}, \quad B = 0, \quad C = \frac{-L\cosh\dfrac{n\pi L'}{L}\cosh\dfrac{n\pi Y}{L}f_n(X)}{n\pi\sinh(n\pi L'/L)},$$

$$D = \frac{L}{n\pi}\cosh\frac{n\pi Y}{L}f_n(X),$$

and hence

$$a_n(y) = \begin{cases} \dfrac{-L\cosh\dfrac{n\pi(L'-Y)}{L}\cosh\dfrac{n\pi y}{L}f_n(X)}{n\pi\sinh(n\pi L'/L)}, & 0 \le y \le Y \\ \dfrac{-L\cosh\dfrac{n\pi L'}{L}\cosh\dfrac{n\pi Y}{L}\cosh\dfrac{n\pi y}{L}f_n(X)}{n\pi\sinh n\pi L'/L)} + \dfrac{L\cosh\dfrac{n\pi Y}{L}\sinh\dfrac{n\pi y}{L}f_n(X)}{n\pi}, & Y \le y \le L' \end{cases}$$

$$= \begin{cases} \dfrac{-L\cosh\dfrac{n\pi(L'-Y)}{L}\cosh\dfrac{n\pi y}{L}f_n(X)}{n\pi\sinh(n\pi L'/L)}, & 0 \le y \le Y \\ \dfrac{-L\cosh\dfrac{n\pi Y}{L}\cosh\dfrac{n\pi(L'-y)}{L}f_n(X)}{n\pi\sinh(n\pi L'/L)}, & Y \le y \le L'. \end{cases}$$

A modified Green's function is therefore

$$N(x,y;X,Y) = \begin{cases} \dfrac{Y}{L} + \dfrac{C}{\sqrt{L}} - \dfrac{y^2}{2LL'} - \displaystyle\sum_{n=1}^{\infty} \dfrac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi y}{L}\cosh\dfrac{n\pi(L'-Y)}{L}}{n\pi\sinh(n\pi L'/L)}\cos\dfrac{n\pi x}{L}, & 0 \le y \le Y \\ \dfrac{y}{L} + \dfrac{C}{\sqrt{L}} - \dfrac{y^2}{2LL'} - \displaystyle\sum_{n=1}^{\infty} \dfrac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi Y}{L}\cosh\dfrac{n\pi(L'-y)}{L}}{n\pi\sinh(n\pi L'/L)}\cos\dfrac{n\pi x}{L}, & Y \le y \le L'. \end{cases}$$

Because $N(x,y;X,Y)$ is not symmetric, we use equation 13.64 to express the solution of the original boundary value problem as a line integral along the edge $C' : y = L'$,

$$V(x,y) = -\int_{C'} N(X,Y;x,y)f(X)\,ds + D = -\int_0^L N(X,L';x,y)f(X)\,dX + D,$$

where D is an arbitrary constant, and

$$N(X,Y;x,y) = \begin{cases} \dfrac{y}{L} + \dfrac{C}{\sqrt{L}} - \dfrac{Y^2}{2LL'} - \displaystyle\sum_{n=1}^{\infty} \dfrac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi Y}{L}\cosh\dfrac{n\pi(L'-y)}{L}}{n\pi\sinh(n\pi L'/L)}\cos\dfrac{n\pi x}{L}, & 0 \le Y \le y \\ \dfrac{Y}{L} + \dfrac{C}{\sqrt{L}} - \dfrac{Y^2}{2LL'} - \displaystyle\sum_{n=1}^{\infty} \dfrac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi y}{L}\cosh\dfrac{n\pi(L'-Y)}{L}}{n\pi\sinh(n\pi L'/L)}\cos\dfrac{n\pi x}{L}, & y \le Y \le L'. \end{cases}$$

When we use the latter of these expressions to evaluate $N(X,L';x,y)$ along C',

$$V(x,y) = -\int_0^L \left[\frac{L'}{L} + \frac{C}{\sqrt{L}} - \frac{L'}{2L} - \sum_{n=1}^{\infty} \frac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi y}{L}}{n\pi\sinh(n\pi L'/L)}\cos\frac{n\pi x}{L}\right] f(X)\,dX + D.$$

Since $f(x)$ must satisfy the consistency condition

$$\int_0^L f(x)\,dx = 0,$$

this solution reduces to

$$V(x,y) = D + \sum_{n=1}^{\infty} a_n \cosh\frac{n\pi y}{L}\cos\frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{n\pi\sinh(n\pi L'/L)}\int_0^L f(X)\cos\frac{n\pi X}{L}dX.$$

Had the nonhomogeneity been along either of the boundaries $x = 0$ or $x = L$, or both, an eigenfunction expansion for $N(x,y;X,Y)$ in terms of functions $g_0(y) = 1/\sqrt{L'}$ and $g_n(y) = \sqrt{2/L'}\cos(n\pi y/L')$ would have been used (see Exercise 2).●

EXERCISES 13.4

1. The analogue of problem 13.62 in ODEs is problem 12.55. Show that when $L\overline{g} = d^2\overline{g}/dx^2$, $0 < x < L$, and the boundary conditions are Neumann, equation 12.55a becomes

$$\frac{d^2\overline{g}}{dx^2} = \delta(x - X) - \frac{1}{L}.$$

Assume a weight function identically equal to unity.

2. (a) Solve Example 13.8 when boundary conditions along $x = L$, $y = 0$, and $y = L'$ are homogeneous and that along $x = 0$ is $V_x(0, y) = f(y)$, $0 < y < L'$.

(b) Find $V(x, y)$ when $f(y) = \delta(y - L'/4) - \delta(y - 3L'/4)$ and $V(0, L'/2) = 0$. What is the value of $V(x, y)$ at all points on the line $y = L'/2$?

3. What is the solution to Example 13.8 if the boundary condition along $y = 0$ is also nonhomogeneous, $V_y(x, 0) = g(x)$?

4. Verify that the steady-state heat conduction problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\frac{1}{\kappa}, \quad 0 < x < L, \quad 0 < y < L,$$
$$U_x(0, y) = \frac{L}{4\kappa}, \quad 0 < y < L,$$
$$U_x(L, y) = \frac{-L}{4\kappa}, \quad 0 < y < L,$$
$$U_y(x, 0) = \frac{L}{4\kappa}, \quad 0 < x < L,$$
$$U_y(x, L) = \frac{-L}{4\kappa}, \quad 0 < x < L,$$

satisfies consistency condition 13.60 and find its solution.

5. In this problem we develop a modified Green's function for the Neumann problem for Poisson's equation on a circle and solve the corresponding boundary value problem,

$$\nabla^2 u = F(r, \theta), \quad 0 < r < a, \quad -\pi < \theta \le \pi,$$
$$\frac{\partial u(a, \theta)}{\partial r} = K(\theta), \quad -\pi < \theta < \pi.$$

(a) What is the boundary value problem characterizing $N(r, \theta; R, \Theta)$ for this problem?

(b) Using a partial eigenfunction expansion identical to that in Exercise 13(a) of Section 13.2, show that coefficient functions $A_0(r)$, $A_n(r)$, and $B_n(r)$ must satisfy

$$\frac{d^2 A_0}{dr^2} + \frac{1}{r}\frac{dA_0}{dr} = \frac{\delta(r - R)}{\sqrt{2\pi}r} - \frac{\sqrt{2}}{\sqrt{\pi}a^2}, \quad 0 < r < a,$$
$$A_0'(a) = 0;$$
$$\frac{d^2 A_n}{dr^2} + \frac{1}{r}\frac{dA_n}{dr} - \frac{n^2}{r^2}A_n = \frac{\delta(r - R)\cos n\theta}{\sqrt{\pi}r}, \quad 0 < r < a,$$
$$A_n'(a) = 0;$$
$$\frac{d^2 B_n}{dr^2} + \frac{1}{r}\frac{dB_n}{dr} - \frac{n^2}{r^2}B_n = \frac{\delta(r - R)\sin n\theta}{\sqrt{\pi}r}, \quad 0 < r < a,$$
$$B_n'(a) = 0.$$

(c) Solve the equations in part (b) and hence show that

$$N(r,\theta;R,\Theta)=\begin{cases}\dfrac{A}{\sqrt{2\pi}}-\dfrac{r^2}{4\pi a^2}-\displaystyle\sum_{n=1}^{\infty}\frac{1}{2\pi n}\left[\left(\frac{rR}{a^2}\right)^n+\left(\frac{r}{R}\right)^n\right]\cos n(\theta-\Theta), & 0\le r\le R\\ \dfrac{A}{\sqrt{2\pi}}+\dfrac{\ln(r/R)}{2\pi}-\dfrac{r^2}{4\pi a^2}-\displaystyle\sum_{n=1}^{\infty}\frac{1}{2\pi n}\left[\left(\frac{rR}{a^2}\right)^n+\left(\frac{R}{r}\right)^n\right]\cos n(\theta-\Theta), & R\le r\le a\end{cases}$$

where A is independent of r and θ.

(d) Use formula 13.33 in Section 13.2 to simplify the modified Green's function to

$$N(r,\theta;R,\Theta)=\frac{A}{\sqrt{2\pi}}-\frac{r^2}{4\pi a^2}+\frac{1}{4\pi}\ln\left\{\frac{[r^2+R^2-2rR\cos(\theta-\Theta)][a^4+r^2R^2-2ra^2R\cos(\theta-\Theta)]}{a^4R^2}\right\}.$$

(e) Find an integral representation for the solution of the boundary value problem in part (a).

6. (a) To satisfy consistency condition 13.60, it is possible to change the boundary condition defining modified Green's functions instead of the PDE. Show that the function $\overline{N}(x,y;X,Y)$ defined by

$$\nabla^2\overline{N}=\delta(x-X,y-Y),\quad (x,y)\text{ in }A,$$
$$\frac{\partial\overline{N}}{\partial n}=\frac{1}{L},\quad (x,y)\text{ on }\beta(A),$$

where L is the length of $\beta(A)$, satisfies 13.60.

(b) Find the solution of problem 13.58 in terms of $\overline{N}(x,y;X,Y)$.

7. Use a modified Green's function of Exercise 6 to find the solution of the problem in Exercise 5.

8. (a) Show that when functions $F(x,y,z)$ and $K(x,y,z)$ for the three-dimensional Neumann problem for Poisson's equation

$$\nabla^2 u=F(x,y,z),\quad (x,y,z)\text{ in }V,$$
$$\frac{\partial u}{\partial n}=K(x,y,z),\quad (x,y,z)\text{ on }\beta(V),$$

satisfy the consistency condition in Exercise 9 of Section 2.1, then the solution of the boundary value problem is

$$u(x,y,z)=\iiint_V N(X,Y,Z;x,y,z)F(X,Y,Z)\,dV-\iint_{\beta(V)}N(X,Y,Z;x,y,z)K(X,Y,Z)\,dS+C,$$

where C is an arbitrary constant and $N(x,y,z;X,Y,Z)$ is a modified Green's function satisfying

$$\nabla^2 N=\delta(x-X,y-Y,z-Z)-\frac{1}{\text{volume}(V)},\quad (x,y,z)\text{ in }V,$$
$$\frac{\partial N}{\partial n}=0,\quad (x,y,z)\text{ on }\beta(V).$$

(b) What is the solution when $N(x,y,z;X,Y,Z)$ is symmetric?

9. What is the three-dimensional analogue of Exercise 6?

10. (a) The Neumann problem for the Helmholtz equation is

$$(\nabla^2 + k^2)u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.65a}$$

$$\frac{\partial u}{\partial n} = K(x,y), \quad (x,y) \text{ on } \beta(A). \tag{13.65b}$$

The homogeneous system

$$(\nabla^2 + k^2)u = 0, \quad (x,y) \text{ in } A, \tag{13.66a}$$

$$\frac{\partial u}{\partial n} = 0, \quad (x,y) \text{ on } \beta(A), \tag{13.66b}$$

has nontrivial solutions. (This is clear when $k = 0$, since $u =$ constant is a solution, and it is also true when $k \neq 0$.) As a result, problem 13.65 does not have a unique solution; if $u(x,y)$ is a solution, then so also is $u(x,y) + Cw(x,y)$, where $w(x,y)$ is any solution of 13.66. In addition, $F(x)$ and $K(x,y)$ must satisfy a consistency condition for there to be a solution of 13.65 at all. Problem 13.65 has solutions if and only if

$$\iint_A w(x,y)F(x,y)\,dA = \oint_{\beta(A)} w(x,y)K(x,y)\,ds \tag{13.67}$$

for every solution $w(x,y)$ of 13.66. Prove the necessity of this condition.

(b) Show that when the consistency condition is satisfied, the solution of problem 13.65 is

$$u(x,y) = \iint_A N(x,y;X,Y)F(X,Y)\,dA - \oint_{\beta(A)} N(x,y;X,Y)K(X,Y)\,ds + Cw(x,y), \tag{13.68}$$

where $w(x,y)$ is the normalized solution of 13.66, C is an arbitrary constant, and $N(x,y;X,Y)$ is a symmetric modified Green's function satisfying

$$(\nabla^2 + k^2)N = \delta(x-X, y-Y) - w(x,y)w(X,Y), \quad (x,y) \text{ in } A, \tag{13.69a}$$

$$\frac{\partial N}{\partial n} = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.69b}$$

11. State and prove the three-dimensional analogue of Exercise 10.

§13.5 Robin and Mixed Boundary Value Problems on Finite Regions

The Robin problem for Poisson's equation is

$$\nabla^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.70a}$$

$$l\frac{\partial u}{\partial n} + hu = K(x,y), \quad (x,y) \text{ on } \beta(A). \tag{13.70b}$$

Its solution can be represented in integral form in terms of the nonhomogeneities and the Green's function for the problem.

Theorem 13.4 The solution of problem 13.70 is

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA - \frac{1}{l}\oint_{\beta(A)} G(x,y;X,Y)K(X,Y)\,ds, \tag{13.71}$$

where $G(x,y;X,Y)$ satisfies

$$\nabla^2 G = \delta(x-X, y-Y), \quad (x,y) \text{ in } A, \tag{13.72a}$$

$$l\frac{\partial G}{\partial n} + hG = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.72b}$$

Proof If in Green's identity 13.14a on A we let $v = G(x,y;X,Y)$ and let $u(x,y)$ be the solution of 13.70,

$$\iint_A (u\nabla^2 G - G\nabla^2 u)\,dA = \oint_{\beta(A)} (u\nabla G - G\nabla u)\cdot \hat{\mathbf{n}}\,ds.$$

Because $\nabla^2 G = \delta(x-X, y-Y)$, $\nabla^2 u = F$, and $l\partial u/\partial n + hu = K$ and $l\partial G/\partial n + hG = 0$ on $\beta(A)$,

$$\iint_A [u(x,y)\delta(x-X,y-Y) - G(x,y;X,Y)F(x,y)]\,dA$$
$$= \oint_{\beta(A)} \left\{\frac{u(x,y)}{l}[-hG(x,y;X,Y)] - \frac{G(x,y;X,Y)}{l}[K(x,y) - hu(x,y)]\right\} ds$$

or,

$$u(X,Y) = \iint_A G(x,y;X,Y)F(x,y)\,dA - \frac{1}{l}\oint_{\beta(A)} G(x,y;X,Y)K(x,y)\,ds.$$

When we interchange (x,y) and (X,Y),

$$u(x,y) = \iint_A G(X,Y;x,y)F(X,Y)\,dA - \frac{1}{l}\oint_{\beta(A)} G(X,Y;x,y)K(X,Y)\,ds$$
$$= \iint_A G(x,y;X,Y)F(X,Y)\,dA - \frac{1}{l}\oint_{\beta(A)} G(x,y;X,Y)K(X,Y)\,ds,$$

since $G(x,y;X,Y)$ must be symmetric (see Exercise 1). ∎

Because $G = -(l/h)\partial G/\partial n$ on $\beta(A)$, we may also express the solution in the form

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \frac{1}{h}\oint_{\beta(A)} \frac{\partial G(x,y;X,Y)}{\partial N}K(X,Y)\,ds, \tag{13.73}$$

where once again $\partial G/\partial N$ indicates the outward normal derivative of G with respect to the (X, Y) variables.

A boundary value problem is said to be **mixed** if all parts of the boundary are not subjected to the same type of condition. For instance, the unknown function may have to satisfy a Dirichlet condition on part of the boundary and a Neumann condition on the remainder.

Example 13.9 Find an integral representation for the solution of the following mixed boundary value problem on a semicircle:

$$\begin{aligned}
\nabla^2 u &= F(r,\theta), \quad 0 < r < a, \quad 0 < \theta < \pi, \\
u(a,\theta) &= K_1(\theta), \quad 0 < \theta < \pi, \\
\frac{\partial u(r,0)}{\partial \theta} &= 0, \quad 0 < r < a, \\
\frac{\partial u(r,\pi)}{\partial \theta} &= 0, \quad 0 < r < a.
\end{aligned}$$

Solution The Green's function for this problem is

$$G(r,\theta;R,\Theta) = \frac{1}{4\pi} \ln \left\{ a^4 \frac{[r^2+R^2-2Rr\cos(\theta+\Theta)][r^2+R^2-2Rr\cos(\theta-\Theta)]}{[R^2r^2+a^4-2a^2rR\cos(\theta+\Theta)][R^2r^2+a^4-2a^2rR\cos(\theta-\Theta)]} \right\}$$

(see Exercise 2). To solve the boundary value problem, we apply identity 13.14a to the semicircle, denoted by A, with $v = G$ and $u = u(r,\theta)$, the solution of the problem,

$$\iint_A (G\nabla^2 u - u\nabla^2 G)\, dA = \oint_{\beta(A)} (G\nabla u - u\nabla G)\cdot \hat{\mathbf{n}}\, ds.$$

With $\nabla^2 G = \delta(r-R, \theta-\Theta)/r$, $\nabla^2 u = F$, and the boundary conditions for G and u,

$$\iint_A \left[G(r,\theta;R,\Theta)F(r,\theta) - u(r,\theta)\frac{\delta(r-R,\theta-\Theta)}{r} \right] r\, dr\, d\theta = \int_0^\pi -K_1(\theta)\frac{\partial G(a,\theta;R,\Theta)}{\partial r} a\, d\theta$$

or

$$u(R,\Theta) = \int_0^\pi \int_0^a G(r,\theta;R,\Theta)F(r,\theta) r\, dr\, d\theta + \int_0^\pi a K_1(\theta)\frac{\partial G(a,\theta;R,\Theta)}{\partial r}\, d\theta.$$

When we interchange (r,θ) and (R,Θ), and note the symmetry in G,

$$u(r,\theta) = \int_0^\pi \int_0^a G(r,\theta;R,\Theta)F(R,\Theta)\, R\, dR\, d\Theta + \int_0^\pi a K_1(\Theta)\frac{\partial G(r,\theta;a,\Theta)}{\partial R}\, d\Theta.$$

EXERCISES 13.5

1. Verify that the Green's function for the Robin problem is symmetric.

2. Show that the Green's function for the boundary value problem of Example 13.9 is

$$G(r,\theta;R,\Theta) = \frac{1}{4\pi} \ln \left\{ a^4 \frac{[r^2+R^2-2Rr\cos(\theta+\Theta)][r^2+R^2-2Rr\cos(\theta-\Theta)]}{[R^2r^2+a^4-2a^2rR\cos(\theta+\Theta)][R^2r^2+a^4-2a^2rR\cos(\theta-\Theta)]} \right\}.$$

3. Use a Green's function to find an integral representation for the solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y), \quad 0 < x < L, \quad 0 < y < L,$$
$$u(0,y) = f(y), \quad 0 < y < L,$$
$$u(L,y) = 0, \quad 0 < y < L,$$
$$u_y(x,0) = 0, \quad 0 < x < L,$$
$$u(x,L) = 0, \quad 0 < x < L.$$

4. Show that the solution of the Robin problem

$$(\nabla^2 + k^2)u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.74a}$$
$$l\frac{\partial u}{\partial n} + hu = K(x,y), \quad (x,y) \text{ on } \beta(A), \tag{13.74b}$$

is given by 13.71 when $G(x,y;X,Y)$ is the associated Green's function.

5. Show that the solution of the three-dimensional Robin problem

$$\nabla^2 u = F(x,y,z), \quad (x,y,z) \text{ in } V, \tag{13.75a}$$
$$l\frac{\partial u}{\partial n} + hu = K(x,y,z), \quad (x,y,z) \text{ on } \beta(V), \tag{13.75b}$$

may be expressed in either of the forms

$$u(x,y,z) = \iiint_V G(x,y,z;X,Y,Z)F(X,Y,Z)\,dV - \frac{1}{l}\iint_{\beta(V)} G(x,y,z;X,Y,Z)K(X,Y,Z)\,dS,$$

and

$$u(x,y,z) = \iiint_V G(x,y,z;X,Y,Z)F(X,Y,Z)\,dV + \frac{1}{h}\iint_{\beta(V)} \frac{\partial G(x,y,z;X,Y,Z)}{\partial N}K(X,Y,Z)\,dS.$$

§13.6 Green's Functions and Unbounded Regions

In this section we use the free space Green's functions of Section 13.2 to solve boundary value problems on unbounded domains. As is usual, the procedure has two steps, find the appropriate Green's function and then use Green's second identity to find the solution of the boundary value problem in terms of the Green's function and nonhomogeneities. Because integrations take place over infinite regions, it is necessary to place restrictions on the behaviour of solutions far from sources. To determine this behaviour for two-dimensional problems, we consider the boundary value problem

$$\nabla^2 u = F(r, \theta), \quad 0 < r < \infty, \quad -\pi \le \theta \le \pi. \tag{13.76}$$

In Green's identity 13.14a, we let $u(r, \theta)$ be the solution of this problem and $v = G(r, \theta; R, \Theta)$ be the two-dimensional Green's function for the Laplacian in Table 13.1 of Section 13.2, and choose A to be a circle of radius a,

$$\iint_A (u\nabla^2 G - G\nabla^2 u)\, dA = \oint_{\beta(A)} (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, ds.$$

Since $\nabla^2 u = F(r, \theta)$ and $\nabla^2 G = (1/r)\delta(r - R)\delta(\theta - \Theta)$,

$$\iint_A \left[u(r, \theta)\frac{1}{r}\delta(r - R)\delta(\theta - \Theta) - G(r, \theta; R, \Theta)F(r, \theta)\right] r\, dr\, d\theta$$
$$= \int_{-\pi}^{\pi} \left[u(a, \theta)\frac{\partial G(a, \theta; R, \Theta)}{\partial r} - G(a, \theta; R, \Theta)\frac{\partial u(a, \theta)}{\partial r}\right] a\, d\theta.$$

Thus,

$$u(R, \Theta) = \iint_A G(r, \theta; R, \Theta)F(r, \theta)\, dA$$
$$+ \int_{-\pi}^{\pi} \left[u(a, \theta)\frac{\partial G(a, \theta; R, \Theta)}{\partial r} - G(a, \theta; R, \Theta)\frac{\partial u(a, \theta)}{\partial r}\right] a\, d\theta. \tag{13.77}$$

We require the second integral to vanish as $a \to \infty$. This will be the case if

$$\lim_{a\to\infty} a \left[u(a, \theta)\frac{\partial G(a, \theta; R, \Theta)}{\partial r} - G(a, \theta; R, \Theta)\frac{\partial u(a, \theta)}{\partial r}\right] = 0.$$

Since a is arbitrary, we can rewrite this requirement as

$$\lim_{r\to\infty} r\left(u\frac{\partial G}{\partial r} - G\frac{\partial u}{\partial r}\right) = 0.$$

If we now substitute $G(r, \theta; R, \Theta) = \dfrac{1}{4\pi}\ln[r^2 + R^2 - 2rR\cos(\theta - \Theta)]$,

$$0 = \lim_{r\to\infty} \frac{r}{4\pi}\left\{u\left[\frac{2r - 2R\cos(\theta - \Theta)}{r^2 + R^2 - 2rR\cos(\theta - \Theta)}\right] - \frac{\partial u}{\partial r}\ln[r^2 + R^2 - 2rR\cos(\theta - \Theta)]\right\}$$
$$= \frac{1}{4\pi}\lim_{r\to\infty}\left\{2u - r\frac{\partial u}{\partial r}\ln r^2\left[1 + \frac{R^2}{r^2} - \frac{2R}{r}\cos(\theta - \Theta)\right]\right\}$$
$$= \frac{1}{2\pi}\lim_{r\to\infty}\left(u - r\ln r\frac{\partial u}{\partial r}\right).$$

Thus, to guarantee vanishing of the terms in equation 13.77 along the artificial boundary $r = a$, we assume that $u(r, \theta)$ satisfies the condition

$$\lim_{r\to\infty}\left(u - r\ln r\frac{\partial u}{\partial r}\right) = 0. \tag{13.78}$$

When this is the case, the solution can be obtained from equation 13.77 by taking limits as $a \to \infty$, interchanging r, θ with R, Θ, and using symmetry of the Green's function,

$$u(r,\theta) = \iint_A G(r,\theta;R,\Theta)F(R,\Theta)\,dA, \tag{13.79}$$

where A now represents the upper half of the xy-plane. Convergence of this improper integral also has implications regarding the behaviour of $F(r, \theta)$ at infinity.

In Exercise 1, it is shown that the analogue of condition 13.78 for three-dimensional problems is

$$\lim_{r\to\infty}\left(u + r\frac{\partial u}{\partial r}\right) = 0. \tag{13.80}$$

We can now proceed to solve other boundary value problems on unbounded domains.

Example 13.10 Use Green's functions to find an integral representation for the solution of the half-plane Dirichlet problem

$$\nabla^2 u = F(x,y), \quad -\infty < x < \infty, \quad y > 0,$$
$$u(x,0) = K(x), \quad -\infty < x < \infty.$$

Solution According to Table 13.1, the infinite space Green's function for the two-dimen-sional Laplacian is $(2\pi)^{-1}\ln\sqrt{(x-X)^2+(y-Y)^2}$. To eliminate its undesirable effect along the x-axis, we introduce a unit negative source at the point $(X, -Y)$. The addition of these sources results in value zero along the x-axis; that is, using the method of images, the Green's function for the half-space problem is

$$\begin{aligned} G(x,y;X,Y) &= \frac{1}{2\pi}\ln\sqrt{(x-X)^2+(y-Y)^2} - \frac{1}{2\pi}\ln\sqrt{(x-X)^2+(y+Y)^2} \\ &= \frac{1}{4\pi}\ln\left[\frac{(x-X)^2+(y-Y)^2}{(x-X)^2+(y+Y)^2}\right]. \end{aligned}$$

We now set set $v = G(x, y; X, Y)$ and let $u(x, y)$ be the solution of the boundary value problem in Green's second identity 13.14a, and choose the region A to be the semicircle $0 \le r \le a$, $0 \le \theta \le \pi$,

$$\iint_A (u\nabla^2 G - G\nabla^2 u)\,dA = \oint_{\beta(A)} (u\nabla G - G\nabla u)\cdot\hat{\mathbf{n}}\,ds.$$

Interior to region A, we have $\nabla^2 G = \delta(x - X, y - Y)$ and $\nabla^2 u = F(x, y)$. If Γ denotes the semi-circular part of $\beta(A)$, then

$$\begin{aligned} \iint_A [u(x,y)\delta(x-X,y-Y) - G(x,y;X,Y)F(x,y)]\,dA &= \int_{-a}^{a} K(x)\left[-\frac{\partial G(x,0;X,Y)}{\partial y}\right]dx \\ &\quad + \int_\Gamma (u\nabla G - G\nabla u)\cdot\hat{\mathbf{n}}\,ds. \end{aligned}$$

If we assume that $u(x,y)$ satisfies condition 13.78, then the line integral along Γ vanishes as $a \to \infty$. With $\dfrac{\partial G(x,0;X,Y)}{\partial y} = \dfrac{-Y}{\pi[(x-X)^2+Y^2]}$,

$$u(X,Y) = \iint_A G(x,y;X,Y)F(x,y)\,dy\,dx - \int_{-\infty}^{\infty} K(x)\left\{\frac{-Y}{\pi[(x-X)^2+Y^2]}\right\}dx,$$

where A now represents the upper half of the xy-plane. When we interchange (x,y) and (X,Y) and use the symmetry of $G(x,y;X,Y)$,

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dY\,dX + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{K(X)}{(x-X)^2+y^2}dX.$$

When $F(x,y) = 0$, this is Poisson's integral formula of Exercise 8 in Section 11.4.•

EXERCISES 13.6

1. Show that for solutions of Poisson's equation in three-space to have vanishing "boundary terms", they must satisfy condition 13.80.

2. Solve the boundary value problem of Example 13.10 when the boundary condition along $y = 0$ is Neumann, namely $\partial u(x,0)/\partial y = K(x)$.

3. (a) Solve the Dirichlet boundary value problem exterior to a circle,

$$\nabla^2 u = F(r,\theta), \quad r > a, \quad -\pi \le \theta \le \pi,$$
$$u(a,\theta) = K(\theta), \quad -\pi < \theta \le \pi.$$

Hint: First convince yourself that the Green's function of Exercise 10 in Section 13.2 for the interior of a circle is also that for the exterior of the circle.

(b) Does the solution become Poisson's integral formula of Exercise 13 in Section 6.3 when $F(r,\theta) = 0$?

4. (a) Solve the first-quadrant Dirichlet problem

$$\nabla^2 u = F(x,y), \quad 0 < x < \infty, \quad 0 < y < \infty,$$
$$u(x,0) = K(x), \quad 0 < x < \infty,$$
$$u(0,y) = H(y), \quad 0 < y < \infty.$$

(b) Does the solution become that of part (e) in Exercise 24 in Section 11.4 when $F(x,y) = 0$?

5. (a) Repeat part (a) of Exercise 4 if $u(x,y)$ must satisfy the Neumann boundary condition $\partial u(x,0)/\partial y = K(x)$ along $y = 0$.

(b) Does the solution become that of Example 11.22 in Section 11.4 when $F(x,y) = 0$?

6. Solve the Dirichlet problem

$$\nabla^2 u = F(r,\phi,\theta), \quad r > a, \quad 0 < \phi < \pi, \quad -\pi < \theta \le \pi,$$
$$u(a,\phi,\theta) = K(\phi,\theta), \quad 0 < \phi < \pi, \quad -\pi < \theta \le \pi,$$

exterior to a sphere of radius a. Does the result become that of Exercise 44 in Section 9.1 when $F(r,\phi,\theta) = 0$?

7. Solve the half-space Dirichlet problem

$$\nabla^2 u = F(x,y,z), \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad z > 0,$$
$$u(x,y,0) = K(x,y), \quad -\infty < x < \infty, \quad -\infty < y < \infty.$$

8. Solve the half-space Neumann problem

$$\nabla^2 u = F(x,y,z), \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad z > 0,$$
$$u_z(x,y,0) = K(x,y), \quad -\infty < x < \infty, \quad -\infty < y < \infty.$$

9. Solve the infinite strip Dirichlet problem

$$\nabla^2 u = F(x,y), \quad -\infty < x < \infty, \quad 0 < y < L',$$
$$u(x,0) = K(x), \quad -\infty < x < \infty,$$
$$u(x,L') = H(x), \quad -\infty < x < \infty.$$

10. Repeat Exercise 9 if the boundary conditions are Neumann.

In Exercises 11–12 use the method of images to find the Green's function for Poisson's equation on the region described with the given boundary conditions.

11. The semi-infinite strip $0 < x < \infty$, $0 < y < L'$, with:
(a) Dirichlet conditions on all three edges $y = 0$, $y = L'$, and $x = 0$;
(b) Dirichlet conditions on edges $y = 0$ and $y = L'$, and a Neumann condition along $x = 0$;
(c) Neumann conditions along $y = 0$ and $y = L'$, and a Dirichlet condition along $x = 0$.

12. The infinite slab $-\infty < x < \infty$, $-\infty < y < \infty$, $0 < z < L''$ with:
(a) Dirichlet conditions on both faces $z = 0$ and $z = L''$;
(b) Neumann conditions on both faces $z = 0$ and $z = L''$;

§13.7 Green's Functions for Heat Conduction Problems

Green's functions can also be defined for initial boundary value problems; they encompass the character of Green's functions for boundary value problems and also the *causal* features of the initial value problems in Section 12.6.

The causal Green's function for the one-dimensional heat conduction problem

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{k}\frac{\partial U}{\partial t} - \frac{g(x,t)}{\kappa}, \quad 0 < x < L, \quad t > 0, \tag{13.81a}$$

$$-l_1 \frac{\partial U}{\partial x} + h_1 U = f_1(t), \quad x = 0, \quad t > 0, \tag{13.81b}$$

$$l_2 \frac{\partial U}{\partial x} + h_2 U = f_2(t), \quad x = L, \quad t > 0, \tag{13.81c}$$

$$U(x,0) = f(x), \quad 0 < x < L, \tag{13.81d}$$

is defined as the solution of the corresponding problem with homogeneous initial and boundary conditions when a unit of heat is inserted at position X and time T,

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{k}\frac{\partial U}{\partial t} - \frac{\delta(x-X)\delta(t-T)}{\kappa}, \quad 0 < x < L, \quad t > T, \tag{13.82a}$$

$$-l_1 \frac{\partial U}{\partial x} + h_1 U = 0, \quad x = 0, \quad t > T, \tag{13.82b}$$

$$l_2 \frac{\partial U}{\partial x} + h_2 U = 0, \quad x = L, \quad t > T, \tag{13.82c}$$

$$U(x,t;X,T) = 0, \quad 0 < x < L, \quad t < T. \tag{13.82d}$$

For $t > T$, it can also be characterized as the solution of

$$\frac{\partial^2 G}{\partial x^2} = \frac{1}{k}\frac{\partial G}{\partial t}, \quad 0 < x < L, \quad t > T, \tag{13.83a}$$

$$-l_1 \frac{\partial G}{\partial x} + h_1 G = 0, \quad x = 0, \quad t > T, \tag{13.83b}$$

$$l_2 \frac{\partial G}{\partial x} + h_2 G = 0, \quad x = L, \quad t > T, \tag{13.83c}$$

$$G(x,T+;X,T) = \frac{k}{\kappa}\delta(x-X), \quad 0 < x < L; \tag{13.83d}$$

that is, the solution of 13.82 is $h(t-T)G(x,t;X,T)$ when $G(x,t;X,T)$ satisfies 13.83. What this means is that the effect of a unit heat source at position X and time T on a rod with zero temperature is equivalent to the effect of suddenly raising the temperature of the rod at point X to k/κ at time T. The causal Green's function for 13.81 is $h(t-T)G(x,t;X,T)$, where $G(x,t;X,T)$ satisfies 13.83. In essence, then, $G(x,t;X,T)$ is the causal Green's function for problem 13.81; we must simply remember to set it equal to zero for $t < T$. Because of this, we shall customarily call $G(x,t;X,T)$, itself, the **causal Green's function**.

Example 13.11 Find the causal Green's function for problem 13.81 in the case that $l_1 = 0 = h_2$.

Solution Separation of variables on problem 13.83 with $l_1 = h_2 = 0$ leads, for $t > T$, to a solution of the form

$$G(x,t;X,T) = \sum_{n=1}^{\infty} C_n e^{-(2n-1)^2\pi^2 kt/(4L^2)} f_n(x),$$

where $f_n(x) = \sqrt{2/L}\sin[(2n-1)\pi x/(2L)]$. If $\delta(x-X)$ is given an eigenfunction expansion in terms of the $\{f_n(x)\}$, the initial condition requires

$$\begin{aligned}\sum_{n=1}^{\infty} C_n e^{-(2n-1)^2\pi^2 kT/(4L^2)} f_n(x) &= \frac{k}{\kappa}\sum_{n=1}^{\infty}\left[\int_0^L \delta(x-X) f_n(x)\,dx\right] f_n(x)\\ &= \frac{k}{\kappa}\sum_{n=1}^{\infty} f_n(X) f_n(x).\end{aligned}$$

It follows, then, that

$$C_n e^{-(2n-1)^2\pi^2 kT/(4L^2)} = \frac{k}{\kappa} f_n(X)$$

and

$$\begin{aligned}G(x,t;X,T) &= \sum_{n=1}^{\infty} \frac{k}{\kappa} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)} f_n(X) f_n(x)\\ &= \frac{2k}{\kappa L}\sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)} \sin\frac{(2n-1)\pi X}{2L}\sin\frac{(2n-1)\pi x}{2L}. \bullet\end{aligned}$$

The solution of problem 13.81 can be expressed in terms of the causal Green's function for the problem as follows,

$$\begin{aligned}U(x,t) = &\int_0^t\int_0^L G(x,t;X,T) g(X,T)\,dX\,dT + \frac{\kappa}{k}\int_0^L G(x,t;X,0) f(X)\,dX\\ &+\kappa\int_0^t\left[G(x,t;L,T)\frac{f_2(T)}{l_2} + G(x,t;0,T)\frac{f_1(T)}{l_1}\right] dT. \qquad (13.84a)\end{aligned}$$

The first term is the contribution of the internal heat source from $t=0$ to present time, the second term is due to the initial temperature distribution in the rod, and the last integral represents the effects of heat transfer at the ends of the rod. Boundary conditions 13.83b,c can be used to rewrite the last integral in the form

$$\begin{aligned}U(x,t) = &\int_0^t\int_0^L G(x,t;X,T) g(X,T)\,dX\,dT + \frac{\kappa}{k}\int_0^L G(x,t;X,0) f(X)\,dX\\ &+\kappa\int_0^t\left[-\frac{\partial G(x,t;L,T)}{\partial X}\frac{f_2(T)}{h_2} + \frac{\partial G(x,t;0,T)}{\partial X}\frac{f_1(T)}{h_1}\right] dT \qquad (13.84b)\end{aligned}$$

(see Exercise 11). This form must be used when $l_1 = l_2 = 0$.

Example 13.12 Solve the heat condution problem in Example 7.3 of Section 7.2.

Solution The Green's function for this problem was obtained in the previous example. With $g(x,t)\equiv 0$, and $f_2(t)$ replaced by $-f_2(t)/\kappa$, we use formulas 13.84a,b to write

$$U(x,t) = \frac{\kappa}{k}\int_0^L G(x,t;X,0)f(X)\,dX + \kappa\int_0^t \left[-G(x,t;L,T)\frac{f_2(T)}{\kappa l_2} + \frac{\partial G(x,t;0,T)}{\partial X}\frac{f_1(T)}{h_1}\right]dT$$
$$= \frac{\kappa}{k}\int_0^L \left[\frac{2k}{\kappa L}\sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 kt/(4L^2)} \sin\frac{(2n-1)\pi X}{2L}\sin\frac{(2n-1)\pi x}{2L}\right] f(X)\,dX$$
$$- \int_0^t \left[\frac{2k}{\kappa L}\sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)} \sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{2L}\right] f_2(T)\,dT$$
$$+ \kappa\int_0^t \left[\frac{2k}{\kappa L}\sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)} \left(\frac{(2n-1)\pi}{2L}\right)\sin\frac{(2n-1)\pi x}{2L}\right] f_1(T)\,dT.$$

When we interchange orders of summation and integration,

$$U(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(X)\sin\frac{(2n-1)\pi X}{2L}dX\right] e^{-(2n-1)^2\pi kt/(4L^2)}\sin\frac{(2n-1)\pi x}{2L}$$
$$+ \frac{2k}{L}\sum_{n=1}^{\infty}\left\{\int_0^t \left[\left(\frac{(2n-1)\pi}{2L}\right)f_1(T) + \frac{(-1)^n}{\kappa}f_2(T)\right]\right.$$
$$\left.\times\ e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)}dT\right\}\sin\frac{(2n-1)\pi x}{2L},$$

and this is solution 7.46 in Section 7.2.●

The causal Green's function for the two-dimensional heat conduction problem

$$\nabla^2 U = \frac{1}{k}\frac{\partial U}{\partial t} - \frac{g(x,y,t)}{\kappa}, \quad (x,y) \text{ in } A, \quad t > 0, \tag{13.85a}$$
$$l\frac{\partial U}{\partial n} + hU = F(x,y,t), \quad (x,y) \text{ on } \beta(A), \quad t > 0, \tag{13.85b}$$
$$U(x,y,0) = f(x,y), \quad (x,y) \text{ in } A, \tag{13.85c}$$

is defined as the solution of

$$\nabla^2 U = \frac{1}{k}\frac{\partial U}{\partial t} - \frac{\delta(x-X,y-Y)\delta(t-T)}{\kappa}, \quad (x,y) \text{ in } A, \quad t > T, \tag{13.86a}$$
$$l\frac{\partial U}{\partial n} + hU = 0, \quad (x,y) \text{ on } \beta(A), \quad t > T, \tag{13.86b}$$
$$U(x,y,t;X,Y,T) = 0 \quad (x,y) \text{ in } A, \quad t < T. \tag{13.86c}$$

It is also given by $h(t-T)G(x,y,t;X,Y,T)$ where $G(x,y,t;X,Y,T)$ satisfies

$$\nabla^2 G = \frac{1}{k}\frac{\partial G}{\partial t}, \quad (x,y) \text{ in } A, \quad t > T, \tag{13.87a}$$
$$l\frac{\partial G}{\partial n} + hG = 0, \quad (x,y) \text{ on } \beta(A), \quad t > T, \tag{13.87b}$$
$$G(x,y,T+;X,Y,T) = \frac{k}{\kappa}\delta(x-X,y-Y) \quad (x,y) \text{ in } A. \tag{13.87c}$$

The solution of problem 13.85 can then be expressed in the form

$$U(x,y,t) = \int_0^t \iint_A G(x,y,t;X,Y,T)g(X,Y,T)\,dA\,dT + \frac{\kappa}{k}\iint_A G(x,y,t;X,Y,0)f(X,Y)\,dA + \frac{\kappa}{l}\int_0^t \oint_{\beta(A)} G(x,y,t;X,Y,T)F(X,Y,T)\,ds\,dT \qquad (13.88a)$$

or

$$U(x,y,t) = \int_0^t \iint_A G(x,y,t;X,Y,T)g(X,Y,T)\,dA\,dT + \frac{\kappa}{k}\iint_A G(x,y,t;X,Y,0)f(X,Y)\,dA - \frac{\kappa}{h}\int_0^t \oint_{\beta(A)} \frac{\partial G(x,y,t;X,Y,T)}{\partial N}F(X,Y,T)\,ds\,dT. \qquad (13.88b)$$

EXERCISES 13.7

In Exercises 1–4 find the causal Green's function for problem 13.81 when the values for l_1, l_2, h_1, and h_2 are as specified.

1. $l_1 = l_2 = 0$, $h_1 = h_2 = 1$

2. $h_1 = h_2 = 0$, $l_1 = l_2 = 1$

3. $l_2 = h_1 = 0$, $l_1 = h_2 = 1$

4. $l_1 = 0$, $h_1 = 1$, $l_2h_2 \neq 0$

In Exercises 5–9 use formulas 13.84a,b to solve the initial boundary value problem.

5. Exercise 8 in Section 4.3

6. Exercise 16 in Section 4.3

7. Exercise 1 in Section 7.2

8. Exercise 7 in Section 7.2

9. Exercise 15 in Section 7.2

10. (a) What is the causal Green's function for problem 13.81?
(b) Use the representation in part (a) to show that $G(x,t;X,T)$ satisfies the *reciprocity principle*

$$G(x,t;X,T) = G(X,t;x,T).$$

What does this mean physically?
(c) Use the representation in part (a) to show that $G(x,t;X,T)$ satisfies the *time-translation* property

$$G(x,t-\overline{T};x,T) = G(x,t;X,T+\overline{T})$$

provided $t - T - \overline{T} > 0$. What does this mean physically?

11. Use the result in Exercise 10(b) to show that solution 13.84a can be expressed in form 13.84b.

In Exercises 12–15 use formulas 13.88a,b to solve the two-dimensional heat conduction problem.

12. Exercise 1 in Section 7.3

13. Exercise 2(a) in Section 7.3

14. Exercise 2(a) in Section 9.2

15. Parts (a) and (b) of Exercise 3 in Section 9.2

16. What are the three-dimensional analogues of equations 13.85–13.88?

§13.8 Green's Functions for the Wave Equation

The causal Green's function $G(x,t;X,T)$ for the one-dimensional vibration problem

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{F(x,t)}{\rho c^2}, \quad 0 < x < L, \quad t > 0, \tag{13.89a}$$

$$-l_1 \frac{\partial y}{\partial x} + h_1 y = f_1(t), \quad x = 0, \quad t > 0, \tag{13.89b}$$

$$l_2 \frac{\partial y}{\partial x} + h_2 y = f_2(t), \quad x = L, \quad t > 0, \tag{13.89c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{13.89d}$$

$$y_t(x,0) = g(x), \quad 0 < x < L, \tag{13.89e}$$

is defined as the solution of

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\delta(x-X)\delta(t-T)}{\rho c^2}, \quad 0 < x < L, \quad t > T, \tag{13.90a}$$

$$-l_1 \frac{\partial y}{\partial x} + h_1 y = 0, \quad x = 0, \quad t > T, \tag{13.90b}$$

$$l_2 \frac{\partial y}{\partial x} + h_2 y = 0, \quad x = L, \quad t > T, \tag{13.90c}$$

$$y(x,t;X,T) = 0, \quad 0 < x < L, \quad t < T. \tag{13.90d}$$

It is also given by $h(t-T)G(x,t;X,T)$, where $G(x,t;X,T)$ satisfies

$$\frac{\partial^2 G}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}, \quad 0 < x < L, \quad t > T, \tag{13.91a}$$

$$-l_1 \frac{\partial G}{\partial x} + h_1 G = 0, \quad x = 0, \quad t > T, \tag{13.91b}$$

$$l_2 \frac{\partial G}{\partial x} + h_2 G = 0, \quad x = L, \quad t > T, \tag{13.91c}$$

$$G(x, T+; X, T) = 0, \quad 0 < x < L, \tag{13.91d}$$

$$G_t(x, T+; X, T) = \frac{\delta(x-X)}{\rho}, \quad 0 < x < L. \tag{13.91e}$$

In other words, the effect of an instantaneous unit force at position X and time T is equivalent to the effect of giving the point at X an instantaneous initial velocity $1/\rho$. Although the causal Green's function for 13.89 is $h(t-T)G(x,y;X,T)$, where $G(x,t;X,T)$ satisfies 13.91, we shall customarily call $G(x,t;X,T)$ itself the Green's function.

Problem 13.91 is easily solved by separation of variables.

Example 13.13 Find the causal Green's function for problem 13.89 when $l_1 = l_2 = 0$.

Solution Separation of variables on 13.91a–d leads, for $t > T$, to

$$G(x,t;X,T) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi c(t-T)}{L} f_n(x),$$

where $f_n(x) = \sqrt{2/L}\sin(n\pi x/L)$. If $\delta(x-X)$ is expanded in terms of the $\{f_n(x)\}$, initial condition 13.91e requires

$$\sum_{n=1}^{\infty}\frac{n\pi c}{L}A_nf_n(x)=\frac{1}{\rho}\sum_{n=1}^{\infty}\left[\int_0^L\delta(x-X)f_n(x)\,dx\right]f_n(x)=\frac{1}{\rho}\sum_{n=1}^{\infty}f_n(X)f_n(x).$$

It follows, then, that $\dfrac{n\pi c}{L}A_n=\dfrac{1}{\rho}f_n(X)$, and

$$\begin{aligned}G(x,t;X,T)&=\sum_{n=1}^{\infty}\frac{L}{n\pi c\rho}f_n(X)\sin\frac{n\pi c(t-T)}{L}f_n(x)\\&=\frac{L}{\rho\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi c(t-T)}{L}f_n(X)f_n(x).\bullet\end{aligned}$$

The solution of problem 13.89 can be expressed in terms of its Green's function as follows,

$$\begin{aligned}y(x,t)=&\int_0^t\int_0^L G(x,t;X,T)F(X,T)\,dX\,dT\\&+\rho\int_0^L\left[g(X)G(x,t;X,0)-f(X)\frac{\partial G(x,t;X,0)}{\partial T}\right]dX\\&+\rho c^2\int_0^t\left[G(x,t;L,T)\frac{f_2(T)}{l_2}+G(x,t;0,T)\frac{f_1(T)}{l_1}\right]dT.\end{aligned}\quad(13.92a)$$

The first integral contains the effect of past external forces, and the second integral contains that of the initial displacement and velocity. The last integral is due to boundary disturbances. Boundary conditions 13.91b,c can be used to rewrite the last integral in the form

$$\begin{aligned}y(x,t)=&\int_0^t\int_0^L G(x,t;X,T)F(X,T)\,dX\,dT\\&+\rho\int_0^L\left[g(X)G(x,t;X,0)-f(X)\frac{\partial G(x,t;X,0)}{\partial T}\right]dX\\&+\rho c^2\int_0^t\left[-\frac{f_2(T)}{h_2}\frac{\partial G(x,t;L,T)}{\partial X}+\frac{f_1(T)}{h_1}\frac{\partial G(x,t;0,T)}{\partial X}\right]dT.\end{aligned}\quad(13.92b)$$

This form must be used when $l_1=l_2=0$.

Example 13.14 Solve the vibration problem of Example 7.4 in Section 7.2.

Solution The Green's function for this problem was derived in Example 13.13. According to formula 13.92b, then,

$$\begin{aligned}y(x,t)&=\rho c^2\int_0^t-\frac{\partial G(x,t;L,T)}{\partial X}g(T)\,dT\\&=\rho c^2\int_0^t\left[\frac{-L}{\rho\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi c(t-T)}{L}f_n'(L)f_n(x)\right]g(T)\,dT\\&=-\frac{Lc}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\int_0^t\sin\frac{n\pi c(t-T)}{L}g(T)\,dT\right]f_n'(L)f_n(x).\end{aligned}$$

When $g(t)=A\sin\omega t$ and $\omega\neq n\pi c/L$ for any integer n,

$$\int_0^t \sin\frac{n\pi c(t-T)}{L} A\sin\omega T\,dT = \frac{AL^2}{n^2\pi^2c^2-\omega^2L^2}\left(\frac{n\pi c}{L}\sin\omega t - \omega\sin\frac{n\pi ct}{L}\right),$$

and therefore

$$y(x,t) = -\frac{Lc}{\pi}\sum_{n=1}^{\infty}\frac{AL^2}{n(n^2\pi^2c^2-\omega^2L^2)}\left(\frac{n\pi c}{L}\sin\omega t-\omega\sin\frac{n\pi ct}{L}\right)\sqrt{\frac{2}{L}}\left(\frac{n\pi}{L}\right)(-1)^n\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

$$= 2cA\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2c^2-\omega^2L^2}\left(\omega L\sin\frac{n\pi ct}{L}-n\pi c\sin\omega t\right)\sin\frac{n\pi x}{L}.$$

When $g(t) = A\sin(m\pi ct/L)$ for some integer m,

$$\int_0^t \sin\frac{n\pi c(t-T)}{L}g(T)\,dT = \begin{cases}\dfrac{AL}{\pi c(n^2-m^2)}\left(n\sin\dfrac{m\pi ct}{L}-m\sin\dfrac{n\pi ct}{L}\right), & n\neq m\\ \dfrac{A}{2m\pi c}\left(L\sin\dfrac{m\pi ct}{L}-m\pi ct\cos\dfrac{m\pi ct}{L}\right), & n=m\end{cases}$$

and

$$y(x,t) = \frac{Lc}{-\pi}\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{AL}{n\pi c(n^2-m^2)}\left(n\sin\frac{m\pi ct}{L}-m\sin\frac{n\pi ct}{L}\right)f_n'(L)f_n(x)$$

$$-\frac{Lc}{\pi}\left[\frac{A}{2m^2\pi c}\left(L\sin\frac{m\pi ct}{L}-m\pi ct\cos\frac{m\pi ct}{L}\right)\right]f_m'(L)f_m(x)$$

$$= \frac{2A}{\pi}\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{(-1)^n}{n^2-m^2}\left(m\sin\frac{n\pi ct}{L}-n\sin\frac{m\pi ct}{L}\right)\sin\frac{n\pi x}{L}$$

$$+\frac{(-1)^m A}{m\pi L}\left(m\pi ct\cos\frac{m\pi ct}{L}-L\sin\frac{m\pi ct}{L}\right)\sin\frac{m\pi x}{L}.\bullet$$

The causal Green's function for the two-dimensional vibration problem

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}-\frac{F(x,y,t)}{\rho c^2},\quad (x,y)\text{ in }A,\quad t>0, \tag{13.93a}$$

$$l\frac{\partial z}{\partial n}+hz = K(x,y,t),\quad (x,y)\text{ on }\beta(A),\quad t>0, \tag{13.93b}$$

$$z(x,y,0) = f(x,y),\quad (x,y)\text{ in }A, \tag{13.93c}$$

$$z_t(x,y,0) = g(x,y),\quad (x,y)\text{ in }A, \tag{13.93d}$$

is defined as the solution of

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}-\frac{\delta(x-X,y-Y)\delta(t-T)}{\rho c^2},\quad (x,y)\text{ in }A,\quad t>T, \tag{13.94a}$$

$$l\frac{\partial z}{\partial n}+hz=0,\quad (x,y)\text{ on }\beta(A),\quad t>T, \tag{13.94b}$$

$$z(x,y,t;X,Y,T)=0,\quad (x,y)\text{ in }A,\quad t>T. \tag{13.94c}$$

It is also given by $h(t-T)G(x,y,t;X,Y,T)$, where $G(x,y,t;X,Y,T)$ satisfies

$$\nabla^2 G = \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}, \quad (x,y) \text{ in } A, \quad t > T, \tag{13.95a}$$

$$\frac{\partial G}{\partial n} + hG = 0, \quad (x,y) \text{ on } \beta(A), \quad t > T, \tag{13.95b}$$

$$G(x,y,T+;X,Y,T) = 0, \quad (x,y) \text{ in } A, \tag{13.95c}$$

$$G_t(x,y,T+;X,Y,T) = \frac{\delta(x-X,y-Y)}{\rho}, \quad (x,y) \text{ in } A. \tag{13.95d}$$

The solution of problem 13.93 can be expressed in the form

$$\begin{aligned} z(x,y,t) = & \int_0^t \iint_A G(x,y,t;X,Y,T)F(X,Y,T)\,dA\,dT \\ & + \rho \iint_A \left[g(X,Y)G(x,y,t;X,Y,0) - f(X,Y)\frac{\partial G(x,y,t;X,Y,0)}{\partial T}\right] dA \\ & + \frac{\rho c^2}{l}\int_0^t \oint_{\beta(A)} G(x,y,t;X,Y,T)K(X,Y,T)\,ds\,dT \end{aligned} \tag{13.96a}$$

or,

$$\begin{aligned} z(x,y,t) = & \int_0^t \iint_A G(x,y,t;X,Y,T)F(X,Y,T)\,dA\,dT \\ & + \rho \iint_A \left[g(X,Y)G(x,y,t;X,Y,0) - f(X,Y)\frac{\partial G(x,y,t;X,Y,0)}{\partial T}\right] dA \\ & - \frac{\rho c^2}{h}\int_0^t \oint_{\beta(A)} K(X,Y,T)\frac{\partial G(x,y,t;X,Y,T)}{\partial N} ds\,dT. \end{aligned} \tag{13.96b}$$

EXERCISES 13.8

In Exercises 1–3 find the causal Green's function for problem 13.89 when values for l_1, l_2, h_1, and h_2 are as specified.

1. $h_1 = h_2 = 0$, $l_1 = l_2 = 1$

2. $l_2 = h_1 = 0$, $l_1 = h_2 = 1$

3. $l_2 = h_1 = 1$, $l_1 = h_2 = 0$

In Exercises 4–6 use formulas 13.92a,b to solve the initial boundary value problem.

4. Exercise 17 in Section 4.3 (see also Exercise 19 in Section 7.2)

5. Exercise 24(a) in Section 7.2

6. Exercise 25 in Section 7.2

7. A taut string initially at rest along the x-axis has its ends fixed at $x = 0$ and $x = L$.
(a) Find displacements in the string for an arbitrary forcing function $F(x,t)$.
(b) Simplify the solution in part (a) when $F(x,t)$ is a time-independent, constant force F_0 concentrated at $x = x_0$.
(c) Simplify the solution in part (b) further if $x_0 = L/2$.
(d) What is the solution in part (b) if x_0 is a node of the m^{th} normal mode of vibration of the string?

8. (a) What is the causal Green's function for problem 13.89?

(b) Use the representation in part (a) to show that $G(x,t;X,T)$ satisfies the *reciprocity principle*

$$G(x,t;X,T) = G(X,t;x,T).$$

What does this mean physically?

(c) Use the representation in part (a) to show that $G(x,t;X,T)$ satisfies the *time-translation* property

$$G(x,t;x,T) = G(x,t+\overline{T};X,T+\overline{T})$$

provided $\overline{T} > 0$. What does this mean physically?

In Exercises 9–10 use formulas 13.96a,b to solve the two-dimensional vibration problem.

9. Exercise 6 in Section 7.3

10. Exercise 22 in Section 9.2

CHAPTER 14 FINITE DIFFERENCES SOLUTIONS

§14.1 Introduction

In Chapters 1–13, we developed what are called analytic techniques for solving initial, boundary value problems; solutions were often in the form of infinite series. These series are considered to be *exact* solutions to their respective problems, although this may be somewhat of a misnomer. If we could sum the series in closed form, then certainly we would have an exact solution (d'Alembert solutions, for example). In general, the likelihood that we can do this is small, and as a result, we cannot evaluate, exactly, the solution to the initial, boundary value problem at a given point and time. Instead, we approximate the solution by truncating the series after sufficiently many terms. Thus, series solutions are not really exact solutions, they are approximations to them. It is worthwhile noting, however, that it is usually possible to ascertain a maximum error in making such truncations. Henceforth we shall call these series solutions **analytic solutions**.

In the remaining chapters, we discuss three techniques for approximating solutions to initial, boundary value problems, namely, finite differences, weighted residuals, and finite elements, and these techniques are clearly approximation schemes from the outset. Unlike treatments of previous topics which have been exhaustive, discussions here are overviews; full treatments of finite differences, weighted residuals, and finite elements are books unto themselves. Our hope is to give the reader an introductory exposition that conveys basic ideas upon which the reader can build with further readings. We concentrate on the methodology of these approximations, but not on their computer implementation.

There are many problems intractable to our analytic techniques that can be handled by finite differences, weighted residuals, and finite elements. One clear restriction on our analytic techniques is the shape of spatial regions. They have always been bounded by coordinate lines or surfaces. For instance, problems in the plane have been rectangles (in Cartesian coordinates), and circles, semi-circles, quarter circles, annuli, etc. (in polar coordinates); and this was out of necessity. Finite differences, weighted residuals, and finite elements can handle problems on arbitrarily shaped regions, and finite elements does it most efficiently.

§14.2 Finite Differences and Finite Difference Equations

Finite Differences

Finite differences can be used to approximate derivatives, and when such differences are used in place of derivatives in a PDE, the result is called a **partial difference equation**, henceforth shortened to pde, (lower case for difference equation and upper case for differential equation). As this is not a text on numerical analysis, we do not show all possible finite difference approximations to a given derivative. Instead, we develop only finite differences that are used in subsequent sections.

The derivative of a function $f(x)$ is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{14.1}$$

When the limiting operation is removed, the result is an approximation to $f'(x)$, the accuracy of the approximation depending on the size of h, the smaller h, the better the approximation,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}. \tag{14.2a}$$

When $h > 0$, the right side of this equation is called the **forward finite difference approximation** to $f'(x)$. When $h < 0$, it is called the **backward finite difference approximation**. Because we always take $h > 0$ in this chapter, the backward difference formula is usually expressed in the form

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}. \tag{14.2b}$$

The right side of

$$f'(x) \approx \frac{f(x+h/2) - f(h-h/2)}{h}, \tag{14.3}$$

is called the **central finite difference approximation** for $f'(x)$. In general, it is a more accurate estimate for $f'(x)$ than the forward and backward differences. To see why, it is necessary to analyze these approximations with Taylor's remainder formula. When a function $f(x)$ has a continuous second derivative on the interval between x and $x+h$, the remainder formula states that there is a value z between x and $x+h$ such that

$$f(x+h) = f(x) + f'(x)h + \frac{f''(z)}{2!}h^2.$$

The difficulty with this result is that it guarantees the existence of z, but does not provide a method for finding it. When this equation is solved for $f'(x)$, the result is

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(z)}{2!}h.$$

The error in using equation 14.2a to approximate $f'(x)$ is therefore

$$\frac{|f''(z)|}{2}h.$$

It is called the **truncation error** in doing so. Essentially the Taylor series for $f(x)$ has been truncated by Taylor's remainder formula in arriving at the formula. If M is the maximum value of $|f''(x)|$ on the interval between x and $x+h$, we can say that the maximum truncation error in using forward difference formula 14.2a to approximate $f'(x)$ is

$$\frac{Mh}{2}. \tag{14.4}$$

What we say is that the local truncation error is of **order** h and we write $O(h)$. order This simply means that the truncation error of forward difference formula 14.2a is less than some constant times h (in this case, the constant is $M/2$). When we do not know $f(x)$, we cannot calculate M, and it might seem that quantity 14.4 is of no value. Certainly, we cannot determine the exact error, but knowing that the error is $O(h)$, we can say the following. If h is cut in half, the truncation error is one half its previous value; if h is reduced to one-tenth its value, the error is one-tenth its previous value. Get the idea? The truncation error of backward difference formula 14.2b is also $O(h)$.

To find the order of the truncation error for central difference 14.3, we once again use Taylor's remainder formula to write

$$f(x+h/2) = f(x) + f'(x)\left(\frac{h}{2}\right) + \frac{f''(x)}{2!}\left(\frac{h}{2}\right)^2 + \frac{f'''(z_1)}{3!}\left(\frac{h}{2}\right)^3,$$

$$f(x-h/2) = f(x) + f'(x)\left(-\frac{h}{2}\right) + \frac{f''(x)}{2!}\left(-\frac{h}{2}\right)^2 + \frac{f'''(z_2)}{3!}\left(-\frac{h}{2}\right)^3.$$

When these are subtracted,

$$f(x+h/2) - f(x-h/2) = f'(x)h + \frac{1}{48}[f'''(z_1) + f'''(z_2)]h^3,$$

or,

$$f'(x) = \frac{f(x+h/2) - f(x-h/2)}{h} - \frac{1}{48}[f'''(z_1) + f'''(z_2)]h^2.$$

This shows that central difference formula 14.3 is $O(h^2)$. When h is decreased to one-tenth its value, the error is one-hundredth its previous value. Now you see why central differences are better approximations for $f'(x)$ than forward or backward differences.

One way to find a finite difference formula for the second derivative $f''(x)$ is to use formula 14.3a three times,

$$\begin{aligned} f''(x) &\approx \frac{f'(x+h/2) - f'(x-h/2)}{h} \approx \frac{1}{h}\left\{\left[\frac{f(x+h) - f(x)}{h}\right] - \left[\frac{f(x) - f(x-h)}{h}\right]\right\} \\ &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \end{aligned} \tag{14.5}$$

This is a **central difference formula** to approximate $f''(x)$. It is $O(h^2)$ (see Exercise 1).

Finite Difference Equations

In Section 14.3, we encounter what are called first- and second-order, homogeneous, constant-coefficient, linear difference equations. We summarize their solutions here. A **first-order, linear, homogeneous difference equation** for a discrete function $f(n)$ of variable $n = 1, 2, \ldots$, is an equation of the form

$$f(n+1) = \lambda f(n). \tag{14.6a}$$

We consider only the case that λ is a constant. If we use the notation $f_n = f(n)$, then this equation can be expressed in the form

$$f_{n+1} = \lambda f_n. \tag{14.6b}$$

By substituting $f_n = \mu^n$, it is straightforward to show that solutions of this equation are

$$f_n = C\lambda^n, \tag{14.7}$$

where C is a constant. If $f(n)$ is known for any particular value of n, then the value of C can be calculated.

A **second-order, linear, homogeneous difference equation** for $f(n)$ is an equation of the form

$$pf(n+2) + qf(n+1) + rf(n) = 0, \tag{14.8a}$$

or, with subscript notation,

$$pf_{n+2} + qf_{n+1} + rf_n = 0. \tag{14.8b}$$

Once again we only consider the case that coefficients p, q, and r are constants. Substitution of $f_n = \mu^n$ leads to what is called the **characteristic equation** of the difference equation, namely,

$$p\mu^2 + q\mu + r = 0. \tag{14.9}$$

The nature of the roots of this quadratic equation dictates the form for solutions of the difference equation.

1. When characteristic equation 14.9 has real, distinct roots μ_1 and μ_2, solutions of equation 14.8 are of the form

$$f_n = A\mu_1^n + B\mu_2^n, \tag{14.10}$$

   where A and B are constants.
2. When characteristic equation 14.9 has a single (real) root μ (of multiplicity 2), solutions of equation 14.8 are of the form

$$f_n = (A + Bn)\mu^n, \tag{14.11}$$

   where A and B are constants.
3. When characteristic equation 14.9 has a pair of complex conjugate roots $Re^{\pm\theta i}$, solutions of equation 14.8 are of the form

$$f_n = R^n(A\cos n\theta + B\sin n\theta), \tag{14.12}$$

where A and B are constants.

Values for A and B can be calculated if f is known at two values of n.

Many readers find it helpful to view difference equations in the context of sequences, f_n is the n^{th} term of a sequence. For example, if $f_n = n^2/(2n+1)$, the first few terms of the sequence are

$$f_1 = \frac{1}{3}, \quad f_2 = \frac{4}{5}, \quad f_3 = \frac{9}{7}, \quad f_4 = \frac{16}{9}, \ldots .$$

This is what is called an **explicitly** defined sequence. It is straightforward to calculate any term of such a sequence simply by substituting the appropriate value of n into the explicit definition. Difference equations represent sequences that are defined **recursively**. For instance, the following defines a sequence whose first term is 3, and does so recursively,

$$f_1 = 3, \qquad f_{n+1} = \frac{f_1}{2}, \quad n \geq 1;$$

each term of the sequence is one-half the previous term. Iteration of the difference equation (or recursive formula) gives an explicit definition of the sequence, namely,

$$f_n = \frac{3}{2^{n-1}}.$$

Formula 14.7 is an explicit formula for all sequences satisfying recursive definition 14.6 (provided λ is constant); each term is defined in terms of its predecessor. Constant C is determined by specifying any term of the sequence. In this context, it is often important to ask whether the sequence has a limit for large n. Clearly, a recursive sequence satisfying 14.6 has limit 0 when $|\lambda| < 1$, has limit C when $\lambda = 1$, and has no limit for any other value of λ (unless trivially $C = 0$).

When the n^{th} term of a sequence satisfies recursive definition 14.8, it is a linear combination of the two terms immediately preceding it. In situation 1., it has a limit when $-1 < \mu_1, \mu_2 \leq 1$. In case 2., it has a limit when $-1 < \mu < 1$. In case 3., it has limit 0 when $R < 1$.

Example 14.1 Find the n^{th} term of the sequence defined recursively by

$$f_1 = 0, \qquad f_2 = 4, \qquad f_{n+1} = \frac{2}{3}f_n + \frac{1}{3}f_{n-1}, \qquad n \geq 2.$$

Solution The characteristic equation is

$$0 = \mu^2 - \frac{2\mu}{3} - \frac{1}{3} = \frac{1}{3}(3\mu^2 - 2\mu - 1) = \frac{1}{3}(3\mu + 1)(\mu - 1),$$

with solutions $\mu = 1$ and $\mu = -1/3$. According to formula 14.10,

$$f_n = A(1)^n + B\left(-\frac{1}{3}\right)^n = A + B\left(-\frac{1}{3}\right)^n.$$

For $f_1 = 0$ and $f_2 = 4$, we must have

$$0 = A - \frac{B}{3}, \qquad 4 = A + \frac{B}{9}.$$

These require $A = 3$ and $B = 9$, and therefore

$$f_n = 3 + 9\left(-\frac{1}{3}\right)^n = 3 + (-1)^n 3^{2-n}.$$

This sequence has limit 3.•

Example 14.2 Find the n^{th} term of the sequence defined recursively by

$$f_1 = 2, \qquad f_2 = -3, \qquad f_{n+1} = -6f_n - 9f_{n-1}, \qquad n \geq 2.$$

Determine whether the sequence as a limit.

Solution The characteristic equation is $\mu^2 + 6\mu + 9 = 0$ with solution $\mu = -3$ of multiplicity 2. According to formula 14.11, the n^{th} term of the sequence is of the form

$$f_n = (A + Bn)(-3)^n.$$

To determine A and B, we use the facts that $f_1 = 2$ and $f_2 = -3$,

$$2 = (A + B)(-3), \qquad -3 = (A + 2B)(-3)^2.$$

These require $A = -1$ and $B = 1/3$, and therefore

$$f_n = \left(-1 + \frac{n}{3}\right)(-3)^n = (n-3)(-1)^n 3^{n-1}.$$

This sequence does not have a limit.•

Example 14.3 Find all solutions of the difference equation

$$f_{n+2} = \frac{3}{5}f_{n+1} - \frac{1}{4}f_n.$$

Is there a limit as $n \to \infty$?

Solution The characteristic equation

$$0 = \mu^2 - \frac{3\mu}{5} + \frac{1}{4} = \frac{1}{20}(20\mu^2 - 12\mu + 5),$$

has solutions

$$\mu = \frac{12 \pm \sqrt{144 - 400}}{40} = \frac{3 \pm 4i}{10}.$$

Since the exponential form of $(3+4i)/10$ is $(1/2)e^{\theta i}$, where $\theta = \text{Tan}^{-1}(4/3)$, solutions of the difference equation are of the form

$$f_n = \frac{1}{2^n}(A\cos n\theta + B\sin n\theta).$$

Since $R = 1/2$, the limit of f_n as $n \to \infty$ is 0.•

EXERCISES 14.2

1. Show that the truncation error associated with central difference formula 14.5 is $O(h^2)$.

In Exercises 2–5 find the solution of the first-order difference equation (or the general term of the sequence). Determine whether $\lim_{n\to\infty} f_n$ exists.

2. $f_1 = 2, \quad f_{n+1} = 3f_n, \quad n \geq 1$

3. $f_1 = 1, \quad f_{n+1} = \frac{1}{3}f_n, \quad n \geq 1$

4. $f_1 = -2, \quad f_{n+1} = -\frac{2}{3}f_n, \quad n \geq 1$

5. $f_1 = 4, \quad f_{n+1} = \frac{3}{2}f_n, \quad n \geq 1$

6. A first-order linear difference equation of the form

$$f_1 = a, \qquad f_{n+1} = \lambda f_n + d, \quad n \geq 1,$$

where d and λ are constants, is said to be **nonhomogeneous** with constant coefficients. Show that solutions are of the form

$$f_n = \begin{cases} a\lambda^{n-1} + d\left(\dfrac{1-\lambda^{n-1}}{1-\lambda}\right), & \lambda \neq 1 \\ a + (n-1)d, & \lambda = 1. \end{cases}$$

In Exercises 7–10 use the result of Exercise 6 to find the solution of the first-order difference equation (or the general term of the sequence). Determine whether $\lim_{n\to\infty} f_n$ **exists.**

7. $c_1 = 0, \quad c_{n+1} = \frac{5}{3}c_n - 2, \quad n \geq 1$

8. $c_1 = -1, \quad c_{n+1} = -\frac{1}{2}c_n + 4, \quad n \geq 1$

9. $c_1 = 2, \quad c_{n+1} = \frac{5}{12}c_n - \frac{1}{3}, \quad n \geq 1$

10. $c_1 = -2, \quad c_{n+1} = -c_n + 5, \quad n \geq 1$

In Exercises 11–15 find the solution of the second-order difference equation (or the general term of the sequence). Determine whether $\lim_{n\to\infty} f_n$ **exists.**

11. $c_1 = 1, \quad c_2 = 2, \quad c_{n+2} = \dfrac{c_{n+1} + c_n}{2}, \quad n \geq 1$

12. $c_1 = 0, \quad c_2 = 2, \quad c_{n+2} = 2c_{n+1} + 2c_n, \quad n \geq 1$

13. $c_1 = 1, \quad c_2 = 1, \quad c_{n+2} = \frac{2}{3}c_{n+1} - \frac{1}{9}c_n, \quad n \geq 1$

14. $c_1 = -1, \quad c_2 = 1, \quad c_{n+2} = -\frac{10}{3}c_{n+1} - \frac{25}{9}c_n, \quad n \geq 1$

15. $c_1 = 1, \quad c_2 = 2, \quad c_{n+2} = -2c_{n+1} - 2c_n, \quad n \geq 1$

16. Find an explicit formula for the n^{th} term of the Fibonacci sequence

$$f_1 = 1, \qquad f_2 = 1, \qquad f_{n+2} = f_{n+1} + f_n, \quad n \geq 1.$$

17. Find an explicit formula for the n^{th} term of the sequence

$$c_1 = a, \quad c_2 = b, \quad c_{n+2} = \frac{c_{n+1} + c_n}{2}, \quad n \geq 1,$$

where a and b are arbitrary numbers. Does the sequence have a limit?

18. Show that when roots of characteristic equation 14.9 have moduli (or absolute values in the case of real roots) less than 1, the solution of equation 14.8 has limit 0 as $n \to \infty$.

19. Show that the solution of equation 14.8 has a nonzero limit if and only if characteristic equation 14.9 has exactly one root equal to 1 and a second root with absolute value less than 1.

§14.3 The Classic Explicit Partial Difference Equation for Parabolic PDEs

We have considered the following homogeneous, one-dimensional heat conduction problem many times

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{14.13a}$$

$$U(0,t) = 0, \quad t > 0, \tag{14.13b}$$

$$U(L,t) = 0, \quad t > 0, \tag{14.13c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{14.13d}$$

Separation of variables leads to the infinite series solution

$$U(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}, \tag{14.14a}$$

where

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{14.14b}$$

To use finite differences to approximate the solution of this initial, boundary value problem, we first replace the partial differential equation (PDE) in continuous variables x and t with a partial difference equation (pde) in discrete counterparts of x and t. We do this by dividing the x-axis between $x = 0$ and $x = L$ into N equal subintervals of length $h = \Delta x = L/N$ by the points

$$x_0 = 0, \quad x_1 = h, \quad x_2 = 2h, \quad \cdots, \quad x_{N-1} = (N-1)h, \quad x_N = L.$$

In addition, we discretize time by choosing a time step $s = \Delta t$. These choices discretize that part of the xt-plane bounded by $0 < x < L$, $t > 0$ as shown in Figure 14.1. Finding the solution $U(x,t)$ to problem 14.13 interior to this region is replaced by finding approximations to $U(x,t)$ at the mesh points $(x_n, t_p) = (nh, ps)$. We denote the approximations of $U(x,t)$ at (x_n,t_p) by $U_{n,p} = U(x_n,t_p)$. These approximations must satisfy a partial difference equation consistent with PDE 14.13a. When we replace $\partial U/\partial t$ at (x,t) with a forward difference at (x_n,t_p), and $\partial^2 U/\partial x^2$ with a central difference, the resulting partial difference equation is

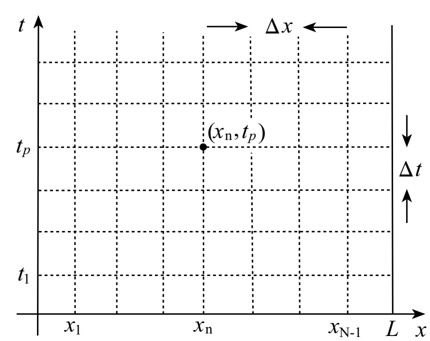


Figure 14.1

$$\frac{U(x_n,t_p+s) - U(x_n,t_p)}{s} = k\left[\frac{U(x_n+h,t_p) - 2U(x_n,t_p) + U(x_n-h,t_p)}{h^2}\right]. \tag{14.15}$$

With our subscript notation $U(x_n,t_p) = U_{n,p}$, this becomes

$$U_{n,p+1} - U_{n,p} = \frac{ks}{h^2}\left(U_{n+1,p} - 2U_{n,p} + U_{n-1,p}\right). \tag{14.16a}$$

This partial difference equation replaces PDE 14.13a; it is to be satisfied at each of the interior mesh points in Figure 14.1. This means for $p = 0, \ldots$ and $n = 1, \ldots, N-1$. It is usually called the **classic explicit** scheme. To replace boundary conditions 14.13b,c, we extend values of n to $0, \ldots, N$, and demand that

$$U_{0,p} = 0, \quad p \geq 0, \tag{14.16b}$$

$$U_{N,p} = 0, \quad p \geq 0. \tag{14.16c}$$

We replace initial condition 14.13d for the PDE with the following initial conditions for the pde,

$$U_{n,0} = f_n = f(nh) = f(x_n), \quad n = 1, \ldots, N-1. \tag{14.16d}$$

If $f(0) = f(L) = 0$, then conditions 14.16b,c are natural. If $f(0) \neq 0$ and/or $f(L) \neq 0$, it may be advisable to modify conditions 14.16b,c, thus creating nonhomogeneous boundary conditions. In summary, we have replaced initial boundary value problem 14.13 for $U(x,t)$ with initial boundary value problem 14.16 for $U_{n,p}$.

The truncation error in replacing the partial derivative $\partial U/\partial t$ with a forward difference is $O(s)$, and the error in replacing $\partial^2 U/\partial x^2$ with a central difference is $O(h^2)$. We say that the truncation error in replacing PDE 14.13a with pde 14.16a is $O(h^2, s)$. Because pde 14.16a approximates PDE 14.13a more and more closely as $h \to 0$ and $s \to 0$, we say that classic explicit scheme 14.16a is **consistent** with the one-dimensional heat equation 14.13a. This is a necessary condition for a partial difference scheme to replace a partial differential equation, but as we shall see, it is not sufficient. Other factors must be taken into account in determining whether a particular pde is a suitable finite difference approximation for a PDE.

In practice, we do not find exact solutions of initial boundary value problems associated with pdes, simply because in applications wherein finite difference techniques are required, it is impossible to do so. Instead, we iterate the pde to find approximate solutions. Nevertheless, in the remainder of this section we derive the exact solution to problem 14.16. We do this for two reasons. Firstly, to demonstrate the complexity of discussions even in this most simple problem, and secondly, and more importantly, to illustrate that choices for h and s are not independent of each other. This leads to the important idea of *stability* for pdes replacing PDEs.

The reader will have no difficulty seeing the similarity of the derivation to separation of variables for PDEs. We begin by searching for solutions of homogeneous pde 14.16a and homogeneous boundary conditions 14.16b,c of the form $U_{n,p} = X_n T_p$; that is, we search for functions $U_{n,p}$ that are the products of function X_n of n, and functions T_p of p. When we substitute this into the pde, we obtain

$$X_n T_{p+1} - X_n T_p = \frac{ks}{h^2}(X_{n+1}T_p - 2X_n T_p + X_{n-1}T_p).$$

Division by $X_n T_p$ gives

$$\frac{T_{p+1} - T_p}{T_p} = \frac{ks}{h^2}\left(\frac{X_{n+1} - 2X_n + X_{n-1}}{X_n}\right) = \lambda - 1,$$

where λ is a constant independent of p and n. (We could have used λ instead of $\lambda - 1$; $\lambda - 1$ facilitates notation in the remainder of the discussion.) This separates the pde into two ordinary difference equations,

$$T_{p+1} = \lambda T_p, \tag{14.17}$$

and

$$X_{n+1} - \left[2 + \frac{h^2}{ks}(\lambda - 1)\right] X_n + X_{n-1} = 0. \tag{14.18a}$$

Boundary conditions 14.16b,c add two boundary conditions to the second of these, namely

$$X_0 = 0, \quad X_N = 0. \tag{14.18b}$$

According to Section 14.2, a general solution of the difference equation in T_p is

$$T_p = C\lambda^p, \tag{14.19}$$

where C is a constant (independent of p). The characteristic equation associated with the difference equation in X_n is

$$\mu^2 - \left[2 + \frac{h^2}{ks}(\lambda - 1)\right] \mu + 1 = 0.$$

Taking for granted that X_n must involve sine and/or cosine functions, it follows that μ must be complex (see Section 14.2). This implies that

$$\left[2 + \frac{h^2}{ks}(\lambda - 1)\right]^2 - 4 < 0. \tag{14.20}$$

If we set $\mu = Re^{\theta i}$, then

$$R^2 e^{2\theta i} - \left[2 + \frac{h^2}{ks}(\lambda - 1)\right] Re^{\theta i} + 1 = 0.$$

When we equate real and imaginary parts, we get

$$R^2 \cos 2\theta - \left[2 + \frac{h^2}{ks}(\lambda - 1)\right] R\cos\theta + 1 = 0, \tag{14.21a}$$

$$R^2 \sin 2\theta - \left[2 + \frac{h^2}{ks}(\lambda - 1)\right] R\sin\theta = 0. \tag{14.21b}$$

The second of these implies that

$$\sin\theta = 0, \qquad \text{or,} \qquad 2R\cos\theta - \left[2 + \frac{h^2}{ks}(\lambda - 1)\right] = 0. \tag{14.22}$$

The first of these requires $\theta = j\pi$, where j is an integer, and when this is substituted into equation 14.21a,

$$R^2 - \left[2 + \frac{h^2}{ks}(\lambda - 1)\right] R(-1)^j + 1 = 0.$$

Equation 14.20 renders this impossible. Thus, we must take the second alternative in equation 14.22. When this is solved for $\cos\theta$ and substituted into 14.21a,

$$\begin{aligned} 0 &= R^2\left(2\cos^2\theta - 1\right) - \left[2 + \frac{h^2}{ks}(\lambda - 1)\right] R\cos\theta + 1 \\ &= \frac{1}{2}\left[2 + \frac{h^2}{ks}(\lambda - 1)\right]^2 - R^2 - \frac{1}{2}\left[2 + \frac{h^2}{ks}(\lambda - 1)\right]^2 + 1 \\ &= -R^2 + 1. \end{aligned}$$

Consequently, R must be 1, in which case $\mu = e^{\theta i}$, where θ is defined by the equation $2\cos\theta - \left[2 + \dfrac{h^2}{ks}(\lambda - 1)\right] = 0$, and

$$X_n = A\cos n\theta + B\sin n\theta.$$

Boundary conditions 14.16b,c require

$$0 = X_0 = A, \quad 0 = X_N = A\cos N\theta + B\sin N\theta.$$

With $A = 0$, the second of these implies that $N\theta = \ell\pi$, where ℓ is a positive integer, and

$$X_n = B\sin\frac{n\pi\ell}{N}.$$

Separated solutions of 14.16a,b,c are now known to be

$$U_{n,p} = C\lambda^p \sin\frac{n\pi\ell}{N},$$

where

$$2\cos\frac{\ell\pi}{N} = 2 + \frac{h^2}{ks}(\lambda - 1) \qquad \Longrightarrow \qquad \lambda_\ell = 1 - \frac{2ks}{h^2}\left(1 - \cos\frac{\ell\pi}{N}\right).$$

Using familiar terminology, we say that the λ_ℓ are eigenvalues of system 14.18 and the X_n are corresponding eigenfunctions. For any integer n,

$$\sin\frac{(N+n)\pi\ell}{N} = \sin\pi\ell\cos\frac{n\pi\ell}{N} + \cos\pi\ell\sin\frac{n\pi\ell}{N} = (-1)^\ell \sin\frac{n\pi\ell}{N}.$$

This shows that, unlike Sturm-Liouville systems where there is an infinity of eigenfunctions, there is only $N-1$ linearly independent eigenfunctions $X_n = \sin(n\pi\ell/N)$, $\ell = 1, 2, \ldots, N-1$, of system 14.18 for each value of n. To satisfy initial conditions 14.16d, we use superposition and take

$$U_{n,p} = \sum_{\ell=1}^{N-1} C_\ell \left[1 - \frac{2ks}{h^2}\left(1 - \cos\frac{\ell\pi}{N}\right)\right]^p \sin\frac{n\pi\ell}{N}.$$

The initial conditions then require coefficients C_ℓ to satisfy

$$U_{n,0} = f_n = \sum_{\ell=1}^{N-1} C_\ell \sin\frac{n\pi\ell}{N}, \qquad n = 1, \ldots, N-1.$$

If we multiply this equation by $\sin(n\pi q/N)$ and add over values of n, we obtain

$$\begin{aligned}\sum_{n=1}^{N-1} f_n \sin\frac{n\pi q}{N} &= \sum_{n=1}^{N-1}\sum_{\ell=1}^{N-1} C_\ell \sin\frac{n\pi\ell}{N}\sin\frac{n\pi q}{N}\\ &= \sum_{\ell=1}^{N-1} C_\ell \sum_{n=1}^{N-1}\sin\frac{n\pi\ell}{N}\sin\frac{n\pi q}{N}\\ &= \sum_{\ell=1}^{N-1} C_\ell\left\{\frac{1}{2}\sum_{n=1}^{N-1}\left[\cos\frac{n\pi(\ell-q)}{N}-\cos\frac{n\pi(\ell+q)}{N}\right]\right\}.\end{aligned}$$

As for the sine functions, when n is an integer,

$$\cos\frac{(N+n)\pi\ell}{N} = \cos\pi\ell\cos\frac{n\pi\ell}{N} - \sin\pi\ell\sin\frac{n\pi\ell}{N} = (-1)^\ell\cos\frac{n\pi\ell}{N}.$$

In other words, only the functions $\cos(n\pi\ell/N)$, $n = 0, \ldots, N-1$ are linearly independent. With this in mind, when $\ell \neq q$, the set of $N-1$ functions $\cos[n\pi(\ell-q)/N]$ is the same as the set of functions $\cos[n\pi(\ell+q)/N]$. In other words, we must have $\ell = q$, in which case

$$\sum_{n=1}^{N-1} f_n\sin\frac{n\pi q}{N} = C_q\sum_{n=1}^{N-1}\sin^2\frac{n\pi q}{N}.$$

Thus,

$$C_q = \frac{1}{\displaystyle\sum_{n=1}^{N-1}\sin^2\frac{n\pi q}{N}}\sum_{n=1}^{N-1} f_n\sin\frac{n\pi q}{N}.$$

Now, with the notation that $\mathrm{Re}\, z$ represents the real part of the complex number z, we may use geometric series to write

$$\begin{aligned}\sum_{n=1}^{N-1}\sin^2\frac{n\pi q}{N} &= \sum_{n=1}^{N-1}\frac{1}{2}\left(1-\cos\frac{2n\pi q}{N}\right) = \frac{N-1}{2}-\frac{1}{2}\sum_{n=1}^{N-1}\cos\frac{2n\pi q}{N}\\ &= \frac{N-1}{2}-\frac{1}{2}\sum_{n=1}^{N-1}\mathrm{Re}[e^{2n\pi qi/N}] = \frac{N-1}{2}-\frac{1}{2}\mathrm{Re}\sum_{n=1}^{N-1}e^{2n\pi qi/N}\\ &= \frac{N-1}{2}-\frac{1}{2}\mathrm{Re}\left\{\frac{e^{2\pi qi/N}\left[1-(e^{2\pi qi/N})^{N-1}\right]}{1-e^{2\pi qi/N}}\right\}\\ &= \frac{N-1}{2}-\frac{1}{2}\mathrm{Re}\left\{\frac{(e^{2\pi qi/N}-e^{2\pi qi})(1-e^{-2\pi qi/N})}{(1-e^{2\pi qi/N})(1-e^{-2\pi qi/N})}\right\}\\ &= \frac{N-1}{2}-\frac{1}{2}\mathrm{Re}\left\{\frac{e^{2\pi qi/N}-e^{2\pi qi}-1+e^{2\pi qi(1-1/N)}}{(1-e^{2\pi qi/N})(1-e^{-2\pi qi/N})}\right\}\\ &= \frac{N-1}{2}-\frac{1}{2}\left\{\frac{\cos(2\pi qN)-1-1+\cos[2\pi q(1-1/N)]}{2[1-\cos(2\pi q/N)]}\right\}\\ &= \frac{N-1}{2}-\frac{1}{2}(-1) = \frac{N}{2}.\end{aligned}$$

Thus, the solution of initial boundary value problem 14.16 is

$$U_{n,p} = \sum_{\ell=1}^{N-1} C_\ell \left[1 - \frac{2ks}{h^2}\left(1 - \cos\frac{\ell\pi}{N}\right)\right]^p \sin\frac{n\pi\ell}{N}, \tag{14.23a}$$

where

$$C_\ell = \frac{2}{N}\sum_{j=1}^{N-1} f_j \sin\frac{\ell\pi j}{N}. \tag{14.23b}$$

In order that the solution not become unbounded in time, we require that

$$\left|1 - \frac{2ks}{h^2}\left(1 - \cos\frac{\ell\pi}{N}\right)\right| < 1, \quad \text{for } \ell = 1, \dots, N-1.$$

This simplifies to

$$0 < \frac{ks}{h^2}\left(1 - \cos\frac{\ell\pi}{N}\right) < 1.$$

Given the interval $h = \Delta x$, this places a restriction on the time step $s = \Delta t$; that is,

$$s < \frac{h^2}{k\left(1 - \cos\dfrac{\ell\pi}{N}\right)}, \quad \text{for } \ell = 1, \dots, N-1.$$

Since the smallest value of $1 - \cos(\ell\pi/N)$ occurs when ℓ is largest, we can say that

$$1 - \cos\frac{\ell\pi}{N} \le 1 - \cos\frac{(N-1)\pi}{N} < 1 - \cos\pi = 2.$$

Consequently, boundedness is guaranteed if

$$s \le \frac{h^2}{2k} \quad \Longrightarrow \quad \Delta t \le \frac{(\Delta x)^2}{2k}. \tag{14.24}$$

Thus, even for an exact solution of the finite difference initial boundary value problem, Δx and Δt cannot be specified independently.

Now that we have the solution of the finite difference problem corresponding to problem 14.13, it is worthwhile comparing solutions for a specific $f(x)$. If the initial temperature is $f(x) = x(L-x)$, then the solution of problem 14.13 is

$$U(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L},$$

where

$$C_n = \frac{2}{L}\int_0^L x(L-x)\sin\frac{n\pi x}{L}dx = \frac{4L^2[1+(-1)^{n+1}]}{n^3\pi^3}.$$

Thus,

$$\begin{aligned} U(x,t) &= \sum_{n=1}^{\infty} \frac{4L^2[1+(-1)^{n+1}]}{n^3\pi^3} e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L} \\ &= \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} e^{-(2n-1)^2\pi^2 kt/L^2}\sin\frac{(2n-1)\pi x}{L}. \end{aligned} \tag{14.25}$$

It is plotted for $t = 5000$ s with $L = 1$ m, and $k = 12.4 \times 10^{-6}$ m^2/s in Figure 14.2. If we choose $N = 10$ subintervals for the finite difference mesh, then $\Delta x = 1/10$. Condition 14.24 requires

$$\Delta t \leq \frac{(1/10)^2}{2(12.4 \times 10^{-6})} \approx 403.2.$$

If we select $\Delta t = 10$ in order to compare the finite difference solution to the analytic solution, then

$$U_{n,p} = \sum_{\ell=1}^{9} C_\ell \left[1 - \frac{2(12.4 \times 10^{-6})(10)}{(1/10)^2}\left(1 - \cos\frac{\ell\pi}{10}\right)\right]^p \sin\frac{n\pi\ell}{10},$$

where

$$C_\ell = \frac{2}{10}\sum_{j=1}^{9}\frac{j}{10}\left(1 - \frac{j}{10}\right)\sin\frac{j\pi\ell}{10}.$$

Thus,

$$U_{n,p} = \frac{1}{500}\sum_{\ell=1}^{9}\sum_{j=1}^{9} j(10-j)\sin\frac{j\pi\ell}{10}\left[1 - 24.8 \times 10^{-3}\left(1 - \cos\frac{\ell\pi}{10}\right)\right]^p \sin\frac{n\pi\ell}{10}. \quad (14.26)$$

When the points $U_{n,500}$, $n = 0, \ldots, 10$ are plotted and joined by straight lines, the graph is also shown in Figure 14.2. It is indistinguishable from the previous plot. Exercise 1 compares these solution numerically. It also investigates what happens as the time step Δt is made larger, but always satisfying criteria 14.24.

Suppose we choose $\Delta t = 1000$, thus, violating condition 14.24, with $p = 5$ to obtain the temperature at $t = 5000$. Then

$$U_{n,5} = \frac{1}{500}\sum_{\ell=1}^{9}\sum_{j=1}^{9} j(10-j)\sin\frac{j\pi\ell}{10}\left[1 - 2.48\left(1 - \cos\frac{\ell\pi}{10}\right)\right]^5 \sin\frac{n\pi\ell}{10}.$$

This is plotted in Figure 14.3. It is erratic.

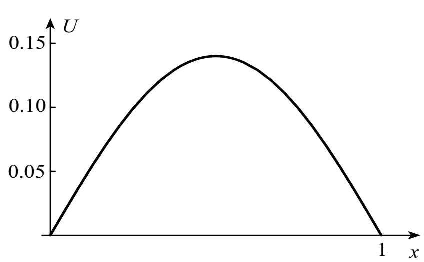


Figure 14.2

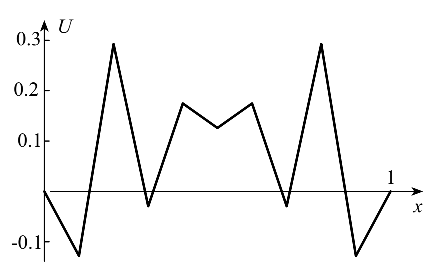


Figure 14.3

EXERCISES 14.3

1. (a) Tabulate solution 14.25 at $t = 5000$ for $x = 0.1, 0.2, \ldots, 0.9$. To compare finite differences (numerically as opposed to graphically as in Figure 14.2), tabulate solution 14.26 for $n = 1, 2, \ldots, 9$.

(b) What function replaces solution 14.26 if Δt is chosen as 50 rather than 10. Repeat part (a) with this function. Do approximations deteriorate with the larger time step?

(c) Repeat part (b) with time step $\Delta t = 200$.

§14.4 Numerical Solution of Partial Difference Equations for Parabolic PDEs

In spite of our calculations in Section 14.3, it is usually impossible to find exact solutions of initial boundary value problems for pdes. Instead, the pde is iterated numerically to find approximations to the unknown function at mesh points. We consider the procedure here for problem 14.16, a difference problem consistent with heat conduction problem 14.13. When we express the classic explicit scheme 14.16a in the form

$$U_{n,p+1} = U_{n,p} + \frac{ks}{h^2}\left(U_{n+1,p} - 2U_{n,p} + U_{n-1,p}\right), \tag{14.27}$$

it defines the solution of the difference equation at position x_n and time t_{p+1} in terms of its values at positions x_{n-1}, x_n, and x_{n+1}, all at time t_p. In other words, once we know the solution at time t_p, we can advance it to time t_{p+1}. With values specified at time $t_0 = 0$ by initial conditions 14.16d, we can find values at t_1; we can then move forward to time t_2, and so on. We have shown this pictorially in Figure 14.4; values at the open circles give the value at the shaded circle.

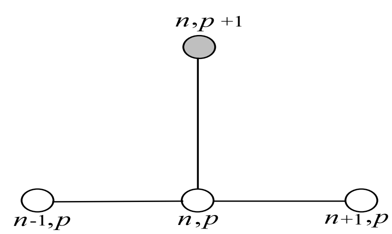


Figure 14.4

To illustrate, suppose that the initial temperature is $f(x) = x(L-x)$, and that we choose a mesh with $N = 10$. According to our results in Section 14.3, when $L = 1$ and $k = 12.4 \times 10^{-6}$, we should choose Δt no larger than 400. Suppose we opt for $s = \Delta t = 10$. Then equation 14.27 becomes

$$U_{n,p+1} = U_{n,p} + \frac{k}{10}\left(U_{n+1,p} - 2U_{n,p} + U_{n-1,p}\right). \tag{14.28a}$$

We add boundary conditions and an initial condition to complete the initial boundary value problem,

$$U_{0,p} = 0, \quad p \geq 0, \tag{14.28b}$$

$$U_{10,p} = 0, \quad p \geq 0, \tag{14.28c}$$

$$U_{n,0} = \frac{n}{10}\left(1 - \frac{n}{10}\right) = \frac{n(10-n)}{100}, \quad n = 1, \ldots, 9. \tag{14.28d}$$

Results:
Convergence and Stability

Because pde 14.27 is consistent with heat equation 14.13a, truncation errors in replacing derivatives with finite differences approach zero as h and s approach zero. When the above procedure is used to approximate the solution of the pde, errors due to calculations are also incurred. Suppose that $U(x,t)$ is the solution of problem 14.13, and suppose for the purposes of this discussion only, that $U(x_n, t_p)$ represents values of this function at mesh points (x_n, t_p), not values $U_{n,p}$. Differences between values $U(x_n, t_p)$ and $U_{n,p}$ encompass both truncation and calculation errors. We say that problem 14.16 represents a **convergent** method if differences between these values approach zero as h and s approach zero; that is,

$$\lim_{h,s\to 0} |U(x_n, t_p) - U_{n,p}| = 0, \quad p = 1, \ldots, \quad n = 1, \ldots, N-1. \tag{14.29}$$

As the mesh size decreases, approximate solutions of initial boundary value problem 14.16 must approach values of the solution to problem 14.13 at every mesh point. It is difficult to determine directly whether a given partial difference problem is convergent. Fortunately, it is not necessary to do so (directly). It follows if the numerical procedure associated with the problem is *stable*, a concept that we now discuss.

Stability is an important aspect of any numerical procedure. We describe it with respect to problem 14.16. Suppose $U_{n,p}$ is a solution of the problem for some given function $f(x)$. Suppose that $V_{n,p}$ is the solution to the same problem corresponding to a small perturbation $\epsilon(x)$ of $f(x)$. Differences $|U_{n,p} - V_{n,p}|$ represent how the difference in initial values is propagated in time. The numerical procedure is said to be **stable** if these differences are bounded as t increases. If values of h and s must be functionally related for bounded differences, the method is said to be **conditionally stable**. We suspect that this is the case for classic explicit scheme 14.16 due to the fact that h and s had to satisfy condition condition 14.24 in Section 14.3.

Convergence and stability go hand-in-hand for consistent systems. This is known as the **Lax Equivalence Theorem**.

Theorem 14.1 (**Lax Equivalence**) Given a well-posed initial boundary value problem associated with a parabolic PDE, a consistent finite difference initial boundary value problem is stable if, and only if, it is convergent.

Once a consistent finite difference scheme is known to be stable, then it must converge to the solution of the associated PDE at mesh points. A common technique to test stability of a pde is called **Von Neumann stability**. It is a necessary condition for numerical stability, in general, and is often sufficient (although we shall not demonstrate that here). The method assumes that the error at each mesh point can be expressed in the form

$$E_{n,p} = E(x_n, t_p) = e^{\gamma t_p} e^{i\beta x_n} = e^{\gamma ps} e^{i\beta nh}. \tag{14.30}$$

With β real, $e^{i\beta nh} = \cos\beta nh + i\sin\beta nh$ is an error introduced at time $t = 0$ with modulus unity. In general, γ is allowed to be complex, (but most often it is real), and the modulus $|e^{\gamma ps}|$ is an amplification factor that represents how errors are propagated in time. The pde is said to be Von Neumann stable if

$$|e^{\gamma s}| \le 1, \tag{14.31}$$

for all choices of γ, and this implies that $|e^{\gamma ps}| \le 1$ for all p. As we shall show, $e^{\gamma s}$ depends on β, s, and h, so that a finite difference scheme is Von Neumann stable if condition 14.31 is valid for all β, s and h. If a condition on h and s must be satisfied in order for inequality 14.31 to hold, a scheme is said to be **conditionally stable**.

We now confirm that the classic explicit scheme is conditionally stable. We do this by substituting 14.30 into pde 14.16a,

$$e^{\gamma(p+1)s} e^{i\beta nh} = e^{\gamma ps} e^{i\beta nh} + \frac{ks}{h^2}[e^{\gamma ps} e^{i\beta(n+1)h} - 2e^{\gamma ps} e^{i\beta nh} + e^{\gamma ps} e^{i\beta(n-1)h}].$$

When we divide by $e^{\gamma ps} e^{i\beta nh}$, we obtain

$$e^{\gamma s} = 1 + \frac{ks}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) = 1 + \frac{2ks}{h^2}(\cos\beta h - 1) = 1 - \frac{4ks}{h^2}\sin^2\frac{\beta h}{2}.$$

For Von Neumann stability of the classic explicit scheme, condition 14.31 requires

$$\left|1 - \frac{4ks}{h^2}\sin^2\frac{\beta h}{2}\right| \le 1.$$

This reduces to

$$\frac{2ks}{h^2} \le \frac{1}{\sin^2(\beta h/2)}.$$

This will be satisfied for all β, if

$$\frac{2ks}{h^2} \le 1, \qquad \text{or,} \qquad \Delta t \le \frac{(\Delta x)^2}{2k}.$$

This is condition 14.24 that we saw in Section 14.3; it shows that the classic explicit scheme is conditionally stable.

There is one aspect of the pde for heat conduction that does not emulate its counterpart for the PDE. We have seen that information is transmitted instantaneously by the PDE for heat conduction. An insertion of any amount of heat at any point is immediately felt at every other point. Such is not the case for the pde. Information is transmitted with finite velocity. To see this suppose the initial temperature for the rod represented by problem 14.16 is zero at every node except at some interior node near the centre of the rod, call it node x_i, where temperature is equal to one. After the first time step, temperature is nonzero at nodes x_{i-1}, x_i, and x_{i+1}; after the second time step, temperature is nonzero at nodes x_{i-2}, x_{i-1}, x_i, x_{i+1}, and x_{i+2}. We have shown this in Figure 14.5 where the x's represent nonzero temperatures at the various time steps. What this shows is that the effect of the initial temperature is transmitted with finite speed $\Delta x/\Delta t$.

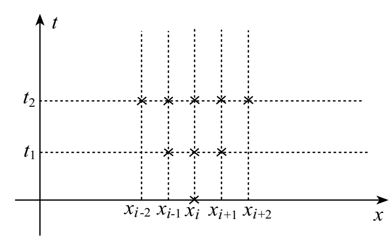


Figure 14.5

EXERCISES 14.4

§14.5 Nonhomogeneous Parabolic Problems

The nonhomogeneous initial boundary value problem associated with heat conduction problem 14.13 is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{kg(x,t)}{\kappa}, \quad 0 < x < L, \quad t > 0, \tag{14.32a}$$

$$U(0,t) = f_1(t), \quad t > 0, \tag{14.32b}$$

$$U(L,t) = f_2(t), \quad t > 0, \tag{14.32c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{14.32d}$$

A source term $g(x,t)$ has been added to the PDE, and end temperatures are no longer zero. The corresponding finite difference initial value problem is

$$U_{n,p+1} = U_{n,p} + \frac{ks}{h^2}(U_{n+1,p} - 2U_{n,p} + U_{n-1,p}) + \frac{ksg_{n,p}}{\kappa},$$
$$p = 0, \ldots, \quad n = 1, \ldots, N-1, \tag{14.33a}$$

$$U_{0,p} = f_{1,p} = f_1(t_p), \quad p \geq 0, \tag{14.33b}$$

$$U_{N,p} = f_{2,p} = f_2(t_p), \quad p \geq 0, \tag{14.33c}$$

$$U_{n,0} = f_n = f(x_n), \quad n = 1, \ldots, N-1. \tag{14.33d}$$

EXERCISES 14.5

§14.6 Neumann and Robin Boundary Conditions

Finite differences can be adapted to handle problems 14.13 and 14.32 when boundary conditions are Neumann and/or Robin. Dirichlet boundary conditions are used as input values for the pde; for Neumann and/or Robin boundary conditions, values of the unknown function are calculated (approximately) at the ends of the rod and then input. For instance, suppose boundary condition 14.32b is Neumann,

$$U_x(0,t) = g(t), \quad t > 0,$$

so that heat flux across the end $x = 0$ of the rod is prescribed. If we replace the partial derivative with a central difference at each time step t_p, we obtain

$$\frac{U(h,t_p) - U(-h,t_p)}{2h} = g(t_p) = g_p,$$

or with our subscript notation,

$$U_{1,p} = U_{-1,p} + 2hg_p.$$

This equation defines temperature $U_{-1,p} = U(-h,t_p)$, at a fictitious point $(-h,t_p)$ outside the rod for each time step t_p. Now, pde 14.16a is to be satisfied for $p = 0, \ldots,$ and $n = 1, \ldots, N-1$. If we also evaluate it at $n = 0$, we get

$$U_{0,p+1} = U_{0,p} + \frac{ks}{h^2}\left(U_{1,p} - 2U_{0,p} + U_{-1,p}\right).$$

If we substitute $U_{-1,p} = U_{1,p} - 2hg_p$, we obtain

$$U_{0,p+1} = U_{0,p} + \frac{ks}{h^2}\left(U_{1,p} - 2U_{0,p} + U_{1,p} - 2hg_p\right),$$

or,

$$U_{0,p+1} = U_{0,p} + \frac{2ks}{h^2}\left(U_{1,p} - U_{0,p} - hg_p\right). \tag{14.34}$$

This equation approximates temperature at the end $x = 0$ of the rod at each time step so that it can be input to calculate temperature at the first mesh point at the next time step. Exercise 1 verifies that when the boundary condition at $x = L$ is Neumann,

$$U_x(L,t) = g(t), \quad t > 0,$$

approximate values of the end of the rod at each time step are

$$U_{N,p+1} = U_{N,p} + \frac{2ks}{h^2}\left(U_{N-1,p} - U_{N,p} + hg_p\right). \tag{14.35}$$

When the boundary condition at $x = 0$ is Robin,

$$-l_1U_x(0,t) + h_1U(0,t) = g(t), \quad t > 0,$$

Exercise 2 derives the following expression for temperature at the end $x = 0$ of the rod at each time step

$$U_{0,p+1} = U_{0,p} + \frac{2ks}{h^2}\left[U_{1,p} - \left(1 + \frac{hh_1}{l_1}\right)U_{0,p} + \frac{hg_p}{l_1}\right]. \tag{14.36}$$

Similarly, for a Robin boundary condition at $x = L$,

$$l_2 x(L,t) + h_2 U(L,t) = g(t), \quad t > 0,$$

Exercise 3 derives the expression for temperature at the end $x = L$ of the rod at each time step

$$U_{N,p+1} = U_{N,p} + \frac{2ks}{h^2}\left[U_{N-1,p} - \left(1 + \frac{hh_2}{l_2}\right)U_{N,p} + \frac{hg_p}{l_2}\right]. \qquad (14.37)$$

EXERCISES 14.6

1. Verify equation 14.35.

2. Derive equation 14.36.

3. Derive equation 14.37.

§14.7 Other Partial Difference Equations for Parabolic PDEs

Because the grid spacing $h = \Delta x$ and the time step $s = \Delta t$ must satisfy condition 14.24, the classic explicit partial difference equation 14.16a is conditionally stable. If h is chosen very small in order to ensure accuracy of approximations, then s might be exceedingly small, resulting in huge numbers of calculations incurred in advancing initial values to large values of time. Other partial difference equations have been developed in an attempt to decrease the number of calculations. We discuss the stability of a few of them here. A method is called **explicit** if evaluation of $U_{n,p}$ at time step t_{p+1} is in terms of $U_{n,p}$ at previous times. For example, evaluation of $U_{n,p+1}$ in the classic explicit scheme 14.16a is in terms of $U_{n+1,p}$, $U_{n,p}$, and $U_{n-1,p}$, all at time t_p.

The Richardson Explicit Approximation

When the forward time difference of the classic explicit approximation is replaced by a central difference, the result is called the **Richardson explicit** scheme, or sometimes the **leapfrog** scheme. It is

$$U_{n,p+1} = U_{n,p-1} + \frac{2ks}{h^2}\left(U_{n+1,p} - 2U_{n,p} + U_{n-1,p}\right), \tag{14.38}$$

shown pictorially in Figure 14.6. Values at the open circles are used to calculate values at the shaded circle. Although more accurate in time ($O(h^2, s^2)$), we demonstrate that the method is unstable, and should not therefore be used. When we substitute $E_{n,p} = e^{\gamma ps}e^{i\beta nh}$ into the Richardson scheme to see how errors are propagated, we obtain

$$e^{\gamma(p+1)s}e^{i\beta nh} = e^{\gamma(p-1)s}e^{i\beta nh} + \frac{2ks}{h^2}[e^{\gamma ps}e^{i\beta(n+1)h} - 2e^{\gamma ps}e^{i\beta nh} + e^{\gamma ps}e^{i\beta(n-1)h}].$$

Division by $e^{\gamma(p-1)s}e^{i\beta nh}$ gives

$$e^{2\gamma s} = 1 + \frac{2kse^{\gamma s}}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) = 1 + \frac{4kse^{\gamma s}}{h^2}(\cos\beta h - 1) = 1 - \frac{8kse^{\gamma s}}{h^2}\sin^2\frac{\beta h}{2}.$$

This is a quadratic equation in $e^{\gamma s}$ with solutions

$$e^{\gamma s} = \frac{-\dfrac{8ks}{h^2}\sin^2\dfrac{\beta h}{2} \pm \sqrt{\dfrac{64k^2s^2}{h^4}\sin^4\dfrac{\beta h}{2} + 4}}{2} = -\frac{4ks}{h^2}\sin^2\frac{\beta h}{2} \pm \sqrt{1 + \frac{16k^2s^2}{h^4}\sin^4\frac{\beta h}{2}}.$$

Since $\left|-\dfrac{4ks}{h^2}\sin^2\dfrac{\beta h}{2} - \sqrt{1 + \dfrac{16k^2s^2}{h^4}\sin^4\dfrac{\beta h}{2}}\right| > 1$, we cannot guarantee Von Neumann's condition 14.31, and therefore Richardson's explicit scheme is unstable.

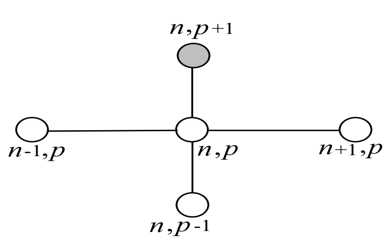


Figure 14.6

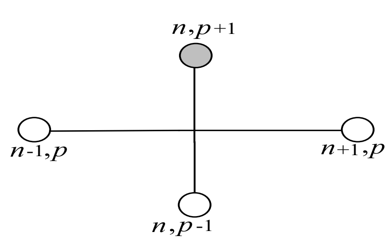


Figure 14.7

The DuFort-Frankel Explicit Approximation

If we replace $U_{n,p}$ in the Richardson model by the average $(U_{n,p+1} + U_{n,p-1})/2$, we obtain

$$U_{n,p+1} = U_{n,p-1} + \frac{2ks}{h^2}\left(U_{n+1,p} - U_{n,p+1} - U_{n,p-1} + U_{n-1,p}\right).$$

Reorganization gives

$$\left(1 + \frac{2ks}{h^2}\right) U_{n,p+1} = \frac{2ks}{h^2}(U_{n+1,p} + U_{n-1,p}) + \left(1 - \frac{2ks}{h^2}\right) U_{n,p-1}. \quad (14.39)$$

This is the DuFort-Frankel explicit scheme. It gives values of $U_{n,p}$ at one time step in terms of values at two previous time steps, shown pictorially in Figure 14.7. Because the technique uses three time steps, it requires initial values at $t = 0$ and $t = s$ to proceed to time $t = 2s$. Values at $t = 0$ are given, and values at $t = s$ could be obtained using say the classic explicit scheme.

To discuss stability of the scheme, we substitute $E_{n,p} = e^{\gamma ps}e^{i\beta nh}$ into the pde,

$$\left(1 + \frac{2ks}{h^2}\right) e^{\gamma(p+1)s}e^{i\beta nh} = \frac{2ks}{h^2}[e^{\gamma ps}e^{i\beta(n+1)h} + e^{\gamma ps}e^{i\beta(n-1)h}] + \left(1 - \frac{2ks}{h^2}\right) e^{\gamma(p-1)s}e^{i\beta nh}.$$

Division by $e^{\gamma(p-1)s}e^{i\beta nh}$ gives

$$\left(1 + \frac{2ks}{h^2}\right) e^{2\gamma s} = \frac{2kse^{\gamma s}}{h^2}(e^{i\beta h} + e^{-i\beta h}) + 1 - \frac{2ks}{h^2} = \frac{4kse^{\gamma a}}{h^2}\cos\beta h + 1 - \frac{2ks}{h^2}.$$

This is a quadratic in $e^{\gamma s}$ with solutions

$$\begin{aligned} e^{\gamma s} &= \frac{\frac{4ks}{h^2}\cos\beta h \pm \sqrt{\frac{16k^2s^2}{h^4}\cos^2\beta h + 4\left(1 - \frac{4k^2s^2}{h^4}\right)}}{2\left(1 + \frac{2ks}{h^2}\right)} \\ &= \frac{\frac{2ks}{h^2}\cos\beta h \pm \sqrt{1 - \frac{4k^2s^2}{h^4} + \frac{4k^2s^2}{h^4}\cos^2\beta h}}{1 + \frac{2ks}{h^2}} \\ &= \frac{\frac{2ks}{h^2}\cos\beta h \pm \sqrt{1 - \frac{4k^2s^2}{h^4}\sin^2\beta h}}{1 + \frac{2ks}{h^2}}. \end{aligned} \quad (14.40)$$

In Exercise 1, the modulus of this quantity is shown to be less than or equal to one for all β, h and s so that the DuFort-Frankel approximation is (unconditionally) stable.

The following two approximations are implicit methods.

The Backward Implicit Approximation

If the central difference for the second derivative in the classic explicit scheme is centred at (x_n, t_{p+1}) instead of (x_n, t_p), then

$$U_{n,p+1} - U_{n,p} = \frac{ks}{h^2}\left(U_{n+1,p+1} - 2U_{n,p+1} + U_{n-1,p+1}\right),$$

or,

$$\left(1 + \frac{2ks}{h^2}\right) U_{n,p+1} - \frac{ks}{h^2}(U_{n+1,p+1} + U_{n-1,p+1}) = U_{n,p}. \tag{14.41}$$

This is the **backward implicit scheme**; it is $O(h^2, s)$. The pictorial representation in Figure 14.8 indicates that three values of $U_{n,p}$ at time step t_{p+1} are defined by the scheme in terms of one value at time step t_p. Hence the adjective implicit. The disadvantage of this method is that at each step forward, it is necessary to solve $N - 1$ linear equations for values of $U_{n,p}$ at mesh points. The advantage of the method is that it is (unconditionally) stable. To show this, we substitute $E_{n,p} = e^{\gamma ps} e^{i\beta nh}$ into the pde,

$$\left(1 + \frac{2ks}{h^2}\right) e^{\gamma(p+1)s} e^{i\beta nh} - \frac{ks}{h^2}[e^{\gamma(p+1)s} e^{i\beta(n+1)h} + e^{\gamma(p+1)s} e^{i\beta(n-1)h}] = e^{\gamma ps} e^{i\beta nh}.$$

Division by $e^{\gamma ps} e^{i\beta nh}$ gives

$$\left(1 + \frac{2ks}{h^2}\right) e^{\gamma s} - \frac{kse^{\gamma s}}{h^2}(e^{i\beta h} + e^{-i\beta h}) = 1,$$

or,

$$\left(1 + \frac{2ks}{h^2}\right) e^{\gamma s} - \frac{2kse^{\gamma s}}{h^2} \cos\beta h = 1.$$

We can solve this for $e^{\gamma s}$,

$$e^{\gamma s} = \frac{1}{1 + \dfrac{2ks}{h^2} - \dfrac{2ks}{h^2}\left(1 - 2\sin^2 \dfrac{\beta h}{2}\right)} = \frac{1}{1 + \dfrac{4ks}{h^2} \sin^2 \dfrac{\beta h}{2}}.$$

Clearly, $|e^{\gamma s}|$ is less than or equal to one for all values of β, h and s, and therefore the scheme is (unconditionally) stable.

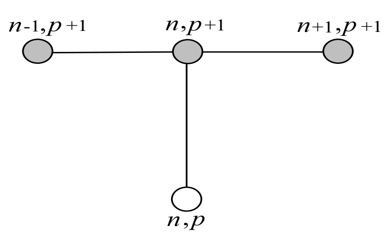


Figure 14.8

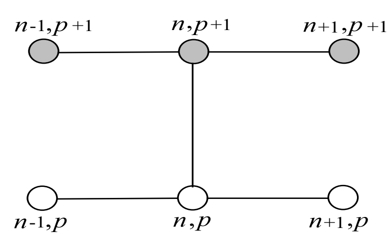


Figure 14.9

The Crank-Nicolson Implicit Approximation

If the central difference for the second derivative in the classic explicit scheme is replaced by the average of central differences at time step t_p and at time step t_{p+1}, the result is

$$U_{n,p+1} = U_{n,p} + \frac{ks}{2h^2}\left[(U_{n+1,p} - 2U_{n,p} + U_{n-1,p}) + (U_{n+1,p+1} - 2U_{n,p+1} + U_{n-1,p+1})\right].$$

This can be rearranged into

$$2\left(1+\frac{ks}{h^2}\right)U_{n,p+1}-\frac{ks}{h^2}(U_{n+1,p+1}+U_{n-1,p+1})$$
$$=2\left(1-\frac{ks}{h^2}\right)U_{n,p}+\frac{ks}{h^2}(U_{n+1,p}+U_{n-1,p}). \qquad (14.42)$$

This is the **Crank-Nicolson implicit scheme**; it is $O(h^2, s^2)$. The pictorial representation in Figure 14.9 indicates that three values of $U_{n,p}$ at time step t_{p+1} are defined by the scheme in terms of three values at time step t_p. Hence the method is implicit. But it is (unconditionally) stable. To show this, we substitute $E_{n,p} = e^{\gamma ps}e^{i\beta nh}$ into the pde,

$$2\left(1+\frac{ks}{h^2}\right)e^{\gamma(p+1)s}e^{i\beta nh}-\frac{ks}{h^2}[e^{\gamma(p+1)s}e^{i\beta(n+1)h}+e^{\gamma(p+1)s}e^{i\beta(n-1)h}]$$
$$=2\left(1-\frac{ks}{h^2}\right)e^{\gamma ps}e^{i\beta nh}+\frac{ks}{h^2}[e^{\gamma ps}e^{i\beta(n+1)h}+e^{\gamma ps}e^{i\beta(n-1)h}].$$

Division by $e^{\gamma ps}e^{i\beta nh}$ gives

$$2\left(1+\frac{ks}{h^2}\right)e^{\gamma s}-\frac{kse^{\gamma s}}{h^2}(e^{i\beta h}+e^{-i\beta h})=2\left(1-\frac{ks}{h^2}\right)+\frac{ks}{h^2}(e^{i\beta h}+e^{-i\beta h}),$$

or,

$$2\left(1+\frac{ks}{h^2}\right)e^{\gamma s}-\frac{2kse^{\gamma s}}{h^2}\cos\beta h=2\left(1-\frac{ks}{h^2}\right)+\frac{2ks}{h^2}\cos\beta h.$$

The solution for $e^{\gamma s}$ is

$$e^{\gamma s}=\frac{2\left(1-\frac{ks}{h^2}\right)+\frac{2ks}{h^2}\left(1-2\sin^2\frac{\beta h}{2}\right)}{2\left(1+\frac{ks}{h^2}\right)-\frac{2ks}{h^2}\left(1-2\sin^2\frac{\beta h}{2}\right)}=\frac{1-\frac{2ks}{h^2}\sin^2\frac{\beta h}{2}}{1+\frac{2ks}{h^2}\sin^2\frac{\beta h}{2}}.$$

The absolute value of this quantity is clearly less than one for all values of h, s, and β, and therefore the method is (unconditionally) stable.

EXERCISES 14.7

1. Show that quantity 14.40 has modulus less than or equal to one for all β, h, and s.

§14.8 Two-dimensional Heat Equation

The homogeneous, two-dimensional heat conduction PDE in some region R of the xy-plane is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right). \tag{14.43}$$

Suppose that R is the square $0 \le x \le L$, $0 \le y \le L$, and we discretize this region with a mesh using N equal subdivisions $h = \Delta x = \Delta y = L/N$ in both the x and y directions. With a time step $s = \Delta t$, we discretize the region $0 < x < L$, $0 < y < L$, $t > 0$ in which equation 14.43 is to hold. If $U_{n,m,p} = U(x_n, y_m, t_p)$ denotes approximate valuess for $U(x, y, t)$ at mesh points (x_n, y_m, t_p), the **classic explicit** partial difference equation corresponding to scheme 14.27 is

$$\begin{aligned} U_{n,m,p+1} = U_{n,m,p} + \frac{ks}{h^2}\big(&U_{n+1,m,p} + U_{n,m+1,p} - 4U_{n,m,p} \\ &+ U_{n-1,m,p} + U_{n,m-1,p}\big), \end{aligned} \tag{14.44}$$

$n = 1, \ldots, N-1$, $m = 1, \ldots, N-1$, $p = 1, \ldots$. It uses a forward difference in time and central differences in x and y. To determine the stability of this pde, we substitute $E_{n,m,p} = e^{\gamma ps}e^{i\beta(nh+mq)}$, where $q = \Delta y$,

$$\begin{aligned} e^{\gamma(p+1)s}e^{i\beta(nh+mq)} = e^{\gamma ps}e^{i\beta(nh+mq)} + \frac{ks}{h^2}\{&e^{\gamma ps}e^{i\beta[(n+1)h+mq]} + e^{\gamma ps}e^{i\beta[nh+(m+1)q]} \\ &- 4e^{\gamma ps}e^{i\beta(nh+mq)} + e^{\gamma ps}e^{i\beta[(n-1)h+mq]} + e^{\gamma ps}e^{i\beta[nh+(m-1)q]}\}. \end{aligned}$$

When we divide by $e^{\gamma ps}e^{i\beta(nh+mq)}$, we obtain

$$\begin{aligned} e^{\gamma s} &= 1 + \frac{ks}{h^2}(e^{i\beta h} + e^{i\beta q} - 4 + e^{-i\beta h} + e^{-i\beta q}) = 1 + \frac{2ks}{h^2}(\cos\beta h + \cos\beta q - 2) \\ &= 1 - \frac{4ks}{h^2}\left(\sin^2\frac{\beta h}{2} + \sin^2\frac{\beta q}{2}\right). \end{aligned}$$

For Von Neumann stability, condition 14.31 requires

$$\left|1 - \frac{4ks}{h^2}\left(\sin^2\frac{\beta h}{2} + \sin^2\frac{\beta q}{2}\right)\right| \le 1.$$

This reduces to

$$\frac{2ks}{h^2} \le \frac{1}{\sin^2(\beta h/2) + \sin^2(\beta q/2)}.$$

This will be satisfied for all β, if

$$\frac{2ks}{h^2} \le \frac{1}{2}, \qquad \text{or,} \qquad \Delta t \le \frac{(\Delta x)^2}{4k}. \tag{14.45}$$

In other words, pde 14.44 is conditionally stable.

Dufort-Frankel Scheme

If we replace the forward time difference with a central difference, and $U_{n,m,p}$ with the average $(U_{n,m,p+1} + U_{n,m,p-1})/2$, we obtain the **Dufort-Frankel** explicit scheme

$$\frac{U_{n,m,p+1} - U_{n,m,p-1}}{2s} = \frac{k}{h^2}\big(U_{n+1,m,p} + U_{n,m+1,p} - 2(U_{n,m,p+1} + U_{n,m,p-1}) + U_{n-1,m,p} + U_{n,m-1,p}\big),$$

or,

$$\left(1 + \frac{4ks}{h^2}\right) U_{n,m,p+1} = \left(1 - \frac{4ks}{h^2}\right) U_{n,m,p-1} + \frac{2ks}{h^2}(U_{n+1,m,p} + U_{n-1,m,p}) + \frac{2ks}{h^2}(U_{n,m+1,p} + U_{n,m-1,p}). \quad (14.46)$$

It is unconditionally stable (Exercise 2). Because it uses values at three time levels, it is necessary to find values at time $t = s$ in order to initiate the scheme. These can be obtained by the classic explicit scheme.

Backward Implicit Scheme

If central differences for second derivatives in the classic explicit scheme are centred at (x_n, y_m, t_{p+1}) instead of (x_n, y_m, t_p), the result is the **backward implicit** scheme

$$U_{n,m,p+1} = U_{n,m,p} + \frac{ks}{h^2}\left(U_{n+1,m,p+1} + U_{n,m+1,p+1} - 4U_{n,m,p+1} + U_{n-1,m,p+1} + U_{n,m-1,p+1}\right),$$

or,

$$\left(1 + \frac{4ks}{h^2}\right) U_{n,m,p+1} = U_{n,m,p} + \frac{ks}{h^2}(U_{n+1,m,p+1} + U_{n,m+1,p+1} + U_{n-1,m,p+1} + U_{n,m-1,p+1}). \quad (14.47)$$

Like its one-dimensional counterpart, it is unconditionally stable (Exercise 3).

Crank-Nicolson Implicit Scheme

If central differences for second derivatives in the classic explicit scheme are replaced by averages of central differences at time step t_p and at time step t_{p+1}, the result is the **Crank-Nicolson implicit** scheme

$$U_{n,m,p+1} = U_{n,m,p} + \frac{ks}{2h^2}\big[(U_{n+1,m,p} + U_{n,m+1,p} - 4U_{n,m,p} + U_{n-1,m,p} + U_{n,m-1,p}) + (U_{n+1,m,p+1} + U_{n,m+1,p+1} - 4U_{n,m,p+1} + U_{n-1,m,p+1} + U_{n,m-1,p+1})\big].$$

Rearrangement gives

$$2\left(1 + \frac{2ks}{h^2}\right) U_{n,m,p+1} - \frac{ks}{h^2}(U_{n+1,m,p+1} + U_{n,m+1,p+1} + U_{n-1,m,p+1} + U_{n,m-1,p+1}) = 2\left(1 - \frac{2ks}{h^2}\right) U_{n,m,p} + \frac{ks}{h^2}(U_{n+1,m,p} + U_{n,m+1,p} + U_{n-1,m,p} + U_{n,m-1,p}). \quad (14.48)$$

It is also unconditionally stable (Exercise 4).

Irregular Shaped Regions

Finite differences, and even more so finite elements, show their indispensability when PDEs are to be considered on regions whose boundaries are not coordinate curves (in the plane) and coordinate surfaces (in space). For example, suppose PDE 14.43 is to describe heat flow in the elliptical plate of Figure 14.10. None of our analytic techniques are applicable to this problem. Finite differences can be adapted to irregular boundaries, with some difficulties, but the difficulties have more to do with computer implementation than with the theoretical aspects of the adaptation.

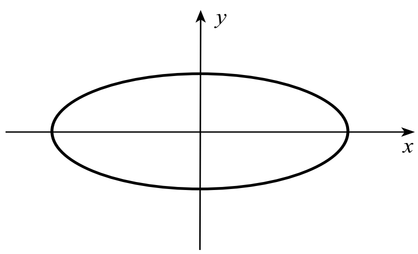


Figure 14.10

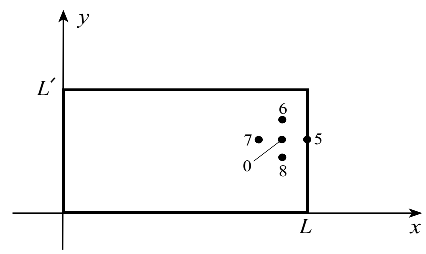


Figure 14.11

When the region is a rectangle (Figure 14.11), a central difference for the Laplacian at node 0 closest to boundary $x = L$ utilizes the boundary data at node 5. When the boundary is curved (Figure 14.12), a discretization of the region with the usual array of points results in very few mesh points on the boundary of the region. A central difference at 0 has node 5 (and node 6) outside the region. We need to replace the "central" difference formula for the Laplacian at node 0 with a difference formula that utilizes boundary data at nodes 1 and 2 in place of nodes 5 and 6. More generally, we need a difference formula that accommodates two horizontal and two vertical nodes at differing distances from node 0. We have shown this in Figure 14.13 where all four surrounding nodes are at different distances from node 0.

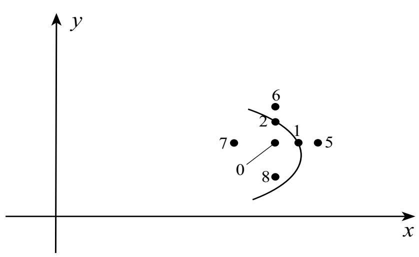


Figure 14.12

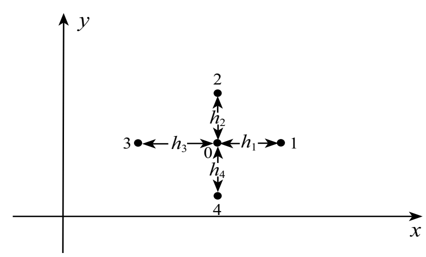


Figure 14.13

Suppose we denote the values of the function $U(x, y)$ at the five nodes by $U(0), \ldots,$ $U(4)$. We seek an approximation to the Laplacian of $U(x, y)$ at node 0 as a linear combination of $U(0), \ldots, U(4)$,

$$(U_{xx} + U_{yy})_{|\text{node } 0} = \sum_{i=0}^{4} \alpha_i U(i). \tag{14.49}$$

To find suitable constants α_i, we represent $U(x, y)$ at nodes 1, 2, 3, and 4 in Taylor series at node 0. If we extend the notation $U(i)$, to include derivatives, such as $U_x(0)$, then

$$U(1) = U(0) + U_x(0)h_1 + \frac{1}{2}U_{xx}(0)h_1^2 + \frac{1}{3!}U_{xxx}(0)h_1^3 + \cdots,$$
$$U(2) = U(0) + U_y(0)h_2 + \frac{1}{2}U_{yy}(0)h_2^2 + \frac{1}{3!}U_{yyy}(0)h_2^3 + \cdots,$$
$$U(3) = U(0) - U_x(0)h_3 + \frac{1}{2}U_{xx}(0)h_3^2 - \frac{1}{3!}U_{xxx}(0)h_3^3 + \cdots,$$
$$U(4) = U(0) - U_y(0)h_4 + \frac{1}{2}U_{yy}(0)h_4^2 - \frac{1}{3!}U_{yyy}(0)h_4^3 + \cdots.$$

When we substitute these into equation 14.49, and gather like terms, the result is

$$(U_{xx} + U_{yy})_{\text{node } 0} = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)U(0) + (\alpha_1 h_1 - \alpha_3 h_3)U_x(0) + (\alpha_2 h_2 - \alpha_4 h_4)U_y(0)$$
$$+ \frac{1}{2}(\alpha_1 h_1^2 + \alpha_3 h_3^2)U_{xx}(0) + \frac{1}{2}(\alpha_2 h_2^2 + \alpha_4 h_4^2)U_{yy}(0) + \cdots.$$

For the the right side to agree with the left, we require

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0,$$
$$\alpha_1 h_1 - \alpha_3 h_3 = 0,$$
$$\alpha_2 h_2 - \alpha_4 h_4 = 0,$$
$$\alpha_1 h_1^2 + \alpha_3 h_3^2 = 2,$$
$$\alpha_2 h_2^2 + \alpha_4 h_4^2 = 2.$$

The solution of these equations is

$$\alpha_0 = -2\left(\frac{1}{h_1 h_3} + \frac{1}{h_2 h_4}\right), \qquad \alpha_1 = \frac{2}{h_1(h_1 + h_3)}, \qquad \alpha_2 = \frac{2}{h_2(h_2 + h_4)},$$

$$\alpha_3 = \frac{2}{h_3(h_1 + h_3)}, \qquad \alpha_4 = \frac{2}{h_4(h_2 + h_4)}.$$

Thus, a difference formula for the Laplacian of $U(x, y)$ at node 0 in Figure 14.13 in terms of values of the function at the five nodes is

$$(U_{xx} + U_{yy})_{|\text{node } 0} = -2\left(\frac{1}{h_1 h_3} + \frac{1}{h_2 h_4}\right)U(0) + \frac{2U(1)}{h_1(h_1 + h_3)} + \frac{2U(2)}{h_2(h_2 + h_4)}$$
$$+ \frac{2U(3)}{h_3(h_1 + h_3)} + \frac{2U(4)}{h_4(h_2 + h_4)}. \tag{14.50}$$

The reader can perhaps appreciate that the computer implementation of this formula at each node of the region of Figure 14.43, wherein it is required, could be a programming nightmare.

EXERCISES 14.8

1. Generalize pde 14.44 to a rectangle $0 \le x \le L$, $0 \le y \le L'$. What is the stability condition replacing inequality 14.45?
2. Verify that the Dufort-Frankel scheme 14.46 is stable.
3. Verify that the backward implicit scheme 14.47 is stable.
4. Verify that the Crank-Nicolson scheme 14.48 is stable.

§14.9 Two-dimensional Heat Equation in Polar and Spherical Coordinates

The homogeneous, two-dimensional heat equation in polar, or cylindrical, coordinates when heat is only a function of time and the radial coordinate r is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right). \tag{14.51}$$

For a similar situation in spherical coordinates, the PDE for $U(r,t)$ is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right). \tag{14.52}$$

Both of these, and the Cartesian equation, are contained in

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{a}{r}\frac{\partial U}{\partial r}\right). \tag{14.53}$$

When $a = 0$ we have the Cartesian equation; for $a = 1$, we have the polar or cylindrical equation; and for $a = 2$, we have the spherical equation. The pde associated with equation 14.53 using a forward time difference and central spatial differences is

$$\frac{U_{n,p+1} - U_{n,p}}{s} = k\left[\left(\frac{U_{n+1,p} - 2U_{n,p} + U_{n-1,p}}{h^2}\right) + \frac{a}{nh}\left(\frac{U_{n+1,p} - U_{n-1,p}}{2h}\right)\right],$$

or,

$$U_{n,p+1} = \left(1 - \frac{2ks}{h^2}\right)U_{n,p} + \frac{ks}{h^2}\left[(U_{n+1,p} + U_{n-1,p}) + \frac{a}{2n}(U_{n+1,p} + U_{n-1,p})\right]. \tag{14.54}$$

In Exercise 1, it is shown that for both $a = 1$ and $a = 2$, this pde is conditionally stable with stability bound

$$\Delta t \le \frac{(\Delta r)^2}{2k}. \tag{14.55}$$

Certainly we could develop backward implicit and Crank-Nicolson implicit methods as alternatives to pde 14.54, both of which are unconditionally stable. If the region of consideration contains $r = 0$ (the origin in polar and spherical coordinates and the z-axis in cylindrical coordinates), pde 14.54 has a singularity ($n = 0$). It can be modified, however, to accommodate this value. First we note that radial symmetry of the solution requires $U_r(0,t)$ to be equal to zero. Furthermore,

$$\lim_{r\to 0}\frac{U_r(r,t)}{r} = \lim_{r\to 0}\frac{U_r(r,t) - U_r(0,t)}{r} = U_{rr}(0,t).$$

Consequently, by taking limits of PDE 14.53 as $r \to 0$, we obtain

$$\frac{\partial U(0,t)}{\partial t} = k\left[\frac{\partial^2 U(0,t)}{\partial r^2} + a\frac{\partial^2 U(0,t)}{\partial r^2}\right] = k(1+a)\frac{\partial^2 U(0,t)}{\partial r^2}.$$

Finite differences now give

$$\frac{U_{0,p+1} - U_{0,p}}{s} = k(1+a)\left(\frac{U_{1,p} - 2U_{0,p} + U_{-1,p}}{h^2}\right).$$

To eliminate the term $U_{-1,p}$, we use the finite difference equivalent of $U_r(0,t) = 0$, namely,

$$\frac{U_{1,p} - U_{-1,p}}{2h} = 0.$$

Thus,

$$\frac{U_{0,p+1} - U_{0,p}}{s} = k(1+a)\left(\frac{U_{1,p} - 2U_{0,p} + U_{1,p}}{h^2}\right),$$

or,

$$U_{0,p+1} = \left[1 - \frac{2ks(1+a)}{h^2}\right] U_{0,p} + \frac{2ks(1+a)}{h^2} U_{1,p}. \tag{14.56}$$

A Von Neumann stability analysis (Exercise 2) shows that this equation is conditionally stable with

$$\Delta t \le \frac{(\Delta r)^2}{2k(1+a)}. \tag{14.57}$$

With $a = 1$ and $a = 2$, stability has deteriorated at $r = 0$.

In summary, we use pde 14.56 when $r = 0$, and equation 14.54 at all other points.

EXERCISES 14.9

1. Verify stability requirement 14.55 for pde 14.54.

2. Verify stability requirement 14.57 for pde 14.56.

§14.10 Partial Difference Equations for the Wave Equation

The homogeneous initial boundary value problem for transverse vibrations of a finite string (or longitudinal vibrations of a bar) is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{14.58a}$$

$$y(0,t) = 0, \quad t > 0, \tag{14.58b}$$

$$y(L,t) = 0, \quad t > 0, \tag{14.58c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{14.58d}$$

$$y_t(x,0) = g(x), \quad 0 < x < L. \tag{14.58e}$$

Separation of variables leads to the infinite series solution

$$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L}, \tag{14.59a}$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \qquad B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L} dx. \tag{14.59b}$$

To find an initial boundary value problem associated with a finite difference equation to replace problem 14.58, we discretize that part of the xt-plane bounded by $0 < x < L$ and $t > 0$ as in Figure 14.1. When both partial derivatives in the PDE are replaced by central difference formulas, a consistent pde is obtained. Its truncation error is $O(h^2, s^2)$. With the usual notation $y(x_n, t_p) = y_{n,p}$, the pde is

$$\frac{y_{n,p+1} - 2y_{n,p} + y_{n,p-1}}{s^2} = c^2 \left(\frac{y_{n+1,p} - 2y_{n,p} + y_{n-1,p}}{h^2} \right), \tag{14.60}$$

or,

$$y_{n,p+1} = 2\left(1 - \frac{c^2 s^2}{h^2}\right) y_{n,p} - y_{n,p-1} + \frac{c^2 s^2}{h^2} \left(y_{n+1,p} + y_{n-1,p} \right). \tag{14.61a}$$

It must be satisfied for $p = 1, \ldots$ and $n = 1, \ldots, N-1$. It is shown pictorially in Figure 14.14. Values at two previous time steps are advanced to the next time step. To replace boundary conditions 14.58b,c, we extend values of n to $0, \ldots, N$, and demand that

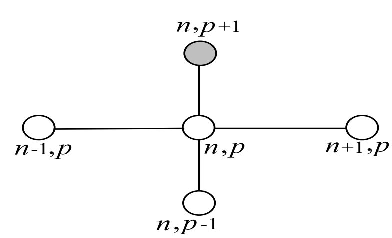


Figure 14.14

$$y_{0,p} = 0, \quad p \geq 0, \tag{14.61b}$$

$$y_{N,p} = 0, \quad p \geq 0. \tag{14.61c}$$

We replace initial condition 14.58d for the PDE with the following initial conditions for the pde,

$$y_{n,0} = f_n = f(x_n), \quad n = 1, \ldots, N-1. \tag{14.61d}$$

For initial condition 14.58e, we can use the same approach as for a Neumann boundary condition in Section 14.6. We replace the partial time derivative with a central difference at each node x_n,

$$\frac{y(x_n, s) - y(x_n, -s)}{2s} = g(x_n) = g_n,$$

or with our subscript notation,

$$y_{n,1} = y_{n,-1} + 2sg_n.$$

This equation defines displacement $y_{n,-1} = y(x_n, -s)$, at a fictitious time $(x_n, -s)$ before $t = 0$ at each node x_n. Now, pde 14.61a is to be satisfied for $p = 1, \ldots,$ and $n = 1, \ldots, N-1$. If we also evaluate it at $p = 0$, we get

$$y_{n,1} - 2y_{n,0} + y_{n,-1} = \frac{c^2 s^2}{h^2}\left(y_{n+1,0} - 2y_{n,0} + y_{n-1,0}\right).$$

If we substitute $y_{n,-1} = y_{n,1} - 2sg_n$, we obtain

$$y_{n,1} - 2y_{n,0} + y_{n,1} - 2sg_n = \frac{c^2 s^2}{h^2}\left(y_{n+1,0} - 2y_{n,0} + y_{n-1,0}\right),$$

or,

$$y_{n,1} = y_{n,0}\left(1 - \frac{c^2 s^2}{h^2}\right) + \frac{c^2 s^2}{2h^2}\left(y_{n+1,0} + y_{n-1,0}\right) + sg_n. \tag{14.62}$$

This equation approximates displacement at the first time step $t = s$ for each node so that it can be input into the pde to calculate displacement at the second time step for each node.

To investigate the stability of pde 14.61a, we substitute $E_{n,p} = e^{\gamma ps}e^{i\beta nh}$,

$$e^{\gamma(p+1)s}e^{i\beta nh} = 2\left(1 - \frac{c^2 s^2}{h^2}\right)e^{\gamma ps}e^{i\beta nh} - e^{\gamma(p-1)s}e^{i\beta nh} + \frac{c^2 s^2}{h^2}[e^{\gamma ps}e^{i\beta(n+1)h} + e^{\gamma ps}e^{i\beta(n-1)h}].$$

Division by $e^{\gamma(p-1)s}e^{i\beta nh}$ gives

$$\begin{aligned}
e^{2\gamma s} &= 2\left(1 - \frac{c^2 s^2}{h^2}\right)e^{\gamma s} - 1 + \frac{c^2 s^2}{h^2}(e^{i\beta h} + e^{-i\beta h})e^{\gamma s} \\
&= 2\left(1 - \frac{c^2 s^2}{h^2}\right)e^{\gamma s} - 1 + \frac{2c^2 s^2 e^{\gamma s}}{h^2}\cos\beta h \\
&= 2\left(1 - \frac{c^2 s^2}{h^2}\right)e^{\gamma s} - 1 + \frac{2c^2 s^2 e^{\gamma s}}{h^2}\left(1 - 2\sin^2\frac{\beta h}{2}\right).
\end{aligned}$$

Thus, $e^{\gamma s}$ must satisfy the quadratic equation

$$e^{2\gamma s} - 2\left(1 - \frac{2c^2 s^2}{h^2}\sin^2\frac{\beta h}{2}\right)e^{\gamma s} + 1 = 0,$$

with solutions

$$\begin{aligned}
e^{\gamma s} &= \frac{2\left(1 - \frac{2c^2 s^2}{h^2}\sin^2\frac{\beta h}{2}\right) \pm \sqrt{4\left(1 - \frac{2c^2 s^2}{h^2}\sin^2\frac{\beta h}{2}\right)^2 - 4}}{2} \\
&= 1 - \frac{2c^2 s^2}{h^2}\sin^2\frac{\beta h}{2} \pm \sqrt{\left(1 - \frac{2c^2 s^2}{h^2}\sin^2\frac{\beta h}{2}\right)^2 - 1}.
\end{aligned}$$

If we let $R = \dfrac{2c^2s^2}{h^2}\sin^2\dfrac{\beta h}{2}$, then

$$e^{\gamma s} = 1 - R \pm \sqrt{(1-R)^2 - 1}.$$

If $(1-R)^2 - 1 < 0$, then

$$|e^{\gamma s}| = |(1-R) \pm \sqrt{1-(1-R)^2}i| = \sqrt{(1-R)^2 + 1 - (1-R)^2} = 1.$$

If $(1-R)^2 - 1 \geq 0$, stability condition 14.31 requires

$$|1 - R \pm \sqrt{(1-R)^2 - 1}| \leq 1 \qquad \text{or,} \qquad -1 \leq 1 - R \pm \sqrt{R^2 - 2R} \leq 1.$$

This is equivalent to

$$-2 \leq -R \pm \sqrt{R^2 - 2R} \leq 0 \qquad \text{or,} \qquad 0 \leq R \pm \sqrt{R^2 - 2R} \leq 2.$$

Since the left inequality is satisfied, we require

$$R \pm \sqrt{R^2 - 2R} \leq 2 \qquad \text{or,} \qquad \pm\sqrt{R^2 - 2R} \leq 2 - R.$$

Squaring gives

$$R^2 - 2R \leq R^2 - 4R + 4 \qquad \text{which leads to} \qquad R \leq 2.$$

For stability we therefore require

$$\frac{2c^2s^2}{h^2}\sin^2\frac{\beta h}{2} \leq 2, \qquad \text{or,} \qquad \frac{c^2s^2}{h^2} \leq \frac{1}{\sin^2(\beta h/2)}.$$

This will be satisfied for all β if we demand that c^2s^2/h^2 be less than 1; that is

$$\frac{c^2s^2}{h^2} \leq 1.$$

Stability is therefore guaranteed if

$$s \leq \frac{h}{c} \qquad \text{or,} \qquad \Delta t \leq \frac{\Delta x}{c}. \tag{14.63}$$

In other words, pde 14.61a is conditionally stable. Condition 14.63 is often called the **Courant-Friedrichs-Lewy condition**; it says that the velocity $\Delta x/\Delta t$ at which information is propagated by the pde must be greater than the velocity c at which information is propagated by the one-dimensional wave equation.

In Section 14.3, we solved the difference equation for the heat equation. We could do the same for problem 14.61, but calculations become formidable. It is, however, instructive to begin the procedure in that it once again yields condition 14.63. The calculations are in Exercise 4.

Neumann and/or Robin boundary conditions can be handled as in Section 14.6.

Other explicit schemes can be developed for the wave equation, but they seem to have little use. Implicit methods can also be derived. We mention one of them. If we replace the second partial derivative $\partial^2 y/\partial x^2$ with the average of central differences at time steps $p-1$ and $p+1$ (a Crank-Nicolson approximation), we obtain

$$\frac{y_{n,p+1} - 2y_{n,p} + y_{n,p-1}}{s^2} = c^2\left[\frac{(y_{n+1,p+1} - 2y_{n,p+1} + y_{n-1,p+1}) + (y_{n+1,p-1} - 2y_{n,p-1} + y_{n-1,p-1})}{2h^2}\right],$$

or,

$$\frac{c^2s^2}{2h^2}(y_{n+1,p+1} + y_{n-1,p+1}) = 2y_{n,p} - \left(1 + \frac{c^2s^2}{h^2}\right)y_{n,p-1} + \frac{c^2s^2}{2h^2}(y_{n+1,p-1} + y_{n-1,p-1}). \qquad (14.64)$$

This scheme is stable (see Exercise 5).

EXERCISES 14.10

1. Modify pde 14.61a if gravity also acts on the string.

2. Modify pde 14.61a if a damping force proportional to velocity also acts on the string.

3. Modify pde 14.61a if a restoring force proportional to displacement also acts on the string.

4. (a) Show that if solutions of 14.61a,b,c are sought in the form $y_{n,p} = X_nT_p$, then X_n and T_p must satisfy

$$T_{p+1} - (\lambda + 2)T_p + T_{p-1} = 0,$$

and

$$X_{n+1} - \left(2 + \frac{h^2\lambda}{c^2s^2}\right)X_n + X_{n-1} = 0, \qquad X_0 = 0, \quad X_N = 0.$$

(b) Show that $X_n = B\sin\frac{n\pi\ell}{N}$, where B is an arbitrary constant, and ℓ is a positive integer.

Furthermore, possible values for λ are $\lambda_\ell = \frac{2c^2s^2}{h^2}\left(\cos\frac{\ell\pi}{N} - 1\right)$.

(c) Use the difference equation for T_p to derive condition 14.63.

5. Verify that implicit scheme 14.64 is stable.

§14.11 Partial Difference Equations for Elliptic PDEs

Application of finite differences to Laplace's equation and Poisson's equation results in a situation quite different from that for the heat and vibration equations. Consider first Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{14.65}$$

on a square $0 \le x \le L$, $0 \le y \le L$. We discretize the square with points $(x_n, y_m) = (nh, mh)$, where $h = \Delta x = \Delta y = L/N$, using therefore equal subdivisions in both directions. When second partial derivatives are replaced by central differences, the result is the pde

$$\frac{(V_{n+1,m} - 2V_{n,m} + V_{n-1,m}) + (V_{n,m+1} - 2V_{n,m} + V_{n,m-1})}{h^2} = 0,$$

or,

$$V_{n,m} = \frac{1}{4}\left(V_{n+1,m} + V_{n-1,m} + V_{n,m+1} + V_{n,m-1}\right). \tag{14.66}$$

We have shown the situation pictorially in Figure 14.15. The value of $V_{n,m}$ at any node is the average of its values at the four adjacent nodes. We cannot "advance" the solution from the x-axis, say, as we did for time-dependent problems. Because pde 14.66 must be satisfied at each interior node, it must be satisfied for $n, m = 1, \ldots, N-1$. In other words, we have a set of $(N-1)^2$ linear equations. Of these, $(N-3)^2$ are homogeneous (when $n, m = 2, \ldots, N-2$) because they do not involve boundary points. When boundary conditions are nonhomogeneous, Dirichlet, there are $2(N-1) + 2(N-3) = 4N-8$ nonhomogeneous equations. This linear system of equations can be solved in many ways, one of which is not Gaussian elimination (the number of equations is simply too large). Discussions on useful techniques and their computer implementations can be found in many references devoted to this aspect of the problem.

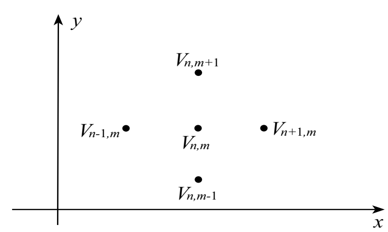


Figure 14.15

Scheme 14.66 is the simplest five-point finite difference approximation for Laplace's equation on a square with equal subdivisions in both directions. Its truncation error is $O(h^2, h^2)$, and it can be shown that no other five-point approximation to Laplace's equation on a square can be more accurate. It can be generalized in various ways, such as using unequal spacings, replacing the square with a rectangle, and using more accurate difference formulas. Suppose, for instance, that the square is replaced by a rectangle $0 \le x \le L$, $0 \le y \le L'$, thus forcing unequal subdivisions. With N equal subdivisions in the x-direction, $h_1 = \Delta x = L/N$, and M equal subdivisions in the y-direction, $h_2 = \Delta y = L'/M$, and central differences, equation 14.66 is replaced by

$$\frac{V_{n+1,m} - 2V_{n,m} + V_{n-1,m}}{h_1^2} + \frac{V_{n,m+1} - 2V_{n,m} + V_{n,m-1}}{h_2^2} = 0,$$

or,

$$V_{n,m} = \frac{h_1^2(V_{n,m+1} + V_{n,m-1}) + h_2^2(V_{n+1,m} + V_{n-1,m})}{2(h_2^2 + h_2^2)}. \qquad (14.67)$$

When $L = L'$ and $h = h_1 = h_2$, this reduces, as it should, to equation 14.66.

EXERCISES 14.11

1. Modify pde 14.66 in the case that Laplace's equation 14.65 is replaced by Poisson's equation with nonhomogeneity $f(x, y)$.

2. Modify pde 14.67 in the case that Laplace's equation 14.65 is replaced by Poisson's equation with nonhomogeneity $f(x, y)$.

3. Explain why pde 14.66 supports the maximum-minimum principle for Laplace's equation discussed in Section 6.7.

CHAPTER 15 Weighted Residuals

§15.1 Introduction

Consider solving Poisson's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y), \tag{15.1}$$

on some region R in the xy-plane such as that in Figure 15.1. With finite differences, the partial differential equation for $V(x, y)$ is replaced by a partial difference equation for approximations $V_{n,m} = V(x_n, y_m)$ at a predetermined set of mesh points (x_n, y_m) (see Section 14.11). Emphasis is shifted from an exact solution at every point in the region to a set of approximations at mesh points. With finite elements, region R is approximated by a set of subregions of various shapes such as triangles and rectangles, and $V(x, y)$ is approximated by a polynomial in x and y on each subregion. Emphasis is shifted to approximating polynomials on subregions, and these subregions are called **finite elements**. For each finite element, there is a polynomial approximation to the solution of the PDE for that element. The flexibility in choosing the subdivision of region R into finite elements, and the opportunity to choose different types of approximating polynomials, gives the method its power and versatility, a fact that we can only hint at in the space of one chapter (Chapter 16).

There is a multitude of ways to develop the theory associated with finite elements, and many of these approaches are discipline dependent. We have chosen an approach that is somewhat general in nature (not directed at any particular branch of engineering), but at the same time is not so abstract as to lose the reader after initial discussions. We approach finite elements through what is called the *method of weighted residuals*, henceforth shortened to MWR. It is important to understand from the outset that the MWR is a stand alone method for approximating solutions to PDEs (and ODEs). It is a technique that we use in finite elements, but it is not an inherent part of finite elements. Because of this, we devote this chapter to a discussion of MWR prior to introducing finite elements in Chapter 16.

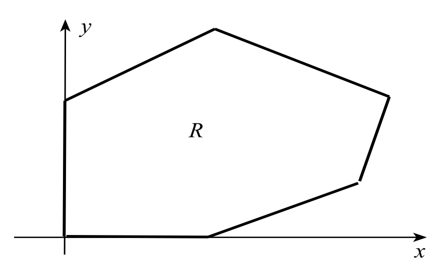


Figure 15.1

§15.2 The Method of Weighted Residuals

The method of weighted residuals is a very general method for approximating solutions to initial, boundary value problems in PDEs (or ODEs). In the case of a PDE, we represent the PDE symbolically by

$$L(U) = F, \tag{15.2}$$

where L is a partial differential operator (which may be linear or nonlinear), U is the unknown function, and F is a given function of the independent variables. For instance, in equation 15.1, L is the Laplacian ∇^2; for the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

L is $\dfrac{\partial^2}{\partial t^2} - c^2 \dfrac{\partial^2}{\partial x^2}$; and for the two-dimensional heat conduction equation

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right),$$

L is $\dfrac{\partial}{\partial t} - k\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$.
Accompanying each of these PDEs will be initial and/or boundary conditions.

In the MWR, the unknown function U in equation 15.2 is approximated by a linear combination of N preselected functions ϕ_n of the independent variables,

$$U_N = \sum_{n=1}^{N} c_n \phi_n, \tag{15.3}$$

where the coefficients c_n are to be determined. The functions ϕ_n are called various names depending on the discipline using them, such as **basis functions**, **shape functions**, or **interpolation functions**. We use the terminology basis functions; they are required to be linearly independent and constitute a complete set of functions. Recall that a set of functions is complete for some space of functions if every function in the space can be expressed as an infinite series of the complete set of functions. (For example, the eigenfunctions of a Sturm-Liouville system on the interval $a \le x \le b$ form a complete set for the space of piecewise smooth functions on the interval. Functions x^n, $n = 0, 1, 2 \ldots$ form a complete set for the space of continuous functions.) By adopting a set of complete functions, we are assured that the solution of the boundary value problem can be represented to any degree of accuracy by choosing sufficiently many terms in summation 15.3.

If basis functions ϕ_n do not satisfy equation 15.2, then neither does linear combination 15.3. As a result, if we substitute it into the PDE, the result will not be F. Alternatively, if we calculate $L(\sum_{n=1}^{N} c_n \phi_n) - F$, it will not be zero; it will be a function of the independent variables and the coefficients c_n. We denote this function by R,

$$R = L\left[\sum_{n=1}^{N} c_n \phi_n\right] - F, \tag{15.4}$$

and call it the **residual** of U_N. Our intuition tells us that the smaller the value of R, the better the linear combination of basis functions approximates the solution of the PDE. As a result, coefficients c_n should be chosen so that R is in some sense minimized. The MWR is an attempt to do this, but it is not immediately obvious how, or in what sense, minimization occurs. The method suggests that N linearly independent test functions w_m be chosen, and demands that integrals of the products of the w_m and R over the domain of the problem vanish. If we generically denote the domain by A, and the integral as a double integral, although it could equally well be a definite integral or a triple integral, then we have the conditions

$$\iint_A R w_m \, dA = 0, \quad m = 1, \dots, N. \tag{15.5}$$

Integrations in equations 15.5 lead to a system of N equations in the N unknown coefficients c_n. These equations will be linear when opertor L is linear, and they will be nonlinear when L is nonlinear. To see them more explicitly in the case that L is linear, we substitute expression 15.4 into equations 15.5,

$$\iint_A \left[L\left(\sum_{n=1}^{N} c_n \phi_n \right) - F \right] w_m \, dA = 0, \quad m = 1, \dots, N,$$

or,

$$\sum_{n=1}^{N} \left[\iint_A L(\phi_n) w_m \, dA \right] c_n = \iint_A F w_m \, dA, \quad m = 1, \dots, N. \tag{15.6}$$

Once integrals are evaluated, these are indeed N linear, nonhomogeneous equations in the unknown coefficients c_n. We could represent the system in matrix form

$$DC = B, \tag{15.7a}$$

where

$$D = \left(\iint_A L(\phi_n) w_m \, dA \right)_{N \times N}, \quad C = (c_n)_{N \times 1}, \quad B = \left(\iint_A F w_m \, dA \right)_{N \times 1}. \tag{15.7b}$$

Various names are given to the MWR depending on the choice of weight functions w_m. We discuss four of them here; there are others (including the least squares method).

Collocation Method

In the collocation method, N points, (x_m, y_m), called collocation points are chosen in the domain of the problem, and weight functions are Dirac-delta functions at these points,

$$w_m = \delta(x - x_m, y - y_m), \quad m = 1, \dots, N. \tag{15.8}$$

In this case, equations 15.6 become

$$\sum_{n=1}^{N} [L(\phi_n)]_{|(x_m, y_m)} c_n = F(x_m, y_m), \quad m = 1, \dots, N. \tag{15.9}$$

The method forces the residual to be zero at the collocation points.

Subdomain or Integral Method

In the subdomain (or integral) method, N subdomains A_m of domain A for the problem are chosen; these subdomains may be distinct, or they may overlap, but in any case, they completely cover the domain for the problem. Weight functions w_m are chosen, such that w_m is equal to unity at every point in subdomain A_m, and is zero otherwise; that is,

$$w_m = \begin{cases} 1, \text{ when evaluated at a point in } A_m \\ 0, \text{ when evaluated at a point not in } A_m. \end{cases} \tag{15.10}$$

In this case, equations 15.6 become

$$\sum_{n=1}^{N} \left[\iint_{A_m} L(\phi_n)\, dA \right] c_n = \iint_{A_m} F\, dA, \quad m = 1, \ldots, N. \tag{15.11}$$

The method requires that the integral of the residual to be zero over each subdomain.

Moment Method

In the moment method, weight functions are chosen to be

$$w_m = x^m, \quad m = 0, 1, 2, \ldots, N - 1. \tag{15.12}$$

With weight functions being only functions of x, the moment method is restricted to ordinary differential equations. If $a \le x \le b$ is the interval of consideration, equations 15.6 become

$$\sum_{n=1}^{N} \left[\int_a^b L(\phi_n) x^m\, dx \right] c_n = \int_a^b F x^m\, dx, \quad m = 0, \ldots, N - 1. \tag{15.13}$$

Galerkin Method

In the Galerkin method, weight functions w_m are chosen to be the basis functions ϕ_n, so that equations 15.5 become

$$\iint_A R\phi_m\, dA = 0, \quad m = 1, \ldots, N. \tag{15.14}$$

More explicitly, equations 15.6 become

$$\sum_{n=1}^{N} \left[\iint_A L(\phi_n)\phi_m\, dA \right] c_n = \iint_A F\phi_m\, dA, \quad m = 1, \ldots, N. \tag{15.15}$$

Before applying these techniques to boundary value problems associated with ODEs and PDEs, some comments are appropriate.

1. The simplicity of the collocation method lies in the fact that no integrations are necessary; the difficulty of the method is the optimum choice of collocation points. For one-dimensional problems, it is customary to distribute collocation points evenly throughout the interval of the problem. Special conditions, however, might suggest a concentration of collocation points in some part of the interval. In addition, a

method called orthogonal collocation has been developed to remove arbitrariness in the choice of collocation points. Basis function are orthogonal polynomials, such as the Legendre polynomials, and collocation points are chosen as the zeros of these polynomials.

2. Next in simplicity after collocation is the subdomain method. Integrations in equations 15.11 are generally simpler than those in equations 15.13 and 15.15.

3. The only condition that we have placed on basis functions is that they be linearly independent and from a complete set of functions. Polynomials x^n, $n = 0, 1, \ldots$, or combinations of them, can always be chosen, but other considerations might suggest a better choice. For instance, symmetry of the problem might suggest a subset of these polynomials, perhaps only the even ones. Alternatively, the physical nature of a problem might indicate a completely different set of basis functions. Finally, an analytic solution of a simplified version of the problem might suggest more suitable basis functions. For example, in Section 15.4, we will approximate eigenvalues and eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda(1 - x^2)X = 0, \quad 0 < x < 1, \quad X(0) = 0, \quad X'(1) = 0.$$

We know from Chapter 5 that when the differential equation is $X'' + \lambda X = 0$, eigenfunctions are $X_n(x) = \sin(2n-1)\pi x/2$. It would be reasonable to use these as basis functions to approximate eigenfunctions of the more complicated system. Galerkin's method is particularly efficient when eigenfunctions are used because of their orthogonality. Integrations become relatively simple, and instead of getting a system of N linear equations each containing all of the coefficients, we obtain N linear equations each containing only one coefficient.

4. No mention has yet been made of boundary and/or initial conditions. It is advantageous to choose basis functions so that approximation 15.3 satisfies some or all of the boundary conditions. After all, the accuracy with which linear combination 15.3 approximates the solution of an initial boundary value problem does not just depend on how well it fits the PDE; it depends on how well it approximates initial and boundary conditions as well. Ensuing discussions will indicate how initial and boundary conditions are incorporated into these methods.

5. **Interior** methods are ones for which approximating functions 15.3 satisfy the boundary conditions, but not the PDE. In this case, the residual, which accounts only for the fact that approximations do not satisfy the differential equation, is often called the **equation residual**. In **boundary** methods, approximations satisfy the PDE, and maybe some, but not all of the boundary conditions. In this case, we have what is called a **boundary residual**. Finally, **mixed** methods are those in which approximations satisfy neither PDE nor all of the boundary conditions. In this situation, there will be both an equation residual and a boundary residual. Initial boundary value problems may also have an **initial residual**.

When dealing with boundary value problems associated with ODEs, it is customary, but not mandatory, to use an interior method, by requiring approximations to satisfy the boundary conditions of the problem. This is usually quite straightforward when boundary conditions are Dirichlet, but it is also possible, albeit more difficult, when boundary conditions are Neumann and/or Robin. On the other hand, when dealing with PDEs, it can be difficult to require approximations to satisfy Neumann and Robin boundary conditions, but it is essential that approximations satisfy

Dirichlet boundary conditions.

6. As mentioned earlier, equations 15.6 for coefficients c_n are linear because operator L is linear. When L is a nonlinear operator, equations for the c_n will also be nonlinear, but they will be algebraic. In other words, the MWR replaces finding the exact solution of a boundary value problem associated with an ODE or PDE with solving an algebraic system of equations for an approximation to the exact solution.
7. Since the MWR approximates the solution of an initial boundary value problem, there is always the question of accuracy. Once coefficients in approximation 15.3 have been calculated and substituted into formula 15.4, the residual is minimized for the criterion chosen, be it collocation, subdomain, moment, or Galerkin. One is tempted to say that the smaller the value of the residual at a point, the better the approximation at that point. But we have no justification for such a claim. We have provided no direct relationship between the value of the residual at a point and the accuracy of the approximation to the solution of the initial boundary value problem at that point. There must be a connection, however, otherwise the MWR would be on a very fragile foundation. We feel that, in some sense, the size of the residual is a measure of the accuracy of the approximation on a point-by-point basis. The problem is "In what sense?" Various *norms* of the residual can be introduced in order to convert from a point-by-point assessment of accuracy to an overall assessment, and this leads discussions into the field of functional analysis. Such discussions are beyond the scope of our presentation of fundamental concepts of the MWR, except to mention one possible approach. If the residual is identically equal to zero, then the approximation is not an approximation, it is an exact solution of the differential equation. The **mean square residual**, defined as the integral

$$\iint_A R^2 \, dA, \tag{15.16}$$

is a measure of how close the residual is to zero over the entire domain A of the problem. It should, in some sense, be a measure of the overall accuracy of an approximation; the smaller the value of the integral, the better the approximation. This is not particularly definitive, but what it does do is give us a criterion by which to compare various approximations. Smaller integrals should correspond to better approximations.
8. Contrast how analytic methods, finite differences, weighted residuals, and finite elements approximate solutions to initial, boundary value problems. The analytic solution provides a sequence of approximations, its sequence of partial sums. Any given partial sum approximates the exact solution over the entire domain of the problem. Finite differences provide approximations to the exact solution at a set of mesh points. Approximations between mesh points can be obtained by interpolation. In weighted residuals, like analytic solutions, each function U_N approximates the exact solution over the entire domain of the problem. Finite elements, on the other hand, yield different approximations for different subdomains of the problem.

§15.3 Method of Weighted Residuals and Ordinary Differential Equations

Our objective is to apply the MWR to boundary value problems and initial boundary value problems associated with PDEs, but because it so much easier to see the principles involved when the technique is applied to ODEs, we apply the MWR to ODEs in this and the next section. We begin with a very simple problem so that ideas are not obscured by excessive calculations. Indeed, the example is so simple that it can be solved exactly, and this gives us the opportunity to compare approximations yielded by the MWR to exact results. The problem is

$$\frac{d^2Y}{dx^2} + Y = x, \quad 0 < x < 1, \tag{15.17a}$$

$$Y(0) = 1, \tag{15.17b}$$

$$Y(1) = 3, \tag{15.17c}$$

with exact solution

$$Y(x) = \cos x + (2\csc 1 - \cot 1)\sin x + x. \tag{15.18}$$

The first decision to be made when using weighted residuals is whether to use an interior method, a boundary method, or a mixed method. For ODEs, it is usually best to use an interior method, but in our second example, we will illustrate a mixed method, and in Section 15.4, we find eigenvalues of a Sturm-Liouville system with a boundary method. Having made the decision to use an interior method on this problem, we must choose basis functions that satisfy the boundary conditions. With nothing to suggest an alternative, we choose polynomials in x. Different approaches lead to different forms for the polynomials, but they are equally acceptable.

Approach 1

The polynomial

$$\sum_{n=0}^{N+1} c_n x^n \tag{15.19}$$

has $N+2$ unknown coefficients, but this will be reduced to N when we require the approximation to satisfy the boundary conditions $Y(0) = 1$ and $Y(1) = 3$. It is not a requirement of the MWR that approximations satisfy the boundary conditions of the problem, but it is once we have chosen to use an interior method. Satisfaction of the boundary conditions requires

$$1 = c_0, \qquad 3 = \sum_{i=0}^{N+1} c_i = c_0 + c_1 + \cdots + c_{N+1}.$$

When the second of these is solved for c_1, the resulting polynomial approximation is

$$\begin{aligned} Y_N(x) &= 1 + (2 - c_2 - \cdots - c_{N+1})x + c_2x^2 + \cdots + c_{N+1}x^{N+1} \\ &= 1 + 2x + c_2(x^2 - x) + c_3(x^3 - x) + \cdots + c_{N+1}(x^{N+1} - x) \\ &= 1 + 2x + \sum_{n=1}^{N} b_n x(1 - x^n). \end{aligned} \tag{15.20}$$

Approach 2

Approach 1 takes a general polynomial and subjects it to the boundary conditions. But notice that the first function $1+2x$ in the approximation satisfies the boundary conditions of the problem and the remaining functions $x(1-x^n)$, and these we regard as the basis functions, satisfy homogeneous versions of the boundary conditions. Based on these remarks, an alternative approach is more intuitive. It is a general method for all types of boundary conditions, but it is probably less work only when boundary conditions are Dirichlet. The idea is to write the N^{th} approximation in the form

$$Y_N(x) = \phi_0(x) + \sum_{n=1}^{N} c_n \phi_n(x), \tag{15.21}$$

and demand that $\phi_0(x)$ satisfy the boundary conditions of the problem, and the $\phi_n(x)$, $i = 1, \ldots, N$ satisfy homogeneous versions of the boundary conditions. Obviously, $Y_N(x)$ will then satisfy the boundary conditions of the problem. Function $\phi_0(x)$ is not regarded as a basis function, the $\phi_n(x)$, $n = 1, \ldots, N$ are. For the present problem, an obvious choice for $\phi_0(x)$ that satisfies $\phi_0(0) = 1$ and $\phi_0(1) = 3$, but not the only one, is $\phi_0(x) = 1 + 2x$. Functions x^n do not satisfy homogeneous versions of the boundary conditions, but $x^n(1-x)$, $n = 1, 2, \ldots$, do. We could therefore choose basis functions as $\phi_n(x) = x^n(1-x)$, and take

$$Y_N(x) = 1 + 2x + \sum_{n=1}^{N} c_n x^n (1-x). \tag{15.22}$$

We will work with approximations in form 15.20, but equally acceptable approximations could be developed with representation 15.22. The first approximation is $Y_1(x) = 1 + 2x + b_1 x(1-x)$, and the equation residual associated with it is

$$R = -2b_1 + [1 + 2x + b_1 x(1-x)] - x = b_1(-x^2 + x - 2) + x + 1.$$

We employ each of the collocation, subdomain, Galerkin, and moment methods to determine b_1. When we choose a single collocation point at the midpoint $x = 1/2$, collocation requires

$$0 = b_1\left(-\frac{1}{4} + \frac{1}{2} - 2\right) + \frac{1}{2} + 1 \qquad \Longrightarrow \qquad b_1 = \frac{6}{7}.$$

The first collocation approximation is therefore $Y_1(x) = 1 + 2x + \dfrac{6}{7}x(1-x)$. The subdomain method, with one subdomain $0 \le x \le 1$, requires

$$0 = \int_0^1 [b_1(-x^2 + x - 2) + x + 1]\,dx = -\frac{11b_1}{6} + \frac{3}{2} \qquad \Longrightarrow \qquad b_1 = \frac{9}{11}.$$

The first subdomain approximation is $Y_1(x) = 1 + 2x + \dfrac{9}{11}x(1-x)$. The moment method gives the same first approximation. Galerkin's method requires

$$\int_0^1 [b_1(-x^2 + x - 2) + x + 1]x(1-x)\,dx = -\frac{3b_1}{10} + \frac{1}{4} \qquad \Longrightarrow \qquad b_1 = \frac{5}{6}.$$

The first Galerkin approximation is $Y_1(x) = 1 + 2x + \dfrac{5}{6}x(1-x)$. We have tabulate the exact solution and the three approximations below for comparisons.

x	Exact	Collocation	Subdomain or moment	Galerkin
0.1	1.268	1.277	1.274	1.275
0.2	1.525	1.537	1.531	1.533
0.3	1.768	1.780	1.772	1.775
0.4	2.000	2.006	1.996	2.000
0.5	2.209	2.214	2.205	2.208
0.6	2.405	2.406	2.396	2.400
0.7	2.582	2.580	2.572	2.575
0.8	2.741	2.737	2.731	2.733
0.9	2.880	2.877	2.874	2.875

Table 15.6

To improve on the Galerkin, subdomain, and moment approximations, our only choice is to take more terms in summation 15.20. We can also refine the collocation method by opting for more terms in this summation, but, what is sometimes done is to maintain the one term approximation and choose more collocation points, find the value of b_1 for each collocation point, and then combine all values of b_1 in some way such as least squares to obtain an optimum value. We shall not pursue this procedure.

We now add a second term to $Y_1(x)$ to see how much better the approximation becomes. The second approximation is $Y_2(x) = 1 + 2x + b_1 x(1-x) + b_2 x(1-x^2)$, with residual

$$\begin{aligned} R &= -2b_1 - 6xb_2 + [1 + 2x + b_1 x(1-x) + b_2 x(1-x^2)] - x \\ &= b_1(-x^2 + x - 2) - b_2(x^3 + 5x) + x + 1. \end{aligned}$$

Collocation with collocation points $x = 1/3$ and $x = 2/3$ requires

$$0 = b_1\left(-\frac{1}{9} + \frac{1}{3} - 2\right) - b_2\left(\frac{1}{27} + \frac{5}{3}\right) + \frac{1}{3} + 1 = -\frac{16b_1}{9} - \frac{46b_2}{27} + \frac{4}{3},$$
$$0 = b_1\left(-\frac{4}{9} + \frac{2}{3} - 2\right) - b_2\left(\frac{8}{27} + \frac{10}{3}\right) + \frac{2}{3} + 1 = -\frac{16b_1}{9} - \frac{98b_2}{27} + \frac{5}{3}.$$

The solution of these equations is $b_1 = 0.584135$ and $b_2 = 0.173077$, and the second collocation approximation is

$$Y_2(x) = 1 + 2x + 0.584135x(1-x) + 0.173077x(1-x^2).$$

The subdomain method with subdomains $0 \le x \le 1/2$ and $1/2 \le x \le 1$ requires

$$0 = \int_0^{1/2} [b_1(-x^2 + x - 2) - b_2(x^3 + 5x) + x + 1]\, dx = -\frac{11b_1}{12} - \frac{41b_2}{64} + \frac{5}{8},$$
$$0 = \int_{1/2}^{1} [b_1(-x^2 + x - 2) - b_2(x^3 + 5x) + x + 1]\, dx = -\frac{11b_1}{12} - \frac{135b_2}{64} + \frac{7}{8}.$$

The solution is $b_1 = 0.562863$ and $b_2 = 0.170213$, and the second subdomain approximation is

$$Y_2(x) = 1 + 2x + 0.562863x(1-x) + 0.170213x(1-x^2).$$

The moment method requires

$$0 = \int_0^1 [b_1(-x^2+x-2) - b_2(x^3+5x) + x + 1]\,dx = -\frac{11b_1}{6} - \frac{11b_2}{4} + \frac{3}{2},$$
$$0 = \int_0^1 [b_1(-x^2+x-2) - b_2(x^3+5x) + x + 1]x\,dx = -\frac{11b_1}{12} - \frac{28b_2}{15} + \frac{5}{6}.$$

The solution is $b_1 = 0.563945$ and $b_2 = 0.169492$, and the second moment approximation is

$$Y_2(x) = 1 + 2x + 0.563945x(1-x) + 0.169492x(1-x^2).$$

Galerkin's method requires

$$0 = \int_0^1 [b_1(-x^2+x-2) - b_2(x^3+5x) + x + 1]x(1-x)\,dx = -\frac{3b_1}{10} - \frac{9b_2}{20} + \frac{1}{4},$$
$$0 = \int_0^1 [b_1(-x^2+x-2) - b_2(x^3+5x) + x + 1]x(1-x^2)\,dx = -\frac{9b_1}{20} - \frac{76b_2}{105} + \frac{23}{60}.$$

The solution is $b_1 = 0.577236$ and $b_2 = 0.170732$, and the second Galerkin approximation is

$$Y_2(x) = 1 + 2x + 0.577236x(1-x) + 0.170732x(1-x^2).$$

These approximations are tabulated below. The improvement over the first approximations is apparent.

x	Exact	Collocation	Subdomain	Moment	Galerkin
0.1	1.268	1.270	1.268	1.268	1.269
0.2	1.525	1.527	1.523	1.523	1.525
0.3	1.768	1.770	1.765	1.765	1.769
0.4	2.000	1.998	1.992	1.992	2.000
0.5	2.209	2.211	2.205	2.205	2.208
0.6	2.405	2.407	2.400	2.400	2.404
0.7	2.582	2.584	2.579	2.579	2.582
0.8	2.741	2.743	2.739	2.739	2.742
0.9	2.880	2.882	2.880	2.880	2.881
MSR		0.029	0.042	0.042	0.034

Table 15.2

We indicated in item 7 of Section 15.2 that the mean square residual is a guide to the accuracy of an approximation. Values are given in the last line of the table.

In the previous example, we used an interior method to approximate the solution of the boundary value problem, approximations satisfied both boundary conditions. In the next example, we consider the same differential equation, but make one of the boundary conditions Neumann. We again find approximations that satisfy both boundary conditions, but we also show that approximations that satisfy only one of the boundary conditions can be developed (a mixed method). We shall see that the interior method is easier to employ and gives better approximation. However, the mixed method becomes important in two- and three-dimensional problems when it is difficult to find basis functions that satisfy the boundary conditions, and we will have at least illustrated the approach in a simpler setting.

Example 15.1 The exact solution of the boundary value problem

$$\frac{d^2Y}{dx^2} + Y = x, \quad 0 < x < 1, \tag{15.23a}$$

$$Y(0) = 2, \tag{15.23b}$$

$$Y'(1) = 1, \tag{15.23c}$$

is

$$Y(x) = 2\cos x + 2\tan 1 \sin x + x. \tag{15.24}$$

Use an interior method, and a mixed method wherein approximations satisfy only the first boundary condition to approximate this solution. Compare approximations to the exact solution.

Solution **Interior Method** For an interior method, Approach 1 subjects the polynomial

$$\sum_{n=0}^{N+1} c_n x^n. \tag{15.25}$$

to the boundary conditions of the problem,

$$2 = c_0, \qquad 1 = c_1 + 2c_2 + \cdots + (N+1)c_{N+1}.$$

This results in

$$\begin{aligned} Y_N(x) &= 2 + [1 - 2c_2 - 3c_3 - \cdots - (N+1)c_{N+1}]x + c_2x^2 + \cdots + c_{N+1}x^{N+1} \\ &= 2 + x + c_2(x^2 - 2x) + c_3(x^3 - 3x) + \cdots + c_{N+1}[x^{N+1} - (N+1)x] \\ &= 2 + x + \sum_{n=1}^{N} b_n x(x^n - n - 1). \end{aligned} \tag{15.26}$$

Notice once again that the function $2 + x$ satisfies both nonhomogeneous boundary conditions, and basis functions $x(x^n - n - 1)$ satisfy homogeneous versions of the boundary conditions. The second approximation is

$$Y_2(x) = 2 + x + b_1(x^2 - 2x) + b_2(x^3 - 3x),$$

with equation residual

$$\begin{aligned} R &= (2b_1 + 6b_2x) + [2 + x + b_1(x^2 - 2x) + b_2(x^3 - 3x)] - x \\ &= 2 + b_1(x^2 - 2x + 2) + b_2(x^3 + 3x). \end{aligned}$$

We could use collocation, subdomains, or Galerkin's method to find b_1 and b_2, but because Galerkin's method is the only one available for the mixed method, we opt for it here also,

$$0 = \int_0^1 [2 + b_1(x^2 - 2x + 2) + b_2(x^3 + 3x)](x^2 - 2x)\,dx = -\frac{4b_1}{5} - \frac{89b_2}{60} - \frac{4}{3},$$
$$0 = \int_0^1 [2 + b_1(x^2 - 2x + 2) + b_2(x^3 + 3x)](x^3 - 3x)\,dx = -\frac{89b_1}{60} - \frac{20b_2}{7} - \frac{5}{2}.$$

The solution of these equations is $b_1 = -1.18439$ and $b_2 = -0.260102$ so that the second Galerkin approximation is

$$Y_2(x) = 2 + x - 1.18439(x^2 - 2x) - 0.260102(x^3 - 3x).$$

We tabulate the exact solution and this approximation after we discuss a mixed method for finding approximations.

Mixed Method

A mixed method for this problem uses approximations that fail to satisfy the differential equation and at least one boundary condition. Suppose we once again use polynomial basis functions and demand that they satisfy boundary condition $Y(0) = 2$, but not $Y'(1) = 1$. Boundary condition $Y'(1) = 1$ will be incorporated at a later stage by introducing what is called a *boundary residual.* It is not evident how to do this once basis functions have been specified and residuals have been formed. Instead, we write

$$Y_N(x) = 2 + \sum_{n=1}^{N} c_n \phi_n(x),$$

where basis functions $\phi_n(x)$ are unspecified (except that they must satisfy $\phi_n(0) = 0$). The equation residual for $Y_N(x)$ is

$$R = 2 - x + \sum_{n=1}^{N} c_n(\phi_n'' + \phi_n).$$

With as yet, unspecified weight functions $w_m(x)$, the MWR requires

$$\begin{aligned} 0 &= \int_0^1 \left[2 - x + \sum_{n=1}^{N} c_n(\phi_n'' + \phi_n)\right] w_m\,dx \\ &= \int_0^1 (2 - x)w_m\,dx + \sum_{n=1}^{N} c_n \left[\int_0^1 \phi_n'' w_m\,dx + \int_0^1 \phi_n w_m\,dx\right]. \end{aligned}$$

We use integration by parts on the first integral in the summation to write

$$0 = \int_0^1 (2 - x)w_m\,dx + \sum_{n=1}^{N} c_n \left[\{\phi_n' w_m\}_0^1 - \int_0^1 \phi_n' w_m'\,dx + \int_0^1 \phi_n w_m\,dx\right]. \quad (15.27)$$

Concentrate, for the moment, on the term $\{\phi_n' w_m\}_0^1$. We can eliminate it at $x = 0$ by choosing only weight functions that satisfy $w_m(0) = 0$. Now consider $\phi_n' w_m$ at $x = 1$. Because approximation $Y_N(x)$ has not been made to satisfy the boundary condition $Y'(1) = 1$, we define a **boundary residual** at $x = 1$,

$$R_{|x=1} = \sum_{n=1}^{N} c_n \phi_n'(1) - 1. \tag{15.28}$$

We implicitly demand that when multiplied by weight functions $w_m(1)$, the result be zero

$$\left[\sum_{n=1}^{N} c_n \phi_n'(1) - 1\right] w_m(1) = 0. \tag{15.29}$$

We have said that we demand this implicitly. What we mean by this is that we use it to remove the term $\phi_n' w_m$ at $x = 1$ in equation 15.27, but not explicitly as one of the conditions to determine coefficients c_n. As a result, approximations $Y_N(x)$ will not satisfy the boundary condition $Y'(1) = 1$, they will only approximate it. Substitution of these two requirements into equation 15.27 results in

$$0 = \int_0^1 (2 - x) w_m\, dx + w_m(1) + \sum_{n=1}^{N} c_n \int_0^1 [\phi_n w_m - \phi_n' w_m']\, dx. \tag{15.30}$$

Once basis functions $\phi_n(x)$ and weight functions $w_m(x)$ are chosen, it is these equations that determine coefficients c_n. Suppose we choose polynomial basis functions $\phi_n(x) = x^n$ which satisfy $Y(0) = 0$, but not $Y'(1) = 0$, so that

$$Y_N(x) = 2 + \sum_{n=1}^{N} c_n x^n.$$

Weight functions associated with the collocation and subdomain methods do not satisfy the requirement that $w_m(0) = 0$. Weight functions $w_m = x^m$ associated with the moment method do satisfy this requirement, and these are the same weight functions for Galerkin's method. Suppose we consider the third approximation $Y_3(x) = 2 + c_1 x + c_2 x^2 + c_3 x^3$ (chosen because the interior method contained a cubic term). Conditions 15.30 with $m = 1, 2, 3$ demand that

$$\begin{aligned}
0 &= \int_0^1 (2 - x) w_1\, dx + w_1(1) + c_1 \int_0^1 (\phi_1 w_1 - \phi_1' w_1')\, dx + c_2 \int_0^1 (\phi_2 w_1 - \phi_2' w_1')\, dx \\
&\qquad + c_3 \int_0^1 (\phi_3 w_1 - \phi_3' w_1')\, dx \\
&= \frac{5}{3} - \frac{2c_1}{3} - \frac{3c_2}{4} - \frac{4c_3}{5}, \\
0 &= \int_0^1 (2 - x) w_2\, dx + w_2(1) + c_1 \int_0^1 (\phi_1 w_2 - \phi_1' w_2')\, dx + c_2 \int_0^1 (\phi_2 w_2 - \phi_2' w_2')\, dx \\
&\qquad + c_3 \int_0^1 (\phi_3 w_2 - \phi_3' w_2')\, dx \\
&= \frac{17}{12} - \frac{3c_1}{4} - \frac{17c_2}{15} - \frac{4c_3}{5}, \\
0 &= \int_0^1 (2 - x) w_3\, dx + w_3(1) + c_1 \int_0^1 (\phi_1 w_3 - \phi_1' w_3')\, dx + c_2 \int_0^1 (\phi_2 w_3 - \phi_2' w_3')\, dx \\
&\qquad + c_3 \int_0^1 (\phi_3 w_3 - \phi_3' w_3')\, dx \\
&= \frac{13}{10} - \frac{4c_1}{5} - \frac{4c_2}{3} - \frac{58c_3}{35}.
\end{aligned}$$

The solution of these is $c_1 = 4.30172$, $c_2 = -1.5885$, and $c_3 = -0.0167087$, and therefore

$$Y_3(x) = 2 + 4.30172x - 1.5885x^2 - 0.0167087x^3.$$

The exact solution and the two approximations $Y_2(x)$ and $Y_3(x)$ are tabulated below. Clearly the interior method with both boundary conditions satisfied gives better approximations. As predicted, $Y_3(x)$ does not satisfy the Neumann boundary condition of the problem, $Y'(1) = 1$; it only approximates it, $Y_3'(1) = 1.075$.•

x	Exact	$Y_2(x)$	$Y_3(x)$
0.1	2.401	2.403	2.414
0.2	2.779	2.780	2.797
0.3	3.131	3.131	3.147
0.4	3.455	3.453	3.465
0.5	3.748	3.746	3.752
0.6	4.009	4.007	4.006
0.7	4.236	4.235	4.227
0.8	4.428	4.428	4.416
0.9	4.583	4.585	4.573

Table 15.3

Choosing Basis Functions

The choice of basis functions is perhaps the most important step in the MWR. A wise choice may lead to good approximations to the solution of the boundary value problem; an unwise choice may lead to poor approximations. So far in this section, we have favoured interior methods wherein approximations satisfy boundary conditions because they are easier to apply, and generally provide better approximations. We have dealt exclusively with polynomial basis functions, but in Section 15.4, we find that trigonometric functions can also be appropriate. Before leaving you to the exercises, we would like to point out two fundamentally different ways to apply an interior MWR with polynomial basis functions to boundary value problems associated with ODEs where boundary conditions are nonhomogeneous. In the two examples discussed here, it was straightforward to apply the boundary conditions directly to a polynomial $\sum_{i=0}^{N+1} c_n x^n$ to reduce it to a polynomial with N undetermined coefficients. The reason for this was that both boundary conditions were Dirichlet in problem 15.20, and in problem 15.23, one boundary condition was Dirichlet and the other was Neumann. When Robin conditions are present, this process can be much more complicated. An alternative is to transform nonhomogeneities from the boundary conditions into the ODE and deal only with homogeneous boundary conditions. We finish this section with a procedure for doing this. We develop transformations that remove nonhomogeneities from all nine combinations of boundary conditions, Dirichlet, Neumann, and Robin, and then provide polynomial approximations for each such combination.

In previous chapters, we consistently wrote boundary conditions for a function $Y(x)$ on the interval $0 < x < L$ in the form

$$-l_1 Y'(0) + h_1 Y(0) = m_1, \tag{15.31a}$$

$$l_2Y'(L) + h_2Y(L) = m_2, \tag{15.31b}$$

and we did this for a good reason. By solving problems with boundary conditions in this form, we were able to specialize results to Neumann and Dirichlet boundary conditions simply by choosing h's and l's to be zero. There seems to be no such advantage in this chapter, so we will write these conditions in the form

$$-Y'(0) + h_1Y(0) = m_1, \tag{15.32a}$$

$$Y'(L) + h_2Y(L) = m_2. \tag{15.32b}$$

Suppose that we have an ODE $L(Y) = F(x)$ on the interval $0 < x < L$ subject to nonhomogeneous boundary conditions at $x = 0$ and $x = L$. Our objective here is to develop transformations that move the nonhomogeneities from the boundary conditions to the ODE for all nine combinations of boundary conditions. With the exception of a Neumann condition at both ends, this can be accomplished with a linear transformation on the dependent variable. We illustrate when the boundary condition is Robin at $x = 0$ and Dirichlet at $x = L$.

$$-Y'(0) + h_1Y(0) = m_1, \qquad Y(L) = m_2,$$

and tabulate the remaining eight possibilities. Consider finding constants a and b so that the change of dependent variable $Z(x) = Y(x) + ax + b$ results in a boundary value problem for $Z(x)$ with homogeneous boundary conditions. These require a and b to satisfy

$$\begin{aligned}0 &= -Z'(0) + h_1Z(0) = -[Y'(0) + a] + h_1[Y(0) + b]\\ &= [-Y'(0) + h_1Y(0)] - a + h_1b = m_1 - a + h_1b,\\ 0 &= Z(L) = Y(L) + aL + b = m_2 + aL + b.\end{aligned}$$

The solution is

$$a = \frac{m_1 - h_1m_2}{1 + h_1L}, \qquad b = -\frac{m_2 + Lm_1}{1 + h_1L}.$$

Hence, the transformation $Z(x) = Y(x) + \left(\dfrac{m_1 - h_1m_2}{1 + h_1L}\right)x - \dfrac{m_2 + Lm_1}{1 + h_1L}$ leads to a boundary value problem for $Z(x)$ with homogeneous boundary conditions. Table 15.4 gives the transformation for all nine combinations of boundary conditions. (See Exercise 19 for further verification.)

If we are willing to make these transformations, we need only consider problems with homogeneous boundary conditions

$$B_0(Y)(0) = 0, \tag{15.33a}$$

$$B_L(Y)(L) = 0. \tag{15.33b}$$

B_0 is one of the differential operators 1, d/dx, or $-d/dx + h_1$, thus representing a Dirichlet, Neumann, or Robin condition at $x = 0$. Operator B_L is 1, d/dx, or $d/dx + h_2$. Accepting this, we now determine polynomial basis functions for the nine sets of boundary conditions. Realize, however, that they are not unique; there may be other equally acceptable sets of polynomial basis functions.

Types of Boundary Conditions	Boundary Conditions	Transformation
Dirichlet Dirichlet	$Y(0) = m_1$, $Y(L) = m_2$	$Z(x) = Y(x) + \left(\frac{m_1 - m_2}{L}\right) x - m_1$
Dirichlet Neumann	$Y(0) = m_1$, $Y'(L) = m_2$	$Z(x) = Y(x) - m_2 x - m_1$
Dirichlet Robin	$Y(0) = m_1$, $Y'(L) + h_2 Y(L) = m_2$	$Z(x) = Y(x) + \left(\frac{h_2 m_1 - m_2}{1 + h_2 L}\right) x - m_1$
Neumann Dirichlet	$Y'(0) = m_1$, $Y(L) = m_2$	$Z(x) = Y(x) - m_1 x + (m_1 L - m_2)$
Neumann Neumann	$Y'(0) = m_1$, $Y'(L) = m_2$	$Z(x) = Y(x) + \left(\frac{m_1 - m_2}{2L}\right) x^2 - m_1 x$
Neumann Robin	$Y'(0) = m_1$, $Y'(L) + h_2 Y(L) = m_2$	$Z(x) = Y(x) - m_1 x + \frac{m_1(1 + h_2 L) - m_2}{h_2}$
Robin Dirichlet	$-Y'(0) + h_1 Y(0) = m_1$, $Y(L) = m_2$	$Z(x) = Y(x) + \left(\frac{m_1 - h_1 m_2}{1 + h_1 L}\right) x - \frac{m_2 + L m_1}{1 + h_1 L}$
Robin Neumann	$-Y'(0) + h_1 Y(0) = m_1$, $Y'(L) = m_2$	$Z(x) = Y(x) - m_2 x - \frac{m_1 + m_2}{h_1}$
Robin Robin	$-Y'(0) + h_1 Y(0) = m_1$, $Y'(L) + h_2 Y(L) = m_2$	$Z(x) = Y(x) + \left(\frac{h_2 m_1 - h_1 m_2}{h_2 + h_1(1 + h_2 L)}\right) x - \frac{m_2 + m_1(1 + h_2 L)}{h_2 + h_1(1 + h_2 L)}$

Table 15.4

We discuss polynomial basis functions for one set of boundary conditions, and tabulate results for the other eight sets. In particular, suppose that the boundary condition at $x = 0$ is Neumann and that at $x = L$ is Robin

$$Y'(0) = 0, \qquad Y'(L) + h_2 Y(L) = 0.$$

The functions x^n, $n = 0, 1, \ldots$ form a complete set of linearly independent basis functions for the space of continuous functions, but they do not satisfy the boundary conditions. Because the function $x^2(L - x)^2$ satisfies both boundary conditions, it follows that the polynomials $x^n[x^2(L - x)^2] = x^{n+2}(L - x)^2$, $n = 0, 1, \ldots$ also satisfy the boundary conditions. Do they form a complete set? The lowest degree polynomial in the set is a quartic. As a result, the set must be augmented with any lower degree polynomials that also satisfy the boundary conditions. It is straightforward to show that no nontrivial linear polynomial satisfies both boundary conditions. For a quadratic $Y(x) = a + bx + cx^2$ to satisfy the boundary conditions, we must have

$$0 = Y'(0) = b, \qquad 0 = Y'(L) + h_2 Y(L) = (b + 2cL) + h_2(a + bL + cL^2).$$

The second implies that $a = -(2L + h_2 L^2)c/h_2$, and if we set $c = 1$, a quadratic polynomial satisfying the boundary conditions is

$$\phi_1(x) = x^2 - \frac{L}{h_2}(2 + h_2 L).$$

For a cubic $Y(x) = a + bx + cx^2 + dx^3$ to satisfy the boundary conditions,

$$0 = Y'(0) = b, \quad 0 = Y'(L) + h_2 Y(L) = (b + 2cL + 3dL^2) + h_2(a + bL + cL^2 + dL^3).$$

The second implies that $a = -c(2L + h_2L^2)/h_2 - d(3L^2 + h_2L^3)/h_2$, and if we set $c = d = 1$, then a cubic polynomial satisfying the boundary conditions is

$$\begin{aligned} Y(x) &= \frac{2L + h_2L^2}{h_2} - \frac{3L^2 + h_2L^3}{h_2} + x^2 + x^3 \\ &= \left[x^2 - \frac{L}{h_2}(2L + h_2L)\right] + \left[x^3 - \frac{L^2}{h_2}(3 + h_2L)\right]. \end{aligned}$$

Since the first function on the right is $\phi_1(x)$, we eliminate it and take

$$\phi_2(x) = x^3 - \frac{L^2}{h_2}(3 + h_2L).$$

These two functions together with

$$\phi_n(x) = x^{n-1}(L - x)^2, \quad n \geq 3,$$

then form a complete set of polynomial basis functions that satisfy both boundary conditions. Table 15.5 gives polynomial basis functions for the other eight combinations of boundary conditions (see Exercise 20).

Types of Boundary Conditions	Boundary Conditions	Basis Polynomials
Dirichlet Dirichlet	$Y(0) = 0,$ $Y(L) = 0$	$\phi_n(x) = x^n(L-x),\ n \geq 1$
Dirichlet Neumann	$Y(0) = 0,$ $Y'(L) = 0$	$\phi_1(x) = x(2L-x)$ $\phi_n(x) = x^{n-1}(L-x)^2,\ n \geq 2$
Dirichlet Robin	$Y(0) = 0,$ $Y'(L) + h_2 Y(L) = 0$	$\phi_1(x) = x\left(x - \dfrac{2L + h_2L^2}{1 + h_2L}\right)$ $\phi_n(x) = x^{n-1}(L-x)^2,\ n \geq 2$
Neumann Dirichlet	$Y'(0) = 0,$ $Y(L) = 0$	$\phi_1(x) = L^2 - x^2$ $\phi_n(x) = x^n(L-x),\ n \geq 2$
Neumann Neumann	$Y'(0) = 0,$ $Y'(L) = 0$	$\phi_1(x) = 1$ $\phi_2(x) = 3Lx^2 - 2x^3$ $\phi_n(x) = x^{n-1}(L-x)^2\ n \geq 3$
Neumann Robin	$Y'(0) = 0,$ $Y'(L) + h_2 Y(L) = 0$	$\phi_1(x) = x^2 - \dfrac{L}{h_2}(2 + h_2L)$ $\phi_2(x) = x^3 - \dfrac{L^2}{h_2}(3 + h_2L)$ $\phi_n(x) = x^{n-1}(L-x)^2,\ n \geq 3$
Robin Dirichlet	$-Y'(0) + h_1Y(0) = 0,$ $Y(L) = 0$	$\phi_1(x) = x^2 - \dfrac{h_1L^2x}{1 + h_1L} - \dfrac{L^2}{1 + h_1L}$ $\phi_2(x) = x^3 - \dfrac{h_1L^3x}{1 + h_1L} - \dfrac{L^3}{1 + h_1L}$ $\phi_n(x) = x^{n-1}(L-x)^2,\ n \geq 3$
Robin Neumann	$-Y'(0) + h_1Y(0) = 0,$ $Y'(L) = 0$	$\phi_1(x) = x^2 - 2Lx - \dfrac{2L}{h_1}$ $\phi_2(x) = x^3 - 3L^2x - \dfrac{3L^2}{h_1}$ $\phi_n(x) = x^{n-1}(L-x)^2,\ n \geq 3$
Robin Robin	$-Y'(0) + h_1Y(0) = 0,$ $Y'(L) + h_2Y(L) = 0$	$\phi_1(x) = x^2 - \dfrac{(L^2 + 2L/h_2)x}{L + 1/h_1 + 1/h_2} - \dfrac{(L^2 + 2L/h_2)/h_1}{L + 1/h_1 + 1/h_2}$ $\phi_2(x) = x^3 - \dfrac{(L^3 + 3L^2/h_2)x}{L + 1/h_1 + 1/h_2} - \dfrac{(L^3 + 3L^2/h_2)/h_1}{L + 1/h_1 + 1/h_2}$ $\phi_n(x) = x^{n-1}(L-x)^2,\ n \geq 3$

Table 15.5

EXERCISES 15.3

1. In each part of this exercise, a boundary value problem in $Y(x)$ on the interval $0 \leq x \leq 1$ is to be approximated by a polynomial. Use **Approach 1** to find the form of the polynomial if the boundary conditions are as specified.

(a) $Y(0) = 2$, $Y'(L) = 1$

(b) $Y'(0) = 0$, $Y'(L) = 0$
(c) $-Y'(0) + Y(0) = 0$, $Y(L) = 3$
(d) $Y(0) = 1$, $Y'(L) + hY(L) = 0$

2. We mentioned that **Approach 2** for finding an approximating polynomial is advantageous only when boundary conditions are Dirichlet. Demonstrate that **Approach 2** to part (a) of Exercise 1, with one Dirichlet and one Neumann condition, does not prove advantageous.

3. The nonlinear problem below describes steady-state temperature in a rod with thermal conductivity that is a linear function of temperature,

$$\frac{d}{dx}\left[(1+U)\frac{dU}{dx}\right] = 0, \quad 0 < x < 1,$$
$$U(0) = 0, \qquad U(1) = 1.$$

There is no heat generation in the rod, and end temperatures are constant.
(a) Show that the exact solution of the problem is $U(x) = -1 + \sqrt{1+3x}$.
(b) Derive the following polynomial approximations that satisfy the boundary conditions

$$U_N(x) = x + \sum_{n=1}^{N} b_n(x^{n+1} - x).$$

(c) Find first approximations using (i) collocation, (ii) subdomains, and the (iii) Galerkin method. Tabulate the exact solution and the approximations at $x = 0.1, 0.25, 0.50, 0.75, 0.90$ for comparison purposes.
(d) Find second approximations using (i) collocation, (ii) subdomains, (iii) moments, and (iv) the Galerkin method. Tabulate the exact solution and the approximations.

4. Find second approximations for the problem in Exercise 3 by first using Table 15.4 to make the second boundary condition homogeneous, and then using Table 15.5 for basis functions. Employ:
(a) collocation,
(b) subdomains,
(c) moments,
(d) Galerkin's method.
Do you get the same second approximations as in Exercise 3?

5. The initial value problem

$$\frac{dT}{dt} = -0.03(T - 20), \qquad T(0) = 100,$$

describes temperature $T(t)$ of a cup of coffee subject to Newton's law of cooling with heat transfer coefficient $k = 0.03$. The coffee is initially at temperature 100° C and the environment is at a constant temperature $20°C$.
(a) Find the analytic solution of the problem.
(b) Find a quadratic approximation $T_2(t) = 100 + c_1 t + c_2 t^2$ using collocation, subdomains, and Galerkin's method on the interval $0 \le t \le 30$.
(c) Tabulate the analytic solution and each approximation for $t = 5, 10, 15, 20,$ and 25.

6. (a) Poisson's equation together with boundary conditions can always be interpreted as a steady state heat conduction problem. The one-dimensional problem

$$\frac{d^2U}{dx^2} = -\sin x, \qquad U(0) = 1, \quad U(1) = -2,$$

can be interpreted as steady state temperature in a rod. Find the exact solution of the problem.

(b) In this one-dimensional setting the result of Exercise 24 in Section 2.2 states that the solution should satisfy the equation

$$\int_0^1 \sin x \, dx = -[U'(1) - U'(0)].$$

Verify that this condition is satisfied.

(c) Polynomial approximations satisfying the boundary conditions are

$$U_N(x) = 1 - 3x + \sum_{n=1}^{N} c_n x^n (1 - x).$$

Use Galerkin's method to find $U_1(x)$, $U_2(x)$, and $U_3(x)$.

(d) Calculate the mean square residual for each of the approximations.

(e) Approximations of the solution to the problem will not satisfy the condition in part (b). The quantity

$$\int_0^1 \sin x \, dx + [U'(1) - U'(0)],$$

is a measure of the extent to which the approximation meets this condition. It is another guide as to the adequacy of the approximation. Calculate the quantity for each of the approximations in part (c).

7. (a) Derive the solution $Y(x) = \dfrac{x^2}{4} - \dfrac{x}{2} + \dfrac{1}{4 \ln 2} \ln(x+1)$ for the boundary value problem

$$(1 + x)Y'' + Y' = x, \qquad Y(0) = 0, \quad Y(1) = 0.$$

(b) Use Approach 1 and Galerkin's method to find first and second polynomial approximations that satisfy the boundary conditions. Tabulate the analytic solution and the two approximations for $x = 0.1, 0.2, \ldots, 0.9$.

(c) Repeat part (b) but use Table 15.5 for basis functions.

8. (a) Derive the solution $V(r) = \dfrac{1}{9}(r^3 - 1) - \dfrac{7}{9 \ln 2} \ln r$ of the boundary value problem

$$r\frac{d^2V}{dr^2} + \frac{dV}{dr} = r^2, \qquad V(1) = V(2) = 0.$$

(b) Translate the independent variable by setting $x = r - 1$ so that $V(x)$ satisfies

$$(x + 1)\frac{d^2V}{dx^2} + \frac{dV}{dx} = (x + 1)^2, \qquad V(0) = V(1) = 0.$$

Use Table 15.5 to determine polynomial basis functions for this problem. Find the first approximation using (i) collocation, (ii) subdomain, and (iii) Galerkin methods.

(c) Tabulate approximations and the exact solution in part (a) for $r = 1.1, 1.2, \ldots, 1.9$.

9. (a) Show that the solution of the boundary value problem

$$r^2R'' + rR' + (r^2 - 1)R = 0, \qquad R(1) = 1, \quad R(2) = 2$$

is $R(r) = 3.60741J_1(r) + 0.751964Y_1(r)$.

(b) Translate the independent variable by setting $x = r - 1$ so that $R(x)$ satisfies

$$(x+1)^2\frac{d^2R}{dx^2} + (x+1)\frac{dR}{dx} + (x^2 + 2x)R = 0, \qquad R(0) = 1, \quad R(1) = 2.$$

Use Table 15.4 to show that the change of dependent variable $Z(x) = R(x) - x - 1$ moves the nonhomogeneities from the boundary conditions to the differential equation. Verify that $Z(x)$ must satisfy

$$(x+1)^2\frac{d^2Z}{dx^2} + (x+1)\frac{dZ}{dx} + (x^2 + 2x)Z = -1 - 3x - 3x^2 - x^3, \qquad Z(0) = 0, \quad Z(1) = 0.$$

(c) Use Table 15.5 to find polynomial basis functions for this problem. Find the first Galerkin approximation.

(d) Compare values of the solution in part (a) to that in part (c) at $r = 1.1, 1.2, \ldots, 1.8, 1.9$.

10. The following problem has a nonhomogeneous differential equation and nonhomogeneous boundary conditions

$$x\frac{d^2V}{dx^2} + \frac{dV}{dx} = \frac{2}{x^2}, \quad 1 < x < 2,$$
$$V(1) = 2,$$
$$V'(2) = -1/4.$$

(a) Show that when an approximation of the form $V_2(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ is required to satisfy the boundary conditions, then

$$V_2(x) = \frac{1}{4}(9 - x) + c_2(x-1)(x-3) + c_3(x-1)(x^2 + x - 11).$$

Notice that $\phi_0(x) = (9-x)/4$ satisfies the boundary conditions of the problem, and $\phi_1(x) = (x-1)(x-3)$ and $\phi_2(x) = (x-1)(x^2+x-11)$ satisfy homogeneous versions of the boundary conditions.

(b) Find values for c_2 and c_3 using (i) collocation, (ii) subdomains, (iii) moments, and (iv) Galerkin's method.

(c) Find the analytic solution to the problem, and tabulate it along with the approximations in part (b) for $x = 1.1, 1.2, \ldots, 1.9$.

11. In this exercise we follow the lead of the mixed method for problem 15.23 to approximate the solution in Exercise 10.

(a) Take approximations of the solution to be

$$V_N(x) = 2 + \sum_{n=0}^{N-1} c_n\phi_n(x) = 2 + \sum_{n=0}^{N-1} c_nx^n(x-1).$$

They satisfy the boundary condition $V(1) = 2$, but not the boundary condition $V'(2) = -1/4$. What is the residual associated with this approximation?

(b) Show that when the residual is multiplied by weight functions $w_m(x)$, and integration by parts is performed on the term involving $\phi_n''(x)$, the result is

$$0 = \sum_{n=0}^{N-1} c_n \left[\{x\phi_n' w_m\}_1^2 - \int_1^2 x\phi_n' w_m' \, dx \right] - \int_1^2 \frac{2}{x^2} w_m \, dx.$$

(c) Suppose weight functions are chosen to satisfy the condition that $w_m(1) = 0$. Furthermore, assume implicitly that the residual at $x = 2$ satisfies

$$\sum_{n=0}^{N-1} c_n \phi_n'(2) = 0.$$

Show that coefficients c_n must then satisfy the equations

$$0 = \sum_{n=0}^{N-1} c_n \int_1^2 x\phi_n' w_m' \, dx + \int_1^2 \frac{2}{x^2} w_m \, dx.$$

(d) Weight functions associated with collocation, subdomains, and moments do not satisfy the condition $w_m(1) = 0$, but those associated with Galerkin's method do. Show that when N is chosen as 3, this method leads to the equations

$$\frac{3c_0}{2} + \frac{19c_1}{6} + \frac{79c_2}{12} = -1 + 2\ln 2,$$
$$\frac{19c_0}{6} + \frac{43c_1}{6} + \frac{937c_2}{60} = 2 - 2\ln 2,$$
$$\frac{79c_0}{12} + \frac{937c_1}{60} + \frac{351c_2}{10} = 1.$$

(e) Solve the equations in part (d) and tabulate $V_3(x)$ for $x = 1.1, 1.2, \ldots, 1.9$. How does it compare to the approximations in Exercise 10?

12. In this exercise we illustrate the mixed method when one of the boundary conditions is Robin. Consider the problem

$$(4 - x^2)\frac{d^2Y}{dx^2} + 2Y = 0, \quad 0 < x < 1, \qquad Y(0) = 0, \quad Y'(1) + 2Y(1) = 1.$$

(a) Show that polynomial approximations that satisfy both boundary conditions are

$$Y_N(x) = \frac{x}{3} + \sum_{n=1}^{N} b_n x \left(x^n - \frac{n+3}{3} \right).$$

Find the second approximation using Galerkin's method.

(b) Use Table 15.4 to move the nonhomogeneity to the differential equation and then use Table 15.5 to determine a second approximation. Calculate coefficients with Galerkin's method.

(c) For polynomial solutions that satisfy $Y(0) = 0$, but not $Y'(1) + 2Y(1) = 1$, take

$$Y_N(x) = \sum_{n=1}^{N} c_n \phi_n(x), \quad \text{where } \phi_n(x) = x^n.$$

Express the residual associated with this approximation in terms of $\phi_n(x)$.

(d) Show that the MWR with weight functions $w_m(x)$, and integration by parts, leads to

$$0 = \sum_{n=1}^{N} c_n \left[\{(4 - x^2)\phi_n' w_m\}_0^1 - \int_0^1 \{\phi_n'[-2xw_m + (4 - x^2)w_m'] - 2\phi_n w_m\} \, dx \right].$$

(e) The boundary residual at $x = 1$ is

$$R_{|x=1} = \sum_{n=1}^{N} c_n[\phi_n'(1) + 2\phi_n(1)] - 1.$$

By implicitly demanding that this be equal to zero, and choosing weight functions that vanish at $x = 0$, show that the equations in part (d) simplify to

$$0 = 3w_m(1) - 6w_m(1)\sum_{n=1}^{N} c_n\phi_n(1) - \sum_{n=1}^{N} c_n \int_0^1 \{\phi_n'[-2xw_m + (4 - x^2)w_m'] - 2\phi_n w_m\}\, dx.$$

(f) Find the second approximation, and tabulate it along with the approximations in parts (a) and (b).

13. (a) Verify that $Y(x) = x + \sin x$ is the exact solution of the nonlinear boundary value problem

$$\sin x\frac{d^2Y}{dx^2} + Y^2 = x^2 + \sin x, \qquad Y(0) = 0, \quad Y(1) = 1 + \sin 1.$$

(b) Show that polynomial approximations that satisfy the boundary conditions are

$$Y_N(x) = (1 + \sin 1)x + \sum_{n=1}^{N} b_n x(1 - x^n).$$

Find the second approximation using Galerkin's method.

(c) Use Table 15.4 to move the nonhomogeneity in the boundary condition to the differential equation, and then use basis functions from Table 15.5 to find a second approximation with Galerkin's method.

(d) Tabulate the exact solution and the approximations in parts (b),(c) for $x = 0.1, 0.2, \ldots, 0.9$.

14. (a) Verify that $V(x) = x^2 - x$ is the solution of the boundary value problem

$$\frac{d^2V}{dx^2} - x\frac{dV}{dx} + 2V = 2 - x, \qquad V(0) = V(1) = 0.$$

(b) Derive polynomial approximations

$$V_N(x) = \sum_{n=1}^{N} b_n x(1 - x^n)$$

that satisfy the boundary conditions of the problem.

(c) The first approximation $V_1(x) = b_1x(1 - x)$ with $b_1 = -1$ is the exact solution. Show that the MWR returns the exact solution.

(d) We now prove this result in general. Suppose that $V(x)$ is the exact solution of a boundary value problem associated with the linear differential equation

$$L(V) = F(x), \quad a < x < b.$$

Suppose further that the basis function $\phi_1(x)$ of approximations $V_N(x) = \sum_{n=1}^{N} c_n\phi_n(x)$, where all basis functions satisfy the boundary conditions, is a scalar multiple of the exact solution $V(x)$. Prove that the MWR returns the exact solution.

15. (a) Verify that $V(x) = 2x^3 + x^2 - 3x$ is the solution of the boundary value problem

$$\frac{d^2V}{dx^2} - x\frac{dV}{dx} + 2V = -2x^3 + 9x + 2, \qquad V(0) = V(1) = 0.$$

(b) Derive polynomial approximations

$$V_N(x) = \sum_{n=1}^{N} b_n x(1 - x^n)$$

that satisfy the boundary conditions of the problem.

(c) The second approximation $V_2(x) = b_1 x(1-x) + b_2 x(1-x^2)$ with $b_1 = -1$ and $b_2 = -2$ is the exact solution. Show that Galerkin's method returns the exact solution. The other methods also return the exact solution.

16. Steady state temperature $T(r)$ in a spherical nuclear-fission fuel element satisfies the boundary value problem

$$-\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = Dr^2\left(1 + \frac{r^2}{a^2}\right), \quad 0 < r < a,$$
$$T'(0) = 0,$$
$$T'(a) + hT(a) = 0,$$

where D and h are constants, and a is the radius of the sphere.

(a) Derive the exact solution

$$T(r) = \frac{Da^2}{60}\left[\left(13 + \frac{32}{ah}\right) - \frac{10r^2}{a^2} - \frac{3r^4}{a^4}\right].$$

(b) Use Table 15.5 to show that a first basis function is $T_1(r) = r^2 - \dfrac{a}{h}(2 + ah)$. Use Galerkin's method and this basis function to find a first approximation to $T(r)$.

(c) Use Table 15.5 to obtain a third approximation. Show that Galerkin's method leads to the exact solution.

17. You may have noticed that Galerkin's method seems to produce symmetric linear equations for the unknown coefficients of an approximating polynomial. In this exercise and the next we show that this is to be expected but only when the differential equation is in self-adjoint form (see Section 12.3). Consider the boundary value problem

$$x\frac{d^2Y}{dx^2} + 2\frac{dY}{dx} + xY = 0, \qquad Y(1) = 1, \quad Y(2) = 0,$$

the exact solution of which is $Y(x) = \dfrac{\sin(2-x)}{(\sin 1)x}$.

(a) Use **Approach 2** to derive the following polynomial approximations to the solution

$$Y_N(x) = 2 - x + \sum_{n=0}^{N-1} c_n x^n (x-1)(x-2).$$

(b) Show that the equations resulting from Galerkin's method for the approximation $Y_2(x) = 2 - x + c_1(x-1)(x-2) + c_2 x(x-1)(x-2)$ are not symmetric. Solve the equations for c_1 and c_2 and tabulate $Y_2(x)$ and $Y(x)$ for $x = 0.1, 0.2, \ldots, 0.9$.

(c) The reason that the equations in part (b) were not symmetric is that the differential equation is not in self-adjoint form. It can be put in self-adjoint form by multiplying by x (see the discussion following equation 5.3 in Section 5.1),

$$x^2\frac{d^2Y}{dx^2}+2x\frac{dY}{dx}+x^2Y=0 \qquad \Longrightarrow \qquad \frac{d}{dx}\left(x^2\frac{dY}{dx}\right)+x^2Y=0.$$

Show that the system of equations for c_1 and c_2 produced by Galerkin's method is now symmetric.

(d) Solve the equations in part (c) for c_1 and c_2. Are solutions the same as in part (b)?

18. Consider the boundary value problem

$$\frac{d^2Y}{dx^2}+\frac{dY}{dx}+Y=2x, \qquad Y(0)=0, \quad Y(1)=1.$$

(a) Show that polynomial approximations that satisfy the boundary conditions are

$$Y_N(x)=x+\sum_{n=1}^{N}b_n x(x^n-1).$$

(b) Show that the equations resulting from Galerkin's method for the approximation $Y_2(x)=x+b_1(x^2-x)+b_2(x^3-x)$ are not symmetric.

(c) The differential equation can be expressed in self-adjoint form by multiplying by e^x,

$$e^x\frac{d^2Y}{dx^2}+e^x\frac{dY}{dx}+e^xY=2xe^x \qquad \Longrightarrow \qquad \frac{d}{dx}\left(e^x\frac{dY}{dx}\right)+e^xY=2xe^x.$$

Show that equations for b_1 and b_2 determined by Galerkin's method for $Y_2(x)$ are now symmetric.

19. Verify the entries in Table 15.4.

20. Verify the basis polynomials in Table 15.5.

21. The boundary value problem

$$EI\frac{d^4Y}{dx^4}+kY=w, \quad 0<x<L,$$
$$Y(0)=Y''(0)=0=Y(L)=Y''(L),$$

describes deflections $Y(x)$ of a static beam of length L on an elastic foundation, simply-supported at both ends. Constant E is Young's modulus of elasticity, I is the moment of inertia of the cross section of the beam, and w is the load per unit length on the beam. According to Exercise 7 in Section 2.5, the exact solution is

$$Y(x)=\frac{w}{k}\left[1-\frac{\cosh\lambda x\cos\lambda(1-x)+\cos\lambda x\cosh\lambda(1-x)}{\cos\lambda L+\cosh\lambda L}\right],$$

when w is constant, where $\lambda^4=k/(4EI)$.

(a) Show that polynomials that satisfy all four boundary conditions are

$$Y_N(x)=\sum_{n=1}^{N}b_n\left[n(n+5)L^{n+2}x-(n+3)(n+2)L^nx^3+6x^{n+3}\right\}.$$

(b) Use Galerkin's method to obtain the first approximation,

$$Y_1(x) = \frac{126w}{3024EI + 31kL^4}(L^3x - 2Lx^3 + x^4).$$

(c) The trigonometric functions $\sin(n\pi x/L)$, where $n \geq 1$ is an integer satisfy all four boundary conditions, so that we could use these as basis functions. Furthermore, the solution should be symmetric about $x = L/2$, and the functions $\sin(2n-1)\pi x/L$ possess this property. In other words, we could take approximations in the form

$$Y_N(x) = \sum_{n=1}^{N} c_n \sin\frac{(2n-1)\pi x}{L}.$$

Use Galerkin's method to find coefficients c_n.

(d) To compare the exact solution to the approximation in part (b) and the second approximation in part (c), let $L = 1$ and $EI = k/4$, and find expressions for $wY(x)/k$, $wY_1(x)/k$, and $wY_2(x)/k$, respectively. Tabulate these functions for $x = 0.1, 0.2, \ldots, 0.9$.

22. The differential equation

$$\frac{d^2X}{dt^2} + X + \epsilon X^3 = \sin\omega t,$$

describes the motion of a nonlinear spring subjected to a periodic forcing function. Suppose we wish to determine whether there exist periodic solutions with the same period as the forcing function. They would satisfy the boundary value problem

$$\frac{d^2X}{dt^2} + X + \epsilon X^3 = \sin\omega t, \quad 0 < t < 2\pi/\omega, \qquad X(0) = 0, \quad X(2\pi/\omega) = 0.$$

A first approximation to such a solution could be taken in the form $X_1(t) = c\sin\omega t$. Use Galerkin's method to find an equation that must be satisfied by c.

§15.4 Method of Weighted Residuals and Sturm-Liouville Systems

The MWR can be used to approximate eigenvalues and eigenfunctions for Sturm-Liouville systems. Consider the system

$$X'' + \lambda(1-x^2)X = 0, \quad 0 < x < 1, \tag{15.34a}$$

$$X(0) = 0, \quad X'(1) = 0. \tag{15.34b}$$

(In previous chapters, we represented eigenvalues as λ^2, rather than λ. We did this to avoid excessive square roots in calculations. In this section wherein we use numerical methods to approximate eigenvalues, there is no necessity for this.) We illustrate three methods for this problem.

Method 1

In the absence of the $1 - x^2$ factor, eigenvalues are $\lambda_n = (2n-1)^2\pi^2/4$ with corresponding eigenfunctions $X_n(x) = \sin{(2n-1)\pi x/2}$ (see Table 5.1 in Chapter 5). We use these as basis functions to approximate eigenvalues and eigenfunctions of Sturm-Liouville system 15.34. Because approximations using these basis functions satisfy the boundary conditions of the SL-system, the MWR is an interior one. Suppose we denote by $\lambda_{1,1}$ and $X_{1,1}(x) = \sin{(\pi x/2)}$ first approximations to the first eigenvalue and eigenfunction of the SL-system. Because the differential equation and boundary conditions are homogeneous, we have not included a multiplicative factor c_1 for $X_{1,1}$, and we will only be finding $\lambda_{1,1}$. To three decimal places, the smallest eigenvalue is known to be $\lambda_1 = 5.122$. We shall see how close each of the MWR methods approximates this value. The (equation) residual obtained by substituting $\lambda_{1,1}$ and $X_{1,1}(x)$ into the differential equation is

$$R = -\frac{\pi^2}{4}\sin\frac{\pi x}{2} + \lambda_{1,1}(1-x^2)\sin\frac{\pi x}{2}.$$

Collocation with collocation point $x = 1/2$ requires

$$0 = -\frac{\pi^2}{4}\sin\frac{\pi}{4} + \lambda_{1,1}\left(1-\frac{1}{4}\right)\sin\frac{\pi}{4} \qquad \Longrightarrow \qquad \lambda_{1,1} = 3.290.$$

The subdomain method (and the moment method) require

$$0 = \int_0^1 \left[-\frac{\pi^2}{4}\sin\frac{\pi x}{2} + \lambda_{1,1}(1-x^2)\sin\frac{\pi x}{2}\right] dx \qquad \Longrightarrow \qquad \lambda_{1,1} = 4.592.$$

With weight function $w_1(x) = \sin{(\pi x/2)}$ for Galerkin's method, we obtain the condition

$$0 = \int_0^1 \left[-\frac{\pi^2}{4}\sin\frac{\pi x}{2} + \lambda_{1,1}(1-x^2)\sin\frac{\pi x}{2}\right]\sin\frac{\pi x}{2}\, dx \qquad \Longrightarrow \qquad \lambda_{1,1} = 5.317.$$

For the second approximation $\lambda_{1,2}$ to the first eigenvalue and corresponding eigenfunction approximation $X_{1,2}(x) = c_1\sin{(\pi x/2)} + c_2\sin{(3\pi x/2)}$, the (equation) residual is

$$R = -\frac{\pi^2}{4}c_1\sin\frac{\pi x}{2} - \frac{9\pi^2}{4}c_2\sin\frac{3\pi x}{2} + \lambda_{1,2}(1-x^2)\left(c_1\sin\frac{\pi x}{2} + c_2\sin\frac{3\pi x}{2}\right).$$

Collocation with collocation points $x = 1/3$ and $x = 2/3$ requires

$$0 = -\frac{\pi^2}{4}c_1 \sin\frac{\pi}{6} - \frac{9\pi^2}{4}c_2 \sin\frac{\pi}{2} + \lambda_{1,2}\left(1 - \frac{1}{9}\right)\left(c_1 \sin\frac{\pi}{6} + c_2 \sin\frac{\pi}{2}\right),$$
$$0 = -\frac{\pi^2}{4}c_1 \sin\frac{\pi}{3} - \frac{9\pi^2}{4}c_2 \sin\pi + \lambda_{1,2}\left(1 - \frac{4}{9}\right)\left(c_1 \sin\frac{\pi}{3} + c_2 \sin\pi\right).$$

The second of these implies that $\lambda_{1,2} = 9\pi^2/20 = 4.441$. The first then implies that $c_2 = 0.041c_1$, so that the second approximation to the first eigenfunction is $X_{1,2}(x) = \sin(\pi x/2) + 0.041\sin(3\pi x/2)$.

The subdomain method requires

$$\begin{aligned}
0 &= \int_0^{1/2}\left[-\frac{\pi^2}{4}c_1 \sin\frac{\pi x}{2} - \frac{9\pi^2}{4}c_2 \sin\frac{3\pi x}{2} + \lambda_{1,2}(1 - x^2)\left(c_1 \sin\frac{\pi x}{2} + c_2 \sin\frac{3\pi x}{2}\right)\right]dx \\
&= c_1\left\{\frac{\pi}{2\sqrt{2}}(1 - \sqrt{2}) + \left[\frac{32(\sqrt{2} - 1) - 8\pi + (4\sqrt{2} - 3)\pi^2}{2\sqrt{2}\pi^3}\right]\lambda_{1,2}\right\} \\
&\qquad + c_2\left\{-\frac{3\pi(1 + \sqrt{2})}{2\sqrt{2}} + \left[\frac{32(\sqrt{2} + 1) - 24\pi + (36\sqrt{2} + 27)\pi^2}{54\sqrt{2}\pi^3}\right]\lambda_{1,2}\right\} \\
0 &= \int_{1/2}^{1}\left[-\frac{\pi^2}{4}c_1 \sin\frac{\pi x}{2} - \frac{9\pi^2}{4}c_2 \sin\frac{3\pi x}{2} + \lambda_{1,2}(1 - x^2)\left(c_1 \sin\frac{\pi x}{2} + c_2 \sin\frac{3\pi x}{2}\right)\right]dx \\
&= c_1\left\{-\frac{\pi}{2\sqrt{2}} + \left[\frac{32 + 8(1 - 2\sqrt{2})\pi + 3\pi^2}{2\sqrt{2}\pi^3}\right]\lambda_{1,2}\right\} \\
&\qquad + c_2\left\{\frac{3\pi}{2\sqrt{2}} + \left[\frac{-32 + 24(2\sqrt{2} + 1)\pi - 27\pi^2}{54\sqrt{2}\pi^3}\right]\lambda_{1,2}\right\}.
\end{aligned}$$

Because nontrivial solutions of this linear system of homogeneous equations in c_1 and c_2 must exist, the determinant of the system must vanish,

$$\begin{aligned}
0 = &\left\{\frac{\pi(1 - \sqrt{2})}{2\sqrt{2}} + \left[\frac{32(\sqrt{2} - 1) - 8\pi + (4\sqrt{2} - 3)\pi^2}{2\sqrt{2}\pi^3}\right]\lambda_{1,2}\right\}\left\{\frac{3\pi}{2\sqrt{2}} + \left[\frac{-32 + 24(2\sqrt{2} + 1)\pi - 27\pi^2}{54\sqrt{2}\pi^3}\right]\lambda_{1,2}\right\} \\
&- \left\{-\frac{3\pi(1 + \sqrt{2})}{2\sqrt{2}} + \left[\frac{32(\sqrt{2} + 1) - 24\pi + (36\sqrt{2} + 27)\pi^2}{54\sqrt{2}\pi^3}\right]\lambda_{1,2}\right\}\left\{-\frac{\pi}{2\sqrt{2}} + \left[\frac{32 + 8(1 - 2\sqrt{2})\pi + 3\pi^2}{2\sqrt{2}\pi^3}\right]\lambda_{1,2}\right\}.
\end{aligned}$$

This is a quadratic equation in $\lambda_{1,2}$. Function $X_{1,2}(x)$ is a second approximation to the first eigenfunction, but it can also be considered an approximation to the second eigenfunction. As a result, the smaller of the solutions of this equation is the second approximation to the first eigenvalue, and the larger solution is a first approximation to the second eigenvalue. They are $\lambda_{1,2} = 5.123$ and $\lambda_{2,1} = 34.76$. The second eigenvalue is known to be $\lambda_2 = 39.66$, accurate to two decimal places. For $\lambda_{1,2} = 5.123$, the second equation implies that $c_2 = 0.059c_1$, and therefore the second approximation to the first eigenfunction is $X_{1,2}(x) = \sin(\pi x/2) + 0.059\sin(3\pi x/2)$.

The moment method requires

$$\begin{aligned}0 &= \int_0^1 \left[-\frac{\pi^2}{4}c_1 \sin\frac{\pi x}{2} - \frac{9\pi^2}{4}c_2\sin\frac{3\pi x}{2} + \lambda_{1,2}(1-x^2)\left(c_1\sin\frac{\pi x}{2} + c_2\sin\frac{3\pi x}{2}\right)\right]dx\\ &= \frac{1}{54\pi^3}\left\{c_1\left[-27\pi^4 + (864 - 432\pi + 108\pi^2)\lambda_{1,2}\right] + c_2\left[-81\pi^4 + (32 + 48\pi + 36\pi^2)\lambda_{1,2}\right]\right\},\\ 0 &= \int_0^1 \left[-\frac{\pi^2}{4}c_1 \sin\frac{\pi x}{2} - \frac{9\pi^2}{4}c_2\sin\frac{3\pi x}{2} + \lambda_{1,2}(1-x^2)\left(c_1\sin\frac{\pi x}{2} + c_2\sin\frac{3\pi x}{2}\right)\right]x\,dx\\ &= \frac{1}{27\pi^4}\left\{c_1\left[-27\pi^4 + (2592 - 216\pi^2)\lambda_{1,2}\right] + c_2\left[27\pi^4 + (24\pi^2 - 32)\lambda_{1,2}\right]\right\}.\end{aligned}$$

When we set the determinant of this system to zero, the resulting quadratic equation has solutions $\lambda_{1,2} = 5.183$ and $\lambda_{2,1} = 40.98$. For $\lambda_{1,2} = 5.183$, the second equation implies that $c_2 = 0.064c_1$, and therefore the second approximation to the first eigenfunction is $X_{1,2}(x) = \sin(\pi x/2) + 0.064\sin(3\pi x/2)$.

Because basis functions for Galerkin's method are orthogonal, integrations are less intensive. With weight functions $w_1 = \sin(\pi x/2)$ and $w_2 = \sin(3\pi x/2)$, the method requires

$$\begin{aligned}0 &= \int_0^1 \left[-\frac{\pi^2}{4}c_1 \sin\frac{\pi x}{2} - \frac{9\pi^2}{4}c_2\sin\frac{3\pi x}{2} + \lambda_{1,2}(1-x^2)\left(c_1\sin\frac{\pi x}{2} + c_2\sin\frac{3\pi x}{2}\right)\right]\sin\frac{\pi x}{2}dx,\\ &= c_1\left[-\frac{\pi^2}{8} + \frac{(\pi^2-3)\lambda_{1,2}}{3\pi^2}\right] + \frac{5c_2\lambda_{1,2}}{4\pi^2},\\ 0 &= \int_0^1 \left[-\frac{\pi^2}{4}c_1 \sin\frac{\pi x}{2} - \frac{9\pi^2}{4}c_2\sin\frac{3\pi x}{2} + \lambda_{1,2}(1-x^2)\left(c_1\sin\frac{\pi x}{2} + c_2\sin\frac{3\pi x}{2}\right)\right]\sin\frac{3\pi x}{2}dx,\\ &= \frac{5c_1\lambda_{1,2}}{4\pi^2} + c_2\left[-\frac{9\pi^2}{8} + \frac{(3\pi^2-1)\lambda_{1,2}}{9\pi^2}\right].\end{aligned}$$

Once again we set the determinant equal to zero, and the resulting quadratic equation has roots $\lambda_{1,2} = 5.125$ and $\lambda_{2,1} = 45.54$.

The above equations also imply that $c_2 = 0.069c_1$, so that the second approximation to the first eigenfunction is $X_{1,2}(x) = \sin(\pi x/2) + 0.069\sin(3\pi x/2)$.

Method 2

We now use polynomial basis functions to approximate the eigenvalues of Sturm-Liouville system 15.34. In particular, suppose we demand that polynomials

$$X_N(x) = \sum_{n=0}^{N+1} c_n x^n$$

satisfy boundary conditions 15.34b,c,

$$0 = X(0) = c_0, \qquad 0 = X'(1) = \sum_{n=0}^{N+1} nc_n.$$

The second of these requires $c_1 = -2c_2 - 3c_3 - \cdots - (N+1)c_{N+1}$, and therefore

$$\begin{aligned}X_N(x) &= [-2c_2 - 3c_3 - \cdots - (N+1)c_{N+1}]x + c_2x^2 + c_3x^3 + \cdots + c_{N+1}x^{N+1}\\ &= c_2(x^2 - 2x) + c_3(x^3 - 3x) + \cdots + c_{N+1}[x^{N+1} - (N+1)x]\\ &= \sum_{n=1}^{N} b_n[x^{n+1} - (n+1)x]. \qquad (15.35)\end{aligned}$$

Thus, basis functions are $x^{n+1} - (n+1)x$. (In Exercise 1, we use basis functions as suggested by Table 15.5.) Because approximations $X_n(x)$ satisfy the boundary conditions of the Sturm-Liouville system, the method is once again an interior one. The first approximation to the first eigenfunction is $X_{1,1}(x) = x^2 - 2x$, and the residual is

$$R = 2 + \lambda_{1,1}(1 - x^2)(x^2 - 2x). \tag{15.36}$$

Collocation with collocation point $x = 1/2$ requires

$$0 = 2 + \lambda_{1,1}\left(1 - \frac{1}{4}\right)\left(\frac{1}{4} - 1\right) \quad \Longrightarrow \quad \lambda_{1,1} = 3.556.$$

The subdomain and moment methods require

$$0 = \int_0^1 [2 + \lambda_{1,1}(1 - x^2)(x^2 - 2x)]\, dx = -2 - \frac{11\lambda_{1,1}}{30} \quad \Longrightarrow \quad \lambda_{1,1} = 5.455.$$

The Galerkin method requires

$$0 = \int_0^1 [2 + \lambda_{1,1}(1 - x^2)(x^2 - 2x)](x^2 - 2x)\, dx = -\frac{4}{3} + \frac{9\lambda_{1,1}}{35} \quad \Longrightarrow \quad \lambda_{1,1} = 5.185.$$

The second approximation to the first eigenfunction is $X_{1,2}(x) = b_1(x^2 - 2x) + b_2(x^3 - 3x)$, with residual

$$R = 2b_1 + 6b_2 x + \lambda_{1,2}(1 - x^2)[b_1(x^2 - 2x) + b_2(x^3 - 3x)].$$

Collocation with collocation points $x = 1/3$ and $x = 2/3$ requires

$$0 = 2b_1 + 6b_2\left(\frac{1}{3}\right) + \lambda_{1,2}\left(1 - \frac{1}{9}\right)\left[b_1\left(\frac{1}{9} - \frac{2}{3}\right) + b_2\left(\frac{1}{27} - 1\right)\right],$$
$$0 = 2b_1 + 6b_2\left(\frac{2}{3}\right) + \lambda_{1,2}\left(1 - \frac{4}{9}\right)\left[b_1\left(\frac{4}{9} - \frac{4}{3}\right) + b_2\left(\frac{8}{27} - 2\right)\right].$$

These simplify to

$$0 = b_1\left(1 - \frac{20\lambda_{1,2}}{81}\right) + b_2\left(1 - \frac{104\lambda_{1,2}}{243}\right),$$
$$0 = b_1\left(1 - \frac{20\lambda_{1,2}}{81}\right) + b_2\left(2 - \frac{115\lambda_{1,2}}{243}\right).$$

When we set the determinant of the system equal to zero, we obtain

$$0 = \begin{vmatrix} 1 - \dfrac{20\lambda_{1,2}}{81} & 1 - \dfrac{104\lambda_{1,2}}{243} \\ 1 - \dfrac{20\lambda_{1,2}}{81} & 2 - \dfrac{115\lambda_{1,2}}{243} \end{vmatrix} = \left(1 - \frac{20\lambda_{1,2}}{81}\right)\left(1 - \frac{115\lambda_{1,2}}{243} + \frac{104\lambda_{1,2}}{243}\right).$$

The smaller solution $\lambda_{1,2} = 4.050$ is the second approximation to the first eigenvalue, and the larger solution $\lambda_{2,1} = 22.09$ is the first approximation to the second eigenvalue.

The subdomain method requires

$$
\begin{aligned}
0 &= \int_0^{1/2} \left\{2b_1 + 6b_2 x + \lambda_{1,2}(1-x^2)[b_1(x^2-2x) + b_2(x^3-3x)]\right\} dx \\
&= b_1\left(1 - \frac{11\lambda_{1,2}}{60}\right) + b_2\left(\frac{3}{4} - \frac{121\lambda_{1,2}}{384}\right), \\
0 &= \int_{1/2}^{1} \left\{2b_1 + 6b_2 x + \lambda_{1,2}(1-x^2)[b_1(x^2-2x) + b_2(x^3-3x)]\right\} dx \\
&= b_1\left(1 - \frac{11\lambda_{1,2}}{60}\right) + b_2\left(\frac{9}{4} - \frac{45\lambda_{1,2}}{128}\right).
\end{aligned}
$$

A vanishing determinant leads to an unimproved estimate of the first eigenvalue $\lambda_{1,2} = 5.455$, and $\lambda_{2,1} = 41.14$.

The moment method demands that

$$
\begin{aligned}
0 &= \int_0^{1} \left\{2b_1 + 6b_2 x + \lambda_{1,2}(1-x^2)[b_1(x^2-2x) + b_2(x^3-3x)]\right\} dx \\
&= b_1\left(2 - \frac{11\lambda_{1,2}}{30}\right) + b_2\left(3 - \frac{2\lambda_{1,2}}{3}\right), \\
0 &= \int_0^{1} \left\{2b_1 + 6b_2 x + \lambda_{1,2}(1-x^2)[b_1(x^2-2x) + b_2(x^3-3x)]\right\} x\, dx \\
&= b_1\left(1 - \frac{11\lambda_{1,2}}{60}\right) + b_2\left(2 - \frac{12\lambda_{1,2}}{35}\right).
\end{aligned}
$$

These yield $\lambda_{1,2} = 5.455$ and $\lambda_{2,1} = 52.50$.

Galerkin's method requires

$$
\begin{aligned}
0 &= \int_0^{1} \left\{2b_1 + 6b_2 x + \lambda_{1,2}(1-x^2)[b_1(x^2-2x) + b_2(x^3-3x)]\right\} (x^2-2x)\, dx \\
&= b_1\left(-\frac{4}{3} + \frac{9\lambda_{1,2}}{35}\right) + b_2\left(-\frac{5}{2} + \frac{401\lambda_{1,2}}{840}\right), \\
0 &= \int_0^{1} \left\{2b_1 + 6b_2 x + \lambda_{1,2}(1-x^2)[b_1(x^2-2x) + b_2(x^3-3x)]\right\} (x^3-3x)\, dx \\
&= b_1\left(-\frac{5}{2} + \frac{401\lambda_{1,2}}{840}\right) + b_2\left(-\frac{24}{5} + \frac{8\lambda_{1,2}}{9}\right).
\end{aligned}
$$

These yield $\lambda_{1,2} = 5.161$ and $\lambda_{2,1} = 42.81$.

Method 3

In this method, we once again use polynomial basis functions to approximate eigenfunctions of the SL-system, but demand that approximations satisfy boundary condition 15.34b, but not 15.34c; that is, we demand that $\phi_n(0) = 0$. The method is therefore a mixed MWR. Boundary condition 15.34c will be incorporated at a later stage by introducing a boundary residual. As we did in Section 15.3, we proceed with unspecified basis functions $\phi_n(x)$ (except that they satisfy $\phi_n(0) = 0$). The residual resulting from the approximation of $X(x)$ by $X_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$ is

$$R = \sum_{n=1}^{N} c_n[\phi_n'' + \lambda(1 - x^2)\phi_n].$$

With as yet unspecified weight functions $w_m(x)$, the MWR requires

$$\begin{aligned} 0 &= \int_0^1 \left\{ \sum_{n=1}^{N} c_n[\phi_n'' + \lambda(1 - x^2)\phi_n] \right\} w_m \, dx \\ &= \sum_{n=1}^{N} c_n \left[\int_0^1 \phi_n'' w_m \, dx + \lambda \int_0^1 (1 - x^2)\phi_n w_m \, dx \right]. \end{aligned}$$

We use integration by parts on the first integral to write

$$0 = \sum_{n=1}^{N} c_n \left[\{\phi_n' w_m\}_0^1 - \int_0^1 \phi_n' w_m' \, dx + \lambda \int_0^1 (1 - x^2)\phi_n w_m \, dx \right]. \quad (15.37)$$

We can eliminate the term $\phi_n' w_m$ at $x = 0$ by choosing only weight functions that satisfy $w_m(0) = 0$. Because basis functions do not satisfy boundary condition 15.34c, neither will $X_N(x)$. We define a boundary residual at $x = 1$,

$$R_{|x=1} = \sum_{n=1}^{N} c_n \phi_n'(1). \quad (15.38)$$

We implicitly demand that when multiplied by weight functions $w_m(1)$, the result be zero

$$\sum_{n=1}^{N} c_n \phi_n'(1) w_m(1) = 0. \quad (15.39)$$

Substitution of these two requirements into equation 15.43 removes the nonintegral terms,

$$0 = \sum_{n=1}^{N} c_n \left[\int_0^1 [\lambda(1 - x^2)\phi_n w_m - \phi_n' w_m'] \, dx \right]. \quad (15.40)$$

Once basis functions $\phi_n(x)$ and weight functions $w_m(x)$ are chosen, it is these equations that determine approximate eigenvalues and eigenfucntions.

Suppose we choose polynomial basis functions $\phi_n(x) = x^n$ which, as required, satisfy boundary condition 15.34b, but not 15.34c. Weight functions associated with the collocation and subdomain methods do not satisfy the requirement that $w_m(0) = 0$. Weight functions $w_m = x^m$ associated with the moment method do satisfy this requirement, and these are the same weight functions for Galerkin's method. The first approximation to the first eigenvalue is denoted as before by $\lambda_{1,1}$, and the first eigenfunction is $X_{1,1}(x) = x$. With these, condition 15.40 requires

$$0 = \int_0^1 [\lambda_{1,1}(1 - x^2)x^2 - 1] \, dx = \frac{2\lambda_{1,1}}{15} - 1 \qquad \Longrightarrow \qquad \lambda_{1,1} = 7.5.$$

The second approximation to the first eigenfunction is $X_{1,2}(x) = c_1 x + c_2 x^2$. With corresponding approximation $\lambda_{1,2}$, conditions 15.40 demand that

$$0 = \sum_{n=1}^{2} c_n \int_0^1 [\lambda_{1,2}(1-x^2)x^{n+m} - nmx^{n+m-2}]\,dx, \qquad m = 1, 2.$$

Integrations with these values of m give

$$\begin{aligned}
0 &= c_1 \int_0^1 [\lambda_{1,2}(1-x^2)x^2 - 1]\,dx + c_2 \int_0^1 [\lambda_{1,2}(1-x^2)x^3 - 2x]\,dx \\
&= c_1\left(\frac{2\lambda_{1,2}}{15} - 1\right) + c_2\left(\frac{\lambda_{1,2}}{12} - 1\right), \\
0 &= c_1 \int_0^1 [\lambda_{1,2}(1-x^2)x^3 - 2x]\,dx + c_2 \int_0^1 [\lambda_{1,2}(1-x^2)x^4 - 4x^2]\,dx \\
&= c_1\left(\frac{\lambda_{1,2}}{12} - 1\right) + c_2\left(\frac{2\lambda_{1,2}}{35} - \frac{4}{3}\right).
\end{aligned}$$

When we set the determinant of this system equal to zero,

$$0 = \left(\frac{2\lambda_{1,2}}{15} - 1\right)\left(\frac{2\lambda_{1,2}}{35} - \frac{4}{3}\right) - \left(\frac{\lambda_{1,2}}{12} - 1\right)^2.$$

Solutions are $\lambda_{1,2} = 5.145$ and $\lambda_{2,1} = 96.03$. The first of these leads to $c_2 = -0.550c_1$, so that the second approximation to the first eigenfunction is $X_{1,2}(x) = x - 0.550x^2$. As suggested earlier, because condition 15.39 was used implicitly, not explicitly, this function does not satisfy boundary condition 15.34c.

Table 15.6 shows first and second approximations to the first eigenvalue of the Sturm-Liouville system for all three methods.

EXERCISES 15.4

1. With basis functions as suggested in Table 15.5, use Galerkin's method to the second approximation to the smallest eigenvalue of system 15.34.

2. Apply Galerkin's method to find first and second approximations for the smallest eigenvalue of Sturm-Liouville system 15.34 when boundary conditions are $X(0) = X(1) = 0$. Use eigenfunctions of the Sturm-Liouville system when the $1 - x^2$ factor is absent as basis functions.

3. Consider the Sturm-Liouville system

$$X'' + \lambda X = 0, \qquad X(0) = 0, \quad X'(1) + 2X(1) = 0.$$

(a) According to Table 5.1 in Section 5.2, values of λ satisfy the equation

$$\cot\sqrt{\lambda} = -\frac{2}{\sqrt{\lambda}}.$$

Use a numerical method to find the smallest value of λ satisfying this equation and hence the smallest eigenvalue of the Sturm-Liouville system.

(b) Use polynomials and Galerkin's method to find first and second approximations to the smallest eigenvalue.

		First Eigenvalue	Second Eigenvalue
Exact (to 4 figures)		5.122	39.66
First Approximation With Trigonometric Basis Functions	Collocation	3.290	
	Subdomain	4.592	34.76
	Moment	4.592	40.98
	Galerkin	5.317	45.54
Second Approximation With Trigonometric Basis Functions	Collocation	4.441	
	Subdomain	5.123	
	Moment	5.183	
	Galerkin	5.125	
First Approximation With Polynomial Basis and Both Boundary Conditions Satisfied	Collocation	3.556	
	Subdomain	5.455	
	Moment	5.183	
	Galerkin	5.125	
Second Approximation With Polynomial Basis and Both Boundary Conditions Satisfied	Collocation	4.050	22.09
	Subdomain	5.455	41.14
	Moment	5.161	52.50
	Galerkin	5.125	42.81
First Approximation With Polynomial Basis and One Boundary Condition Satisfied	Galerkin	7.5	
Second Approximation With Polynomial Basis and One Boundary Condition Satisfied	Galerkin	5.145	96.03

Table 15.6

4. To four figures, the smallest eigenvalue of the Sturm-Liouville system

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r R = 0, \quad 0 < r < a,$$
$$R(a) = 0,$$

is $\lambda_1 = 5.783/a^2$ (see Section 8.4). Use basis functions $\phi_n(r) = \cos\dfrac{(2n-1)\pi r}{2a}$, $n = 1, 2, \ldots$, to find a first approximation to this value with:
(a) collocation;
(b) the subdomain (or moment) method;
(c) Galerkin's method.

5. (a) Repeat Exercise 4 with polynomial basis functions that satisfy the boundary condition.
(b) Find second approximations to the smallest eigenvalue using collocation, subdomain, and Galerkin methods.

6. The smallest nonnegative eigenvalue of the Sturm-Liouville system

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r R = 0, \quad 0 < r < a,$$
$$R'(a) = 0,$$

is $\lambda_0 = 0$ (see Section 8.4). To four figures, the smallest positive eigenvalue is $\lambda_1 = 14.68/a^2$.
(a) Show that polynomial approximations that satisfy the boundary conditions are

$$R_N(r) = b_0 + \sum_{n=1}^{N} b_n r[r^n - (n+1)a^n].$$

(b) Use Galerkin's method with $R_1(r) = b_0 + b_1(r^2 - 2ar)$ to find a first approximation to λ_1.
(c) Use Galerkin's method with $R_2(r) = b_0 + b_1(r^2 - 2ar) + b_2(r^3 - 3a^2 r)$ to find a second approximation to λ_1, and a first approximation to λ_2.

7. To four figures, the smallest eigenvalue of the Sturm-Liouville system

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r R = 0, \quad 0 < r < 1,$$
$$R'(1) + 2R(1) = 0,$$

is $\lambda_1 = 2.558$ (see Section 8.4).
(a) Show that polynomial approximations that satisfy the boundary conditions are

$$R_N(r) = \sum_{n=1}^{N} b_n(2r^n - n - 2).$$

(b) Use Galerkin's method with $R_1(r) = b_1(2r - 3)$ to find a first approximation to λ_1.
(c) Use Galerkin's method with $R_2(r) = b_1(2r - 3) + b_2(2r^2 - 4)$ to find a second approximation to λ_1, and a first approximation to λ_2.

8. (a) Suppose the Neumann boundary condition at $x = 1$ in Sturm-Liouville system 15.34 is replaced by a Robin condition $X'(1) + 2X(1) = 0$. Show that polynomial approximations that satisfy both boundary conditions are

$$X_N(x) = \sum_{n=1}^{N} c_n x\left(x^n - \frac{n+3}{3}\right).$$

(b) Find first and second approximations to the smallest eigenvalue of the Sturm-Liouville system using Galerkin's method.

9. Eigenvalues of Legendre's differential equation

$$(1 - x^2)\frac{d^2Y}{dx^2} - 2x\frac{dY}{dx} + \lambda Y = 0, \quad -1 < x < 1,$$

are $\lambda_n = i(i+1)$, where $i \geq 0$ is an integer. Suppose that polynomials $Y_N(x) = \sum_{n=0}^{N} c_n x^n$ are used to approximate the solution of the equation. Show that with Galerkin's method:
(a) $Y_0(x)$ yields λ_0;

(b) $Y_1(x)$ yields λ_0 and λ_1;
(c) $Y_2(x)$ yields λ_0, λ_1, and λ_2;
(d) $Y_3(x)$ yields λ_0, λ_1, λ_2, and λ_3.

§15.5 Method of Weighted Residuals and Dirichlet Boundary Value Problems

We now apply the MWR to boundary value problems associated with partial differential equations, and in this section, we deal with Dirichlet problems as they are the easiest to handle. We begin with a general discussion to outline one possible procedure, but other approaches may be advantageous, such as *reduction of dimensionality*, a method that we also introduce in this section. Consider the two-dimensional problem

$$L(V) = F(x, y), \quad (x, y) \text{ in } R, \tag{15.41a}$$

$$V(x, y) = G(x, y), \quad (x, y) \text{ on } \beta(R), \tag{15.41b}$$

where L is some partial differential operator (which may be linear, such as the Laplacian, or nonlinear), $\beta(R)$ is the boundary of some region R in the xy-plane, and $F(x, y)$ and $G(x, y)$ are given functions. For an interior method, we could take approximations in the form

$$V_N(x, y) = \phi_0(x, y) + \sum_{n=1}^{N} c_n \phi_n(x, y), \tag{15.42}$$

where $\phi_0(x, y)$ satisfies the nonhomogeneous boundary condition, and basis functions $\phi_n(x, y)$ for $n = 1, \dots, N$ satisfy the homogeneous version of the boundary condition; that is $\phi_n(x, y) = 0$ on $\beta(R)$. Approximations $V_N(x, y)$ then satisfy boundary condition 15.41b, and the resulting (equation) residual need only account for $V_N(x, y)$ not satisfying the PDE,

$$\begin{aligned} R = L(V_N) - F(x, y) &= L\left[\phi_0(x, y) + \sum_{n=1}^{N} c_n \phi_n(x, y)\right] - F(x, y) \\ &= L(\phi_0) + \sum_{n=1}^{N} c_n L(\phi_n) - F(x, y). \end{aligned} \tag{15.43}$$

(This calculation has assumed that L is linear.) When N weight functions $w_m(x, y)$ are chosen, the MWR requires

$$0 = \iint_R \left[L(\phi_0) + \sum_{n=1}^{N} c_n L(\phi_n) - F(x, y)\right] w_m(x, y)\, dA, \quad m = 1, \dots, N.$$

We can express these equations, which determine the c_n, in the form

$$\sum_{n=1}^{N} c_n \iint_R L(\phi_n) w_m\, dA = \iint_R [F - L(\phi_0)] w_m\, dA, \quad m = 1, \dots, N.$$

As was the case for boundary value problems associated with ODEs, basis functions must be linearly independent and from a complete set. Possible choices once again include eigenfunctions of associated Sturm-Liouville problems, and polynomials. The polynomials $x^n y^m$, $n, m = 0, 1, 2, \dots$ form a complete set for the space of continuous functions, but in the above approach, it is unlikely that they will satisfy the boundary condition. However, if $\omega(x, y)$ is a positive, continuously differentiable function in R that vanishes on the boundary of R, then the functions $\omega(x, y)x^n y^m$

constitute a complete set and they do satisfy the boundary condition. We use this idea in our first example which has a simple nonhomogeneity in the differential equation, and homogeneous boundary conditions.

Example 15.2 Find polynomial approximations to the solution of the boundary value problem

$$\begin{aligned}\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= k, \quad -L < x < L, \quad -L < y < L,\\ V(-L, y) = V(L, y) &= 0, \quad -L < y < L,\\ V(x, -L) = V(x, L) &= 0, \quad -L < x < L,\end{aligned}$$

where k is a constant. Use an interior method and a boundary method.

Solution Interior Method

The function $\omega(x, y) = (L^2 - x^2)(L^2 - y^2)$ is positive, continuously differentiable, and vanishes on the edges of the square. Polynomial basis functions can therefore be taken as $x^n y^m (L^2 - x^2)(L^2 - y^2)$, $n, m = 0, 1, \ldots$. Furthermore, because the solution of the problem should be an even function of both x and y, and be symmetric in x and y, we can further restrict the choices for $x^n y^m$. First and second approximations that satisfy the boundary conditions and symmetry requirements are $V_1(x, y) = c(L^2 - x^2)(L^2 - y^2)$ and $V_2(x, y) = (L^2 - x^2)(L^2 - y^2)[c + d(x^2 + y^2)]$. We work with the second approximation. The equation residual is

$$\begin{aligned}R(x, y) &= -2(L^2 - y^2)[c + d(x^2 + y^2)] + 2(-2x)(L^2 - y^2)(2dx)\\ &\quad + (L^2 - x^2)(L^2 - y^2)(2d) - 2(L^2 - x^2)[c + d(x^2 + y^2)]\\ &\quad + 2(-2y)(L^2 - x^2)(2dy) + (L^2 - x^2)(L^2 - y^2)(2d) - k\\ &= -2[c + d(x^2 + y^2)](2L^2 - x^2 - y^2) - 8d[x^2(L^2 - y^2) + y^2(L^2 - x^2)]\\ &\quad + 4d(L^2 - x^2)(L^2 - y^2) - k.\end{aligned}$$

We use collocation, subdomains, and Galerkin's method to find values for c and d. In addition, to four decimal places, the solution of the boundary value problem at the centre of the square is $-0.2947L^2k$ (see Exercise 32 in Section 4.3). We compare this value to that predicted by each approximation.

Collocation

Due to the symmetry of the problem, we choose two collocation points in the first quadrant part of the square, namely $(0, 0)$ and $(L/2, L/2)$. These yield the equations

$$0 = -4cL^2 + 4dL^4 - k, \qquad 0 = -3cL^2 - \frac{9}{4}dL^4 - k.$$

The solution is $c = -\dfrac{25k}{84L^2}$ and $d = -\dfrac{k}{21L^4}$, and therefore the second collocation approximation is

$$V_2(x, y) = -\frac{k}{84L^4}(L^2 - x^2)(L^2 - y^2)[25L^2 + 4(x^2 + y^2)].$$

It predicts a value of $V_2(0, 0) = -25L^2k/84 \approx -0.2976L^2k$ for the centre of the region.

Subdomain

We choose two symmetric subdomains, namely the square $A_1 : 0 \le x, y \le L/2$ and A_2 as the remainder of the original square in the first quadrant. These require

$$0 = \int_0^{L/2}\int_0^{L/2} R(x,y)\,dy\,dx = -\frac{kL^2}{4} - \frac{11cL^4}{12} + \frac{31dL^6}{80},$$
$$0 = \int_0^{L/2}\int_{L/2}^{L} R(x,y)\,dy\,dx + \int_{L/2}^{L}\int_0^{L} R(x,y)\,dy\,dx = -\frac{3kL^2}{4} - \frac{7cL^4}{4} - \frac{287dL^6}{80}.$$

The solution of these equations is $c = -\dfrac{285k}{952L^2}$ and $d = -\dfrac{15k}{238L^4}$, and the second subdomain approximation is

$$V_2(x,y) = -\frac{15k}{952L^4}(L^2 - x^2)(L^2 - y^2)[17L^2 - 4(x^2 + y^2)].$$

It predicts a value of $V_2(0,0) = -285kL^2/952 \approx -0.2994L^2k$ for the centre of the region.

Galerkin

Galerkin's method requires

$$0 = \int_0^L\int_0^L R(x,y)(L^2 - x^2)(L^2 - y^2)\,dy\,dx = -\frac{4kL^6}{9} - \frac{64cL^8}{45} - \frac{256dL^{10}}{525},$$
$$0 = \int_0^L\int_0^L R(x,y)(L^2 - x^2)(L^2 - y^2)(x^2 + y^2)\,dy\,dx = -\frac{8kL^8}{45} - \frac{256cL^{10}}{525} - \frac{2816dL^{12}}{4725}.$$

The solution of these equations is $c = -\dfrac{1295k}{4432L^2}$ and $d = -\dfrac{525k}{8864L^4}$. The second Galerkin approximation is therefore

$$V_2(x) = -\frac{5k}{8864L^2}(L^2 - x^2)(L^2 - y^2)\left[518L^2 + 105(x^2 + y^2)\right].$$

It predicts a value of $V_2(0,0) = -1295kL^2/4432 \approx -0.2922L^2k$ for the centre of the region.

Boundary Method

In a boundary method, polynomial approximations must satisfy the PDE. The function $k(x^2 + y^2)/4$ satisfies the PDE. To find polynomials that satisfy the homogeneous version of the PDE, namely, Laplace's equation, we use the fact that real and imaginary parts of every complex analytic function satisfy Laplace's equation; in particular, real and imaginary parts of the function $z^n = (x + yi)^n$ give polynomial solutions of Laplace's equation. The first few, from $n = 1, 2, 3$, and 4 are

$$1,\ x,\ y,\ x^2 - y^2,\ xy,\ x^3 - 3xy^2,\ 3x^2y - y^3,\ x^4 - 6x^2y^2 + y^4,\ 4x^3 - 4xy^3, \ldots.$$

As already noted, the solution of the problem must be even in x and y, and symmetric with respect to x and y. The first two such polynomials are 1 and $x^4 - 6x^2y^2 + y^4$. We therefore take as an approximating polynomial that satisfies the PDE

$$V_2(x,y) = c + \frac{k}{4}(x^2 + y^2) + d(x^4 - 6x^2y^2 + y^4).$$

The residual of this approximation along each of the four edges of the square is identical, and we therefore consider it along $x = L$,

$$R = c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4).$$

We now use collocation, subdomains, and Galerkin's method to determine c and d.

Collocation

Collocation with $y = L/3$ and $y = 2L/3$ requires

$$0 = c + \frac{k}{4}\left(L^2 + \frac{L^2}{9}\right) + d\left(L^4 - \frac{2L^4}{3} + \frac{L^4}{81}\right),$$
$$0 = c + \frac{k}{4}\left(L^2 + \frac{4L^2}{9}\right) + d\left(L^4 - \frac{8L^4}{3} + \frac{16L^4}{81}\right).$$

These imply that $c = -37kL^2/126$ and $d = 9k/(196L^2)$, and the second collocation approximation is

$$V_2(x, y) = -\frac{37kL^2}{126} + \frac{k}{4}(x^2 + y^2) + \frac{9k}{196L^2}(x^4 - 6x^2y^2 + y^4).$$

It predicts a value of $-37kL^2/126 \approx -0.2937L^2k$ at $(0, 0)$.

Subdomain

Subdomains require

$$0 = \int_0^{L/2}\left[c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4)\right]dy = \frac{Lc}{2} + \frac{13L^3k}{96} + \frac{41L^5d}{160},$$
$$0 = \int_{L/2}^{L}\left[c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4)\right]dy = \frac{Lc}{2} + \frac{19L^3k}{96} - \frac{169L^5d}{160}.$$

These give $c = -31kL^2/105$ and $d = k/(21L^2)$, and the second subdomain approximation is

$$V_2(x, y) = -\frac{31kL^2}{105} + \frac{k}{4}(x^2 + y^2) + \frac{k}{21L^2}(x^4 - 6x^2y^2 + y^4).$$

It predicts $V_2(0, 0) = -31kL^2/105 \approx -0.2952L^2k$.

Galerkin

Galerkin's method requires

$$0 = \int_0^{L}\left[c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4)\right]dy = Lc + \frac{kL^3}{3} - \frac{4L^5d}{5},$$
$$0 = \int_0^{L}\left[c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4)\right](L^4 - 6L^2y^2 + y^4)\,dy$$
$$= -\frac{4L^5}{5} - \frac{8L^7k}{21} + \frac{944L^9d}{315}.$$

The solution is $c = -205kL^2/696$ and $d = 45k/(928L^2)$, and the second Galerkin approximation is

$$V_2(x, y) = -\frac{205kL^2}{696} + \frac{k}{4}(x^2 + y^2) + \frac{45k}{928L^2}(x^4 - 6x^2y^2 + y^4).$$

Its prediction at the centre is $-205kL^2/696 \approx -0.2945L^2k$.●

The next example has a more general nonhomogeneity in the differential equation and two nonhomogeneous boundary conditions. We also use it to introduce the method of *reduction of dimensionality*.

Example 15.3 Use polynomials and eigenfunctions to approximate the solution of the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x,y), \quad 0 < x < L, \quad 0 < y < L', \tag{15.44a}$$

$$V(0,y) = V(L,y) = 0, \quad 0 < y < L', \tag{15.44b}$$

$$V(x,0) = g(x), \quad 0 < x < L, \tag{15.44c}$$

$$V(x,L') = h(x), \quad 0 < x < L. \tag{15.44d}$$

For continuity of boundary conditions at the corners of the rectangle, we assume that nonhomogeneities $g(x)$ and $h(x)$ satisfy the conditions $g(0) = g(L) = h(0) = h(L) = 0$.

Solution Consider using approximations of the form

$$V_N(x,y) = \phi_0(x,y) + \sum_{n=1}^{N} c_n \phi_n(x,y), \tag{15.45}$$

where $\phi_0(x,y)$ satisfies all boundary conditions, homogeneous and nonhomogeneous, and basis function $\phi_n(x,y)$ for $n = 1, \ldots, N$ satisfy homogeneous versions of the boundary conditions. A convenient choice for $\phi_0(x,y)$ is $g(x)(1 - y/L') + h(x)y/L'$. (Can you see the difficulty at this point were $g(0)$, $g(L)$, $h(0)$, and/or $h(L)$ nonzero?) For polynomial approximations, we choose $\omega(x,y) = xy(L-x)(L'-y)$, in which case basis functions that satisfy homogeneous versions of the boundary conditions are $\phi_{nm}(x,y) = x^n y^m (L-x)(L'-y)$, $n, m = 1, 2, \ldots$. Approximate solutions of the problem are therefore

$$V_N(x,y) = g(x)\left(1 - \frac{y}{L'}\right) + h(x)\frac{y}{L'} + \sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm} x^n y^m (L-x)(L'-y). \tag{15.46}$$

The first approximation is

$$V_1(x,y) = g(x)\left(1 - \frac{y}{L'}\right) + h(x)\frac{y}{L'} + c_{11}xy(L-x)(L'-y), \tag{15.47}$$

with equation residual

$$R = g''(x)\left(1 - \frac{y}{L'}\right) + h''(x)\frac{y}{L'} + 2c_{11}(x^2 - Lx + y^2 - L'y) - F(x,y).$$

Galerkin's method requires

$$0 = \int_0^L \int_0^{L'} \left[g''(x)\left(1 - \frac{y}{L'}\right) + h''(x)\frac{y}{L'} + 2c_{11}(x^2 - Lx + y^2 - L'y) \right.$$
$$\left. - F(x,y)\right] xy(L-x)(L'-y)\, dy\, dx,$$

and integrations lead to

$$c_{11} = \frac{-90}{L^3 L'^3 (L^2 + L'^2)} \left[\frac{L'^3}{6} \int_0^L g(x)\,dx + \frac{L'^3}{6} \int_0^L h(x)\,dx \right.$$
$$\left. + \int_0^L \int_0^{L'} F(x,y)xy(L-x)(L'-y)\,dy\,dx \right].$$

We now consider using eigenfunctions of the associated eigenvalue problem

$$\phi_{nm}(x,y) = \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'}$$

as basis functions. Approximations are

$$V_{NM}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'}. \tag{15.48}$$

These approximations satisfy homogeneous boundary conditions 15.44b, but not nonhomogeneous conditions 15.44c,d. The solution is pursued when all boundary conditions are homogeneous in Exercise 5. Nonhomogeneous conditions 15.44c,d can be handled by transforming them into the PDE. Suppose we make a change of dependent variable by $U(x,y) = V(x,y) + \phi_0(x,y)$ where $\phi_0(x,y)$ is any function that has value $g(x)$ along $y = 0$ and $h(x)$ along $y = L'$. The obvious choice is $g(x)(1 - y/L') + h(x)y/L'$. With this change, the boundary value problem for $U(x,y)$ is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F(x,y) - g''(x)\left(1 - \frac{y}{L'}\right) - h''(x)\frac{y}{L'}, \quad 0 < x < L, \quad 0 < y < L',$$
$$U(0,y) = U(L,y) = 0, \quad 0 < y < L',$$
$$U(x,0) = U(x,L') = 0, \quad 0 < x < L.$$

Approximations

$$U_{NM}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'}$$

satisfy the homogeneous boundary conditions. The equation residual is

$$R = -\pi^2 \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \left(\frac{n^2}{L^2} + \frac{m^2}{L'^2}\right) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'} - F(x,y)$$
$$+ g''(x)\left(1 - \frac{y}{L'}\right) + h''(x)\frac{y}{L'}.$$

Galerkin's method requires

$$0 = \int_0^L \int_0^{L'} \left[-\pi^2 \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \left(\frac{n^2}{L^2} + \frac{m^2}{L'^2}\right) \sin\frac{k\pi x}{L} \sin\frac{l\pi y}{L'} - F(x,y) \right.$$
$$\left. + g''(x)\left(1 - \frac{y}{L'}\right) + h''(x)\frac{y}{L'} \right] \sin\frac{k\pi x}{L} \sin\frac{l\pi y}{L'} dy\,dx.$$

Due to the orthogonality of the eigenfunctions, this immediately leads to

$$c_{nm} = \frac{-4LL'}{\pi^2(n^2L'^2 + m^2L^2)} \int_0^L \int_0^{L'} \left[F(x,y) - g''(x)\left(1 - \frac{y}{L'}\right) - h''(x)\frac{y}{L'}\right] * \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'} dy\, dx. \qquad (15.49a)$$

Multiple integrations by parts on the terms involving $g(x)$ and $h(x)$ leads to the alternative formula

$$c_{nm} = \frac{-4LL'}{\pi^2(n^2L'^2 + m^2L^2)} \left\{ \int_0^L \int_0^{L'} F(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'} dy\, dx + \frac{n^2\pi L'}{mL^2} \int_0^L [g(x) + (-1)^{m+1}h(x)] \sin\frac{n\pi x}{L} dx \right\}. \qquad (15.49b)$$

Finally then

$$V_{NM}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'} - g(x)\left(1 - \frac{y}{L'}\right) - h(x)\frac{y}{L'}. \bullet$$

Reduction of Dimensionality

The MWR can be used to reduce the dimensionality of a problem; for problem 15.44, the PDE is reduced to an ODE. We represent approximations as sums of separated functions

$$V_N(x,y) = \sum_{n=1}^{N} \phi_n(x,y) = \sum_{n=1}^{N} c_n(y)\psi_n(x),$$

where basis functions $\psi_n(x)$ must be specified, and coefficients $c_n(y)$ will be determined by the MWR. (Approximations were separated in the previous approach, but they need not have been so.) According to Table 15.5, polynomial basis functions satisfying boundary conditions 15.44b are $\psi_n(x) = x^n(L-x)$, $n = 1, 2, \ldots$. We therefore take approximations in the form

$$V_N(x,y) = \sum_{n=1}^{N} c_n(y)x^n(L-x),$$

the first being

$$V_1(x,y) = c_1(y)x(L-x).$$

The equation residual is

$$R = -2c_1 + x(L-x)c_1'' - F(x,y).$$

Galerkin's method requires

$$0 = \int_0^L [-2c_1 + x(L-x)c_1'' - F(x,y)]x(L-x)\, dx.$$

Integrations lead to

$$c_1'' - \frac{10c_1}{L^2} = \frac{30}{L^5}\int_0^L F(x,y)x(L-x)\,dx.$$

So that the remainder of the procedure can be illustrated without unduly complicated calculations, we assume that $F(x,y) = k$, a constant. In this case, $c_1(y)$ must satisfy the ODE

$$c_1'' - \frac{10c_1}{L^2} = \frac{5k}{L^2}.$$

A general solution of this equation is

$$c_1(y) = A\cosh\frac{\sqrt{10}y}{L} + B\sinh\frac{\sqrt{10}y}{L} - \frac{k}{2}.$$

The first approximation is therefore

$$V_1(x,y) = \left[A\cosh\frac{\sqrt{10}y}{L} + B\sinh\frac{\sqrt{10}y}{L} - \frac{k}{2}\right]x(L-x).$$

We associate boundary residuals with this approximation due to the fact that it does not satisfy boundary conditions 15.44c,d,

$$R_{|y=0} = \left(A - \frac{k}{2}\right)x(L-x) - g(x),$$

$$R_{|y=L'} = \left[A\cosh\frac{\sqrt{10}L'}{L} + B\sinh\frac{\sqrt{10}L'}{L} - \frac{k}{2}\right]x(L-x) - h(x).$$

We apply Galerkin's method to find A and B,

$$0 = \int_0^L \left[\left(A - \frac{k}{2}\right)x(L-x) - g(x)\right]x(L-x)\,dx,$$

$$0 = \int_0^L \left\{\left[A\cosh\frac{\sqrt{10}L'}{L} + B\sinh\frac{\sqrt{10}L'}{L} - \frac{k}{2}\right]x(L-x) - h(x)\right\}x(L-x)\,dx.$$

Integrations lead to

$$A = \frac{k}{2} + \frac{30}{L^5}\int_0^L g(x)x(L-x)\,dx,$$

$$B = \frac{30}{L^5}\mathrm{csch}\frac{\sqrt{10}L'}{L}\int_0^L h(x)x(L-x)\,dx - \coth\frac{\sqrt{10}L'}{L}\left[\frac{k}{2} + \frac{30}{L^5}\int_0^L g(x)x(L-x)\,dx\right].$$

We can also reduce the dimensionality of the problem using eigenfunctions $\psi_n(x) = \sin(n\pi x/L)$ as basis functions,

$$V_N(x,y) = \sum_{n=1}^{N} c_n(y)\sin\frac{n\pi x}{L}.$$

The equation residual is

$$R = \sum_{n=1}^{N}\left(-\frac{n^2\pi^2}{L^2}\right)c_n \sin\frac{i\pi x}{L} + \sum_{n=1}^{N} c_n'' \sin\frac{n\pi x}{L} - F(x,y)$$
$$= \sum_{n=1}^{N}\left(c_n'' - \frac{n^2\pi^2}{L^2}c_n\right)\sin\frac{n\pi x}{L} - F(x,y).$$

Galerkin's method requires

$$0 = \int_0^L \left[\sum_{n=1}^{N}\left(c_n'' - \frac{n^2\pi^2}{L^2}c_n\right)\sin\frac{n\pi x}{L} - F(x,y)\right]\sin\frac{m\pi x}{L}\,dx,$$

and due to orthogonality of eigenfunctions, this reduces to

$$c_m'' - \frac{m^2\pi^2}{L^2}c_m = \frac{2}{L}\int_0^L F(x,y)\sin\frac{m\pi x}{L}\,dx.$$

When $F(x,y) = k$, a constant, integration gives

$$c_m'' - \frac{m^2\pi^2}{L^2}c_m = \frac{2}{L}\int_0^L k\sin\frac{m\pi x}{L}\,dx = \frac{2k[1+(-1)^{m+1}]}{m\pi}.$$

A general solution of this ODE is

$$c_m(y) = A_m\cosh\frac{m\pi y}{L} + B_m\sinh\frac{m\pi y}{L} - \frac{2kL^2[1+(-1)^{m+1}]}{m^3\pi^3},$$

and the N^{th} approximation is

$$V_N(x,y) = \sum_{n=1}^{N}\left[A_n\cosh\frac{n\pi y}{L} + B_n\sinh\frac{n\pi y}{L} - \frac{2kL^2[1+(-1)^{n+1}]}{n^3\pi^3}\right]\sin\frac{n\pi x}{L}.$$

To evaluate A_n and B_n, we form boundary residuals along $y=0$ and $y=L'$,

$$R_{|y=0} = \sum_{n=1}^{N}\left[A_n - \frac{2kL^2[1+(-1)^{n+1}]}{n^3\pi^3}\right]\sin\frac{n\pi x}{L} - g(x),$$
$$R_{|y=L'} = \sum_{n=1}^{N}\left[A_n\cosh\frac{n\pi L'}{L} + B_n\sinh\frac{n\pi L'}{L} - \frac{2kL^2[1+(-1)^{n+1}]}{n^3\pi^3}\right]\sin\frac{n\pi x}{L} - h(x).$$

Application of Galerkin's method gives

$$0 = \int_0^L\left\{\sum_{n=1}^{N}\left[A_n - \frac{2kL^2[1+(-1)^{n+1}]}{n^3\pi^3}\right]\sin\frac{n\pi x}{L} - g(x)\right\}\sin\frac{m\pi x}{L}\,dx,$$
$$0 = \int_0^L\left\{\sum_{n=1}^{N}\left[A_n\cosh\frac{n\pi L'}{L} + B_n\sinh\frac{n\pi L'}{L} - \frac{2kL^2[1+(-1)^{n+1}]}{n^3\pi^3}\right]\sin\frac{n\pi x}{L} - h(x)\right\}\sin\frac{m\pi x}{L}\,dx.$$

These give

$$A_m = \frac{2kL^2[1+(-1)^{m+1}]}{m^3\pi^3} + \frac{2}{L}\int_0^L g(x)\sin\frac{m\pi x}{L}\,dx,$$

$$B_m = \operatorname{csch}\frac{m\pi L'}{L}\left\{\cosh\frac{m\pi L'}{L}\left[-\frac{2kL^2[1+(-1)^{m+1}]}{m^3\pi^3} - \frac{2}{L}\int_0^L g(x)\sin\frac{m\pi x}{L}\,dx\right]\right.$$
$$\left. + \frac{2kL^2[1+(-1)^{m+1}]}{m^3\pi^3} + \frac{2}{L}\int_0^L h(x)\sin\frac{m\pi x}{L}\,dx.\right.$$

This is the N^{th} partial sum of the analytic solution obtained by separation of variables.

The next example cannot be solved with separation of variables; one edge of the region under consideration is not a coordinate curve.

Example 15.4 Use Galerkin's method to find a first approximation to the solution to the following problem involving Poisson's equation on the triangle R in Figure 15.2,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x,y), \quad (x,y) \text{ in } R, \tag{15.50a}$$

$$V(0,y) = 0, \quad 0 < y < L, \tag{15.50b}$$

$$V(x,0) = 0, \quad 0 < x < L, \tag{15.50c}$$

$$V(x,y) = 0, \quad (x,y) \text{ on } x+y=L. \tag{15.50d}$$

Simplify the approximation when $F(x,y) = k$, a constant.

Solution Since boundary conditions are homogeneous, we take approximations in the form

$$V_N(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\phi_{nm}(x,y),$$

where basis functions $\phi_{nm}(x,y)$ must be linearly independent and from a complete set of functions, and satisfy the boundary conditions. With $\omega(x,y) = xy(L-x-y)$, one possible choice is $\phi_{nm}(x,y) = x^n y^m (L-x-y)$, in which case

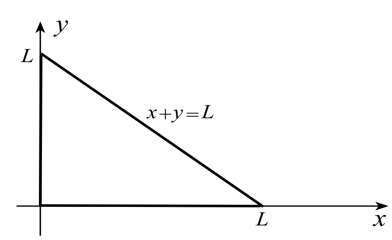


Figure 15.2

$$V_N(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}x^n y^m (L-x-y).$$

The first approximation is

$$V_1(x,y) = c_{11}xy(L-x-y),$$

with (equation) residual

$$R = -2c_{11}(x+y) - F(x,y).$$

Galerkin's method requires

$$0 = \int_0^L\int_0^{L-x} [-2c_{11}(x+y) - F(x,y)]xy(L-x-y)\,dy\,dx.$$

Integrations lead to

$$c_{11} = -\frac{90}{L^6}\int_0^L\int_0^{L-x} F(x,y)xy(L-x-y)\,dy\,dx.$$

In the special case that $F(x,y) = k$, a constant, we obtain $c_{11} = -3k/(4L)$, and the first approximation is

$$V_1(x,y) = -\frac{3k}{4L}xy(L-x-y).\bullet$$

In the event that any of the boundary conditions in this example are nonhomogeneous, calculations become more intensive. See Exercise 8 for the case when the nonhomogeneity is along the hypotenuse of the triangle.

EXERCISES 15.5

1. In this exercise we discuss a number of possible ways to approximate the solution to the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L',$$
$$U(0,y) = U(L,y) = 0, \quad 0 < y < L',$$
$$U(x,0) = x(L-x), \quad 0 < x < L,$$
$$U(x,L') = 0, \quad 0 < x < L.$$

(a) Since the function $\phi_0(x,y) = x(L-x)(1-y/L')$ satisfies all four boundary conditions, and the functions $\phi_n(x,y) = x^n y^m (L-x)(L'-y)$, $n,m = 1,2,\ldots$ satisfy homogeneous versions of the boundary conditions, we could take as a first approximation

$$U_1(x,y) = x(L-x)\left(1 - \frac{y}{L'}\right) + cxy(L-x)(L'-y).$$

Use Galerkin's method to determine c.

(b) Since the function $x(L-x)$ satisfies the first three boundary conditions, we could use reduction of dimensionality with $U_1(x,y) = c(y)x(L-x)$. Use Galerkin's method to determine $c(y)$.

(c) The functions $\sin(n\pi x/L)$ satisfy the first two boundary conditions so that we could take approximations in the form

$$U_N(x,y) = \sum_{n=1}^{N} c_n(y)\sin\frac{n\pi x}{L}.$$

Use Galerkin's method to determine the $c_n(y)$. Are approximations partial sums of the analytic solution obtained by separation of variables?

2. The approximation in part (a) of Exercise 1 was available because of the form of the boundary condition along $y = 0$. In addition, this made the calculations in part (b) simpler than they might otherwise be. In this exercise, we replace this boundary condition with $U(x,0) = f(x)$, $0 < x < L$.

(a) Since the function $x(L-x)$ satisfies the first two boundary conditions, we could take a first approximation of the form $U_1(x,y) = c(y)x(L-x)$. Use reduction of dimensionality and Galerkin's method to find $c(y)$. Compare the procedure to that in part (b) of Exercise

1.

(b) Show that approximations of the form in part (c) of Exercise 1 are once again partial sums of the analytic solution obtained by separation of variables.

3. Repeat parts (a) and (b) of Exercise 2 if the boundary condition along $y = L'$ is also nonhomogeneous, $U(x, L') = g(x)$, $0 < x < L$.

4. (a) Could the square in Example 15.2 be divided into two triangles, one above the line $y = x$ and the other below the line, for the subdomain method? Explain.

(b) Divide the square into two triangles one above the line $x + y = L$ and one below for the subdomain method. What does the resulting approximation predict for $V_2(0,0)$?

5. (a) Pursue approximation 15.48 of problem 15.44 in the case that all boundary conditions are homogeneous ($g(x) = h(x) = 0$).

(b) Confirm that it is the N^{th} partial sum of the analytic solution as obtained by finite Fourier transforms with respect to x and y (see Exercise 54 in Section 7.2).

6. The boundary value problem occurs

$$\frac{\partial^2 V}{\partial x^2} + \epsilon^2 \frac{\partial^2 V}{\partial y^2} = -1, \quad -L < x < L, \quad -L < y < L,$$
$$V(-L, y) = V(L, y) = 0, \quad -L < y < L,$$
$$V(x, -L) = V(x, L) = 0, \quad -L < x < L,$$

where ϵ is a constant, occurs in fluid flow. With $\omega(x, y) = (L^2 - x^2)(L^2 - y^2)$, which vanishes on the boundary of the square, a first polynomial approximation for $V(x, y)$ is $V_1(x, y) = c(L^2 - x^2)(L^2 - y^2)$. Use collocation and Galerkin's method to find c.

7. Use reduction of order to find a first polynomial approximation to problem 15.44 when $F(x, y) = xy$.

8. Find the first approximation to the solution of problem 15.50 when the boundary condition along the hypotenuse of the triangle is $V(x, y) = h(x, y) = h(x, L-x) = g(x)$, where $g(0) = g(L) = 0$.

9. The boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -2, \quad -L < x < L, \quad -L < y < L,$$
$$V(-L, y) = V(L, y) = 0, \quad -L < y < L,$$
$$V(x, -L) = V(x, L) = 0, \quad -L < x < L,$$

arises in the study of torsion for a square prismatic rod.

(a) With $\omega(x, y) = (L^2 - x^2)(L^2 - y^2)$, which vanishes on the boundary of the square, a first polynomial approximation for $V(x, y)$ is $V_1(x, y) = c(L^2 - x^2)(L^2 - y^2)$. Use Galerkin's method to find c.

(b) The solution $V(x, y)$ must be an even function of x and y, and be symmetric in x and y. Taking this into account, a second polynomial approximation would be $V_2(x, y) = (L^2 - x^2)(L^2 - y^2)[c + d(x^2 + y^2)]$. Use Galerkin's method to find c and d.

(c) Use reduction of dimensionality with $V_1(x) = f(x)(L^2 - y^2)$, where $f(-L) = f(L) = 0$ to approximate the solution.

(d) Use trigonometric basis functions $\cos\dfrac{(2n-1)\pi x}{2L}\cos\dfrac{(2m-1)\pi y}{2L}$, $n, m = 1, 2, \ldots$, to find approximations to $V(x, y)$.

(e) An important integral in this application is

$$M = \int_{-L}^{L}\int_{-L}^{L} V(x, y)\,dy\,dx.$$

To four decimal places, its value is $1.1248L^4$. We can use it to gauge the accuracy of the various approximations. Calculate M for the approximations in parts (a), (b) and (c).

10. We use reduction of dimensionality to approximate solutions of the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad y > 0,$$
$$V(0, y) = 0, \quad y > 0,$$
$$V(L, y) = 0, \quad y > 0,$$
$$V(x, 0) = x(L - x), \quad 0 < x < L.$$

Because solutions of this problem must be symmetric about $x = L/2$, choose basis functions $\psi_n(x) = x^n(L - x)^n$ in

$$V_N(x, y) = \sum_{n=1}^{N} c_n(y)\psi_n(x).$$

Such approximations satisfy the homogeneous boundary conditions for arbitrary $c_n(y)$, and satisfy the nonhomogeneous condition provided $c_1(0) = 1$ and $c_n(0) = 0$, $n = 2, \ldots, N$. Find the first approximation $V_1(x, y) = c_1(y)x(L - x)$ using:

(a) collocation;

(b) the subdomain (or moment) method;

(c) Galerkin's method.

11. Use Galerkin's method to find the second approximation in Exercise 10.

12. Since the PDE in Exercise 10 is homogeneous, we might consider a boundary method by choosing basis functions that satisfy the PDE, in particular, $\phi_n(x, y) = e^{-(2n-1)\pi y}\sin\dfrac{(2n-1)\pi x}{L}$. They are also symmetric about $x = L/2$. Approximations are then

$$V_N(x, y) = \sum_{n=1}^{N} c_n e^{-(2n-1)\pi y}\sin\frac{(2n-1)\pi x}{L}.$$

(a) Show that Galerkin's method gives the partial sums of the analytic solution.

(b) Find the first approximation using collocation.

(c) Find the first approximation using the subdomain (or moment) method.

13. Because the nonhomogeneity in Exercise 10 corresponded to the first term in the approximations, it was possible to incorporate the nonhomogeneity into boundary conditions for coefficients $c_n(y)$. This may not always be the case. For instance, consider the same problem where $x(L - x)$ is replaced by an arbitrary function $g(x)$ except that it satisfy $g(0) = g(L) = 0$. With no symmetry about $x = L/2$, basis functions are chosen as $\psi_n(x) = x^n(L - x)$, $n = 1, 2, \ldots$; they satisfy the homogeneous boundary conditions (see Table 15.5). Use Galerkin's method to find:

(a) the first approximation,
(b) the second approximation.

14. Show that when Galerkin's method is used in Exercise 13, with basis functions chosen as eigenfunctions $\psi_n(x) = \sin(n\pi x/L)$ of the associated Sturm-Liouville system, approximations are the partial sums of the analytic solution obtained by separation of variables.

15. Because the PDE in Exercise 13 is homogeneous, we might consider a boundary method by choosing basis functions that satisfy the PDE, in particular, $\phi_n(x,y) = e^{-n\pi y}\sin(n\pi x/L)$. Approximations are then

$$V_N(x,y) = \sum_{n=1}^{N} c_n e^{-n\pi y} \sin \frac{n\pi x}{L}.$$

(a) Show that Galerkin's method gives the partial sums of the analytic solution.
(b) Find the first and second approximations using collocation.
(c) Find the first and second approximations using the subdomain method.
(d) Find the first and second approximations using the moment method.

16. In some developments of the MWR, it is suggested that nonhomogeneous boundary conditions need never be considered; the nonhomogeneity can always be transformed into the PDE. In this exercise we show that whether this is done or not, the same residual to which the MWR would be applied is the same. The residual for problem 15.41 when approximations are taken in form 15.42 where $\phi_0(x,y)$ satisfies the nonhomogeneous boundary condition, and the $\phi_n(x,y)$, $n = 1, \cdots, N$, satisfy the homogeneous version of the boundary condition, is given in equation 15.43. The nonhomogeneity can be removed from the boundary condition with the transformation $V(x,y) = U(x,y) - \phi_0(x,y)$, where again $\phi_0(x,y)$ is a function satisfying the nonhomogeneous boundary condition. With this transformation, problem 15.41 is replaced by

$$L(U) = F(x,y) - L(\phi_i), \quad (x,y) \text{ in } R,$$
$$U(x,y) = 0, \quad (x,y) \text{ on } \beta(R).$$

Show that the residual for approximation $U_N(x,y) = \sum_{n=1}^{N} c_n\phi_n(x,y)$ is also that given in equation 15.43.

§15.6 Method of Weighted Residuals and Neumann Boundary Value Problems

In this section, we apply the MWR to boundary value problems with Neumann boundary conditions. When the boundary conditions are homogeneous, we may be able to follow the procedure of Section 15.5, namely produce basis functions that satisfy the boundary conditions and use an internal method. The following example illustrates this.

Example 15.5 Find a first approximation to the solution of the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -F(x, y), \quad 0 < x < L, \quad 0 < y < L', \tag{15.51a}$$

$$U_x(0, y) = U(L, y) = 0, \quad 0 < y < L', \tag{15.51b}$$

$$U_y(x, 0) = U(x, L') = 0, \quad 0 < x < L. \tag{15.51c}$$

Solution The function $U_1(x, y) = c(L^2 - x^2)(L'^2 - y^2)$ satisfies all four boundary conditions and therefore serves as a first approximation. Its equation residual is

$$R = -2c(L'^2 - y^2) - 2c(L^2 - x^2) + F(x, y) = 2c(x^2 + y^2 - L^2 - L'^2) + F(x, y).$$

Galerkin's method requires

$$0 = \int_0^L \int_0^{L'} [2c(x^2 + y^2 - L^2 - L'^2) + F(x, y)](L^2 - x^2)(L'^2 - y^2)\, dy\, dx.$$

Integrations lead to

$$c = \frac{45}{32L^3 L'^3 (L^2 + L'^2)} \int_0^L \int_0^{L'} F(x, y)(L^2 - x^2)(L'^2 - y^2)\, dy\, dx.$$

In the special case that $F(x, y) = k$, a constant,

$$c = \frac{45}{32L^3 L'^3 (L^2 + L'^2)} \int_0^L \int_0^{L'} k(L^2 - x^2)(L'^2 - y^2)\, dy\, dx = \frac{5k}{8(L^2 + L'^2)}.$$

The first Galerkin approximation is therefore

$$U_1(x, y) = \frac{5k}{8(L^2 + L'^2)}(L^2 - x^2)(L'^2 - y^2).$$

According to Exercise 52 in Section 7.2, an analytic solution of the problem is

$$U(x, y) = \frac{k}{2}(L^2 - x^2) + \frac{16kL^2}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n \cosh \dfrac{(2n-1)\pi y}{2L}}{(2n-1)^3 \cosh \dfrac{(2n-1)\pi L'}{2L}} \cos \frac{(2n-1)\pi x}{2L}.$$

We compare values of these functions at the centre of a square plate,

$$U_1(L/2, L/2) = \frac{5k}{16L^2}\left(L^2 - \frac{L^2}{4}\right)^2 = \frac{45kL^2}{256} \approx 0.176kL^2,$$

and

$$U(L/2, L/2) = \frac{k}{2}\left(L^2 - \frac{L^2}{4}\right) + \frac{16kL^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n\cosh\dfrac{(2n-1)\pi}{4}}{(2n-1)^3\cosh\dfrac{(2n-1)\pi}{2}}\cos\frac{(2n-1)\pi}{4}$$
$$\approx 0.181kL^2.$$

If Galerkin's method is used with the eigenfunctions $X_n(x) = \cos\dfrac{(2n-1)\pi x}{2L}$ to reduce the problem to one variable, the partial sums of the analytic solution are obtained as approximations (see Exercise 1).•

When Neumann boundary conditions are nonhomogeneous, we must adopt a completely different approach, an approach similar to that in Example 15.23 wherein we used integration by parts due to a Neumann boundary condition. Taking the place of integration by parts will be Green's first identity (see Appendix C). We begin with a general discussion on how to apply the method to such problems, followed by specific examples. Consider Poisson's equation in a region R of the xy-plane with boundary $\beta(R)$ (Figure 15.3),

$$\nabla^2 V = F(x,y), \quad (x,y) \text{ in } R, \tag{15.52a}$$
$$\frac{\partial V(x,y)}{\partial n} = G(x,y), \quad (x,y) \text{ on } \beta(R). \tag{15.52b}$$

The derivative in condition 15.52b is assumed to be in the outwardly normal direction to the boundary. If basis functions $\phi_n(x,y)$, which for the moment are arbitrary, except that they must be linearly independent and from a complete set, are used to approximate $V(x,y)$, then the N^{th} approximation is

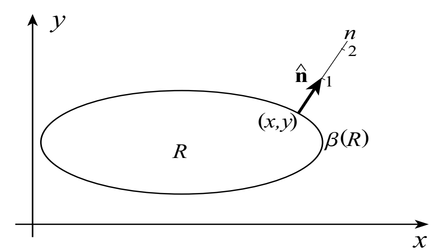


Figure 15.3

$$V_N(x,y) = \sum_{n=1}^{N} c_n\phi_n(x,y). \tag{15.53}$$

The equation residual is

$$R = \nabla^2 V_N - F(x,y) = \sum_{n=1}^{N} c_n\nabla^2\phi_n - F(x,y).$$

If N weight functions $w_m(x,y)$, $m = 1,\ldots,N$, are chosen, we might take the following N equations as defining coefficients c_n,

$$0 = \iint_R R w_m\,dA = \iint_R (\nabla^2 V_N - F)w_m\,dA$$
$$= \iint_R \nabla^2 V_N\, w_m\,dA - \iint_R F\, w_m\,dA. \tag{15.54}$$

But this does not take into account the boundary condition, and we certainly want $V_N(x,y)$ to approximate condition 15.52b as well as the PDE. To see how to do this, we use Green's first identity on the first of the integrals in equation 15.54,

$$0 = \oint_{\beta(R)} (w_m \nabla V_N) \cdot \hat{\mathbf{n}}\, ds - \iint_R (\nabla V_N \cdot \nabla w_m)\, dA - \iint_R F\, w_m\, dA$$
$$= \oint_{\beta(R)} \left(w_m \frac{\partial V_N}{\partial n} \right) ds - \iint_R (\nabla V_N \cdot \nabla w_m + F\, w_m)\, dA. \qquad (15.55)$$

We now have a line integral over the boundary and a double integral over R. To incorporate boundary condition 15.52b into equation 15.55, we form a boundary residual

$$R_{|\beta(R)} = \frac{\partial V_N}{\partial n} - G, \quad (x,y) \text{ on } \beta(R).$$

We implicitly demand that integrals of weighted residuals vanish; that is, we demand that

$$\oint_{\beta(R)} \left[\frac{\partial V_N}{\partial n} - G \right] w_m\, ds = 0. \qquad (15.56)$$

We substitute from this into equation 15.55, along with $V_N = \sum_{n=1}^N c_n \phi_n$,

$$0 = \oint_{\beta(R)} G\, w_m\, ds - \iint_R \left[\nabla \left(\sum_{n=1}^N c_n \phi_n \right) \cdot \nabla w_m + F\, w_m \right] dA,$$

or,

$$\sum_{n=1}^N c_n \iint_R (\nabla \phi_n \cdot \nabla w_m)\, dA = \oint_{\beta(R)} G\, w_m\, ds - \iint_R F\, w_m\, dA. \qquad (15.57)$$

These equations are used to determine coefficients c_n. Because conditions 15.56 have been incorporated implicitly, but not demanded explicitly, approximation $V_N(x,y)$ will not satisfy the boundary condition; the boundary condition will only be approximated. The following example illustrates this approach.

Example 15.6 Find a first approximation to the solution of the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -F(x,y), \quad 0 < x < L, \quad 0 < y < L', \qquad (15.58a)$$
$$U(0,y) = 0, \quad 0 < y < L', \qquad (15.58b)$$
$$U_x(L,y) = f(y), \quad 0 < y < L', \qquad (15.58c)$$
$$U_y(x,0) = U(x,L') = 0, \quad 0 < x < L. \qquad (15.58d)$$

Solution Since $x(L'^2 - y^2)$ satisfies the homogeneous boundary conditions, we take $U_1(x,y) = cx(L'^2 - y^2)$. Equation 15.57 gives

$$c \int_0^L \int_0^{L'} [(L'^2 - y^2)\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}] \cdot [(L'^2 - y^2)\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}]\, dy\, dx$$
$$= \int_0^{L'} f(y) L(L'^2 - y^2)\, dy - \int_0^L \int_0^{L'} x(L'^2 - y^2) F(x,y)\, dy\, dx.$$

Integration on the left leads to

$$c = \frac{45}{4LL'^3(5L^2 + 6L'^2)} \left[L \int_0^{L'} f(y)(L'^2 - y^2)\, dy - \int_0^L \int_0^{L'} x(L'^2 - y^2) F(x,y)\, dy\, dx \right].$$

For further discussion of this example see Exercise 3.•

EXERCISES 15.6

1. Show that if Galerkin's method is used with the eigenfunctions $X_n(x) = \cos \dfrac{(2n-1)\pi x}{2L}$ to reduce problem 15.51 to one of one variable, approximations are partial sums of the analytic solution.

2. (a) Find the solution to problem 15.58 when $L = L' = 1$, $F(x,y) = y$, and $f(y) = y^4(L - y)$.
(b) Show that it does not satisfy boundary condition 15.58c.

3. An analytic solution of problem 15.58 when $L = L' = 1$, $F(x,y) = k_1$, a constant, and $f(y) = k_2$, also a constant, is

$$U(x,y) = (k_1 + k_2)x - \frac{k_1 x^2}{2} - \frac{8}{\pi^3} \sum_{n=1}^{\infty} \left\{ \frac{[k_2(-1)^{n+1}\pi(2n-1) + 2k_1]\text{sech}[(2n-1)\pi/2}{(2n-1)^3} \right\} * \cosh \frac{(2n-1)\pi y}{2} \sin \frac{(2n-1)\pi x}{2}.$$

(See Exercise 31 in Section 4.3.)
(a) Use the series to find the solution at the centre $(1/2, 1/2)$ of the square.
(b) What does the first approximation $U_1(x,y)$ predict for the value of $U(x,y)$ at the centre of the square?
(c) Find a second approximation in the form $U_2(x,y) = cx(1 - y^2) + dx^2(1 - y^2)$.
(d) What is $U_2(1/2, 1/2)$?
(e) According to Exercise 24 in Section 2.2 the solution should satisfy the equation

$$\iint_R F(x,y)\, dA = \oint_{\beta(R)} \frac{\partial U}{\partial n} ds,$$

where R is the square $0 < x, y < 1$, $\beta(R)$ is its boundary, and the derivative on the right is in the outward normal direction. An approximation to the solution will not satisfy this condition, and as a result, the quantity

$$Q = \iint_R F(x,y)\, dA - \oint_{\beta(R)} \frac{\partial U}{\partial n} ds$$

will not be equal to zero. The size of Q is a measure of the extent to which the approximation meets this condition. It is another guide as to the adequacy of the approximation. Calculate Q for $U_1(x,y)$ and $U_2(x,y)$. Is Q_2 less than Q_1?

4. Find a first approximation for the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -F(x,y), \quad 0 < x < L, \quad 0 < y < L',$$
$$U(0,y) = 0, \quad 0 < y < L',$$

$$U_x(L,y)=f(y),\quad 0<y<L',$$
$$U_y(x,0)=g(x),\quad 0<x<L,$$
$$U(x,L')=0,\quad 0<x<L.$$

5. Find an approximation of the form $U_4(x,y) = (L^2-x^2)(L^2-y^2)[c+d(x^2+y^2)+fxy]$ to problem 15.51 in the case that $F(x,y)$ is a constant function, and $L=L'$.

6. Consider the Neumann problem

$$\nabla^2 U = xy,\quad -L<x<L,\quad -L'<y<L',$$
$$U_x(-L,y)=U_x(L,y)=0,\quad -L'<y<L',$$
$$U_y(x,-L')=U_y(x,L')=0,\quad -L<x<L.$$

The solution of the problem must be an odd function of both x and y. This would imply that if a third degree polynomial in x and y were used to approximate the solution, it would contain only the terms $ax+by+cxy+dx^3+fy^3$.
(a) Show that if this polynomial is subjected to the boundary conditions, it reduces to

$$U_3(x,y)=d(x^3-3L^2x)+f(y^3-3L'^2y).$$

(b) Use Galerkin's method to find d and f.

7. Show that if the nonhomogeneity xy in Exercise 6 is replaced by $x+y$, then Galerkin's method with a third degree polynomial gives the exact solution $U(x,y)=[(x^3-3L^2x)+(y^3-3L'^2y)]/6$ of the problem.

8. (a) If the nonhomogeneity xy in Exercise 6 is replaced by xy^2, then a fifth degree polynomial that is odd in x and even in y is $bx+cy^2+dx^3+exy^2+fy^4+gx^5+hx^3y^2+kxy^4$. Show that for such a polynomial to satisfy the boundary conditions, it must be of the form

$$U_5(x,y)=d(x^3-3L^2x)+g(x^5-5L^4x)+f(y^4-2L'^2y^2).$$

(b) Use Galerkin's method to find d, g, and f.

§15.7 Method of Weighted Residuals and Robin Boundary Value Problems

In this section, we apply the MWR to boundary value problems with Robin boundary conditions; the procedure is identical to that of Section 15.6. Consider Poisson's equation in a region R of the xy-plane with boundary $\beta(R)$ (Figure 15.3),

$$\nabla^2 V = F(x,y), \quad (x,y) \text{ in } R, \tag{15.59a}$$

$$\frac{\partial V(x,y)}{\partial n} + hV(x,y) = G(x,y), \quad (x,y) \text{ on } \beta(R). \tag{15.59b}$$

The derivative in condition 15.59b is assumed to be in the outwardly normal direction to the boundary. If basis functions $\phi_n(x,y)$, which for the moment are arbitrary, except that they must be linearly independent and from a complete set, are used to approximate $V(x,y)$, then

$$V_N(x,y) = \sum_{n=1}^{N} c_n \phi_n(x,y). \tag{15.60}$$

The equation residual is

$$R = \nabla^2 V_N - F(x,y) = \sum_{n=1}^{N} c_n \nabla^2 \phi_n - F(x,y).$$

If N weight functions $w_m(x,y)$, $m = 1,\ldots,N$, are chosen, we might take the following N equations as defining coefficients c_n,

$$\begin{aligned} 0 &= \iint_R R w_m \, dA = \iint_R (\nabla^2 V_N - F) w_m \, dA \\ &= \iint_R \nabla^2 V_N w_m \, dA - \iint_R F\, w_m \, dA. \end{aligned} \tag{15.61}$$

But this does not take into account the boundary condition, and we certainly want $V_N(x,y)$ to approximate condition 15.59b as well as the PDE. To see how to do this, we use Green's first identity on the first of the integrals in equation 15.61 (see Appendix C),

$$\begin{aligned} 0 &= \oint_{\beta(R)} (w_m \nabla V_N) \cdot \hat{\mathbf{n}} \, ds - \iint_R (\nabla V_N \cdot \nabla w_m) \, dA - \iint_R F\, w_m \, dA \\ &= \oint_{\beta(R)} \left(w_m \frac{\partial V_N}{\partial n} \right) ds - \iint_R (\nabla V_N \cdot \nabla w_m + F\, w_m) \, dA. \end{aligned} \tag{15.62}$$

We now have a line integral over the boundary and a double integral over R. To incorporate boundary condition 15.59b into equation 15.62, we implicitly demand that integrals of weighted residuals vanish on this boundary; that is, we demand that

$$\oint_{\beta(R)} \left[\frac{\partial V_N}{\partial n} + hV_N - G \right] w_m \, ds = 0. \tag{15.63}$$

We substitute from this into equation 15.62, along with $V_N = \sum_{n=1}^{N} c_n \phi_n$,

$$0 = \oint_{\beta(R)} (G - hV_N)\, w_m\, ds - \iint_R [\nabla V_N \cdot \nabla w_m + F\, w_m]\, dA,$$

or,

$$\sum_{n=1}^{N} c_n \left[\iint_R (\nabla \phi_n \cdot \nabla w_m)\, dA + h \oint_{\beta(R)} \phi_n\, w_m\, ds \right] = \oint_{\beta(R)} G\, w_m\, ds - \iint_R F\, w_m\, dA. \qquad (15.64)$$

These equations are used to determine coefficients c_i. Because condition 15.63 has been incorporated implicitly, but not demanded explicitly, approximation $V_N(x,y)$ will not satisfy the boundary condition; the boundary condition will only be approximated. The following example illustrates this approach.

Example 15.7 Find a first approximation to the solution of the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F(x,y), \quad 0 < x < L, \quad 0 < y < L', \qquad (15.65a)$$

$$U(0,y) = 0, \quad 0 < y < L', \qquad (15.65b)$$

$$U_x(L,y) + 2U(L,y) = f(y), \quad 0 < y < L', \qquad (15.65c)$$

$$U(x,0) = U(x,L') = 0, \quad 0 < x < L. \qquad (15.65d)$$

Solution Since $xy(L'-y)$ satisfies the homogeneous boundary conditions, we take $U_1(x,y) = cxy(L'-y)$. Equation 15.64 gives

$$c \int_0^L \int_0^{L'} [y(L'-y)\hat{\mathbf{i}} + x(L'-2y)\hat{\mathbf{j}}] \cdot [y(L'-y)\hat{\mathbf{i}} + x(L'-2y)\hat{\mathbf{j}}]\, dy\, dx + 2c \int_0^{L'} L^2 y^2 (L'-y)^2\, dy$$

$$= \int_0^{L'} Ly(L'-y) f(y)\, dy - \int_0^L \int_0^{L'} xy(L'-y) F(x,y)\, dy\, dx.$$

Integrations on the left lead to

$$c = \frac{90}{LL'^3(10L^2 + 3L'^2 + 6LL'^2)} \left[L \int_0^{L'} y(L'-y) f(y)\, dy - \int_0^L \int_0^{L'} xy(L'-y) F(x,y)\, dy\, dx \right]. \bullet$$

In the next example, we specialize the nonhomogeneities in the previous example, but find a third approximation.

Example 15.8 Find a third approximation to the solution of the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = k_1, \quad 0 < x < 1, \quad 0 < y < 1, \qquad (15.66a)$$

$$U(0,y) = 0, \quad 0 < y < 1, \qquad (15.66b)$$

$$U_x(1,y) + 2U(1,y) = k_2, \quad 0 < y < 1, \qquad (15.66c)$$

$$U(x,0) = U(x,1) = 0, \quad 0 < x < 1, \qquad (15.66d)$$

where k_1 and k_2 are constants.

Solution Since $xy(1-y)$ satisfies the homogeneous boundary conditions, we take $U_3(x,y) = cxy(1-y) + dx^2y(1-y) + fxy^2(1-y)$. Equations 15.64 give

$$c\left\{\int_0^1\int_0^1 [y(1-y)\hat{\mathbf{i}} + x(1-2y)\hat{\mathbf{j}}]\cdot[y(1-y)\hat{\mathbf{i}} + x(1-2y)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^2(1-y)^2\,dx\right\}$$
$$+\,d\left\{\int_0^1\int_0^1 [2xy(1-y)\hat{\mathbf{i}} + x^2(1-2y)\hat{\mathbf{j}}]\cdot[y(1-y)\hat{\mathbf{i}} + x(1-2y)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^2(1-y)^2\,dy\right\}$$
$$+\,f\left\{\int_0^1\int_0^1 [y^2(1-y)\hat{\mathbf{i}} + x(2y-3y^2)\hat{\mathbf{j}}]\cdot[y(1-y)\hat{\mathbf{i}} + x(1-2y)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^3(1-y)^2\,dy\right\}$$
$$= \int_0^1 k_2y(1-y)\,dy - \int_0^1\int_0^1 k_1xy(1-y)\,dy\,dx,$$

$$c\left\{\int_0^1\int_0^1 [y(1-y)\hat{\mathbf{i}} + x(1-2y)\hat{\mathbf{j}}]\cdot[2xy(1-y)\hat{\mathbf{i}} + x^2(1-2y)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^2(1-y)^2\,dy\right\}$$
$$+\,d\left\{\int_0^1\int_0^1 [2xy(1-y)\hat{\mathbf{i}} + x^2(1-2y)\hat{\mathbf{j}}]\cdot[2xy(1-y)\hat{\mathbf{i}} + x^2(1-2y)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^2(1-y)^2\,dy\right\}$$
$$+\,f\left\{\int_0^1\int_0^1 [y^2(1-y)\hat{\mathbf{i}} + x(2y-3y^2)\hat{\mathbf{j}}]\cdot[2xy(1-y)\hat{\mathbf{i}} + x^2(1-2y)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^3(1-y)^2\,dy\right\}$$
$$= \int_0^1 k_2y(1-y)\,dy - \int_0^1\int_0^1 k_1x^2y(1-y)\,dy\,dx,$$

$$c\left\{\int_0^1\int_0^1 [y(1-y)\hat{\mathbf{i}} + x(1-2y)\hat{\mathbf{j}}]\cdot[y^2(1-y)\hat{\mathbf{i}} + x(2y-3y^2)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^3(1-y)^2\,dy\right\}$$
$$+\,d\left\{\int_0^1\int_0^1 [2xy(1-y)\hat{\mathbf{i}} + x^2(1-2y)\hat{\mathbf{j}}]\cdot[y^2(1-y)\hat{\mathbf{i}} + x(2y-3y^2)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^3(1-y)^2\,dy\right\}$$
$$+\,f\left\{\int_0^1\int_0^1 [y^2(1-y)\hat{\mathbf{i}} + x(2y-3y^2)\hat{\mathbf{j}}]\cdot[y^2(1-y)\hat{\mathbf{i}} + x(2y-3y^2)\hat{\mathbf{j}}]\,dy\,dx + 2\int_0^1 y^4(1-y)^2\,dy\right\}$$
$$= \int_0^1 k_2y^2(1-y)\,dy - \int_0^1\int_0^1 k_1xy^2(1-y)\,dy\,dx.$$

Integrations lead to the equations

$$\frac{19c}{90} + \frac{11d}{60} + \frac{19f}{180} = \frac{k_2}{6} - \frac{k_1}{12},$$
$$\frac{11c}{60} + \frac{8d}{45} + \frac{11f}{120} = \frac{k_2}{6} - \frac{k_1}{18},$$
$$\frac{19c}{180} + \frac{11d}{120} + \frac{23f}{315} = \frac{k_2}{12} - \frac{k_1}{24},$$

the solution of which is $c = -30(5k_1 + k_2)/127$, $d = 5(23k_1 + 30k_2)/127$, and $f = 0$. The third approximation is therefore

$$U_3(x,y) = -\frac{30}{127}(5k_1 + k_2)xy(1-x) + \frac{5}{127}(23k_1 + 30k_2)x^2y(1-x).\bullet$$

EXERCISES 15.7

1. Find a first approximation to the solution of the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F(x,y), \quad 0 < x < L, \quad 0 < y < L',$$
$$-U_x(0,y) + 3U(0,y) = f(y), \quad 0 < y < L',$$
$$U(L,y) = 0, \quad 0 < y < L',$$
$$U(x,0) = U(x,L') = 0, \quad 0 < x < L.$$

2. Find a first approximation to the solution of the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = xy, \quad 0 < x < L, \quad 0 < y < L,$$
$$U(0,y) = 0, \quad 0 < y < L,$$
$$U_x(L,y) + U(L,y) = y, \quad 0 < y < L,$$
$$U(x,0) = 0, \quad 0 < x < L,$$
$$U_y(x,L) + 2U(x,L) = L - x, \quad 0 < x < L.$$

3. An analytic solution to the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$U(0,y) = 0, \quad 0 < y < 1,$$
$$U(L,y) = 0, \quad 0 < y < 1,$$
$$U(x,0) = 0, \quad 0 < x < 1,$$
$$U_y(x,1) + 2U(x,1) = k, \quad 0 < x < 1,$$

where k is a constant, is

$$U(x,y) = \frac{4k}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)[(2n-1)\pi\cosh(2n-1)\pi + 2\sinh(2n-1)\pi]} \sinh(2n-1)\pi y \sin(2n-1)\pi x.$$

(a) What does it predict for $U(1/2,1/2)$?

(b) Since $xy(1-x)$ satisfies the homogeneous boundary conditions, find a first approximation $U_1(x,y) = cxy(1-x)$. What is $U_1(1/2,1/2)$?

(c) Find a third approximation $U_3(x,y) = cxy(1-x) + dx^2y(1-x) + fxy^2(1-x)$. What is $U_3(1/2,1/2)$?

§15.8 Method of Weighted Residuals and Initial Boundary Value Problems

For initial boundary value problems, we use the MWR to reduce the dimensionality of the problem. In particular, for an initial boundary value problem in x and t, we reduce the problem to an ordinary differential in t. We illustrate with the following heat conduction problem,

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \tag{15.67a}$$

$$U(0,t) = 1, \quad t > 0, \tag{15.67b}$$

$$U(1,t) = 0, \quad t > 0, \tag{15.67c}$$

$$U(x,0) = f(x), \quad 0 < x < 1. \tag{15.67d}$$

We could have made the problem more general by making boundary condition 15.67c nonhomogeneous, using the more general Robin boundary conditions, and including a heat source term in the PDE. Doing so would not change the argument; it would only complicate the calculations.

To use weighted residuals to reduce problem 15.67 to an initial value problem in t only, we choose basis functions that are separated

$$U_N(x,t) = \phi_0(x) + \sum_{n=1}^{N} A_n(t)\phi_n(x),$$

and require $\phi_0(x)$ to satisfy boundary conditions 15.67b,c, and the $\phi_n(x)$ to satisfy the homogeneous version of 15.67b and condition 15.67c (which is already homogeneous). A simple choice for $\phi_0(x)$ is $1 - x$, and the $\phi_n(x)$ must satisfy $\phi_n(0) = \phi_n(1) = 0$. Among the many possible choices for the $\phi_n(x)$ are eigenfunctions $\sin n\pi x$ of the associated Sturm-Liouville system and the polynomials $x^n(1-x)$. In Exercise 1, it is shown that Galerkin's method with the eigenfunctions of the Sturm-Liouville system leads to partial sums of the analytic solution. If we choose basis functions $\phi_n(x) = x^n(1 - x)$, then

$$U_N(x,t) = 1 - x + \sum_{n=1}^{N} A_n(t)x^n(1 - x). \tag{15.68}$$

The first approximation is

$$U_1(x,t) = 1 - x + A_1(t)x(1 - x),$$

with equation residual

$$R = \frac{\partial U_1}{\partial t} - k\frac{\partial^2 U_1}{\partial x^2} = A_1'(t)x(1 - x) + 2kA_1(t).$$

Galerkin's method applied to this residual gives

$$0 = \int_0^1 [A_1'(t)x(1 - x) + 2kA_1(t)]x(1 - x)\,dx = \frac{1}{30}A_1'(t) + \frac{k}{3}A_1(t).$$

Hence, $A_1(t) = Ce^{-10kt}$, and

$$U_1(x,t) = 1 - x + Ce^{-10kt}x(1 - x).$$

To find C, we form an (initial) residual of this approximation at $t = 0$,

$$R_{|t=0} = 1 - x + Cx(1-x) - f(x).$$

Once again we apply Galerkin's method,

$$0 = \int_0^1 [1 - x + Cx(1-x) - f(x)]x(1-x)\,dx.$$

For instance, if $f(x) = 1 - x^3$, then this equation yields $C = 3/2$, and the first approximation to problem 15.67 is

$$U_1(x,t) = 1 - x + \frac{3}{2}e^{-10kt}x(1-x).$$

The analytic solution of problem 15.67 with $f(x) = 1 - x^3$ is

$$U(x,t) = 1 - x + \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}e^{-n^2\pi^2kt}\sin n\pi x.$$

We have tabulated $U(x,t)$ and $U_1(x,t)$ in Table 15.7 for $t = 10$ and $t = 10\,000$. The approximation of $U(x,t)$ by $U_1(x,t)$ improves as t increases; that is, as we approach the steady state solution $1 - x$, but it is woefully inadequate for small times.

x	$U(x,10)$	$U_1(x,10)$	$U(x,10\,000)$	$U_1(x,10\,000)$
0.1	0.9990	1.0348	0.9350	0.9391
0.2	0.9918	1.0397	0.8666	0.8695
0.3	0.9729	1.0146	0.7914	0.7912
0.4	0.9356	0.9596	0.7080	0.7042
0.5	0.8748	0.8745	0.6138	0.6085
0.6	0.7833	0.7596	0.5085	0.5042
0.7	0.6568	0.6146	0.3924	0.3912
0.8	0.4870	0.4397	0.2672	0.2695
0.9	0.2709	0.2348	0.1354	0.1391

Table 15.7

Let us see what improvement is achieved with the second approximation

$$U_2(x,t) = 1 - x + A_1(t)x(1-x) + A_2(t)x^2(1-x),$$

with equation residual

$$\begin{aligned} R &= \frac{\partial U_2}{\partial t} - k\frac{\partial^2 U_2}{\partial x^2} \\ &= A_1'(t)x(1-x) + 2kA_1(t) + A_2'(t)x^2(1-x) - kA_2(t)(2-6x). \end{aligned}$$

When we apply Galerkin's method, we obtain

$$\begin{aligned}
0 &= \int_0^1 [A_1'(t)x(1-x) + 2kA_1(t) + A_2'(t)x^2(1-x) - kA_2(t)(2-6x)]x(1-x)\,dx \\
&= \frac{1}{60}[2A_1'(t) + 20kA_1(t) + A_2'(t) + 10kA_2(t)], \\
0 &= \int_0^1 [A_1'(t)x(1-x) + 2kA_1(t) + A_2'(t)x^2(1-x) - kA_2(t)(2-6x)]x^2(1-x)\,dx \\
&= \frac{1}{420}[7A_1'(t) + 70kA_1(t) + 4A_2'(t) + 56kA_2(t)].
\end{aligned}$$

When orthogonal eigenfunctions are used as basis functions, differential equations for the $A_n(t)$ are uncoupled (see Exercise 1); polynomial basis functions, not being orthogonal, lead to coupled differential equations in $A_1(t)$ and $A_2(t)$. Solutions are

$$A_1(t) = Ce^{-10kt} + De^{-42kt}, \qquad A_2(t) = -2De^{-42kt}.$$

The second Galerkin approximation is therefore

$$U_2(x,t) = 1 - x + (Ce^{-10kt} + De^{-42kt})x(1-x) - 2De^{-42kt}x^2(1-x).$$

The residual of this approximation at $t = 0$ is

$$R_{|t=0} = 1 - x + (C+D)x(1-x) - 2Dx^2(1-x) - f(x),$$

and if we once again apply Galerkin's method, equations defining C and D are

$$\begin{aligned}
0 &= \int_0^1 [1 - x + (C+D)x(1-x) - 2Dx^2(1-x) - f(x)]x(1-x)\,dx, \\
0 &= \int_0^1 [1 - x + (C+D)x(1-x) - 2Dx^2(1-x) - f(x)]x^2(1-x)\,dx.
\end{aligned}$$

In the case that $f(x) = 1 - x^3$, these equations yield $C = 3/2$ and $D = -1/2$, and the second Galerkin approximation to problem 15.67 is

$$U_2(x,t) = 1 - x + \frac{1}{2}(3e^{-10kt} - e^{-42kt})x(1-x) + e^{-42kt}x^2(1-x).$$

Table 15.8 shows values of $U(x,t)$ and $U_2(x,t)$ at $t = 10$ and $t = 10\,000$. The agreement at $t = 10$ is much better than that for $U_1(x,t)$.

x	$U(x,10)$	$U_2(x,10)$	$U(x,10\,000)$	$U_2(x,10\,000)$
0.1	0.9990	0.9990	0.9350	0.9389
0.2	0.9918	0.9920	0.8666	0.8692
0.3	0.9729	0.9728	0.7917	0.7909
0.4	0.9356	0.9357	0.7080	0.7040
0.5	0.8748	0.8745	0.6138	0.6085
0.6	0.7833	0.7834	0.5085	0.5043
0.7	0.6568	0.6564	0.3924	0.3914
0.8	0.4870	0.4875	0.2672	0.2697
0.9	0.2709	0.2706	0.1354	0.1393

Table 15.8

Example 15.9 The problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < a, \quad t > 0, \tag{15.69a}$$
$$U(a,t) = 0, \quad t > 0, \tag{15.69b}$$
$$U(r,0) = a^2 - r^2, \quad 0 < r < a, \tag{15.69c}$$

describes radially symmetric temperature $U(r,t)$ in a circular plate. According to Exercise 1 in Section 9.1, the analytic solution of this problem is

$$U(r,t) = \frac{8}{a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}e^{-k\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)},$$

where the first ten eigenvalues λ_n can be found in Example 8.3 of Section 8.4. Use Galerkin's method to approximate the solution with a quadratic in r multiplied by a function of t; that is, $U_1(r,t) = d(t)(c_0 + c_1 r + c_2 r^2)$.

Solution For $U_1(r,t)$ to satisfy the boundary condition, we must have $c_0 + c_1 a + c_2 a^2 = 0$. We also impose the condition that $U_r(0,t) = 0$ which results in $c_1 = 0$. Thus, $U_1(r,t)$ will be of the form $U_1(r,t) = c_2 d(t)(r^2 - a^2)$, or absorbing $-c_2$ into $d(t)$, we take $U_1(r,t) = d(t)(a^2 - r^2)$. The equation residual for this approximation is

$$R = d'(t)(a^2 - r^2) - kd(t)(-2 - 2) = d'(t)(a^2 - r^2) + 4kd(t).$$

Galerkin's method requires

$$0 = \int_0^a [d'(t)(a^2 - r^2) + 4kd(t)](a^2 - r^2)\,dr = \frac{8a^5}{15}d'(t) + \frac{8ka^3}{3}d(t).$$

Thus, $d(t)$ must satisfy the ODE

$$d'(t) + \frac{5k}{a^2}d(t) = 0, \qquad \text{from which} \qquad d(t) = Ae^{-5kt/a^2}.$$

The approximate solution of the boundary value problem is therefore $U_1(r,t) = Ae^{-5kt/a^2}(a^2 - r^2)$. To satisfy the initial condition, we create a residual at $t = 0$,

$$R_{|t=0} = A(a^2 - r^2) - (a^2 - r^2) = (A - 1)(a^2 - r^2).$$

When we apply Galerkin's method to this residual, we obtain

$$0 = \int_0^a (A - 1)(a^2 - r^2)^2\,dr,$$

and this implies that $A = 1$. Our final approximation is $U_1(r,t) = e^{-5kt/a^2}(a^2 - r^2)$. We have tabulated the analytic solution and this approximation below at $t = 100$, using $a = 1$ and $k = 10^{-4}$. The agreement is excellent, but this is clearly due to the fact that the initial temperature is of the same form as in $U_1(r,t) = d(t)(a^2 - r^2)$. In addition, there is agreement between the boundary condition at $t = 0$ and the initial condition at $r = a$. See Exercise 3 when these conditions are not met; the approximation is unacceptable.●

r	$U(r,100)$	$U_1(r,100)$
0.1	0.950	0.942
0.2	0.920	0.913
0.3	0.870	0.866
0.4	0.800	0.799
0.5	0.710	0.713
0.6	0.600	0.609
0.7	0.470	0.485
0.8	0.323	0.342
0.9	0.162	0.181

Table 15.9

EXERCISES 15.8

1. Show that Galerkin's method for problem 15.67 with eigenfunctions of the associated Sturm-Liouville system as basis functions leads to the partial sums of the analytic solution.

2. Consider the nonlinear problem

$$\begin{aligned}\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + \epsilon U^2, \quad 0 < x < L, \quad t > 0,\\ U(0,t) &= 0, \quad t > 0,\\ U(L,t) &= 0, \quad t > 0,\\ U(x,0) &= 1, \quad 0 < x < L.\end{aligned}$$

(a) Show that if a first Galerkin approximation is chosen in the form $U_1(x,t) = c_1(t)X_1(x)$, where $X_1(t) = \sin(\pi x/L)$ is the first eigenfunction of the associated Sturm-Liouville system, then $c_1(t)$ must satisfy the ODE

$$c_1'(t) + \frac{\pi^2}{L^2}c_1(t) = \frac{8\epsilon\sqrt{2/L}}{3\pi}c_1^2(t).$$

(b) Solve this Bernoulli equation for $c_1(t)$ and use Galerkin's method on the initial residual to determine the constant of integration.

3. (a) Find approximation $U_1(r,t)$ to problem 15.69 when the initial temperature of the plate is 1 throughout.

(b) An analytic solution is

$$U(r,t) = \frac{2}{a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}e^{-k\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)},$$

(see Exercise 1 of Section 9.1). With $a = 1$, $k = 10^{-4}$, tabulate, or plot, $U_1(r,100)$ and $U(r,100)$ to show that $U_1(r,t)$ is not a good approximation to $U(r,t)$ for this initial temperature.

4. Consider the following heat conduction problem where constants have been eliminated for simplicity,

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < 1, \quad t > 0,$$
$$-U_x(0,t) + U(0,t) = 0, \quad t > 0,$$
$$U_x(1,t) = 0, \quad t > 0,$$
$$U(x,0) = 1, \quad 0 < x < 1.$$

To reduce this problem to an initial value problem in t only, we take approximations in the form

$$U_N(x,t) = \sum_{n=1}^{N} d_n(t)\phi_n(x),$$

where basis functions $\phi_n(x)$ satisfy the boundary conditions: that is,

$$-\phi_n'(0) + \phi_n(0) = 0, \qquad \phi_n'(1) = 0.$$

(a) Show that when basis functions are assumed in the form $\phi_n(x) = c_0 + c_1 x + c_{n+1}x^{n+1}$, then

$$\phi_n(x) = 1 + x - \frac{x^{n+1}}{n+1}.$$

Does Table 15.5 in Section 15.3 give the same $\phi_1(x)$?

(b) Use Galerkin's method to find the first approximation $U_1(x,t) = d_1(t)\phi_1(x)$.

(c) Use Galerkin's method to find the second approximation $U_2(x,t) = d_1(t)\phi_1(x) + d_2(t)\phi_2(x)$.

5. The initial boundary value problem

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$U(0,t) = 0, \quad t > 0,$$
$$U(L,t) = 0, \quad t > 0,$$
$$U(x,0) = f(x), \quad 0 < x < L,$$

involves the nonlinear Burger equation. Use Galerkin's method to find a first approximation if the basis function is (a) $\sin(\pi x/L)$, (b) $x(L-x)$.

CHAPTER 16 Finite Element Solutions

§16.1 Introduction

You may have noticed that the boundary value problems of Sections 15.5–15.7 were always posed on regions with simple boundaries such as rectangles, triangles, and circles. This is a major deficiency of the MWR; it can be very difficult to use the MWR on regions R with complex boundaries such as that in Figure 16.1a. There are other difficulties associated with the method. We usually worked with relatively low order approximations, first, second, and sometimes a third; higher order approximations can increase calculations dramatically. In addition, polynomials were often the choice for basis functions, yielding therefore approximations which were continuous functions. But many physical problems have discontinuous material properties, and it would be unreasonable to expect continuous functions to accurately describe such situations.

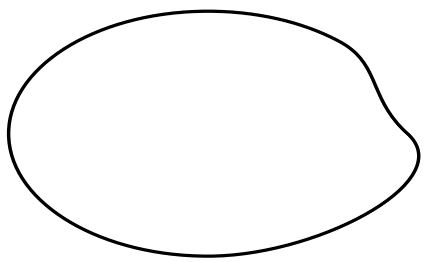


Figure 16.1a

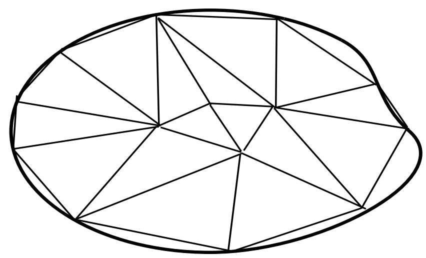


Figure 16.1b

Finite elements can overcome these difficulties. Region R is approximated by a set of subregions, usually triangles and quadrilaterals, perhaps like that in Figure 16.1b. Each subregion is called an **element**, and the totality of elements is called the **mesh** for region R. Needless to say, straight edges of the elements of the mesh can approximate the boundary of R arbitrarily closely. In addition, triangles and quadrilaterals with curved edges are also possible, and these could approximate the boundary of R even more closely. On each element, the unknown function is approximated by a polynomial in x and y, and the form of this polynomial is the same for each element (of the same type). Better and better approximations are not achieved by increasing the number of terms in the polynomial (as was the case for the MWR). Instead polynomials remain fixed in form, and the number of elements is increased. Finally, elements can be adapted to changing material properties by choosing boundaries of elements to coincide with, or approximate, curves along which such changes take place.

In the space of one chapter, we cannot give a complete treatment of finite elements; it is a vast subject with many texts devoted to it, and it alone. Most readers will likely be users, not developers, of prewritten finite element programs, but in order to appreciate these programs and be able to adapt to them, it is wise to have some understanding of the underlying mathematics. This is the purpose of the present chapter — present the underlying concept of finite elements. We make no attempt to assemble elements into an efficient program, how to write finite element code, or how to use commercially prepared programs. Hopefully, with the ideas presented here, the reader will find assembly, coding, and commercial programs easier to follow.

As was done in Chapter 15, we begin with boundary value problems associated with ordinary differential equations, and then move on to multi-dimensional boundary and initial boundary value problems.

§16.2 Finite Elements for Ordinary Differential Equations

In this section we illustrate the central idea of the method of finite elements (henceforth shortened to MFE), by applying it to boundary value problems associated with ODEs. Finite elements always use polynomials to approximate solutions of boundary value problems, unlike the MWR which may also use other basis functions such as trigonometric functions.

Linear Approximations

The simplest polynomial is the linear one, $y(x) = a + bx$, but finite elements requires them to be expressed in a different form. If the function has values $y(x_1)$ and $y(x_2)$ at x_1 and x_2, then

$$y(x_1) = a + bx_1, \quad y(x_2) = a + bx_2.$$

When these are solved for a and b, we get

$$a = \frac{x_2 y(x_1) - x_1 y(x_2)}{x_2 - x_1}, \quad b = \frac{y(x_2) - y(x_1)}{x_2 - x_1}.$$

The linear function can therefore be expressed in the form

$$\begin{aligned} y(x) &= \frac{x_2 y(x_1) - x_1 y(x_2)}{x_2 - x_1} + \left[\frac{y(x_2) - y(x_1)}{x_2 - x_1}\right] x \\ &= y(x_1)\left(\frac{x - x_2}{x_1 - x_2}\right) + y(x_2)\left(\frac{x - x_1}{x_2 - x_1}\right). \end{aligned} \tag{16.1}$$

What we have shown is the following. If a linear approximation is sought for a boundary value problem associated with an ordinary differential equation, instead of taking the approximation in the form $y = a + bx$ and determining values for a and b, finite elements takes the approximation in the form

$$y = c_1\phi_1(x) + c_2\phi_2(x), \tag{16.2a}$$

where

$$\phi_1(x) = \frac{x - x_2}{x_1 - x_2} \qquad \phi_2(x) = \frac{x - x_1}{x_2 - x_1}, \tag{16.2b}$$

and determines values for c_1 and c_2. These are values of the approximation at x_1 and x_2, called *nodes*. This is one of the essential ideas of finite elements; it finds approximations to the solution of the boundary value problem at predetermined nodes. This is reminiscent of finite differences in Chapter 14 which also approximates the solution at nodes. Finite elements goes further; it provides, through its interpolating polynomial (in this case a linear one), approximations between nodes. Basis functions 1 and x are replaced by $\phi_1(x)$ and $\phi_2(x)$. They have the property

$$\phi_n(x_m) = \delta_{nm} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m; \end{cases} \tag{16.3}$$

each function has value 1 at one node and value 0 at the other. We have shown the functions $\phi_1(x)$ and $\phi_2(x)$ in Figure 16.2a. Figure 16.2b shows a linear function with values $y(x_1)$ and $y_2(x)$ in terms of $\phi_1(x)$ and $\phi_2(x)$.

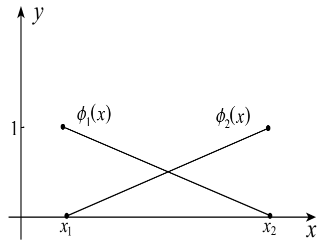


Figure 16.2a

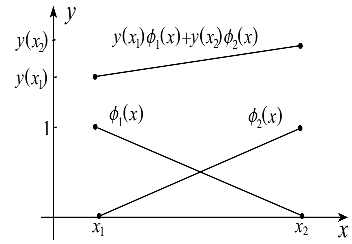


Figure 16.2b

You may feel that representation 16.2 is excessively complicated compared to $a+bx$, but the advantage of this representation will soon become apparent. In fact, the MFE with multiple elements would be impossible to program on a computer without it.

We illustrate the essentials of the MFE with problem 15.23 of Section 15.3

$$\frac{d^2Y}{dx^2} + Y = x, \quad 0 < x < 1, \tag{16.4a}$$

$$Y(0) = 2, \tag{16.4b}$$

$$Y'(1) = 1, \tag{16.4c}$$

the exact solution of which is

$$Y(x) = 2\cos x + 2\tan 1 \sin x + x. \tag{16.5}$$

We choose this problem so that we can make comparisons with the MWR applied to the same problem.

The first step is to discretize the domain; in this case, divide the interval $0 \le x \le 1$ into subintervals, or elements. We begin with one element $0 \le x \le 1$. With only one element, we essentially repeat what we did in Chapter 15, but we rewrite equations in a form that facilitates the transition to multiple elements. The two points $x = 0$ and $x = 1$ are called **nodes** of our discretization; we have two nodes for the element. Later we will see that additional nodes can be attached to elements. In other words, the number of nodes is not necessarily just one more than the number of elements. Next to consider is the order of the polynomial to be used to approximate the solution on an element. It is tied to the number of nodes in the element. The number of terms in the polynomial must be equal to the number of nodes for the element; otherwise the number of nodes would not uniquely define the polynomial. For two nodes, a linear polynomial must be used; for three nodes, a quadratic, etc. In other words, if the number of nodes per element is chosen, it determines the order of the polynomial; conversely, if the order of the polynomial is specified, it determines the number of nodes per element. With our beginning choice of two nodes for the single element, our approximation must be linear. We intend writing it in form 16.2 by setting $x_1 = 0$ and $x_2 = 1$, and using the MWR to find values for $c_1 = Y(0)$ and $c_2 = Y(1)$. Because we want to establish a framework that can be used on any subinterval when we next subdivide the interval $0 \le x \le 1$ into multiple subintervals, we work on a generic interval $x_1 \le x \le x_2$. At the appropriate point in the calculations, we will set $x_1 = 0$ and $x_2 = 1$. On the interval $x_1 \le x \le x_2$, then, we take the linear approximation of $Y(x)$ to be

$$Y_1(x) = c_1\phi_1(x) + c_2\phi_2(x) = c_1\left(\frac{x-x_2}{x_1-x_2}\right) + c_2\left(\frac{x-x_1}{x_2-x_1}\right), \tag{16.6}$$

where c_1 and c_2 represent $Y_1(x_1)$ and $Y_1(x_2)$. The (equation) residual of this approximation is

$$R = Y_1''(x) + Y_1(x) - x.$$

Notice that no account has been taken of the boundary conditions. The MWR and the MFE handle boundary conditions in fundamentally different ways. In Example 15.1, where we used an interior MWR on problem 16.4, the lowest possible order for a polynomial approximation was quadratic because we forced the polynomial to satisfy the boundary conditions, and did so at the beginning of the process. We also used a mixed method where the initial approximation satisfied the Dirichlet condition $Y(0) = 2$, but not the Neumann condition $Y'(1) = 1$. The MFE does not subject approximating polynomials to any boundary conditions prior to forming the residual. Integration by parts is always applied to integrals of residuals, and boundary conditions are introduced into the resulting equations.

Galerkin's method applied to the residual requires

$$0 = \int_{x_1}^{x_2} [Y_1''(x) + Y_1(x) - x]\phi_1(x)\,dx, \qquad 0 = \int_{x_1}^{x_2} [Y_1''(x) + Y_1(x) - x]\phi_2(x)\,dx.$$

Integration by parts in the first of these gives

$$0 = \left\{Y_1'\phi_1\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} Y_1'\phi_1'\,dx + \int_{x_1}^{x_2} (Y_1 - x)\phi_1\,dx.$$

If we now substitute $Y_1 = c_1\phi_1 + c_2\phi_2$ in the integrals,

$$0 = \left\{Y_1'\phi_1\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} (c_1\phi_1' + c_2\phi_2')\phi_1'\,dx + \int_{x_1}^{x_2} (c_1\phi_1 + c_2\phi_2 - x)\phi_1\,dx.$$

This can be written in the form

$$c_1\int_{x_1}^{x_2} [(\phi_1')^2 - \phi_1^2]\,dx + c_2\int_{x_1}^{x_2} (\phi_1'\phi_2' - \phi_1\phi_2)\,dx = \left\{Y_1'\phi_1\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} x\phi_1\,dx.$$

Similarly, the second Galerkin requirement can be expressed in the form

$$c_1\int_{x_1}^{x_2} (\phi_2'\phi_1' - \phi_2\phi_1)\,dx + c_2\int_{x_1}^{x_2} [(\phi_2')^2 - \phi_2^2]\,dx = \left\{Y_1'\phi_2\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} x\phi_2\,dx.$$

These are linear equations in c_1 and c_2 which we can write in matrix form

$$\begin{pmatrix} \displaystyle\int_{x_1}^{x_2} [(\phi_1')^2 - \phi_1^2]\,dx & \displaystyle\int_{x_1}^{x_2} (\phi_1'\phi_2' - \phi_1\phi_2)\,dx \\ \displaystyle\int_{x_1}^{x_2} (\phi_2'\phi_1' - \phi_2\phi_1)\,dx & \displaystyle\int_{x_1}^{x_2} [(\phi_2')^2 - \phi_2^2]\,dx \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \left\{Y_1'\phi_1\right\}_{x_1}^{x_2} \\ \left\{Y_1'\phi_2\right\}_{x_1}^{x_2} \end{pmatrix} + \begin{pmatrix} -\displaystyle\int_{x_1}^{x_2} x\phi_1\,dx \\ -\displaystyle\int_{x_1}^{x_2} x\phi_2\,dx \end{pmatrix}.$$

Symbolically, we write

$$KC = B + N, \tag{16.7a}$$

where

$$K = (K_{ij}) = \begin{pmatrix} \int_{x_1}^{x_2} [(\phi_1')^2 - \phi_1^2]\, dx & \int_{x_1}^{x_2} (\phi_1'\phi_2' - \phi_1\phi_2)\, dx \\ \int_{x_1}^{x_2} (\phi_2'\phi_1' - \phi_2\phi_1)\, dx & \int_{x_1}^{x_2} [(\phi_2')^2 - \phi_2^2]\, dx \end{pmatrix}, \qquad (16.7b)$$

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \{Y_1'\phi_1\}_{x_1}^{x_2} \\ \{Y_1'\phi_2\}_{x_1}^{x_2} \end{pmatrix}, \qquad N = \begin{pmatrix} -\int_{x_1}^{x_2} x\phi_1\, dx \\ -\int_{x_1}^{x_2} x\phi_2\, dx \end{pmatrix}. \qquad (16.7c)$$

Matrix K is symmetric; it is called the **stiffness matrix**. Vector C is a vector of nodal values, called the **displacement vector**. The remaining two vectors are called **load vectors**; B is due to boundary terms, and N is due to the internal non-homogeneity. The terminology had its origin in structural analysis, but it persists even in other applications such as heat conduction and electromagnetics. Matrix equation 16.7a represents a set of linear equations for the nodal values c_1 and c_2; they are called the **system equations**. When we move to multiple elements, they will be called the **element equations**, and the assembly of all element equations will be called the system equations.

It is important to point out that these equations are very general. They apply to any approximation $Y_1(x) = c_1\phi_1(x) + c_2\phi_2(x)$ of differential equation16.4a, where $\phi_1(x)$ and $\phi_2(x)$ are any two basis functions, and boundary conditions 16.4b,c have not yet been invoked. In other words, they are valid for any two-term approximation to differential equation 16.4a on the interval $x_1 < x < x_2$. Let us now calculate entries in these matrices when $\phi_1(x)$ and $\phi_2(x)$ are the linear polynomials in approximation 16.6,

$$K_{11} = \int_{x_1}^{x_2} [(\phi_1')^2 - \phi_1^2]\, dx = \int_{x_1}^{x_2} \left[\frac{1}{(x_1 - x_2)^2} - \frac{(x - x_2)^2}{(x_1 - x_2)^2}\right] dx = \frac{3 - (x_2 - x_1)^2}{3(x_2 - x_1)},$$

$$K_{12} = \int_{x_1}^{x_2} (\phi_2'\phi_1' - \phi_2\phi_1)\, dx = \int_{x_1}^{x_2} \left[\frac{-1}{(x_2 - x_1)^2} + \frac{(x - x_2)(x - x_1)}{(x_2 - x_1)^2}\right] dx$$

$$= -\frac{6 + (x_2 - x_1)^2}{6(x_2 - x_1)},$$

$$K_{22} = \int_{x_1}^{x_2} [(\phi_2')^2 - \phi_2^2]\, dx = \int_{x_1}^{x_2} \left[\frac{1}{(x_2 - x_1)^2} - \frac{(x - x_1)^2}{(x_2 - x_1)^2}\right] dx = \frac{3 - (x_2 - x_1)^2}{3(x_2 - x_1)},$$

$$N_1 = -\int_{x_1}^{x_2} x\phi_1\, dx = \int_{x_1}^{x_2} x\left(\frac{x - x_2}{x_1 - x_2}\right) dx = \frac{1}{6}(x_2^2 + x_1x_2 - 2x_1^2),$$

$$N_2 = -\int_{x_1}^{x_2} x\phi_2\, dx = \int_{x_1}^{x_2} x\left(\frac{x - x_1}{x_2 - x_1}\right) dx = \frac{1}{6}(x_1^2 + x_1x_2 - 2x_2^2),$$

$$B_1 = Y_1'(x_2)\phi_1(x_2) - Y_1'(x_1)\phi_1(x_1) = -Y_1'(x_1),$$

$$B_2 = Y_1'(x_2)\phi_2(x_2) - Y_1'(x_1)\phi_2(x_1) = Y_1'(x_2).$$

For our single element, we set $x_1 = 0$ and $x_2 = 1$, in which case system equations 16.7a become

$$\begin{pmatrix} 2/3 & -7/6 \\ -7/6 & 2/3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -Y_1'(0) \\ Y_1'(1) \end{pmatrix} + \begin{pmatrix} 1/6 \\ -1/3 \end{pmatrix}. \qquad (16.8)$$

These equations define nodal values $c_1 = Y(0)$ and $c_2 = Y(1)$ once boundary conditions have been incorporated. The boundary residual at $x = 1$ is $R_{|x=1} = Y_1'(1) - 1$. We demand that it vanish when multiplied by basis functions; that is, $[Y_1'(1) - 1]\phi_1(1) = 0$ and $[Y_1'(1) - 1]\phi_2(1) = 0$. The first requires nothing (since $\phi_1(1) = 0$) whereas the second requires $Y_1'(1) = 1$ (since $\phi_2(1) = 1$). In other words, the boundary condition $Y'(1) = 1$ forces the approximation to satisfy this condition. We demand this implicitly by substituting it into the right side of the second equation, and because it is an implicit requirement, $Y_1(x)$ will only approximate this condition. (You might want to review Example 15.1 in Section 15.3 for what we mean by an implicit requirement.) The Dirichlet condition at $x = 0$ is different; there is no way to incorporate it into the boundary load vector B; that is, it does not specify a value for $Y_1'(0)$. Instead, we impose it on the trial solution; that is, we demand that $Y_1(0) = 2$. This implies that $2 = c_1\phi_1(0) + c_2\phi_2(0) = c_1$. When we substitute this into the left side of the equations, it is an explicit requirement of the solution; it will be satisfied,

$$\begin{pmatrix} 2/3 & -7/6 \\ -7/6 & 2/3 \end{pmatrix}\begin{pmatrix} 2 \\ c_2 \end{pmatrix} = \begin{pmatrix} -Y_1'(0) \\ 1 \end{pmatrix} + \begin{pmatrix} 1/6 \\ -1/3 \end{pmatrix}.$$

The second equation now determines c_2,

$$-\frac{7}{6}(2) + \frac{2}{3}c_2 = 1 - \frac{1}{3} \quad \Longrightarrow \quad c_2 = \frac{9}{2}.$$

Thus, the one-element linear polynomial approximating the solution of problem 16.4 is

$$Y_1(x) = 2\left(\frac{x-1}{-1}\right) + \frac{9}{2}\left(\frac{x-0}{1}\right) = 2 + \frac{5x}{2}.$$

We have plotted the exact solution along with this approximation in Figure 16.3. As expected, it does not satisfy the Neumann boundary condition.

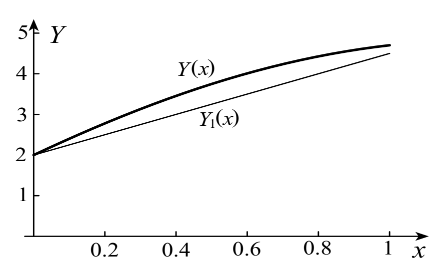


Figure 16.3

We spent an inordinate amount of time formatting the calculations for the single element, but it will now pay dividends as we move to multiple elements. We begin with two elements $0 \le x \le 1/2$ and $1/2 \le x \le 1$, but for forthcoming extensions to more than two elements, we replace $x = 0$, $x = 1/2$, and $x = 1$ with $x = x_1$, $x = x_2$, and $x = x_3$, respectively. At the appropriate juncture, we substitute specific values for the two elements. On each element, we approximate the solution of problem 16.4 with a linear polynomial,

$$Y_2(x) = \begin{cases} Y^{(1)}(x) = c_1^{(1)}\phi_1^{(1)}(x) + c_2^{(1)}\phi_2^{(1)}(x) = c_1^{(1)}\dfrac{x - x_2}{x_1 - x_2} + c_2^{(1)}\dfrac{x - x_1}{x_2 - x_1}, & x_1 \le x \le x_2 \\ Y^{(2)}(x) = c_1^{(2)}\phi_1^{(2)}(x) + c_2^{(2)}\phi_2^{(2)}(x) = c_1^{(2)}\dfrac{x - x_3}{x_2 - x_3} + c_2^{(2)}\dfrac{x - x_2}{x_3 - x_2}, & x_2 < x \le x_3. \end{cases}$$

The subscript in $Y_2(x)$ indicates that we have a two-element approximation. The superscripts (1) and (2) represent element numbers, (1) for element 1, $0 \le x \le 1/2$, and (2) for element 2, $1/2 \le x \le 1$. We now set up element equations for nodal values $(c_1^{(1)}, c_2^{(1)})$ and $(c_1^{(2)}, c_2^{(2)})$. Because we treated the single element situation in complete generality, there is no need to perform the same analysis for each element; we use element equations 16.7 for each element with appropriate changes in notation — add superscripts to identify elements, and adjust subscripts. Furthermore, because the polynomials are, for the moment, unrelated, we combine the two sets of element equations into one set of equations, the system equations,

$$KC = B + N, \tag{16.9a}$$

where,

$$K = \begin{pmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & 0 & 0 \\ 0 & 0 & K_{11}^{(2)} & K_{12}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} \end{pmatrix}, \tag{16.9b}$$

$$\begin{pmatrix} K_{11}^{(1)} & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} \int_{x_1}^{x_2} [(\phi_1^{(1)'})^2 - (\phi_1^{(1)})^2]\,dx & \int_{x_1}^{x_2} [\phi_1^{(1)'}\phi_2^{(1)'} - \phi_1^{(1)}\phi_2^{(1)}]\,dx \\ \int_{x_1}^{x_2} [\phi_2^{(1)'}\phi_1^{(1)'} - \phi_2^{(1)}\phi_1^{(1)}]\,dx & \int_{x_1}^{x_2} [(\phi_2^{(1)'})^2 - (\phi_2^{(1)})^2]\,dx \end{pmatrix}, \tag{16.9c}$$

$$\begin{pmatrix} K_{11}^{(2)} & K_{12}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} \int_{x_2}^{x_3} [(\phi_1^{(2)'})^2 - (\phi_1^{(2)})^2]\,dx & \int_{x_2}^{x_3} [\phi_1^{(2)'}\phi_2^{(2)'} - \phi_1^{(2)}\phi_2^{(2)}]\,dx \\ \int_{x_2}^{x_3} [\phi_2^{(2)'}\phi_1^{(2)'} - \phi_2^{(2)}\phi_1^{(2)}]\,dx & \int_{x_2}^{x_3} [(\phi_2^{(2)'})^2 - (\phi_2^{(2)})^2]\,dx \end{pmatrix}, \tag{16.9d}$$

$$C = \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix}, \qquad B = \begin{pmatrix} \left\{Y^{(1)'}\phi_1^{(1)}\right\}_{x_1}^{x_2} \\ \left\{Y^{(1)'}\phi_2^{(1)}\right\}_{x_1}^{x_2} \\ Y^{(2)'}\phi_1^{(2)}\big\}_{x_2}^{x_3} \\ \left\{Y^{(2)'}\phi_2^{(2)}\right\}_{x_2}^{x_3} \end{pmatrix}, \qquad N = \begin{pmatrix} -\int_{x_1}^{x_2} x\phi_1^{(1)}\,dx \\ -\int_{x_1}^{x_2} x\phi_2^{(1)}\,dx \\ -\int_{x_2}^{x_3} x\phi_1^{(2)}\,dx \\ -\int_{x_2}^{x_3} x\phi_2^{(2)}\,dx \end{pmatrix}. \tag{16.9e}$$

When we set $x_1 = 0$, $x_2 = 1/2$ and $x_3 = 1$, and evaluate integrals, we get

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 0 & 0 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ Y^{(1)'}(1/2) \\ -Y^{(2)'}(1/2) \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/12 \\ -1/6 \\ -5/24 \end{pmatrix}. \tag{16.10}$$

These (system) equations define nodal values $c_1^{(1)}$, $c_2^{(1)}$, $c_1^{(2)}$, and $c_2^{(2)}$ once boundary conditions have been incorporated. We delay them for the moment and discuss what are called **interelement boundary conditions**. The approximations must, in some way, "match" at the node $x = 1/2$. We certainly want the two-element approximation to be continuous at $x = 1/2$. Because $c_2^{(1)}$ and $c_1^{(2)}$ both represent the value of the approximation at $x = 1/2$, we make the explicit demand that $c_2^{(1)} = c_1^{(2)}$. Quantities $Y^{(1)'}(1/2)$ and $Y^{(2)'}(1/2)$ represent slopes of the linear approximations in the two elements at $x = 1/2$. We implicitly demand that they be the same, $Y^{(1)'}(1/2) = Y^{(2)'}(1/2)$. When we make these substitutions in system equations 16.10, we obtain

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 0 & 0 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ Y^{(1)'}(1/2) \\ -Y^{(1)'}(1/2) \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/12 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

If we add the third equation to the second, we eliminate the unknown quantity ${Y^{(1)}}'(1/2)$ from the second equation.

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 11/6 & -25/12 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \begin{pmatrix} -{Y^{(1)}}'(0) \\ 0 \\ -{Y^{(1)}}'(1/2) \\ {Y^{(2)}}'(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/4 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

We are now ready to incorporate the boundary conditions. As in the single element case, the Neumann condition at $x = 1$ is (implicitly) incorporated by replacing ${Y^{(2)}}'(1)$ with 1. For the Dirichlet condition at $x = 0$, we (explicitly) demand, as in the single element case, that $Y^{(1)}(0) = 2$. This implies that $2 = c_1^{(1)}\phi_1^{(1)}(0) + c_2^{(1)}\phi_2^{(1)}(0) = c_1^{(1)}$. The system equations now read

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 11/6 & -25/12 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} 2 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \begin{pmatrix} -{Y^{(1)}}'(0) \\ 0 \\ -{Y^{(1)}}'(1/2) \\ 1 \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/4 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

The second and fourth equations determine $c_1^{(2)}$ and $c_2^{(2)}$,

$$-\frac{25}{12}(2) + \frac{11c_1^{(2)}}{6} + \frac{11c_1^{(2)}}{6} - \frac{25c_2^{(2)}}{12} = -\frac{1}{4}, \qquad -\frac{25c_1^{(2)}}{12} + \frac{11c_2^{(2)}}{6} = \frac{19}{24}.$$

The solution is $c_1^{(2)} = 3.707$ and $c_2^{(2)} = 4.644$. Thus, the approximate solution of problem 16.4 with two elements each with a linear polynomial is

$$Y_2(x) = \begin{cases} 2\left(\dfrac{x-1/2}{-1/2}\right) + 3.707\left(\dfrac{x}{1/2}\right), & 0 \le x \le 1/2 \\ 3.707\left(\dfrac{x-1}{-1/2}\right) + 4.644\left(\dfrac{x-1/2}{1/2}\right), & 1/2 < x \le 1. \end{cases}$$

We have shown the exact solution and this approximation in Figure 16.4. It should be compared to that in Figure 16.3.

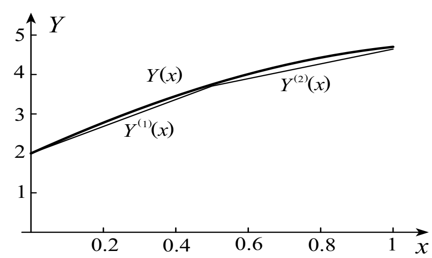


Figure 16.4

We do a final subdivision into four elements, $0 \le x \le 1/4$, $1/4 \le x \le 1/2$, $1/2 \le x \le 3/4$, and $3/4 \le x \le 1$, but give a summarized version. Working first on a general subdivision with points x_1, x_2, x_3, x_4, and x_5, we approximate $Y(x)$ with a linear polynomial on each element,

$$Y_4(x) = \begin{cases} Y^{(1)}(x), & x_1 \le x \le x_2, \\ Y^{(2)}(x), & x_2 < x \le x_3, \\ Y^{(3)}(x), & x_3 < x \le x_4, \\ Y^{(4)}(x), & x_4 < x \le x_5, \end{cases}$$

where

$$Y^{(i)}(x) = c_1^{(i)}\phi_1^{(i)}(x) + c_2^{(i)}\phi_2^{(i)}(x) = c_1^{(i)}\frac{x - x_{i+1}}{x_i - x_{i+1}} + c_2^{(i)}\frac{x - x_i}{x_{i+1} - x_i}, \quad i = 1, \ldots, 4.$$

Nodal values $(c_1^{(i)}, c_2^{(i)})$, $i = 1, \ldots, 4$ satisfy the system equations

$$KC = B + N, \tag{16.11a}$$

where

$$K = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{11}^{(2)} & K_{12}^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{11}^{(3)} & K_{12}^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{11}^{(4)} & K_{12}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{21}^{(4)} & K_{22}^{(4)} \end{bmatrix}, \tag{16.11b}$$

$$\begin{bmatrix} K_{11}^{(i)} & K_{12}^{(i)} \\ K_{21}^{(i)} & K_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} \int_{x_i}^{x_{i+1}} [({\phi_1^{(i)}}')^2 - (\phi_1^{(i)})^2]\,dx & \int_{x_i}^{x_{i+1}} [{\phi_1^{(i)}}'{\phi_2^{(i)}}' - \phi_1^{(i)}\phi_2^{(i)}]\,dx \\ \int_{x_i}^{x_{i+1}} [{\phi_2^{(i)}}'{\phi_1^{(i)}}' - \phi_2^{(i)}\phi_1^{(i)}]\,dx & \int_{x_i}^{x_{i+1}} [({\phi_2^{(i)}}')^2 - (\phi_2^{(i)})^2]\,dx \end{bmatrix}, \tag{16.11c}$$

$$C = \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_1^{(3)} \\ c_2^{(3)} \\ c_1^{(4)} \\ c_2^{(4)} \end{pmatrix}, \qquad B = \begin{bmatrix} \left\{{Y^{(1)}}'\phi_1^{(1)}\right\}_{x_1}^{x_2} \\ \left\{{Y^{(1)}}'\phi_2^{(1)}\right\}_{x_1}^{x_2} \\ \left\{{Y^{(2)}}'\phi_1^{(2)}\right\}_{x_2}^{x_3} \\ \left\{{Y^{(2)}}'\phi_2^{(2)}\right\}_{x_2}^{x_3} \\ \left\{{Y^{(3)}}'\phi_1^{(3)}\right\}_{x_3}^{x_4} \\ \left\{{Y^{(3)}}'\phi_2^{(3)}\right\}_{x_3}^{x_4} \\ \left\{{Y^{(4)}}'\phi_1^{(4)}\right\}_{x_4}^{x_5} \\ \left\{{Y^{(4)}}'\phi_2^{(4)}\right\}_{x_4}^{x_5} \end{bmatrix}, \qquad N = \begin{bmatrix} -\int_{x_1}^{x_2} x\phi_1^{(1)}\,dx \\ -\int_{x_1}^{x_2} x\phi_2^{(1)}\,dx \\ -\int_{x_2}^{x_3} x\phi_1^{(2)}\,dx \\ -\int_{x_2}^{x_3} x\phi_2^{(2)}\,dx \\ -\int_{x_3}^{x_4} x\phi_1^{(3)}\,dx \\ -\int_{x_3}^{x_4} x\phi_2^{(3)}\,dx \\ -\int_{x_4}^{x_5} x\phi_1^{(4)}\,dx \\ -\int_{x_4}^{x_5} x\phi_2^{(4)}\,dx \end{bmatrix}. \tag{16.11e}$$

When we set $x_1 = 0$, $x_2 = 1/4$, $x_3 = 1/2$, $x_4 = 3/4$, and $x_5 = 1$, and evaluate integrals, we get

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -97/24 & 47/12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -97/24 & 47/12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -97/24 & 47/12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_1^{(3)} \\ c_2^{(3)} \\ c_1^{(4)} \\ c_2^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} -{Y^{(1)}}'(0) \\ {Y^{(1)}}'(1/4) \\ -{Y^{(2)}}'(1/4) \\ {Y^{(2)}}'(1/2) \\ -{Y^{(3)}}'(1/2) \\ {Y^{(3)}}'(3/4) \\ -{Y^{(4)}}'(3/4) \\ {Y^{(4)}}'(1) \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/48 \\ -1/24 \\ -5/96 \\ -7/96 \\ -1/12 \\ -5/48 \\ -11/96 \end{pmatrix}.$$

The boundary conditions at $x = 0$ and $x = 1$ along with the interelement boundary conditions at $x = 1/4$, $x = 1/2$, and $x = 3/4$ give

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -97/24 & 47/12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -97/24 & 47/12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -97/24 & 47/12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} 2 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_1^{(3)} \\ c_1^{(3)} \\ c_1^{(4)} \\ c_1^{(4)} \\ c_2^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)\prime}(0) \\ Y^{(1)\prime}(1/4) \\ -Y^{(1)\prime}(1/4) \\ Y^{(2)\prime}(1/2) \\ -Y^{(2)\prime}(1/2) \\ Y^{(3)\prime}(3/4) \\ -Y^{(3)\prime}(3/4) \\ 1 \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/48 \\ -1/24 \\ -5/96 \\ -7/96 \\ -1/12 \\ -5/48 \\ -11/96 \end{pmatrix}.$$

We now add the third equation to the second, the fifth to the fourth, and the seventh to the sixth,

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -97/24 & 47/12 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -97/24 & 47/12 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -97/24 & 47/12 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} 2 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_1^{(3)} \\ c_1^{(3)} \\ c_1^{(4)} \\ c_1^{(4)} \\ c_2^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)\prime}(0) \\ 0 \\ -Y^{(1)\prime}(1/4) \\ 0 \\ -Y^{(2)\prime}(1/2) \\ 0 \\ -Y^{(3)\prime}(3/4) \\ 1 \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/16 \\ -1/24 \\ -1/8 \\ -7/96 \\ -3/16 \\ -5/48 \\ -11/96 \end{pmatrix}.$$

The second, fourth, sixth, and eighth equations reduce to

$$\frac{47c_1^{(2)}}{6} - \frac{97c_1^{(3)}}{24} = \frac{385}{48},$$

$$-\frac{97c_1^{(2)}}{24} + \frac{47c_1^{(3)}}{6} - \frac{97c_1^{(4)}}{24} = -\frac{1}{8},$$

$$-\frac{97c_1^{(3)}}{24}+\frac{47c_1^{(4)}}{6}-\frac{97c_2^{(4)}}{24}=-\frac{3}{16},$$
$$-\frac{97c_1^{(4)}}{24}+\frac{47c_2^{(4)}}{12}=\frac{85}{96}.$$

The solution is $c_1^{(2)}=2.95246$, $c_1^{(3)}=3.73775$, $c_1^{(4)}=4.32277$, and $c_2^{(4)}=4.68680$. Thus, the four-element, linear approximate to the solution of problem 16.4 is

$$Y_4(x)=\begin{cases}2(-4)(x-1/4)+2.95246(4)x, & 0\le x\le 1/4\\ 2.95246(-4)(x-1/2)+3.73775(4)(x-1/4), & 1/4<x\le 1/2\\ 3.73775(-4)(x-3/4)+4.32277(4)(x-1/2), & 1/2<x\le 3/4\\ 4.32277(-4)(x-1)+4.68680(4)(x-3/4), & 3/4<x\le 1\end{cases}$$

It is plotted along with the exact solution in Figure 16.5. They are barely distinguishable.

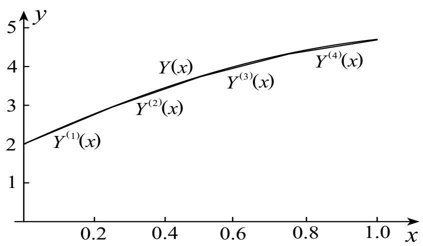


Figure 16.5

In the next example, we indicate how to handle Robin boundary conditions.

Example 16.1 Use one- and two-element, linear approximations on problem 16.4 when the boundary condition at $x=1$ is Robin,

$$\frac{d^2Y}{dx^2}+Y=x, \quad 0<x<1, \tag{16.12a}$$
$$Y(0)=2, \tag{16.12b}$$
$$Y'(1)+3Y(1)=1. \tag{16.12c}$$

Solution It is straightforward to generate the exact solution

$$Y(x)=2\cos x+\left[\frac{2(\sin 1-3\cos 1)-3}{3\sin 1+\cos 1}\right]\sin x+x.$$

System (or element) equations for a one-element, linear approximation are equations 16.8,

$$\begin{pmatrix}2/3 & -7/6\\ -7/6 & 2/3\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}-Y_1'(0)\\ Y_1'(1)\end{pmatrix}+\begin{pmatrix}1/6\\ -1/3\end{pmatrix}.$$

The Dirichlet boundary condition $Y(0)=2$ is handled as before, set $c_1=2$. The boundary residual at $x=1$ is $R_{|x=1}=Y_1'(1)+3Y_1(1)-1$. When we demand that this be zero, the approximation must implicitly satisfy

$$Y_1'(1)=1-3Y_1(1)=1-3[c_1\phi_1(1)+c_2\phi_2(1)]=1-3c_2.$$

Substituting these requirements in the system equations yields

$$\begin{pmatrix}2/3 & -7/6\\ -7/6 & 2/3\end{pmatrix}\begin{pmatrix}2\\ c_2\end{pmatrix}=\begin{pmatrix}-Y_1'(0)\\ 1-3c_2\end{pmatrix}+\begin{pmatrix}1/6\\ -1/3\end{pmatrix}.$$

The second equation gives

$$-\frac{7}{6}(2)+\frac{2c_2}{3}=1-3c_2-\frac{1}{3} \qquad \Longrightarrow \qquad c_2=\frac{9}{11}.$$

The one-element, linear approximation is therefore

$$Y_1(x)=2\left(\frac{x-1}{-1}\right)+\frac{9}{11}\left(\frac{x-0}{1}\right)=2-\frac{12x}{11}.$$

Graphs of the exact solution and $Y_1(x)$ are shown in Figure 16.6a. System equations for a two-element, linear approximation

$$Y_2(x)=\begin{cases} Y^{(1)}(x)=c_1^{(1)}\phi_1^{(1)}(x)+c_2^{(1)}\phi_2^{(1)}(x)=c_1^{(1)}\dfrac{x-x_2}{x_1-x_2}+c_2^{(1)}\dfrac{x-x_1}{x_2-x_1}, & x_1\le x\le x_2 \\ Y^{(2)}(x)=c_1^{(2)}\phi_1^{(2)}(x)+c_2^{(2)}\phi_2^{(2)}(x)=c_1^{(2)}\dfrac{x-x_3}{x_2-x_3}+c_2^{(2)}\dfrac{x-x_2}{x_3-x_2}, & x_2< x\le x_3, \end{cases}$$

before application of the boundary conditions, are given by equations 16.10,

$$\begin{bmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 0 & 0 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{bmatrix}\begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix}=\begin{pmatrix} -{Y^{(1)}}'(0) \\ {Y^{(1)}}'(1/2) \\ -{Y^{(2)}}'(1/2) \\ {Y^{(2)}}'(1) \end{pmatrix}+\begin{pmatrix} -1/24 \\ -1/12 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

We incorporate the Dirichlet condition by demanding that $c_1^{(1)}=2$. The Robin condition at $x=1$ is handled implicitly by setting $Y_1'(1)=1-3c_2^{(2)}$. The interelement boundary conditions are $c_2^{(1)}=c_1^{(2)}$ and ${Y^{(1)}}'(1/2)={Y^{(2)}}'(1/2)$. When we make these substitutions in the system equations, we obtain

$$\begin{bmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 0 & 0 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{bmatrix}\begin{pmatrix} 2 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix}=\begin{pmatrix} -{Y^{(1)}}'(0) \\ {Y^{(1)}}'(1/2) \\ -{Y^{(1)}}'(1/2) \\ 1-3c_2^{(2)} \end{pmatrix}+\begin{pmatrix} -1/24 \\ -1/12 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

If we add the third equation to the second, we obtain

$$\begin{bmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 11/6 & -25/12 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{bmatrix}\begin{pmatrix} 2 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix}=\begin{pmatrix} -{Y^{(1)}}'(0) \\ 0 \\ {Y^{(1)}}'(1/2) \\ 1-3c_2^{(2)} \end{pmatrix}+\begin{pmatrix} -1/24 \\ -1/4 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

The second and fourth equations are

$$-\frac{25}{12}(2)+\frac{11c_1^{(2)}}{6}+\frac{11c_1^{(2)}}{6}-\frac{25c_2^{(2)}}{12}=-\frac{1}{4}, \qquad -\frac{25c_1^{(2)}}{12}+\frac{11c_2^{(2)}}{6}=\frac{19}{24}-3c_2^{(2)}.$$

The solution is $c_1^{(2)}=1.538$ and $c_2^{(2)}=0.8267$. Thus, the two-element, linear approximate to the solution of problem 16.4 is

$$Y_2(x)=\begin{cases} 2\left(\dfrac{x-1/2}{-1/2}\right)+1.538\left(\dfrac{x}{1/2}\right), & 0\le x\le 1/2 \\ 1.538\left(\dfrac{x-1}{-1/2}\right)+0.8267\left(\dfrac{x-1/2}{1/2}\right), & 1/2< x\le 1. \end{cases}$$

The exact solution and this two-element, linear approximation are shown in Figure 16.6b.●

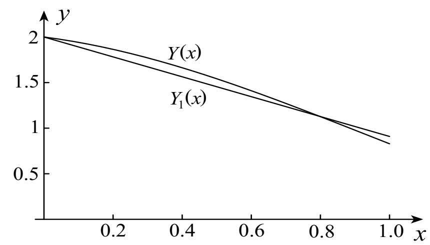


Figure 16.6a

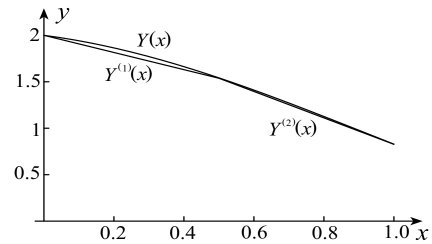


Figure 16.6b

Quadratic Approximation

We now use quadratic polynomials to approximate the solution to problem 16.4. It is anticipated that we should need fewer elements to obtain a comparable approximation derived with linear functions. As was the case for linear functions, finite elements requires quadratic functions to be written in a specific way. If a quadratic function $y(x) = a + bx + cx^2$ is to have values $y(x_1)$, $y(x_2)$, and $y(x_3)$ at three points $x_1 < x_2 < x_3$, then

$$y(x_1) = a + bx_1 + cx_1^2, \quad y(x_2) = a + bx_2 + cx_2^2, \quad y(x_3) = a + bx_3 + cx_3^2. \tag{16.13}$$

These equations can be solved for

$$a = \frac{y(x_1)x_2x_3(x_3 - x_2) + y(x_2)x_1x_3(x_1 - x_3) + y(x_3)x_1x_2(x_2 - x_1)}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)}, \tag{16.14a}$$

$$b = \frac{-y(x_1)(x_3^2 - x_2^2) + y(x_2)(x_3^2 - x_1^2) - y(x_3)(x_2^2 - x_1^2)}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)}, \tag{16.14b}$$

$$c = \frac{y(x_1)(x_3 - x_2) - y(x_2)(x_3 - x_2) + y(x_3)(x_2 - x_1)}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)}, \tag{16.14c}$$

(see Exercise 1). When these values are substituted into $y = a + bx + cx^2$, the result can be rearranged into the form

$$\begin{aligned} y(x) = y(x_1)\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} &+ y(x_2)\frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} \\ &+ y(x_3)\frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}. \end{aligned} \tag{16.15}$$

If we denote the three quadratic functions on the right by

$$\begin{aligned} \phi_1(x) &= \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \quad \phi_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}, \\ \phi_3(x) &= \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}, \end{aligned} \tag{16.16}$$

then we have represented a quadratic function in terms of three other quadratic functions

$$y(x) = y(x_1)\phi_1(x) + y(x_2)\phi_2(x) + y(x_3)\phi_3(x). \tag{16.17}$$

Those who have studied numerical analysis will recognize the $\phi_i(x)$ as Lagrange interpolation formulas. They satisfy property 16.3; each function has value one at one of the nodes and value zero at the other two nodes. When a quadratic approximation is sought for a boundary value problem associated with an ordinary differential equation, instead of taking the approximation in the form $y = a + bx + cx^2$, it will be taken in the form

$$y = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x). \tag{16.18}$$

Basis functions 1, x, and x^2 are replaced by $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$. We have shown these functions in Figure 16.7a. Figure 16.7b shows a quadratic function with values $y(x_1)$, $y(x_2)$, and $y(x_3)$ in terms of $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$.

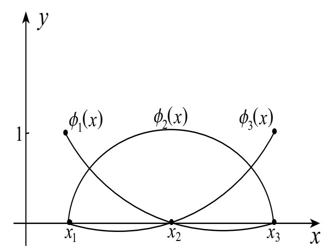


Figure 16.7a

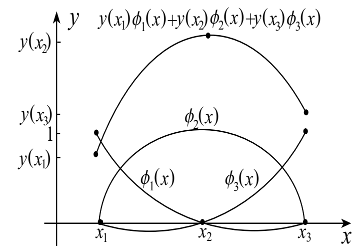


Figure 16.7b

Constants c_1, c_2 and c_3 are values of the approximation at the nodes x_1, x_2, and x_3, respectively. Finite elements approximates these nodal values, and approximates the solution at other values of x with approximating quadratic 16.18.

Element equations 16.7 were derived for a (one-element) linear approximation $c_1\phi_1(x) + c_2\phi_2(x)$ on the interval $x_1 \le x \le x_2$. Based on these, it is not difficult to write element equations for a quadratic approximation $Y_1(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$ on the interval $x_1 \le x \le x_2$, where

$$\phi_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}, \quad \phi_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}, \quad \phi_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.$$

Those who feel that it would be instructive to derive them with the MWR, see Exercise 2. The element equations are

$$KC = B + N, \tag{16.19a}$$

where

$$K = \begin{bmatrix} \int_{x_1}^{x_3} [(\phi_1')^2 - \phi_1^2]\,dx & \int_{x_1}^{x_3} (\phi_1'\phi_2' - \phi_1\phi_2)\,dx & \int_{x_1}^{x_3} (\phi_1'\phi_3' - \phi_1\phi_3)\,dx \\ \int_{x_1}^{x_3} (\phi_2'\phi_1' - \phi_2\phi_1)\,dx & \int_{x_1}^{x_3} [(\phi_2')^2 - \phi_2^2]\,dx & \int_{x_1}^{x_3} (\phi_2'\phi_3' - \phi_2\phi_3)\,dx \\ \int_{x_1}^{x_3} (\phi_3'\phi_1' - \phi_3\phi_1)\,dx & \int_{x_1}^{x_3} (\phi_3'\phi_2' - \phi_3\phi_2)\,dx & \int_{x_1}^{x_3} [(\phi_3')^2 - \phi_3^2]\,dx \end{bmatrix}, \tag{16.19b}$$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \qquad B = \begin{bmatrix} \{Y_1'\phi_1\}_{x_1}^{x_3} \\ \{Y_1'\phi_2\}_{x_1}^{x_3} \\ \{Y_1'\phi_3\}_{x_1}^{x_3} \end{bmatrix}, \qquad N = \begin{bmatrix} -\int_{x_1}^{x_3} x\phi_1\,dx \\ -\int_{x_1}^{x_3} x\phi_2\,dx \\ -\int_{x_1}^{x_3} x\phi_3\,dx \end{bmatrix}. \tag{16.19c}$$

Evaluation of terms when $x_1 = 0$, $x_2 = 1/2$, and $x_3 = 1$ gives

$$\begin{pmatrix} 11/5 & -41/15 & 11/30 \\ -41/15 & 24/5 & -41/15 \\ 11/30 & -41/15 & 11/5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -Y_1'(0) \\ 0 \\ Y_1'(1) \end{pmatrix} + \begin{pmatrix} 0 \\ -1/3 \\ -1/6 \end{pmatrix}.$$

These equations define nodal values c_1, c_2, and c_3 once boundary conditions have been incorporated. To include the Neumann condition at $x = 1$, we once again, replace $Y_1'(1)$ with 1, and this is an implicit requirement. We impose the Dirichlet condition at $x = 0$ on the trial solution; that is, we demand that $Y_1(0) = 2$. This implies that $2 = c_1\phi_1(0) + c_2\phi_2(0) + c_3\phi_3(0) = c_1$, an explicit requirement. When we substitute these in the element equations,

$$\begin{pmatrix} 11/5 & -41/15 & 11/30 \\ -41/15 & 24/5 & -41/15 \\ 11/30 & -41/15 & 11/5 \end{pmatrix} \begin{pmatrix} 2 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -Y_1'(0) \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/3 \\ -1/6 \end{pmatrix}.$$

The second and third equations determine c_2 and c_3,

$$2\left(-\frac{41}{15}\right) + \frac{24c_2}{5} - \frac{41c_3}{15} = -\frac{1}{3}, \quad 2\left(\frac{11}{30}\right) - \frac{41c_2}{15} + \frac{11c_3}{5} = \frac{5}{6}.$$

The solution is $c_2 = 1041/278$ and $c_3 = 658/139$. Thus, the one-element, quadratic approximation is

$$\begin{aligned} Y_1(x) &= 2\phi_1(x) + \frac{1041}{278}\phi_2(x) + \frac{658}{139}\phi_3(x) \\ &= 2(2x-1)(x-1) - \frac{4164}{278}x(x-1) + \frac{658}{139}x(2x-1). \end{aligned}$$

The graph of this approximation and the exact solution are almost indistinguishable. They are tabulated below for comparison. For a two-element, quadratic approximation, see Exercise 6.

x	Exact	$Y_1(x)$
0.1	2.401	2.409
0.2	2.779	2.788
0.3	3.131	3.137
0.4	3.455	3.456
0.5	3.748	3.745
0.6	4.009	4.003
0.7	4.236	4.231
0.8	4.428	4.429
0.9	4.583	4.596
1.0	4.701	4.734

Table 16.1

The stiffness matrix for the MFE is always symmetric provided the differential equation is in self-adjoint form (see Section 12.3 for the definition of self-adjoint). Any second-order differential equation can be put in self-adjoint form with a suitable multiplicative factor (see the discussion following equation 5.3 in Section 5.1). For a typical problem using finite elements, the stiffness matrix can be very large, and with symmetry, computer storage is kept to a minimum and numerical procedures for solving the system equations are enhanced.

Examination of system equations 16.11 illustrates another feature of the stiffness matrix; it is banded, nonzero entries cluster around the diagonal. This is due to the fact that we numbered elements sequentially from left to right. This is natural; why would we number elements in some random fashion? This becomes an important issue in two- and three-dimensional problems where there is no natural way to number elements.

EXERCISES 16.2

1. Verify that the solution of equations 16.13 is given by equations 16.14.

2. Use the MWR to derive system equations 16.19.

3. Find one- and two-element linear approximations to the boundary value problem of Exercise 16 in Section 15.3. Use values $a = 1$, $h = 2$, and $D = 1$. Compare the exact solution and the approximations graphically.

4. The stiffness matrix is symmetric when the differential equation is in self-adjoint form, and nonsymmetric otherwise. Illustrate this with a one-element linear approximation for the differential equation in Exercise 17 of Section 15.3.

5. (a) Find one- and two-element, linear approximations to the boundary value problem of Exercise 10 in Section 15.3. Compare the exact solution and the approximations graphically.
(b) Find a one-element, quadratic approximation to the problem. Tabulate it and the exact solution for $x = 1.0, 1.1, 1.2, \ldots, 2.0$.

6. Find a two-element, quadratic approximation to boundary value problem 16.4. Tabulate this approximation along with the one-element approximation and the exact solution.

7. The self-adjoint form for second-order linear differential equations is

$$\frac{d}{dx}\left[\alpha(x)\frac{dY}{dx}\right] + \beta(x)Y = F(x).$$

Let $Y_1(x)$, as defined in equation 16.6, be a one-element, linear approximation of the solution on the interval $x_1 < x < x_2$. Find the element equations for c_1 and c_2.

8. Repeat Exercise 7 for a one-element quadratic approximation $Y_1(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$, on the interval $x_1 \leq x \leq x_2$, where the $\phi_i(x)$ are given in equations 16.16.

9. (a) Can we use a one-element, linear approximation for the boundary value problem in Exercise 9 of Section 15.3? Explain.
(b) Use Exercise 7 to find a two-element, linear approximation. (You must express the differential equation in self-adjoint form.) Draw graphs of the exact solution and the approximation.

10. We developed the Lagrange quadratic basis functions 16.16 by requiring that a quadratic polynomial be expressible in terms of nodal values. Subsequently, we noted that they satisfy conditions

16.3. Show that they can be developed by requiring a quadratic polynomial to satisfy conditions
16.3.

§16.3 Toward a More comprehensive Notation

In Section 16.2, we developed system equations for finite element approximations to boundary value problems associated with ordinary differential equations. On the basis of these discussions, we can take a more direct route to the same equations, and at the same time, we can simplify the notation, bringing us a step closer to the customary notation of finite element programs. In the exercises, we provide a template for system equations of any self-adjoint second-order differential equation. We illustrate with problem 16.4

$$\frac{d^2Y}{dx^2} + Y = x, \quad 0 < x < 1, \tag{16.20a}$$
$$Y(0) = 2, \tag{16.20b}$$
$$Y'(1) = 1, \tag{16.20c}$$

System equations for a two-element, linear approximation

$$Y_2(x) = \begin{cases} Y^{(1)}(x) = c_1^{(1)}\phi_1^{(1)}(x) + c_2^{(1)}\phi_2^{(1)}(x), & 0 \le x \le 1/2 \\ Y^{(2)}(x) = c_1^{(2)}\phi_1^{(2)}(x) + c_2^{(2)}\phi_2^{(2)}(x), & 1/2 < x \le 1 \end{cases}$$
$$= \begin{cases} c_1^{(1)}\dfrac{x-x_2}{x_1-x_2} + c_2^{(1)}\dfrac{x-x_1}{x_2-x_1} = c_1^{(1)}(1-2x) + c_2^{(1)}(2x), & 0 \le x \le 1/2 \\ c_1^{(2)}\dfrac{x-x_3}{x_2-x_3} + c_2^{(2)}\dfrac{x-x_2}{x_3-x_2} = c_1^{(2)}(2-2x) + c_2^{(2)}(2x-1), & 1/2 < x \le 1 \end{cases}$$

were developed in Section 16.2. They were denoted by $KC = B + N$ with entries (see equations 16.10),

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 0 & 0 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ Y^{(1)'}(1/2) \\ -Y^{(2)'}(1/2) \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/12 \\ -1/6 \\ -5/24 \end{pmatrix}. \tag{16.21}$$

When interelement boundary conditions $c_2^{(1)} = c_1^{(2)}$ and $Y^{(1)'}(1/2) = Y^{(2)'}(1/2)$ are imposed, these become

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 0 & 0 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ Y^{(2)'}(1/2) \\ -Y^{(2)'}(1/2) \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/12 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

If we now add the third equation to the second,

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 11/6 & -25/12 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ -Y^{(2)'}(1/2) \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/4 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

The objective is find $c_1^{(1)}$, $c_1^{(2)}$, and $c_2^{(2)}$, and these are the values of the linear polynomials approximating the solution at the three nodes $x = 0$, $x = 1/2$ and $x = 1$. Suppose we denote these values simply by a_1, a_2, and a_3,

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/6 & 11/6 & -25/12 \\ 0 & 0 & 11/6 & -25/12 \\ 0 & 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ -Y^{(2)'}(1/2) \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/4 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

The value a_2 appears in the second and third equations, and we intend using the second equation, not the third, because the third equation contains the unknown quantity $Y^{(2)'}(1/2)$. The question then is how to eliminate the third equation, without losing any information contained in it. We cannot simply delete the third row of each matrix. You should check that if this were done, the remaining equations would not be the same as the first, second, and third of the above equations. However, the same equations are obtained if we follow two steps. Add the third column of K to the second column,

$$\begin{pmatrix} 11/6 & -25/12 & 0 & 0 \\ -25/12 & 11/3 & 11/6 & -25/12 \\ 0 & 11/6 & 11/6 & -25/12 \\ 0 & -25/12 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ -Y^{(2)'}(1/2) \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/4 \\ -1/6 \\ -5/24 \end{pmatrix}.$$

Now delete the third equation by deleting the third row and third column of K and the third entry of C, B and N,

$$\begin{pmatrix} 11/6 & -25/12 & 0 \\ -25/12 & 11/3 & -25/12 \\ 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/4 \\ -5/24 \end{pmatrix}. \tag{16.22}$$

There are three points to notice about these equations.

1. The derivative at the interior node has been eliminated, only the value a_2 of the approximation at this node remains.
2. The stiffness matrix in system equations 16.21 was symmetric. We have made various changes to this matrix, but the stiffness matrix in equations 16.22 is also symmetric.
3. The $(1, 3)$ (and $(3, 1)$) entry is zero because there is no coupling between nodes 1 and 3.

The final step is to incorporate boundary conditions 16.20b,c. They require $a_1 = 2$ and $Y(2)'(1) = 1$,

$$\begin{pmatrix} 11/6 & -25/12 & 0 \\ -25/12 & 11/3 & -25/12 \\ 0 & -25/12 & 11/6 \end{pmatrix} \begin{pmatrix} 2 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/24 \\ -1/4 \\ -5/24 \end{pmatrix}.$$

The last two equations are now solved for a_2 and a_3.

To emphasize what has happened, we repeat it with the four-element linear approximation in Section 16.2. System equations were

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -97/24 & 47/12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -97/24 & 47/12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -97/24 & 47/12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_1^{(3)} \\ c_2^{(3)} \\ c_1^{(4)} \\ c_2^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)\prime}(0) \\ Y^{(1)\prime}(1/4) \\ -Y^{(2)\prime}(1/4) \\ Y^{(2)\prime}(1/2) \\ -Y^{(3)\prime}(1/2) \\ Y^{(3)\prime}(3/4) \\ -Y^{(4)\prime}(3/4) \\ Y^{(4)\prime}(1) \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/48 \\ -1/24 \\ -5/96 \\ -7/96 \\ -1/12 \\ -5/48 \\ -11/96 \end{pmatrix}.$$

Interelement boundary conditions $c_2^{(1)} = c_1^{(2)}$, $c_2^{(2)} = c_1^{(3)}$, $c_2^{(3)} = c_1^{(4)}$, $Y^{(2)\prime}(1/4) = Y^{(1)\prime}(1/4)$, $Y^{(3)\prime}(1/2) = Y^{(2)\prime}(1/2)$, and $Y^{(4)\prime}(3/4) = Y^{(3)\prime}(3/4)$ yield

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -97/24 & 47/12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -97/24 & 47/12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -97/24 & 47/12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} c_1^{(1)} \\ c_1^{(2)} \\ c_1^{(2)} \\ c_1^{(3)} \\ c_1^{(3)} \\ c_1^{(4)} \\ c_1^{(4)} \\ c_2^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)\prime}(0) \\ Y^{(1)\prime}(1/4) \\ -Y^{(1)\prime}(1/4) \\ Y^{(2)\prime}(1/2) \\ -Y^{(2)\prime}(1/2) \\ Y^{(3)\prime}(3/4) \\ -Y^{(3)\prime}(3/4) \\ Y^{(4)\prime}(1) \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/48 \\ -1/24 \\ -5/96 \\ -7/96 \\ -1/12 \\ -5/48 \\ -11/96 \end{pmatrix}.$$

We now add the third equation to the second, the fifth to the fourth, and the seventh to the sixth,

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -97/24 & 47/12 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -97/24 & 47/12 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -97/24 & 47/12 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} c_1^{(1)} \\ c_1^{(2)} \\ c_1^{(2)} \\ c_1^{(3)} \\ c_1^{(3)} \\ c_1^{(4)} \\ c_1^{(4)} \\ c_2^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ -Y^{(1)'}(1/4) \\ 0 \\ -Y^{(2)'}(1/2) \\ 0 \\ -Y^{(3)'}(3/4) \\ Y^{(4)'}(1) \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/16 \\ -1/24 \\ -1/8 \\ -7/96 \\ -3/16 \\ -5/48 \\ -11/96 \end{pmatrix}.$$

Suppose we denote $c_1^{(1)}$, $c_1^{(2)}$, $c_1^{(3)}$, $c_1^{(3)}$, $c_1^{(4)}$, and $c_2^{(4)}$, the values of the linear polynomials approximating the solution at the five nodes $x = 0$, $x = 1/4$, $x = 1/2$, $x = 3/4$, and $x = 1$ by a_1, a_2, a_3, a_4, and a_5,

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -97/24 & 47/12 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 0 & -97/24 & 47/12 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -97/24 & 47/12 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_2 \\ a_3 \\ a_3 \\ a_4 \\ a_4 \\ a_5 \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ -Y^{(1)'}(1/4) \\ 0 \\ -Y^{(2)'}(1/2) \\ 0 \\ -Y^{(3)'}(3/4) \\ Y^{(4)'}(1) \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/16 \\ -1/24 \\ -1/8 \\ -7/96 \\ -3/16 \\ -5/48 \\ -11/96 \end{pmatrix}.$$

In order to eliminate the third, fifth, and seventh equations, we first add the third column of the stiffness matrix to the second, the fifth column to the fourth, and the seventh column to the sixth,

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -97/24 & 47/6 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & 47/12 & 47/12 & -97/24 & 0 & 0 & 0 & 0 \\ 0 & -97/24 & -97/24 & 47/6 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & 47/12 & 47/12 & -97/24 & 0 & 0 \\ 0 & 0 & 0 & -97/24 & -97/24 & 47/6 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & 47/12 & 47/12 & -97/24 \\ 0 & 0 & 0 & 0 & 0 & -97/24 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_2 \\ a_3 \\ a_3 \\ a_4 \\ a_4 \\ a_5 \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ -Y^{(1)'}(1/4) \\ 0 \\ -Y^{(2)'}(1/2) \\ 0 \\ -Y^{(3)'}(3/4) \\ Y^{(4)'}(1) \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/16 \\ -1/24 \\ -1/8 \\ -7/96 \\ -3/16 \\ -5/48 \\ -11/96 \end{pmatrix}.$$

Now delete the third, fifth, and seventh equations by deleting the third, fifth, and seventh rows and columns of K and the third, fifth, and seventh entries of C, B and N,

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 \\ -97/24 & 47/6 & -97/24 & 0 & 0 \\ 0 & -97/24 & 47/6 & -97/24 & 0 \\ 0 & 0 & -97/24 & 47/6 & -97/24 \\ 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ 0 \\ 0 \\ Y^{(4)'}(1) \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/16 \\ -1/8 \\ -3/16 \\ -11/96 \end{pmatrix}.$$

Points to notice again are that the stiffness matrix is symmetric and zero entries correspond to uncoupled nodes. Derivatives at interior nodes have been eliminated and only the values a_2, a_3, and a_4 of the approximation at these nodes remain. The final step is to incorporate boundary conditions 16.20b,c. They require $a_1 = 2$ and $Y(2)'(1) = 1$,

$$\begin{bmatrix} 47/12 & -97/24 & 0 & 0 & 0 \\ -97/24 & 47/6 & -97/24 & 0 & 0 \\ 0 & -97/24 & 47/6 & -97/24 & 0 \\ 0 & 0 & -97/24 & 47/6 & -97/24 \\ 0 & 0 & 0 & -97/24 & 47/12 \end{bmatrix} \begin{pmatrix} 2 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/96 \\ -1/16 \\ -1/8 \\ -3/16 \\ -11/96 \end{pmatrix}.$$

These are now solved for a_2, a_3, a_4, and a_5.

In the exercises we provide a template for system equations for linear approximations to self-adjoint, second-order differential equations.

EXERCISES 16.3

1. In Exercise 7 of Section 16.2, element equations for a linear approximation

$$Y_1(x) = c_1\phi_1(x) + c_2\phi_2(x) = c_1\left(\frac{x - x_2}{x_1 - x_2}\right) + c_2\left(\frac{x - x_1}{x_2 - x_1}\right),$$

to the solution of the self-adjoint, second-order differential equation

$$\frac{d}{dx}\left[\alpha(x)\frac{dY}{dx}\right] + \beta(x)Y = F(x).$$

on the interval $x_1 \le x \le x_2$ were shown to be

$$KC = B + N,$$

where

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{x_1}^{x_2} [\alpha(\phi_1')^2 - \beta(\phi_1)^2]\,dx & \displaystyle\int_{x_1}^{x_2} (\alpha\phi_1'\phi_2' - \beta\phi_1\phi_2)\,dx \\ \displaystyle\int_{x_1}^{x_2} (\alpha\phi_2'\phi_1' - \beta\phi_1\phi_2)\,dx & \displaystyle\int_{x_1}^{x_2} [\alpha(\phi_2')^2 - \beta(\phi_2)^2]\,dx \end{pmatrix},$$

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \{\alpha Y_1'\phi_1\}_{x_1}^{x_2} \\ \{\alpha Y_1'\phi_2\}_{x_1}^{x_2} \end{pmatrix}, \qquad N = \begin{pmatrix} -\displaystyle\int_{x_1}^{x_2} F\phi_1\,dx \\ -\displaystyle\int_{x_1}^{x_2} F\phi_2\,dx \end{pmatrix}.$$

Show that system equations for a two-element, linear approximation

$$Y_2(x) = \begin{cases} Y^{(1)}(x) = c_1^{(1)}\phi_1^{(1)}(x) + c_2^{(1)}\phi_2^{(1)}(x) = c_1^{(1)}\dfrac{x - x_2}{x_1 - x_2} + c_2^{(1)}\dfrac{x - x_1}{x_2 - x_1}, & x_1 \le x \le x_2 \\ Y^{(2)}(x) = c_1^{(2)}\phi_1^{(2)}(x) + c_2^{(2)}\phi_2^{(2)}(x) = c_1^{(2)}\dfrac{x - x_3}{x_2 - x_3} + c_2^{(2)}\dfrac{x - x_2}{x_3 - x_2}, & x_2 < x \le x_3, \end{cases}$$

on the interval $x_1 \le x \le x_3$ can be written in the form

$$\begin{pmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -\alpha(x_1)Y^{(1)'}(x_1) \\ 0 \\ \alpha(x_3)Y^{(2)'}(x_3) \end{pmatrix} + \begin{pmatrix} -\int_{x_1}^{x_2} F\phi_1^{(1)}dx \\ -\int_{x_1}^{x_2} F\phi_2^{(1)}dx - \int_{x_2}^{x_3} F\phi_1^{(2)}dx \\ -\int_{x_2}^{x_3} F\phi_2^{(2)}dx \end{pmatrix},$$

where a_1, a_2, and a_3 are values of $Y_2(x)$ at nodes x_1, x_2, and x_3, respectively.

2. Show that system equations for a four-element, linear approximation

$$Y_4(x) = \begin{cases} Y^{(1)}(x), & x_1 \le x \le x_2, \\ Y^{(2)}(x), & x_2 < x \le x_3, \\ Y^{(3)}(x), & x_3 < x \le x_4, \\ Y^{(4)}(x), & x_4 < x \le x_5, \end{cases}$$

where

$$Y^{(i)}(x) = c_1^{(i)}\phi_1^{(i)}(x) + c_2^{(i)}\phi_2^{(i)}(x) = c_1^{(i)}\frac{x - x_{i+1}}{x_i - x_{i+1}} + c_2^{(i)}\frac{x - x_i}{x_{i+1} - x_i}, \quad i = 1, \ldots, 4.$$

to the differential equation in Exercise 1 on the interval $x_1 \le x \le x_5$ can be written in the form

$$\begin{pmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} & 0 \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} + K_{11}^{(4)} & K_{12}^{(4)} \\ 0 & 0 & 0 & K_{21}^{(4)} & K_{22}^{(4)} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha(x_1)Y^{(1)\prime}(x_1) \\ 0 \\ 0 \\ 0 \\ \alpha(x_5)Y^{(4)\prime}(x_5) \end{pmatrix} + \begin{pmatrix} -\int_{x_1}^{x_2} F\phi_1^{(1)}dx \\ -\int_{x_1}^{x_2} F\phi_2^{(1)}dx - \int_{x_2}^{x_3} F\phi_1^{(2)}dx \\ -\int_{x_2}^{x_3} F\phi_2^{(2)}dx - \int_{x_3}^{x_4} F\phi_1^{(3)}dx \\ -\int_{x_3}^{x_4} F\phi_2^{(3)}dx - \int_{x_4}^{x_5} F\phi_1^{(4)}dx \\ -\int_{x_4}^{x_5} F\phi_2^{(4)}dx \end{pmatrix}.$$

3. (a) Find the exact solution of the boundary value problem

$$\frac{d}{dx}\left(x\frac{dY}{dx}\right) = x^3, \quad 1 < x < 2,$$
$$Y'(1) = 1, \quad Y(2) = 2,$$

(b) Use the template of Exercise 1 to show that system equations for a two-element, linear approximation to the solution of the boundary value problem are

$$\begin{pmatrix} 5/2 & -5/2 & 0 \\ -5/2 & 6 & -7/2 \\ 0 & -7/2 & 7/2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -Y^{(1)\prime}(1) \\ 0 \\ 2Y^{(2)\prime}(2) \end{pmatrix} + \begin{pmatrix} -131/320 \\ -22/15 \\ -499/320 \end{pmatrix}.$$

(c) Show that the stiffness matrix is singular. Can you explain why this must be so?
(d) Apply the boundary conditions to find a_1, a_2, and a_3. Plot the approximation and the exact solution.

4. (a) Use the template of Exercise 2 to show that system equations for a four-element, linear approximation to the solution of the boundary value problem in Exercise 3 are

$$\begin{pmatrix} 9/2 & -9/2 & 0 & 0 & 0 \\ -9/2 & 10 & -11/2 & 0 & 0 \\ 0 & -11/2 & 12 & -13/2 & 0 \\ 0 & 0 & -13/2 & 14 & -15/2 \\ 0 & 0 & 0 & -15/2 & 15/2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} -Y^{(1)\prime}(1) \\ 0 \\ 0 \\ 0 \\ 2Y^{(4)\prime}(2) \end{pmatrix} + \begin{pmatrix} -625/1024 \\ -763/2560 \\ -219/256 \\ -693/512 \\ -4519/5120 \end{pmatrix}.$$

(b) Show that the stiffness matrix is singular. The reason for this was explained in Exercise 3.
(c) Apply the boundary conditions to find a_1, a_2, a_3, a_4, and a_5. Plot the approximation and the exact solution.

5. (a) Find the exact solution of the boundary value problem in Exercise 3 if the Dirichlet boundary condition $Y(2) = 2$ is replaced by the Robin condition $Y'(2) + 3Y(2) = 1$.
(b) Use the template of Exercise 1 to find a two-element, linear approximation. Plot the exact

solution and the approximation.

6. Repeat Exercise 5 with the four-element, linear approximation of Exercise 2.

7. According to Exercise 6 in Section 16.2, system equations for a two-element, quadratic approximation to differential equation 16.4a are

$$\begin{pmatrix} 23/5 & -161/30 & 41/60 & 0 & 0 & 0 \\ -161/30 & 52/5 & -161/30 & 0 & 0 & 0 \\ 41/60 & -161/30 & 23/5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 23/5 & -161/30 & 41/60 \\ 0 & 0 & 0 & -161/30 & 52/5 & -161/30 \\ 0 & 0 & 0 & 41/60 & -161/30 & 23/5 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_3^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_3^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ Y^{(1)'}(1/2) \\ -Y^{(2)'}(1/2) \\ 0 \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} 0 \\ -1/12 \\ -1/24 \\ -1/24 \\ -1/4 \\ -1/12 \end{pmatrix}.$$

Incorporate interelement boundary conditions to show that these equations can be written in terms of nodal values of the approximation as follows:

$$\begin{pmatrix} 23/5 & -161/30 & 41/60 & 0 & 0 \\ -161/30 & 52/5 & -161/30 & 0 & 0 \\ 41/60 & -161/30 & 46/5 & -161/30 & 41/60 \\ 0 & 0 & -161/30 & 52/5 & -161/30 \\ 0 & 0 & 41/60 & -161/30 & 23/5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$$

$$= \begin{pmatrix} -Y^{(1)'}(0) \\ 0 \\ 0 \\ 0 \\ Y^{(2)'}(1) \end{pmatrix} + \begin{pmatrix} 0 \\ -1/12 \\ -1/12 \\ -1/4 \\ -1/12 \end{pmatrix}$$

8. In Exercise 8 of Section 16.2, element equations for a quadratic approximation

$$Y_1(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x),$$

to the solution of the self-adjoint, second-order differential equation

$$\frac{d}{dx}\left[\alpha(x)\frac{dY}{dx}\right] + \beta(x)Y = F(x).$$

on the interval $x_1 \le x \le x_2$ were shown to be

$$KC = B + N,$$

where

$$K = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \int_{x_1}^{x_2} [\alpha(\phi_1')^2 - \beta(\phi_1)^2]\,dx & \int_{x_1}^{x_2} (\alpha\phi_1'\phi_2' - \beta\phi_1\phi_2)\,dx & \int_{x_1}^{x_2} (\alpha\phi_1'\phi_3' - \beta\phi_1\phi_3)\,dx \\ \int_{x_1}^{x_2} (\alpha\phi_1'\phi_2' - \beta\phi_1\phi_2)\,dx & \int_{x_1}^{x_2} [\alpha(\phi_2')^2 - \beta(\phi_2)^2]\,dx & \int_{x_1}^{x_2} (\alpha\phi_2'\phi_3' - \beta\phi_2\phi_3)\,dx \\ \int_{x_1}^{x_2} (\alpha\phi_1'\phi_3' - \beta\phi_1\phi_3)\,dx & \int_{x_1}^{x_2} (\alpha\phi_2'\phi_3' - \beta\phi_2\phi_3)\,dx & \int_{x_1}^{x_2} [\alpha(\phi_3')^2 - \beta(\phi_3)^2]\,dx \end{pmatrix},$$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \qquad B = \begin{pmatrix} \{\alpha Y_1'\phi_1\}_{x_1}^{x_2} \\ \{\alpha Y_1'\phi_2\}_{x_1}^{x_2} \\ \{\alpha Y_1'\phi_3\}_{x_1}^{x_2} \end{pmatrix}, \qquad N = \begin{pmatrix} -\int_{x_1}^{x_2} F\phi_1\,dx \\ -\int_{x_1}^{x_2} F\phi_2\,dx \\ -\int_{x_1}^{x_2} F\phi_3\,dx \end{pmatrix}.$$

Basis functions are given in equations 16.16. Show that system equations for a two-element, quadratic approximation

$$Y_2(x) = \begin{cases} Y^{(1)}(x) = c_1^{(1)}\phi_1^{(1)}(x) + c_2^{(1)}\phi_2^{(1)}(x) + c_3^{(1)}\phi_3^{(1)}(x), & x_1 \le x \le x_3 \\ Y^{(2)}(x) = c_1^{(2)}\phi_1^{(2)}(x) + c_2^{(2)}\phi_2^{(2)}(x) + c_3^{(2)}\phi_3^{(2)}(x), & x_3 < x \le x_5, \end{cases}$$

on the interval $x_1 \le x \le x_5$ can be written in the form

$$\begin{pmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} -\alpha(x_1)Y^{(1)'}(x_1) \\ 0 \\ 0 \\ 0 \\ \alpha(x_5)Y^{(2)'}(x_5) \end{pmatrix} + \begin{pmatrix} -\int_{x_1}^{x_3} F\phi_1^{(1)}dx \\ -\int_{x_1}^{x_3} F\phi_2^{(1)}dx \\ -\int_{x_1}^{x_3} F\phi_3^{(1)}dx - \int_{x_3}^{x_5} F\phi_1^{(2)}dx \\ -\int_{x_3}^{x_5} F\phi_2^{(2)}dx \\ -\int_{x_3}^{x_5} F\phi_3^{(2)}dx \end{pmatrix},$$

where a_1, a_2, a_3, a_4, and a_5 are values of $Y_2(x)$ at nodes x_1, x_2, x_3, x_4, and x_5, respectively.

9. Use the template of Exercise 8 to find a two-element, quadratic approximation to the solution of the boundary value problem in Exercise 3. Tabulate the exact solution and the approximation.

§16.4 One-dimensional Isoparametric Elements

In Section 16.2, we employed what might be called a direct method for constructing polynomial basis functions which turned out to be Lagrange interpolation formulas. In this section, we develop what are called **isoparametric elements**; they are indispensable in higher dimensional problems. Essentially what we do is develop polynomial approximations in an abstract coordinate, and then map them to each element in the mesh. To illustrate with quadratic approximations, we begin by setting up quadratic basis functions $\phi_i(\xi)$ on the interval $-1 \le \xi \le 1$, called the **parent element**, that satisfy conditions 16.3. Using equations 16.16, they are

$$\phi_1(\xi) = \frac{\xi(\xi-1)}{2}, \qquad \phi_2(\xi) = (1+\xi)(1-\xi), \qquad \phi_3(\xi) = \frac{\xi(\xi+1)}{2}. \tag{16.23}$$

These basis functions, called the **parent basis functions**, are to be mapped to basis functions on each element in some mesh. This is accomplished through a coordinate transformation between ξ and x on each element. Suppose the i^{th} element has nodes identified by $x_1 < x_2 < x_3$. We could denote these by $x_1^{(i)}$, $x_2^{(i)}$, and $x_3^{(i)}$, but calculations are less cumbersome without the superscripts, and we can insert them later. Keep in mind, however, that what we are doing is being done on each element of the mesh. If the coordinate transformation is expressed in the form $x = f(\xi)$, then for ξ-nodes to be mapped to x-nodes, it must satisfy

$$x_1 = f(-1), \qquad x_2 = f(0), \qquad x_3 = f(1). \tag{16.24}$$

One transformation, among others, that accomplishes this is

$$\begin{aligned} x &= \phi_1(\xi)x_1 + \phi_2(\xi)x_2 + \phi_3(\xi)x_3 \\ &= \frac{\xi(\xi-1)}{2}x_1 + (1+\xi)(1-\xi)x_2 + \frac{\xi(\xi+1)}{2}x_3. \end{aligned} \tag{16.25}$$

It is called the **isoparametric transformation**; the parent element is mapped to every x-element in some mesh in exactly the same way. All that varies from x-element to x-element is specification of the nodes x_1, x_2, and x_3. Still unspecified in this transformation is the position of x_2. We will make the choice here of x_2 being the midpoint of the interval; other possibilities are considered in the exercises. If we substitute $x_2 = (x_1 + x_3)/2$ in equation 16.25, we obtain

$$\begin{aligned} x &= \frac{\xi(\xi-1)}{2}x_1 + (1+\xi)(1-\xi)\left(\frac{x_1+x_3}{2}\right) + \frac{\xi(\xi+1)}{2}x_3 \\ &= \frac{x_1}{2}[\xi(\xi-1) + (1+\xi)(1-\xi)] + \frac{x_3}{2}[\xi(\xi+1) + (1+\xi)(1-\xi)] \\ &= \frac{1}{2}(1-\xi)x_1 + \frac{1}{2}(1+\xi)x_3. \end{aligned}$$

Thus, mapping 16.25, which generally is quadratic in ξ, is linear when node x_2 is chosen as the midpoint of the interval. The inverse mapping is

$$\xi = \frac{2x - x_1 - x_3}{x_3 - x_1}. \tag{16.26}$$

Substitution of this into equations 16.23 yields basis functions on the interval $x_1 \le x \le x_3$ corresponding to the parental basis functions. For instance,

$$\begin{aligned}\phi_1(x) &= \frac{1}{2}\left(\frac{2x-x_1-x_3}{x_3-x_1}\right)\left(\frac{2x-x_1-x_3}{x_3-x_1}-1\right)\\ &= \frac{1}{2}\left(\frac{2x-x_1-x_3}{x_3-x_1}\right)\left(\frac{2x-x_1-x_3-x_3+x_1}{x_3-x_1}\right)\\ &= \left[\frac{2x-2x_2}{2(x_2-x_1)}\right]\left(\frac{x-x_3}{x_3-x_1}\right) = \frac{(x-x_2)(x-x_3)}{(x_2-x_1)(x_3-x_1)}.\end{aligned}$$

This is the Lagrange interpolation formula that we developed directly in Section 16.2. Basis functions $\phi_2(\xi)$ and $\phi_3(\xi)$ in equations 16.23 lead to the other two Lagrange formulas in Section 16.2. This is a direct result of the choice of x_2 as the midpoint of the interval. Basis functions $\phi_i(x)$ have the same shape as the $\phi_i(\xi)$; parabolas have simply been rescaled from the interval $-1 \le \xi \le 1$ to $x_1 \le x \le x_3$. Other choices of x_2 lead to very different basis functions $\phi_i(x)$ (see Exercise 1).

EXERCISES 16.4

1. Draw basis functions $\phi_i(x)$ corresponding to functions 16.23 when $x_2 = x_1 + (x_3 - x_1)/4$.

2. (a) What are the linear parent functions $\phi_1(\xi)$ and $\phi_2(\xi)$ on the interval $-1 \le \xi \le 1$ satisfying equations 16.3?

(b) Use the isoparametric transformation $x = \phi_1(\xi)x_1 + \phi_2(\xi)x_2$ to show that Lagrange interpolation formulas are obtained for the basis functions $\phi_1(x)$ and $\phi_2(x)$.

3. (a) What are the cubic parent functions $\phi_1(\xi)$, $\phi_2(\xi)$, $\phi_3(\xi)$, and $\phi_4(\xi)$ on the interval $-1 \le \xi \le 1$ satisfying equations 16.3?

(b) Use the isoparametric transformation

$$x = \phi_1(\xi)x_1 + \phi_2(\xi)x_2 + \phi_3(\xi)x_3 + \phi_4(\xi)x_4$$

with equally spaced nodes $x_2 = x_1 + (x_4 - x_1)/3$ and $x_3 = x_1 + 2(x_4 - x_1)/3$ to show that Lagrange interpolation formulas are obtained for the basis functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, and $\phi_4(x)$.

§16.5 Finite Elements and Sturm-Liouville Systems

In Section 15.4, we used the MWR to approximate eigenvalues and eigenfunctions of Sturm-Liouville systems. We can do the same with finite elements. The method reduces the Sturm-Liouville eigenvalue problem to a matrix eigenvalue problem. We illustrate with system 15.34,

$$X'' + \lambda(1-x^2)X = 0, \quad 0 < x < 1, \tag{16.27a}$$

$$X(0) = 0, \quad X'(1) = 0. \tag{16.27b}$$

Suppose we seek linear approximations on multiple elements for the eigenfunctions of the system. We adopt the procedure in Section 16.2 of first deriving element equations on an arbitrary interval $x_1 \le x \le x_2$, and then using these element equations to write system equations for multiple elements. The (equation) residual associated with a linear approximation

$$X_1(x) = c_1\phi_1(x) + c_2\phi_2(x) = c_1\left(\frac{x-x_2}{x_1-x_2}\right) + c_2\left(\frac{x-x_1}{x_2-x_1}\right)$$

is

$$R = X_1'' + \lambda(1-x^2)X_1.$$

Galerkin's method requires

$$0 = \int_{x_1}^{x_2} [X_1'' + \lambda(1-x^2)X_1]\phi_1\,dx, \qquad 0 = \int_{x_1}^{x_2} [X_1'' + \lambda(1-x^2)X_1]\phi_2\,dx.$$

Integration by parts on the second derivative term in the first equation gives

$$\begin{aligned} 0 &= \left\{X_1'\phi_1\right\}_{x_1}^{x_2} + \int_{x_1}^{x_2} [-X_1'\phi_1' + \lambda(1-x^2)X_1\phi_1]\,dx \\ &= X_1'(x_2)\phi_1(x_2) - X_1'(x_1)\phi_1(x_1) + \int_{x_1}^{x_2} [-X_1'\phi_1' + \lambda(1-x^2)X_1\phi_1]\,dx. \end{aligned}$$

The first term vanishes because $\phi_1(x_2) = 0$. We now replace X_1 by $c_1\phi_1 + c_2\phi_2$ in the integral,

$$0 = -X_1'(x_1) + \int_{x_1}^{x_2} [-(c_1\phi_1' + c_2\phi_2')\phi_1' + \lambda(1-x^2)(c_1\phi_1 + c_2\phi_2)\phi_1]\,dx.$$

A similar calculation with the second Galerkin requirement yields

$$0 = X_1'(x_2) + \int_{x_1}^{x_2} [-(c_1\phi_1' + c_2\phi_2')\phi_2' + \lambda(1-x^2)(c_1\phi_1 + c_2\phi_2)\phi_2]\,dx.$$

These equations can be expressed in the form

$$KC = \lambda MC + B, \tag{16.28a}$$

where

$$K = \begin{pmatrix} \displaystyle\int_{x_1}^{x_2} (\phi_1')^2\,dx & \displaystyle\int_{x_1}^{x_2} \phi_1'\phi_2'\,dx \\ \displaystyle\int_{x_1}^{x_2} \phi_1'\phi_2'\,dx & \displaystyle\int_{x_1}^{x_2} (\phi_2')^2\,dx \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{16.28b}$$

and

$$M = \begin{pmatrix} \displaystyle\int_{x_1}^{x_2} (1-x^2)\phi_1^2\,dx & \displaystyle\int_{x_1}^{x_2} (1-x^2)\phi_1\phi_2\,dx \\ \displaystyle\int_{x_1}^{x_2} (1-x^2)\phi_1\phi_2\,dx & \displaystyle\int_{x_1}^{x_2} (1-x^2)\phi_2^2\,dx \end{pmatrix}, \quad B = \begin{pmatrix} -X_1'(x_1) \\ X_1'(x_2) \end{pmatrix}. \tag{16.29c}$$

When we set $x_1 = 0$ and $x_2 = 1$, and evaluate integrals, the element equations become

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} 3/10 & 7/60 \\ 7/60 & 7/15 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -X_1'(0) \\ X_1'(1) \end{pmatrix}.$$

To satisfy the second of boundary conditions 16.27b, we implicitly set $X_1'(1) = 0$. To satisfy the first of the boundary conditions, we explicitly demand that $X_1(0) = 0$, and this implies that $c_1 = 0$. The system equations now read

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} 3/10 & 7/60 \\ 7/60 & 7/15 \end{pmatrix}\begin{pmatrix} 0 \\ c_2 \end{pmatrix} + \begin{pmatrix} -X_1'(0) \\ 0 \end{pmatrix}.$$

The second equation requires $c_2 = 7\lambda c_2/15$, from which $\lambda = 15/7$. This is the first approximation to the smallest eigenvalue (which we know to be 5.122). Suppose we now use a two-element, linear approximation

$$X_2(x) = \begin{cases} X^{(1)}(x) = c_1^{(1)}\phi_1^{(1)}(x) + c_2^{(1)}\phi_2^{(1)}(x) = c_1^{(1)}\dfrac{x-1/2}{-1/2} + c_2^{(1)}\dfrac{x-0}{1/2}, & 0 \le x \le 1/2 \\ X^{(2)}(x) = c_1^{(2)}\phi_1^{(2)}(x) + c_2^{(2)}\phi_2^{(2)}(x) = c_1^{(2)}\dfrac{x-1}{-1/2} + c_2^{(2)}\dfrac{x-1/2}{1/2}, & 1/2 < x \le 1 \end{cases}$$

$$= \begin{cases} c_1^{(1)}(1-2x) + c_2^{(1)}(2x), & 0 \le x \le 1/2 \\ c_1^{(2)}(2-2x) + c_2^{(2)}(2x-1), & 1/2 < x \le 1. \end{cases}$$

System equations are

$$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} 13/80 & 37/480 & 0 & 0 \\ 37/480 & 17/120 & 0 & 0 \\ 0 & 0 & 1/10 & 17/480 \\ 0 & 0 & 17/480 & 3/80 \end{pmatrix}\begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} + \begin{pmatrix} -X^{(1)\prime}(0) \\ X^{(1)\prime}(1/2) \\ -X^{(2)\prime}(1/2) \\ X^{(2)\prime}(1) \end{pmatrix}.$$

To satisfy boundary conditions 16.27b we explicitly demand that $c_1^{(1)} = 0$ and implicitly require $X^{(2)\prime}(1) = 0$. Interelement boundary conditions are $c_1^{(2)} = c_2^{(1)}$ and $X^{(2)\prime}(1/2) = X^{(1)\prime}(1/2)$. With these, system equations reduce to

$$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} 13/80 & 37/480 & 0 & 0 \\ 37/480 & 17/120 & 0 & 0 \\ 0 & 0 & 1/10 & 17/480 \\ 0 & 0 & 17/480 & 3/80 \end{pmatrix}\begin{pmatrix} 0 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} + \begin{pmatrix} -X^{(1)\prime}(0) \\ X^{(1)\prime}(1/2) \\ -X^{(1)\prime}(1/2) \\ 0 \end{pmatrix}.$$

We now add the third equation to the second,

$$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 2 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} 13/80 & 37/480 & 0 & 0 \\ 37/480 & 17/120 & 1/10 & 17/480 \\ 0 & 0 & 1/10 & 17/480 \\ 0 & 0 & 17/480 & 3/80 \end{pmatrix} \begin{pmatrix} 0 \\ c_1^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} + \begin{pmatrix} -X^{(1)\prime}(0) \\ 0 \\ -X^{(1)\prime}(1/2) \\ 0 \end{pmatrix}.$$

The second and fourth equations give

$$\begin{aligned} 4c_1^{(2)} - 2c_2^{(2)} &= \lambda \left(\frac{29}{120} c_1^{(2)} + \frac{17}{480} c_2^{(2)} \right), \\ -2c_1^{(2)} + 2c_2^{(2)} &= \lambda \left(\frac{17}{480} c_1^{(2)} + \frac{3}{80} c_2^{(2)} \right), \end{aligned}$$

which we could write matrically as

$$\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1^{(2)} \\ c_2^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} 29/120 & 17/480 \\ 17/480 & 3/80 \end{pmatrix} \begin{pmatrix} c_1^{(2)} \\ c_2^{(2)} \end{pmatrix}.$$

This is a matrix eigenvalue problem. Eigenvalues are given by

$$0 = \begin{vmatrix} 4 - 29\lambda/120 & -2 - 17\lambda 480 \\ -2 - 17\lambda/480 & 2 - 3\lambda/80 \end{vmatrix} \qquad \Longrightarrow \lambda = 5.462,\ 93.79.$$

The smaller value is a second approximation to the smallest eigenvalue of Sturm-Liouville system 16.27; the larger value is a first approximation to the second eigenvalue of the Sturm-Liouville system. (See Exercise 1 for a four-element, linear approximation, and Exercise 2 for a one-element, quadratic approximation.)

EXERCISES 16.5

1. Find a third approximation for the smallest eigenvalue of Sturm-Liouville system 16.27 using a four-element, linear approximation.

2. Find an approximation for the smallest eigenvalue of Sturm-Liouville system 16.27 using a one-element, quadratic approximation.

3. The smallest eigenvalue of the Sturm-Liouville system

$$\begin{aligned} X'' + \lambda X &= 0, \quad 0 < x < 1, \\ X'(0) = X(1) &= 0, \end{aligned}$$

is known to be $\pi^2/4$ (see Table 5.1 in Section 5.2).
(a) What is the estimate of this value using a one-element linear approximation?
(b) What is the estimate using a two-element linear approximation?
(c) What is the estimate using a one-element quadratic approximation?

4. Repeat Exercise 3 for the Sturm-Liouville system

$$X'' + \lambda X = 0, \quad 0 < x < 1,$$
$$-X'(0) + 2000X(0) = 0, \quad X'(1) = 0.$$

To two decimals, the smallest eigenvalue is 2.46 (see Exercise 21 in Section 5.2).

5. Repeat Exercise 3 for the Sturm-Liouville system

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r R = 0, \quad 0 < r < 1,$$
$$R(1) = 0.$$

To two decimals, the smallest eigenvalue is 5.78 (see Example 8.3 in Section 8.4).

§16.6 Approximations With Continuous First Derivatives

Basis functions in Sections 16.2 and 16.4 were infinitely differentiable polynomials, resulting in infinitely differentiable approximations on each element. When elements were assembled, we explicitly demanded continuity of the approximation at all interior nodes. As a result, the global approximation was a continuous function. Such approximations are said to be C^0, meaning continuous. At interior nodes, we implicitly demanded equality of the first derivative from adjoining elements, and as a result, the global approximation did not have a continuous first derivative at interior nodes. In this section, we discuss how this can be achieved; that is, how to find global approximations that have continuous first derivatives everywhere, including at interior nodes. These approximations will be said to be C^1, having continuous first derivatives. We take the direct approach in developing appropriate basis functions, but an isoparametric approach is also possible.

Suppose the i^{th} element of some mesh has nodes identified by its end points x_1 and x_2. As noted in Section 16.4, we could denote these by $x_1^{(i)}$ and $x_2^{(i)}$, but calculations are less cumbersome without the superscripts, and we can insert them later. Keep in mind, however, that what we are doing is being done on each element of the mesh. Linear basis functions $\phi_1(x)$ and $\phi_2(x)$ in Section 16.2 were such that an approximation on the interval $x_1 < x < x_2$ could be expressed in the form

$$c_1\phi_1(x) + c_2\phi_2(x),$$

where c_1 and c_2 were values of the approximation at x_1 and x_2, respectively. Basis functions satisfied property 16.3. We now want basis functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, and $\phi_4(x)$ so that approximations can be expressed in the form

$$c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + c_4\phi_4(x),$$

where c_1 and c_3 represent values of the approximation at nodes x_1 and x_2, and c_2 and c_4 represent values of the derivative of the approximation at these nodes. This requires the basis functions to satisfy the following sixteen conditions

$$\phi_1(x_1) = 1, \quad \phi_2(x_1) = 0, \quad \phi_3(x_1) = 0, \quad \phi_4(x_1) = 0, \tag{16.30a}$$

$$\phi_1(x_2) = 0, \quad \phi_2(x_2) = 0, \quad \phi_3(x_2) = 1, \quad \phi_4(x_2) = 0, \tag{16.30b}$$

$$\phi_1'(x_1) = 0, \quad \phi_2'(x_1) = 1, \quad \phi_3(x_1) = 0, \quad \phi_4'(x_1) = 0, \tag{16.30c}$$

$$\phi_1'(x_2) = 0, \quad \phi_2'(x_2) = 0, \quad \phi_3'(x_2) = 0, \quad \phi_4'(x_2) = 1. \tag{16.30d}$$

Since each function must satisfy four conditions, they must be cubic polynomials. For instance, if $\phi_1(x) = a_1 + a_2x + a_3x^2 + a_4x^3$, then coefficients must satisfy

$$1 = a_1 + a_2x_1 + a_3x_1^2 + a_4x_1^3, \quad 0 = a_1 + a_2x_2 + a_3x_2^2 + a_4x_2^3,$$

$$0 = a_2 + 2a_3x_1 + 3a_4x_1^2, \qquad 0 = a_2 + 2a_3x_2 + 3a_4x_2^2.$$

The solution is $a_1 = \dfrac{x_2^2(x_2 - 3x_1)}{(x_2 - x_1)^3}$, $a_2 = \dfrac{6x_1x_2}{(x_2 - x_1)^3}$, $a_3 = \dfrac{-3(x_1 + x_2)}{(x_2 - x_1)^3}$, and $a_4 = \dfrac{2}{(x_2 - x_1)^3}$, so that

$$\phi_1(x) = \frac{1}{(x_2 - x_1)^3}[x_2^2(x_2 - 3x_1) + 6x_1x_2x - 3(x_1 + x_2)x^2 + 2x^3].$$

This can be rewritten in the form

$$\phi_1(x) = 1 - 3\left(\frac{x - x_1}{x_2 - x_1}\right)^2 + 2\left(\frac{x - x_1}{x_2 - x_1}\right)^3. \tag{16.31a}$$

Similar calculations lead to

$$\phi_2(x) = (x - x_1)\left(1 - \frac{x - x_1}{x_2 - x_1}\right)^2, \tag{16.31b}$$

$$\phi_3(x) = \left(\frac{x - x_1}{x_2 - x_1}\right)^2\left[3 - \frac{2(x - x_1)}{x_2 - x_1}\right], \tag{16.31c}$$

$$\phi_4(x) = -\frac{(x - x_1)^2}{x_2 - x_1}\left(1 - \frac{x - x_1}{x_2 - x_1}\right). \tag{16.31d}$$

By introducing the quantity $s = \dfrac{x - x_1}{x_2 - x_1}$, the basis functions can also be expressed in the form

$$\phi_1(x) = 1 - 3s^2 + 2s^3, \qquad \phi_2(x) = (x_2 - x_1)s(1 - s)^2,$$
$$\phi_3(x) = s^2(3 - 2s), \qquad \phi_4(x) = (x_2 - x_1)s^2(s - 1).$$

They are called **Hermite interpolation polynomials**. Notice how their graphs in Figures 16.8 adhere to conditions 16.30.

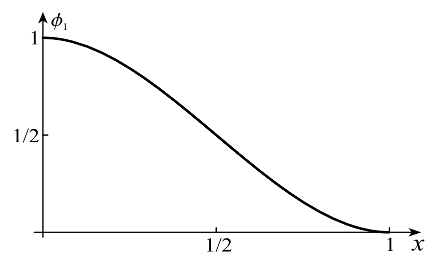


Figure 16.8a

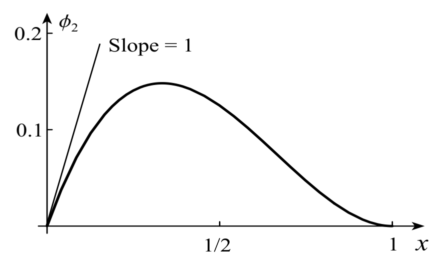


Figure 16.8b

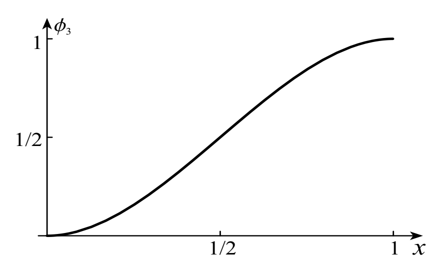


Figure 16.8c

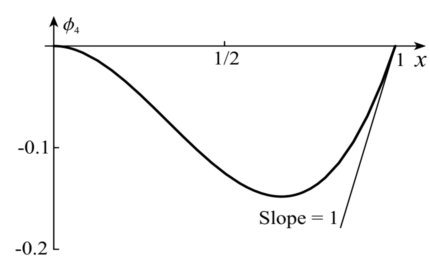


Figure 16.8d

Let us use these basis functions to find a two-element, C^1 approximation to the solution of problem 16.4,

$$\frac{d^2Y}{dx^2} + Y = x, \quad 0 < x < 1, \tag{16.32a}$$

$$Y(0) = 2, \tag{16.32b}$$

$$Y'(1) = 1, \tag{16.32c}$$

and compare it to the two-element, linear, C^0 approximation in Section 16.2. In preparation for this, we develop element equations for a one-element, C^1 approximation

$$Y_1(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + c_4\phi_4(x) \tag{16.33}$$

on an arbitrary interval $x_1 \le x \le x_2$. Galerkin's method applied to the residual

$$R = Y_1''(x) + Y_1(x) - x$$

requires

$$0 = \int_{x_1}^{x_2} [Y_1''(x) + Y_1(x) - x]\phi_i(x)\,dx, \quad i = 1, 2, 3, 4.$$

Integration by parts gives

$$0 = \left\{Y_1'\phi_i\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} Y_1'\phi_i'\,dx + \int_{x_1}^{x_2} (Y_1 - x)\phi_i\,dx.$$

If we now substitute $Y_1 = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$ in the integrals,

$$0 = \left\{Y_1'\phi_i\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} (c_1\phi_1' + c_2\phi_2' + c_3\phi_3' + c_4\phi_4')\phi_i'\,dx + \int_{x_1}^{x_2} (c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4 - x)\phi_i\,dx.$$

This can be written in the form

$$c_1\int_{x_1}^{x_2} (\phi_1'\phi_i' - \phi_1\phi_i)\,dx + c_2\int_{x_1}^{x_2} (\phi_2'\phi_i' - \phi_2\phi_i)\,dx + c_3\int_{x_1}^{x_2} (\phi_3'\phi_i' - \phi_3\phi_i)\,dx + c_4\int_{x_1}^{x_2} (\phi_4'\phi_i' - \phi_4\phi_i)\,dx = \left\{Y_1'\phi_i\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} x\phi_i\,dx.$$

When all four equations ($i = 1, 2, 3, 4$) are assembled into one matrix equation, the result is

$$KC = B + N, \tag{16.34a}$$

where K is the matrix

$$\begin{bmatrix}
\int_{x_1}^{x_2} [(\phi_1')^2 - \phi_1^2]\,dx & \int_{x_1}^{x_2} (\phi_1'\phi_2' - \phi_1\phi_2)\,dx & \int_{x_1}^{x_2} (\phi_1'\phi_3' - \phi_1\phi_3)\,dx & \int_{x_1}^{x_2} (\phi_1'\phi_4' - \phi_1\phi_4)\,dx \\
\int_{x_1}^{x_2} (\phi_2'\phi_1' - \phi_2\phi_1)\,dx & \int_{x_1}^{x_2} [(\phi_2')^2 - \phi_2^2]\,dx & \int_{x_1}^{x_2} (\phi_3'\phi_2' - \phi_3\phi_2)\,dx & \int_{x_1}^{x_2} (\phi_4'\phi_2' - \phi_4\phi_2)\,dx \\
\int_{x_1}^{x_2} (\phi_3'\phi_1' - \phi_3\phi_1)\,dx & \int_{x_1}^{x_2} (\phi_3'\phi_2' - \phi_3\phi_2)\,dx & \int_{x_1}^{x_2} [(\phi_3')^2 - \phi_3^2]\,dx & \int_{x_1}^{x_2} (\phi_3'\phi_4' - \phi_3\phi_4)\,dx \\
\int_{x_1}^{x_2} (\phi_4'\phi_1' - \phi_4\phi_1)\,dx & \int_{x_1}^{x_2} (\phi_4'\phi_2' - \phi_4\phi_2)\,dx & \int_{x_1}^{x_2} (\phi_4'\phi_3' - \phi_4\phi_3)\,dx & \int_{x_1}^{x_2} [(\phi_4')^2 - \phi_4^2]\,dx
\end{bmatrix}, \tag{16.34b}$$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}, \qquad B = \begin{bmatrix} \{Y_1'\phi_1\}_{x_1}^{x_2} \\ \{Y_1'\phi_2\}_{x_1}^{x_2} \\ \{Y_1'\phi_3\}_{x_1}^{x_2} \\ \{Y_1'\phi_4\}_{x_1}^{x_2} \end{bmatrix}, \qquad N = \begin{bmatrix} -\int_{x_1}^{x_2} x\phi_1\,dx \\ -\int_{x_1}^{x_2} x\phi_2\,dx \\ -\int_{x_1}^{x_2} x\phi_3\,dx \\ -\int_{x_1}^{x_2} x\phi_4\,dx \end{bmatrix}. \qquad (16.34c)$$

These are the element equations for four-term C^1 approximation 16.33 to the solution of problem 16.32 on an arbitrary interval $x_1 \le x \le x_2$. There will be one set for each of the elements in a two-element C^1-approximation

$$Y_2(x) = \begin{cases} Y^{(1)}(x) = c_1^{(1)}\phi_1^{(1)}(x) + c_2^{(1)}\phi_2^{(1)}(x) + c_3^{(1)}\phi_3^{(1)}(x) + c_4^{(1)}\phi_4^{(1)}(x), & 0 \le x \le 1/2 \\ Y^{(2)}(x) = c_1^{(2)}\phi_1^{(2)}(x) + c_2^{(2)}\phi_2^{(2)}(x) + c_3^{(2)}\phi_3^{(2)}(x) + c_4^{(2)}\phi_4^{(2)}(x), & 1/2 < x \le 1. \end{cases}$$

When integrals are evaluated, the resulting system equations are

$$\begin{bmatrix} 31/14 & 73/840 & -69/28 & 181/1680 & 0 & 0 & 0 & 0 \\ 73/840 & 11/168 & -181/1680 & -53/3360 & 0 & 0 & 0 & 0 \\ -69/28 & -181/1680 & 31/14 & -73/840 & 0 & 0 & 0 & 0 \\ 181/1680 & -53/3360 & -73/840 & 11/168 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 31/14 & 73/840 & -69/28 & 181/1680 \\ 0 & 0 & 0 & 0 & 73/840 & 11/168 & -181/1680 & -53/3360 \\ 0 & 0 & 0 & 0 & -69/28 & -181/1680 & 31/14 & -73/840 \\ 0 & 0 & 0 & 0 & 181/1680 & -53/3360 & -73/840 & 11/168 \end{bmatrix} *$$

$$\begin{bmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_3^{(1)} \\ c_4^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_3^{(2)} \\ c_4^{(2)} \end{bmatrix} = \begin{bmatrix} \left\{Y^{(1)'}\phi_1^{(1)}\right\}_0^{1/2} \\ \left\{Y^{(1)'}\phi_2^{(1)}\right\}_0^{1/2} \\ \left\{Y^{(1)'}\phi_3^{(1)}\right\}_0^{1/2} \\ \left\{Y^{(1)'}\phi_4^{(1)}\right\}_0^{1/2} \\ \left\{Y^{(2)'}\phi_1^{(2)}\right\}_{1/2}^{1} \\ \left\{Y^{(2)'}\phi_2^{(2)}\right\}_{1/2}^{1} \\ \left\{Y^{(2)'}\phi_3^{(2)}\right\}_{1/2}^{1} \\ \left\{Y^{(2)'}\phi_4^{(2)}\right\}_{1/2}^{1} \end{bmatrix} + \begin{bmatrix} -3/80 \\ -1/240 \\ -7/80 \\ 1/160 \\ -13/80 \\ -7/480 \\ -17/80 \\ 1/60 \end{bmatrix}.$$

Interelement boundary conditions are $c_3^{(1)} = c_1^{(2)}$, $c_4^{(1)} = c_2^{(2)}$, and $Y^{(1)'}(1/2) = Y^{(2)'}(1/2)$. With these, the system equations reduce to

$$\begin{bmatrix} 31/14 & 73/840 & -69/28 & 181/1680 & 0 & 0 & 0 & 0 \\ 73/840 & 11/168 & -181/1680 & -53/3360 & 0 & 0 & 0 & 0 \\ -69/28 & -181/1680 & 31/14 & -73/840 & 0 & 0 & 0 & 0 \\ 181/1680 & -53/3360 & -73/840 & 11/168 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 31/14 & 73/840 & -69/28 & 181/1680 \\ 0 & 0 & 0 & 0 & 73/840 & 11/168 & -181/1680 & -53/3360 \\ 0 & 0 & 0 & 0 & -69/28 & -181/1680 & 31/14 & -73/840 \\ 0 & 0 & 0 & 0 & 181/1680 & -53/3360 & -73/840 & 11/168 \end{bmatrix} *$$

$$\begin{bmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_3^{(2)} \\ c_4^{(2)} \end{bmatrix} = \begin{bmatrix} -Y^{(1)\prime}(0) \\ 0 \\ Y^{(1)\prime}(1/2) \\ 0 \\ -Y^{(1)\prime}(1/2) \\ 0 \\ Y^{(2)\prime}(1) \\ 0 \end{bmatrix} + \begin{bmatrix} -3/80 \\ -1/240 \\ -7/80 \\ 1/160 \\ -13/80 \\ -7/480 \\ -17/80 \\ 1/60 \end{bmatrix}.$$

There are now six unknowns, values of the approximation and its first derivative at the three nodes x_1, x_2, and x_3. If we denote these by a_1 and a_2, a_3 and a_4, and a_5 and a_6, respectively, the equations are

$$\begin{bmatrix} 31/14 & 73/840 & -69/28 & 181/1680 & 0 & 0 & 0 & 0 \\ 73/840 & 11/168 & -181/1680 & -53/3360 & 0 & 0 & 0 & 0 \\ -69/28 & -181/1680 & 31/14 & -73/840 & 0 & 0 & 0 & 0 \\ 181/1680 & -53/3360 & -73/840 & 11/168 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 31/14 & 73/840 & -69/28 & 181/1680 \\ 0 & 0 & 0 & 0 & 73/840 & 11/168 & -181/1680 & -53/3360 \\ 0 & 0 & 0 & 0 & -69/28 & -181/1680 & 31/14 & -73/840 \\ 0 & 0 & 0 & 0 & 181/1680 & -53/3360 & -73/840 & 11/168 \end{bmatrix} *$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -Y^{(1)\prime}(0) \\ 0 \\ Y^{(1)\prime}(1/2) \\ 0 \\ -Y^{(1)\prime}(1/2) \\ 0 \\ Y^{(2)\prime}(1) \\ 0 \end{bmatrix} + \begin{bmatrix} -3/80 \\ -1/240 \\ -7/80 \\ 1/160 \\ -13/80 \\ -7/480 \\ -17/80 \\ 1/60 \end{bmatrix}.$$

We now add the fifth equation to the third, and the sixth equation to the fourth,

$$\begin{bmatrix} 31/14 & 73/840 & -69/28 & 181/1680 & 0 & 0 & 0 & 0 \\ 73/840 & 11/168 & -181/1680 & -53/3360 & 0 & 0 & 0 & 0 \\ -69/28 & -181/1680 & 31/14 & -73/840 & 31/14 & 73/840 & -69/28 & 181/1680 \\ 181/1680 & -53/3360 & -73/840 & 11/168 & 73/840 & 11/168 & -181/1680 & -53/3360 \\ 0 & 0 & 0 & 0 & 31/14 & 73/840 & -69/28 & 181/1680 \\ 0 & 0 & 0 & 0 & 73/840 & 11/168 & -181/1680 & -53/3360 \\ 0 & 0 & 0 & 0 & -69/28 & -181/1680 & 31/14 & -73/840 \\ 0 & 0 & 0 & 0 & 181/1680 & -53/3360 & -73/840 & 11/168 \end{bmatrix} *$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -Y^{(1)'}(0) \\ 0 \\ 0 \\ 0 \\ -Y^{(1)'}(1/2) \\ 0 \\ Y^{(2)'}(1) \\ 0 \end{bmatrix} + \begin{bmatrix} -3/80 \\ -1/240 \\ -1/4 \\ -1/120 \\ -13/80 \\ -7/480 \\ -17/80 \\ 1/60 \end{bmatrix}.$$

In preparation for elimination of the fifth and sixth equations, we add column five of the stiffness matrix to column three, and column six to column four,

$$\begin{bmatrix} 31/14 & 73/840 & -69/28 & 181/1680 & 0 & 0 & 0 & 0 \\ 73/840 & 11/168 & -181/1680 & -53/3360 & 0 & 0 & 0 & 0 \\ -69/28 & -181/1680 & 31/7 & 0 & 31/14 & 73/840 & -69/28 & 181/1680 \\ 181/1680 & -53/3360 & 0 & 11/84 & 73/840 & 11/168 & -181/1680 & -53/3360 \\ 0 & 0 & 31/14 & 73/840 & 31/14 & 73/840 & -69/28 & 181/1680 \\ 0 & 0 & 73/840 & 11/168 & 73/840 & 11/168 & -181/1680 & -53/3360 \\ 0 & 0 & -69/28 & -181/1680 & -69/28 & -181/1680 & 31/14 & -73/840 \\ 0 & 0 & 181/1680 & -53/3360 & 181/1680 & -53/3360 & -73/840 & 11/168 \end{bmatrix} *$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -Y^{(1)'}(0) \\ 0 \\ 0 \\ 0 \\ -Y^{(1)'}(1/2) \\ 0 \\ Y^{(2)'}(1) \\ 0 \end{bmatrix} + \begin{bmatrix} -3/80 \\ -1/240 \\ -1/4 \\ -1/120 \\ -13/80 \\ -7/480 \\ -17/80 \\ 1/60 \end{bmatrix}.$$

Now eliminate rows five and six of the stiffness matrix, and columns five and six, and fifth and sixth entries of the remaining matrices,

$$\begin{bmatrix} 31/14 & 73/840 & -69/28 & 181/1680 & 0 & 0 \\ 73/840 & 11/168 & -181/1680 & -53/3360 & 0 & 0 \\ -69/28 & -181/1680 & 31/7 & 0 & -69/28 & 181/1680 \\ 181/1680 & -53/3360 & 0 & 11/84 & -181/1680 & -53/3360 \\ 0 & 0 & -69/28 & -181/1680 & 31/14 & -73/840 \\ 0 & 0 & 181/1680 & -53/3360 & -73/840 & 11/168 \end{bmatrix} *$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -Y^{(1)'}(0) \\ 0 \\ 0 \\ 0 \\ Y^{(2)'}(1) \\ 0 \end{bmatrix} + \begin{bmatrix} -3/80 \\ -1/240 \\ -1/4 \\ -1/120 \\ -17/80 \\ 1/60 \end{bmatrix}.$$

The boundary conditions require $a_1 = 2$ and $Y^{(2)'}(1) = 1$ so that

$$\begin{bmatrix} 31/14 & 73/840 & -69/28 & 181/1680 & 0 & 0 \\ 73/840 & 11/168 & -181/1680 & -53/3360 & 0 & 0 \\ -69/28 & -181/1680 & 31/7 & 0 & -69/28 & 181/1680 \\ 181/1680 & -53/3360 & 0 & 11/84 & -181/1680 & -53/3360 \\ 0 & 0 & -69/28 & -181/1680 & 31/14 & -73/840 \\ 0 & 0 & 181/1680 & -53/3360 & -73/840 & 11/168 \end{bmatrix} *$$

$$\begin{bmatrix} 2 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -Y^{(1)'}(0) \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3/80 \\ -1/240 \\ -1/4 \\ -1/120 \\ -17/80 \\ 1/60 \end{bmatrix}.$$

The last five equations are

$$\begin{aligned}
\frac{73}{420} + \frac{11}{168}a_2 - \frac{181}{1680}a_3 - \frac{53}{3360}a_4 &= -\frac{1}{240}, \\
-\frac{69}{14} - \frac{181}{1680}a_2 + \frac{31}{7}a_3 - \frac{69}{28}a_5 + \frac{181}{1680}a_6 &= \frac{1}{4}, \\
\frac{181}{840} - \frac{53}{3360}a_2 + \frac{11}{84}a_4 - \frac{181}{1680}a_5 - \frac{53}{3360}a_6 &= -\frac{1}{120}, \\
-\frac{69}{28}a_3 - \frac{181}{1680}a_4 + \frac{31}{14}a_5 - \frac{73}{840}a_6 &= \frac{63}{80}, \\
\frac{181}{1680}a_3 - \frac{53}{3360}a_4 - \frac{73}{840}a_5 + \frac{11}{168}a_6 &= \frac{1}{60}.
\end{aligned}$$

The solution is $a_2 = 4.11862$, $a_3 = 3.74867$, $a_4 = 2.77505$, $a_5 = 4.70163$, and $a_6 = 0.995161$. The two-element, C^1 approximation is

$$Y_2(x) = \begin{cases} 2\phi_1^{(1)}(x) + 4.11862\phi_2^{(1)}(x) + 3.74867\phi_3^{(1)}(x) + 2.77505\phi_4^{(1)}(x), & 0 \le x \le 1/2 \\ 3.74867\phi_1^{(2)}(x) + 2.77505\phi_2^{(2)}(x) + 4.70163\phi_3^{(2)}(x) + 0.995161\phi_4^{(2)}(x), & 1/2 < x \le 1. \end{cases}$$

Its graph in Figure 16.9 shows continuity of its first derivative at $x = 1/2$. The graph is indistinguishable from that of the exact solution $Y(x) = 2\cos x + 2\tan 1\, \sin x + x$. We have also tabulated $Y(x)$ and $Y_2(x)$ below for comparison.

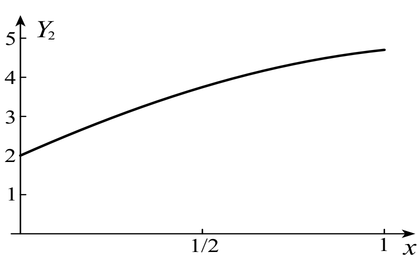


Figure 16.9

x	$Y(x)$	$Y_2(x)$
0.0	2.00000	2.00000
0.1	2.40097	2.40105
0.2	2.77895	2.77887
0.3	3.13116	3.13103
0.4	3.45509	3.45510
0.5	3.74849	3.74867
0.6	4.00943	4.00946
0.7	4.23630	4.23615
0.8	4.42785	4.42774
0.9	4.58314	4.58323
1.0	4.70163	4.70163

Table 16.2

The work in evaluation of integrals in K and N was formidable. Often the same accuracy is achieved with less work using more elements with C^0 continuity. C^1 approximations become more attractive when applied to fourth-order differential equations that are encountered in the bending of beams, plates, and shells. We illustrate in the next example.

Example 16.2 Find the exact solution of the boundary value problem

$$\frac{d^4Y}{dx^4} = -5, \quad 0 < x < 1,$$
$$Y(0) = Y'(0) = 0,$$
$$Y(1) = Y'(1) = 0.$$

(A boundary value problem of this form would arise for static deflections of a beam that is built in horizontally at each end.) Find a two-element C^1-approximation and compare it to the exact solution.

Solution It is straightforward to integrate the differential equation four times and apply the boundary conditions to obtain the exact solution

$$Y(x) = -\frac{5}{24}x^2(x-1)^2.$$

In preparation for system equations for a two-element C^1-approximation, we develop element equations for a one-element, C^1-approximation

$$Y_1(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + c_4\phi_4(x) \tag{16.35}$$

on an arbitrary interval $x_1 \le x \le x_2$. Basis functions are defined in equations 16.31. Galerkin's method applied to the residual

$$R = Y_1''''(x) + 5$$

requires

$$0 = \int_{x_1}^{x_2} [Y_1''(x) + 5]\phi_i(x)\,dx, \quad i = 1, 2, 3, 4.$$

Two integration by parts give

$$\begin{aligned} 0 &= \left\{Y_1'''\phi_i\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} (Y_1'''\phi_i' - 5\phi_i)\,dx \\ &= \left\{Y_1'''\phi_i - Y_1''\phi_i'\right\}_{x_1}^{x_2} + \int_{x_1}^{x_2} (Y_1''\phi_i'' + 5\phi_i)\,dx. \end{aligned}$$

If we now substitute $Y_1 = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$ in the integral,

$$0 = \left\{Y_1'''\phi_i - Y_1''\phi_i'\right\}_{x_1}^{x_2} + \int_{x_1}^{x_2} (c_1\phi_1'' + c_2\phi_2'' + c_3\phi_3'' + c_4\phi_4'')\phi_i''\,dx + \int_{x_1}^{x_2} 5\phi_i\,dx.$$

This can be written in the form

$$\begin{aligned} c_1\int_{x_1}^{x_2} \phi_1''\phi_i''\,dx + c_2\int_{x_1}^{x_2} \phi_2''\phi_i''\,dx + c_3\int_{x_1}^{x_2} \phi_3''\phi_i''\,dx + c_4\int_{x_1}^{x_2} \phi_4''\phi_i''\,dx \\ = -\left\{Y_1'''\phi_i - Y_1''\phi_i'\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} 5\phi_i\,dx. \end{aligned}$$

When all four equations ($i = 1, 2, 3, 4$) are assembled into one matrix equation, the element equations are

$$\left(\int_{x_1}^{x_2}\phi_i''\phi_j''\,dx\right)\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}\{Y_1''\phi_1'-Y_1'''\phi_1\}_{x_1}^{x_2}\\\{Y_1''\phi_2'-Y_1'''\phi_2\}_{x_1}^{x_2}\\\{Y_1''\phi_3'-Y_1'''\phi_3\}_{x_1}^{x_2}\\\{Y_1''\phi_4'-Y_1'''\phi_4\}_{x_1}^{x_2}\end{pmatrix}+\begin{pmatrix}-\int_{x_1}^{x_2}5\phi_1\,dx\\-\int_{x_1}^{x_2}5\phi_2\,dx\\-\int_{x_1}^{x_2}5\phi_3\,dx\\-\int_{x_1}^{x_2}5\phi_4\,dx\end{pmatrix}.$$

For a two-element C^1-approximation,

$$Y_2(x)=\begin{cases}Y^{(1)}(x)=c_1^{(1)}\phi_1^{(1)}(x)+c_2^{(1)}\phi_2^{(1)}(x)+c_3^{(1)}\phi_3^{(1)}(x)+c_4^{(1)}\phi_4^{(1)}(x), & 0\le x\le 1/2\\Y^{(2)}(x)=c_1^{(2)}\phi_1^{(2)}(x)+c_2^{(2)}\phi_2^{(2)}(x)+c_3^{(2)}\phi_3^{(2)}(x)+c_4^{(2)}\phi_4^{(2)}(x), & 1/2< x\le 1.\end{cases}$$

there will be one set for each of the elements. When they are assembled and integrals are evaluated, system equations are

$$\begin{bmatrix}96&24&-96&24&0&0&0&0\\24&8&-24&4&0&0&0&0\\-96&-24&96&-24&0&0&0&0\\24&4&-24&8&0&0&0&0\\0&0&0&0&96&24&-96&24\\0&0&0&0&24&8&-24&4\\0&0&0&0&-96&-24&96&-24\\0&0&0&0&24&4&-24&8\end{bmatrix}\begin{bmatrix}c_1^{(1)}\\c_2^{(1)}\\c_3^{(1)}\\c_4^{(1)}\\c_1^{(2)}\\c_2^{(2)}\\c_3^{(2)}\\c_4^{(2)}\end{bmatrix}$$

$$=\begin{bmatrix}\{Y^{(1)''}\phi_1^{(1)'}-Y^{(1)'''}\phi_1^{(1)}\}_0^{1/2}\\\{Y^{(1)''}\phi_2^{(1)'}-Y^{(1)'''}\phi_2^{(1)}\}_0^{1/2}\\\{Y^{(1)''}\phi_3^{(1)'}-Y^{(1)'''}\phi_3^{(1)}\}_0^{1/2}\\\{Y^{(1)''}\phi_4^{(1)'}-Y^{(1)'''}\phi_4^{(1)}\}_0^{1/2}\\\{Y^{(2)''}\phi_1^{(2)'}-Y^{(2)'''}\phi_1^{(2)}\}_{1/2}^{1}\\\{Y^{(2)''}\phi_2^{(2)'}-Y^{(2)'''}\phi_2^{(2)}\}_{1/2}^{1}\\\{Y^{(2)''}\phi_3^{(2)'}-Y^{(2)'''}\phi_3^{(2)}\}_{1/2}^{1}\\\{Y^{(2)''}\phi_4^{(2)'}-Y^{(2)'''}\phi_4^{(2)}\}_{1/2}^{1}\end{bmatrix}+\begin{bmatrix}-5/4\\-5/48\\-5/4\\5/48\\-5/4\\-5/48\\-5/4\\5/48\end{bmatrix}.$$

Interelement boundary conditions certainly include $c_3^{(1)}=c_1^{(2)}$, $c_4^{(1)}=c_2^{(2)}$, to ensure continuity. We have built-in continuity of the first derivative at $x=1/2$ (and notice that the first derivative at $x=1/2$ does not appear in the system equations). We implicitly demand continuity of the second and third derivatives at $x=1/2$. This implies that $Y^{(1)''}(1/2)=Y^{(2)''}(1/2)$ and $Y^{(1)'''}(1/2)=Y^{(2)'''}(1/2)$. With these, the system equations reduce to

$$\begin{bmatrix} 96 & 24 & -96 & 24 & 0 & 0 & 0 & 0 \\ 24 & 8 & -24 & 4 & 0 & 0 & 0 & 0 \\ -96 & -24 & 96 & -24 & 0 & 0 & 0 & 0 \\ 24 & 4 & -24 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 96 & 24 & -96 & 24 \\ 0 & 0 & 0 & 0 & 24 & 8 & -24 & 4 \\ 0 & 0 & 0 & 0 & -96 & -24 & 96 & -24 \\ 0 & 0 & 0 & 0 & 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_3^{(2)} \\ c_4^{(2)} \end{bmatrix} = \begin{bmatrix} {Y^{(1)}}'''(0) \\ -{Y^{(1)}}''(0) \\ -{Y^{(1)}}'''(1/2) \\ {Y^{(1)}}''(1/2) \\ {Y^{(1)}}'''(1/2) \\ -{Y^{(1)}}''(1/2) \\ -{Y^{(2)}}'''(1) \\ {Y^{(2)}}''(1) \end{bmatrix} + \begin{bmatrix} -5/4 \\ -5/48 \\ -5/4 \\ 5/48 \\ -5/4 \\ -5/48 \\ -5/4 \\ 5/48 \end{bmatrix}.$$

There are now six unknowns, values of the approximation and its first derivative at the three nodes x_1, x_2, and x_3. If we denote these by a_1 and a_2, a_3 and a_4, and a_5 and a_6, respectively, the equations are

$$\begin{bmatrix} 96 & 24 & -96 & 24 & 0 & 0 & 0 & 0 \\ 24 & 8 & -24 & 4 & 0 & 0 & 0 & 0 \\ -96 & -24 & 96 & -24 & 0 & 0 & 0 & 0 \\ 24 & 4 & -24 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 96 & 24 & -96 & 24 \\ 0 & 0 & 0 & 0 & 24 & 8 & -24 & 4 \\ 0 & 0 & 0 & 0 & -96 & -24 & 96 & -24 \\ 0 & 0 & 0 & 0 & 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} {Y^{(1)}}'''(0) \\ -{Y^{(1)}}''(0) \\ -{Y^{(1)}}'''(1/2) \\ {Y^{(1)}}''(1/2) \\ {Y^{(1)}}'''(1/2) \\ -{Y^{(1)}}''(1/2) \\ -{Y^{(2)}}'''(1) \\ {Y^{(2)}}''(1) \end{bmatrix} + \begin{bmatrix} -5/4 \\ -5/48 \\ -5/4 \\ 5/48 \\ -5/4 \\ -5/48 \\ -5/4 \\ 5/48 \end{bmatrix}.$$

We now add the fifth equation to the third, and the sixth equation to the fourth,

$$\begin{bmatrix} 96 & 24 & -96 & 24 & 0 & 0 & 0 & 0 \\ 24 & 8 & -24 & 4 & 0 & 0 & 0 & 0 \\ -96 & -24 & 96 & -24 & 96 & 24 & -96 & 24 \\ 24 & 4 & -24 & 8 & 24 & 8 & -24 & 4 \\ 0 & 0 & 0 & 0 & 96 & 24 & -96 & 24 \\ 0 & 0 & 0 & 0 & 24 & 8 & -24 & 4 \\ 0 & 0 & 0 & 0 & -96 & -24 & 96 & -24 \\ 0 & 0 & 0 & 0 & 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} {Y^{(1)}}'''(0) \\ -{Y^{(1)}}''(0) \\ 0 \\ 0 \\ {Y^{(1)}}'''(1/2) \\ -{Y^{(1)}}''(1/2) \\ -{Y^{(2)}}'''(1) \\ {Y^{(2)}}''(1) \end{bmatrix} + \begin{bmatrix} -5/4 \\ -5/48 \\ -5/2 \\ 0 \\ -5/4 \\ -5/48 \\ -5/4 \\ 5/48 \end{bmatrix}.$$

In preparation for elimination of the fifth and sixth equations, we add column five of the stiffness matrix to column three, and column six to column four,

$$\begin{bmatrix} 96 & 24 & -96 & 24 & 0 & 0 & 0 & 0 \\ 24 & 8 & -24 & 4 & 0 & 0 & 0 & 0 \\ -96 & -24 & 192 & 0 & 96 & 24 & -96 & 24 \\ 24 & 4 & 0 & 16 & 24 & 8 & -24 & 4 \\ 0 & 0 & 96 & 24 & 96 & 24 & -96 & 24 \\ 0 & 0 & 24 & 8 & 24 & 8 & -24 & 4 \\ 0 & 0 & -96 & -24 & -96 & -24 & 96 & -24 \\ 0 & 0 & 24 & 4 & 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} {Y^{(1)}}'''(0) \\ -{Y^{(1)}}''(0) \\ 0 \\ 0 \\ {Y^{(1)}}'''(1/2) \\ -{Y^{(1)}}''(1/2) \\ -{Y^{(2)}}'''(1) \\ {Y^{(2)}}''(1) \end{bmatrix} + \begin{bmatrix} -5/4 \\ -5/48 \\ -5/2 \\ 0 \\ -5/4 \\ -5/48 \\ -5/4 \\ 5/48 \end{bmatrix}.$$

Now eliminate rows five and six of the stiffness matrix, and columns five and six, and fifth and sixth entries of the remaining matrices,

$$\begin{bmatrix} 96 & 24 & -96 & 24 & 0 & 0 \\ 24 & 8 & -24 & 4 & 0 & 0 \\ -96 & -24 & 192 & 0 & -96 & 24 \\ 24 & 4 & 0 & 16 & -24 & 4 \\ 0 & 0 & -96 & -24 & 96 & -24 \\ 0 & 0 & 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} Y^{(1)'''}(0) \\ -Y^{(1)''}(0) \\ 0 \\ 0 \\ -Y^{(2)'''}(1) \\ Y^{(2)''}(1) \end{bmatrix} + \begin{bmatrix} -5/4 \\ -5/48 \\ -5/2 \\ 0 \\ -5/4 \\ 5/48 \end{bmatrix}.$$

The (displacement)boundary conditions $Y(0) = Y(1) = 0$ require $a_1 = a_5 = 0$, and the (slope) conditions necessitate $a_2 = a_6 = 0$ so that

$$\begin{bmatrix} 96 & 24 & -96 & 24 & 0 & 0 \\ 24 & 8 & -24 & 4 & 0 & 0 \\ -96 & -24 & 192 & 0 & -96 & 24 \\ 24 & 4 & 0 & 16 & -24 & 4 \\ 0 & 0 & -96 & -24 & 96 & -24 \\ 0 & 0 & 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a_3 \\ a_4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Y^{(1)'''}(0) \\ -Y^{(1)''}(0) \\ 0 \\ 0 \\ -Y^{(2)'''}(1) \\ Y^{(2)''}(1) \end{bmatrix} + \begin{bmatrix} -5/4 \\ -5/48 \\ -5/2 \\ 0 \\ -5/4 \\ 5/48 \end{bmatrix}.$$

The third and fourth equations now give $a_3 = -5/384$ and $a_4 = 0$. The two-element approximation is therefore

$$Y_2(x) = \begin{cases} -(5/384)(12x^2 - 16x^3), & 0 \le x \le 1/2 \\ -(5/384)(-4 + 24x - 36x^2 + 16x^3), & 1/2 < x \le 1 \end{cases}.$$

The exact solution and this approximation are shown to the right.

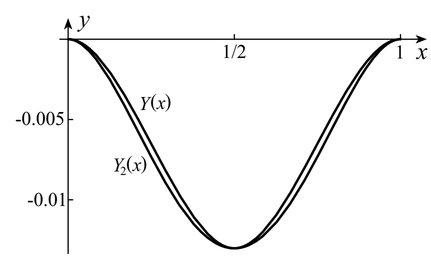


Figure 16.10

EXERCISES 16.6

In Exercise 1–3 repeat Example 16.2 with the given boundary conditions.

1. $Y(0) = Y''(0) = 0$, $Y(1) = Y''(1) = 0$

2. $Y(0) = Y'(0) = 0$, $Y''(1) = Y'''(1) = 0$

3. $Y(0) = Y'(0) = 0$, $Y(1) = Y''(1) = 0$

§16.7 Finite Elements for Boundary Value Problems

With the fundamental ideas of finite elements introduced in Sections 16.2 and 16.4, we are prepared to apply finite elements to two-dimensional boundary value problems. As our vehicle of illustration, we consider a boundary value problem associated with Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y), \tag{16.36}$$

in some region R of the xy-plane with one or more boundary conditions on the boundary $\beta(R)$ of R. Particular forms for boundary conditions will be introduced later since, as we know from our one-dimensional discussions, the FEM first addresses the differential equation. We follow the same approach as in Section 16.2 by considering a one-element region, and then moving to multiple elements. Elements are always triangles and quadrilaterals, three and four sided elements with straight sides. With isoparametric elements, triangles and quadrilaterals with curved sides are also available. We discuss them in Sections 16.8 and 16.9.

Triangular Elements

We begin then with a region consisting of one triangle (Figure 16.11). The solution of the boundary value problem is approximated by a polynomial on the triangle, and as we know, the degree of the polynomial is tied to the number of nodes in the triangle. For a linear polynomial $V_1(x, y) = a + bx + cy$, three nodes are required for the triangle, and these are naturally chosen as the vertices.

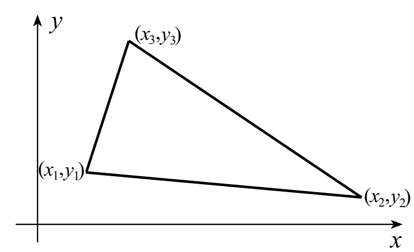


Figure 16.11

Before applying Galerkin's method to $V_1(x, y)$ on the triangle, we rewrite $V_1(x, y)$ in terms of different basis functions, just as we did in Section 16.2. The three nodes in Figure 16.11 uniquely determine the linear approximation $V_1(x, y) = a + bx + cy$ of $V(x, y)$ on the triangle. If $V_1(x_1, y_1)$, $V_1(x_2, y_2)$, and $V_1(x_3, y_3)$ are values of the approximation at the vertices, then

$$V_1(x_1, y_1) = a + bx_1 + cy_1, \quad V_1(x_2, y_2) = a + bx_2 + cy_2, \quad V_1(x_3, y_3) = a + bx_3 + cy_3.$$

The solution of these for a, b, and c in terms of $V_1(x_1, y_1)$, $V_1(x_2, y_2)$, and $V_1(x_3, y_3)$ is

$$a = \frac{1}{\Delta}[(x_2y_3 - x_3y_2)V_1(x_1, y_1) + (x_3y_1 - x_1y_3)V_1(x_2, y_2) + (x_1y_2 - x_2y_1)V_1(x_3, y_3)], \tag{16.37a}$$

$$b = \frac{1}{\Delta}[(y_2 - y_3)V_1(x_1, y_1) + (y_3 - y_1)V_1(x_2, y_2) + (y_1 - y_2)V_1(x_3, y_3)], \tag{16.37b}$$

$$c = \frac{1}{\Delta}[(x_3 - x_2)V_1(x_1, y_1) + (x_1 - x_3)V_1(x_2, y_2) + (x_2 - x_1)V_1(x_3, y_3)], \tag{16.37c}$$

where $\Delta = (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)$ is twice the area of the triangle. When we substitute these into $V_1(x, y) = a + bx + cy$, we obtain

$$\begin{aligned}V_1(x,y) &= \frac{1}{\Delta}\big\{[(x_2y_3 - x_3y_2)V_1(x_1,y_1) + (x_3y_1 - x_1y_3)V_1(x_2,y_2) + (x_1y_2 - x_2y_1)V_1(x_3,y_3)] \\ &\quad + [(y_2 - y_3)V_1(x_1,y_1) + (y_3 - y_1)V_1(x_2,y_2) + (y_1 - y_2)V_1(x_3,y_3)]x \\ &\quad + [(x_3 - x_2)V_1(x_1,y_1) + (x_1 - x_3)V_1(x_2,y_2) + (x_2 - x_1)V_1(x_3,y_3)]y\big\} \\ &= V_1(x_1,y_1)\left[\frac{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y}{\Delta}\right] \\ &\quad + V_1(x_2,y_2)\left[\frac{(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y}{\Delta}\right] \\ &\quad + V_1(x_3,y_3)\left[\frac{(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y}{\Delta}\right]. \qquad (16.38)\end{aligned}$$

In other words, with basis functions

$$\phi_1(x,y) = \frac{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y}{\Delta}, \qquad (16.39a)$$

$$\phi_2(x,y) = \frac{(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y}{\Delta}, \qquad (16.39b)$$

$$\phi_3(x,y) = \frac{(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y}{\Delta}, \qquad (16.39c)$$

we can express the linear approximation in the form

$$V_1(x,y) = c_1\phi_1(x,y) + c_2\phi_2(x,y) + c_3\phi_3(x,y), \qquad (16.40)$$

where coefficients are values of $V_1(x,y)$ at the nodes (vertices) of the triangle.

Linear approximation $V_1(x,y) = a + bx + cy$ approximates the solution surface of the boundary value problem with a plane. Representation 16.40 also does this, but as the linear combination of three planes $V = \phi_1(x,y)$, $V = \phi_2(x,y)$, and $V = \phi_3(x,y)$. We have shown these planes in Figures 16.12a,b,c. Each plane has value 1 at one vertex of the triangle and value 0 at the other two vertices. Each plane is therefore zero on one side of the base triangle in the xy-plane. Basis functions satisfy the equivalent of property 16.3;

$$\phi_i(x_j,y_j) = \delta_{ij}. \qquad (16.41)$$

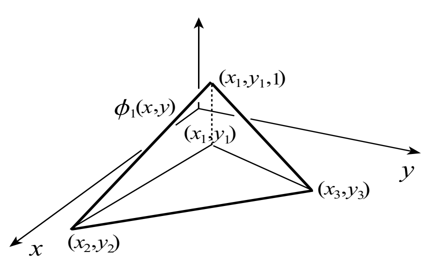


Figure 16.12a

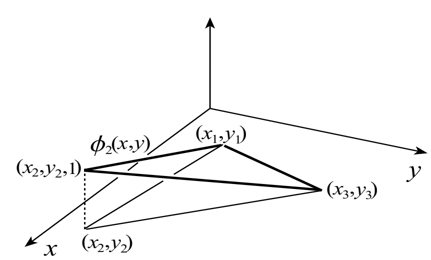


Figure 16.12b

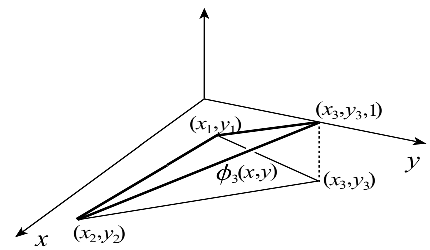


Figure 16.12c

We are now ready to apply the MWR to linear approximation 16.40 of the solution of Poisson's equation 16.36. Galerkin's method on the residual

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} - F(x,y)$$

demands that for $i = 1, 2, 3$,

$$0 = \iint_R \left[\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} - F(x,y)\right] \phi_i(x,y)\, dA.$$

For one-dimensional problems, we applied integation by parts to the second derivative. For PDEs, we apply Green's first identity to the derivative terms (see Appendix C),

$$0 = \oint_{\beta(R)} (\phi_i \nabla V_1) \cdot \hat{\mathbf{n}}\, ds - \iint_R \nabla V_1 \cdot \nabla \phi_i\, dA - \iint_R F\, \phi_i\, dA,$$

or,

$$\iint_R \nabla V_1 \cdot \nabla \phi_i\, dA = \oint_{\beta(R)} \left(\phi_i \frac{\partial V_1}{\partial n}\right) ds - \iint_R F\, \phi_i\, dA. \qquad (16.42)$$

Because $\hat{\mathbf{n}}$ is the outward normal to the boundary $\beta(R)$ of R, the derivative $\partial V_1/\partial n$ is the outwardly normal derivative of $V_1(x,y)$ along the three sides of the triangle. When we substitute representation 16.40 into the double integral, we get

$$\iint_R \nabla(c_1\phi_1 + c_2\phi_2 + c_3\phi_3) \cdot \nabla \phi_i\, dA = \oint_{\beta(R)} \left(\phi_i \frac{\partial V_1}{\partial n}\right) ds - \iint_R F\, \phi_i\, dA,$$

or,

$$c_1 \iint_R \nabla\phi_1 \cdot \nabla\phi_i\, dA + c_2 \iint_R \nabla\phi_2 \cdot \nabla\phi_i\, dA + c_3 \iint_R \nabla\phi_3 \cdot \nabla\phi_i\, dA$$
$$= \oint_{\beta(R)} \left(\phi_i \frac{\partial V_1}{\partial n}\right) ds - \iint_R F\, \phi_i\, dA. \qquad (16.43)$$

Since this must be valid for $i = 1, 2, 3$, the element equations for c_1, c_2, and c_3 are

$$KC = B + N, \qquad (16.44a)$$

where

$$K = \begin{pmatrix} \iint_R \nabla\phi_1 \cdot \nabla\phi_1\, dA & \iint_R \nabla\phi_2 \cdot \nabla\phi_1\, dA & \iint_R \nabla\phi_3 \cdot \nabla\phi_1\, dA \\ \iint_R \nabla\phi_1 \cdot \nabla\phi_2\, dA & \iint_R \nabla\phi_2 \cdot \nabla\phi_2\, dA & \iint_R \nabla\phi_3 \cdot \nabla\phi_2\, dA \\ \iint_R \nabla\phi_1 \cdot \nabla\phi_3\, dA & \iint_R \nabla\phi_2 \cdot \nabla\phi_3\, dA & \iint_R \nabla\phi_3 \cdot \nabla\phi_3\, dA \end{pmatrix}, \qquad (16.44b)$$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad B = \begin{pmatrix} \oint_{\beta(R)} \left(\phi_1 \dfrac{\partial V_1}{\partial n}\right) ds \\ \oint_{\beta(R)} \left(\phi_2 \dfrac{\partial V_1}{\partial n}\right) ds \\ \oint_{\beta(R)} \left(\phi_3 \dfrac{\partial V_1}{\partial n}\right) ds \end{pmatrix}, \quad N = \begin{pmatrix} -\iint_R F\,\phi_1\,dA \\ -\iint_R F\,\phi_2\,dA \\ -\iint_R F\,\phi_3\,dA \end{pmatrix}. \quad (16.44c)$$

Because Poisson's equation is in self-adjoint form, the stiffness matrix K is symmetric; load vector N is once again due to the nonhomogeneity in the PDE; and load vector B is more complicated than in the case of ODEs because the boundary now consists of curves (the edges of the triangle) instead of two end points of an interval. We should now evaluate integrals and impose boundary conditions. Let us simplify calculations by choosing a very simple triangle, one with vertices $(0,0)$, $(2,0)$, and $(0,3)$, and boundary conditions as shown in (Figure 16.13). In addition, suppose that $F(x,y) = -10$. For this triangle, basis functions are

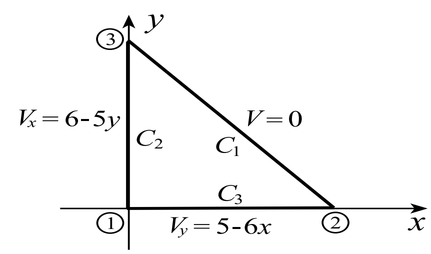


Figure 16.13

$$\phi_1(x,y) = \frac{1}{6}(6 - 3x - 2y), \qquad \phi_2(x,y) = \frac{x}{2}, \qquad \phi_3(x,y) = \frac{y}{3}.$$

They can be obtained from equations 16.39, or simply by inspection, since each function must be one at a node of the triangle and zero along the opposite side. When integrals are evaluated, element equations 16.44a become

$$\begin{pmatrix} 13/12 & -3/4 & 1/3 \\ -3/4 & 3/4 & 0 \\ 1/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} + \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}. \qquad (16.45)$$

The Dirichlet condition along the hypotenuse of the triangle requires

$$0 = c_1\left(\frac{6 - 3x - 2y}{6}\right) + c_2\left(\frac{x}{2}\right) + c_3\left(\frac{y}{3}\right) = c_2\left(\frac{x}{2}\right) + c_3\left(\frac{6 - 3x}{2}\right),$$

and this is satisfied if we take $c_2 = c_3 = 0$. When these are substituted into equations 16.45, the first equation determines c_1, and therefore we need only calculate B_1,

$$\begin{aligned} B_1 &= \oint_{C_1+C_2+C_3} \frac{\partial V}{\partial n}\phi_1\,ds = \int_{C_2+C_3} \frac{\partial V}{\partial n}\phi_1\,ds \\ &= \int_{C_2} -\frac{\partial V(0,y)}{\partial x}\left(\frac{6-2y}{6}\right) ds + \int_{C_3} -\frac{\partial V(x,0)}{\partial y}\left(\frac{6-3x}{6}\right) ds \\ &= \frac{1}{6}\int_0^3 -(6-5y)(6-2y)\,dy + \frac{1}{6}\int_0^2 -(6-5x)(6-3x)\,dx = -\frac{25}{6}. \end{aligned}$$

Consequently, element equations are

$$\begin{pmatrix} 13/12 & -3/4 & 1/3 \\ -3/4 & 3/4 & 0 \\ 1/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -25/6 \\ B_2 \\ B_3 \end{pmatrix} + \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}.$$

The first equation yields $c_1 = 70/13$, and the one-element, linear approximation is

$$V_1(x,y) = c_1\phi_1(x,y) = \frac{35}{39}(6-3x-2y).$$

The exact solution of the boundary value problem is

$$V(x,y) = (x+y)(6-3x-2y).$$

A sketch of this function and the linear approximation are shown in Figure 16.14.

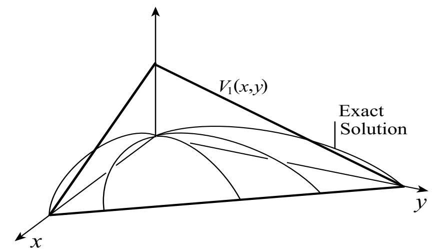


Figure16.14

It is now time to move on to what the MFE is all about, multiple elements. We adopt the procedure of Section 16.2 by beginning with two elements, and progressing to more and more elements. We continue with triangle elements, but in Section 16.9, we introduce quadrilateral elements. Consider solving Poisson's equation 16.36, with $F(x,y) = xy$, on the four-sided figure in Figure 16.15a, subject to the boundary conditions shown. We can divide the triangle into two triangles in two ways. Suppose we choose the triangles in Figure 16.15b.

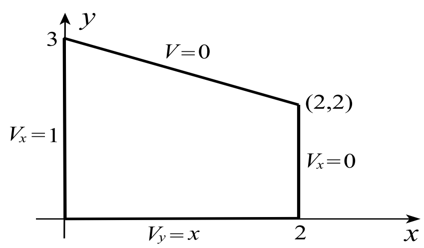


Figure 16.15a

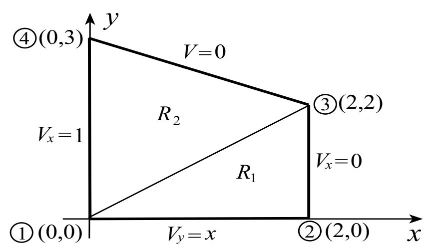


Figure 16.15b

Using the process that has become quite familiar to us, we write element equations 16.44 for each triangle and then combine them into system equations. For element (1) in Figure 16.16a, basis functions are

$$\phi_1^{(1)}(x,y) = \frac{1}{2}(2-x), \qquad \phi_2^{(1)}(x,y) = \frac{1}{2}(x-y), \qquad \phi_3^{(1)}(x,y) = \frac{y}{2}.$$

Element equations 16.44 are

$$\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_3^{(1)} \end{pmatrix} = \begin{pmatrix} B_1^{(1)} \\ B_2^{(1)} \\ B_3^{(1)} \end{pmatrix} + \begin{pmatrix} -2/5 \\ -8/15 \\ -16/15 \end{pmatrix},$$

where

$$\begin{aligned} B_1^{(1)} &= \oint_{C_1^{(1)}+C_2^{(1)}+C_3^{(1)}} \left(\phi_1^{(1)}\frac{\partial V_1}{\partial n}\right) ds = \int_{C_2^{(1)}+C_3^{(1)}} \left(\phi_1^{(1)}\frac{\partial V_1}{\partial n}\right) ds \\ &= \int_{C_2^{(1)}} \left(\phi_1^{(1)}\frac{\partial V_1}{\partial n}\right) ds + \int_0^2 \frac{1}{2}(2-x)(-x)\,dx = \int_{C_2^{(1)}} \left(\phi_1^{(1)}\frac{\partial V_1}{\partial n}\right) ds - \frac{2}{3}, \end{aligned}$$

$$\begin{aligned}B_2^{(1)} &= \oint_{C_1^{(1)}+C_2^{(1)}+C_3^{(1)}} \left(\phi_2^{(1)}\frac{\partial V_1}{\partial n}\right) ds = \int_{C_1^{(1)}+C_3^{(1)}} \left(\phi_2^{(1)}\frac{\partial V_1}{\partial n}\right) ds \\ &= \int_{C_1^{(1)}} \frac{1}{2}(2-y)(0)\,ds + \int_0^2 \frac{1}{2}(x)(-x)\,dx = -\frac{4}{3}, \\ B_3^{(1)} &= \oint_{C_1^{(1)}+C_2^{(1)}+C_3^{(1)}} \left(\phi_3^{(1)}\frac{\partial V_1}{\partial n}\right) ds = \int_{C_1^{(1)}+C_2^{(1)}} \left(\phi_3^{(1)}\frac{\partial V_1}{\partial n}\right) ds \\ &= \int_{C_1^{(1)}} \frac{y}{2}(0)\,ds + \int_{C_2^{(1)}} \left(\phi_3^{(1)}\frac{\partial V_1}{\partial n}\right) ds = \int_{C_2^{(1)}} \left(\phi_3^{(1)}\frac{\partial V_1}{\partial n}\right) ds.\end{aligned}$$

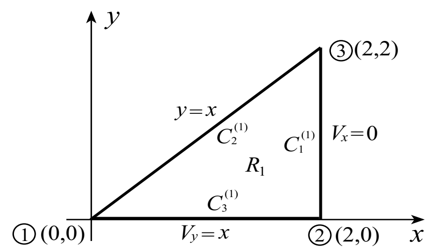


Figure 16.16a

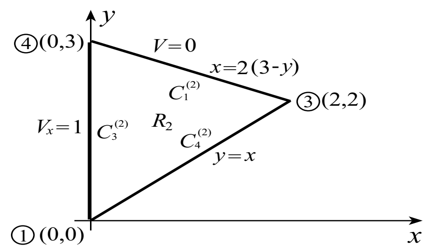


Figure 16.16b

Basis functions for element 2 in Figure 16.16b are

$$\phi_1^{(2)}(x,y) = \frac{1}{6}(6-x-2y), \qquad \phi_3^{(2)}(x,y) = \frac{x}{2}, \qquad \phi_4^{(2)}(x,y) = \frac{1}{3}(y-x).$$

Element equations are

$$\begin{pmatrix} 5/12 & -1/4 & -1/6 \\ -1/4 & 3/4 & -1/2 \\ -1/6 & -1/2 & 2/3 \end{pmatrix} \begin{pmatrix} c_1^{(2)} \\ c_3^{(2)} \\ c_4^{(2)} \end{pmatrix} = \begin{pmatrix} B_1^{(2)} \\ B_3^{(2)} \\ B_4^{(2)} \end{pmatrix} + \begin{pmatrix} -7/10 \\ -9/5 \\ -1 \end{pmatrix},$$

where

$$\begin{aligned}B_1^{(2)} &= \oint_{C_1^{(2)}+C_3^{(2)}+C_4^{(2)}} \left(\phi_1^{(2)}\frac{\partial V_1}{\partial n}\right) ds = \int_{C_3^{(2)}+C_4^{(2)}} \left(\phi_1^{(2)}\frac{\partial V_1}{\partial n}\right) ds \\ &= \int_0^2 \frac{1}{6}(6-2y)(-1)\,dy + \int_{C_4^{(2)}} \left(\phi_1^{(2)}\frac{\partial V_1}{\partial n}\right) ds = \int_{C_4^{(2)}} \left(\phi_1^{(2)}\frac{\partial V_1}{\partial n}\right) ds + \frac{4}{3}, \\ B_3^{(2)} &= \oint_{C_1^{(2)}+C_3^{(2)}+C_4^{(2)}} \left(\phi_3^{(2)}\frac{\partial V_1}{\partial n}\right) ds = \int_{C_1^{(2)}} \left(\phi_3^{(2)}\frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)}\frac{\partial V_1}{\partial n}\right) ds, \\ B_4^{(2)} &= \oint_{C_1^{(2)}+C_3^{(2)}+C_4^{(2)}} \left(\phi_4^{(2)}\frac{\partial V_1}{\partial n}\right) ds = \int_{C_1^{(2)}+C_3^{(2)}} \left(\phi_4^{(2)}\frac{\partial V_1}{\partial n}\right) ds \\ &= \int_{C_1^{(2)}} \left(\phi_4^{(2)}\frac{\partial V_1}{\partial n}\right) ds + \int_0^2 \frac{1}{3}(y)(-1)\,dy = \int_{C_1^{(2)}} \left(\phi_4^{(2)}\frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3}.\end{aligned}$$

We put both sets of element equations together to get the system equations for the two-element, linear approximation

$$V_2(x,y) = \begin{cases} c_1^{(1)}\phi_1^{(1)}(x,y) + c_2^{(1)}\phi_2^{(1)}(x,y) + c_3^{(1)}\phi_3^{(1)}(x,y), & (x,y) \text{ in } R_1 \\ c_1^{(2)}\phi_1^{(2)}(x,y) + c_3^{(2)}\phi_3^{(2)}(x,y) + c_4^{(2)}\phi_4^{(2)}(x,y), & (x,y) \text{ in } R_2. \end{cases}$$

They are

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5/12 & -1/4 & -1/6 \\ 0 & 0 & 0 & -1/4 & 3/4 & -1/2 \\ 0 & 0 & 0 & -1/6 & -1/2 & 2/3 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_3^{(1)} \\ c_1^{(2)} \\ c_3^{(2)} \\ c_4^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \int_{C_2^{(1)}} \left(\phi_1^{(1)} \frac{\partial V_1}{\partial n}\right) ds - \frac{2}{3} \\ -\frac{4}{3} \\ \int_{C_2^{(1)}} \left(\phi_3^{(1)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_4^{(2)}} \left(\phi_1^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{4}{3} \\ \int_{C_1^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_1^{(2)}} \left(\phi_4^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -2/5 \\ -8/15 \\ -16/15 \\ -7/10 \\ -9/5 \\ -1 \end{pmatrix}.$$

Interelement boundary conditions are $c_1^{(2)} = c_1^{(1)}$ and $c_3^{(2)} = c_3^{(1)}$. If we denote values of $V_2(x, y)$ at the four vertices of the quadrilateral by a_1, a_2, a_3, and a_4, then system equations for these nodal values are

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5/12 & -1/4 & -1/6 \\ 0 & 0 & 0 & -1/4 & 3/4 & -1/2 \\ 0 & 0 & 0 & -1/6 & -1/2 & 2/3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_1 \\ a_3 \\ a_4 \end{pmatrix}$$

$$= \begin{pmatrix} \int_{C_2^{(1)}} \left(\phi_1^{(1)} \frac{\partial V_1}{\partial n}\right) ds - \frac{2}{3} \\ -\frac{4}{3} \\ \int_{C_2^{(1)}} \left(\phi_3^{(1)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_4^{(2)}} \left(\phi_1^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{4}{3} \\ \int_{C_1^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_1^{(2)}} \left(\phi_4^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -2/5 \\ -8/15 \\ -16/15 \\ -7/10 \\ -9/5 \\ -1 \end{pmatrix}.$$

We now add the fourth equation to the first, and the fifth to the third,

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5/12 & -1/4 & -1/6 \\ 0 & 0 & 0 & -1/4 & 3/4 & -1/2 \\ 0 & 0 & 0 & -1/6 & -1/2 & 2/3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_1 \\ a_3 \\ a_4 \end{pmatrix}$$

$$= \begin{pmatrix} \int_{C_4^{(2)}} \left(\phi_1^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_2^{(1)}} \left(\phi_1^{(1)} \frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3} \\ -\frac{4}{3} \\ \int_{C_1^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_2^{(1)}} \left(\phi_3^{(1)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_4^{(2)}} \left(\phi_1^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{4}{3} \\ \int_{C_1^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_1^{(2)}} \left(\phi_4^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -11/10 \\ -8/15 \\ -43/15 \\ -7/10 \\ -9/5 \\ -1 \end{pmatrix}.$$

The first entry of the B matrix is equal to

$$\int_{C_4^{(2)}} \left[\frac{1}{6}(6-3x)\frac{\partial V_1}{\partial n}\right] ds + \int_{C_2^{(1)}} \left[\frac{1}{2}(2-x)\frac{\partial V_1}{\partial n}\right] ds + \frac{2}{3}.$$

Along $C_4^{(2)}$, the derivative $\partial V_1/\partial n$ is in the normal direction downward from the line $y = x$, whereas along $C_2^{(1)}$, which is the same line, the derivative is in the normal direction upward from the line. In other words, these derivatives are negatives of each other, and the integrals cancel, leaving $2/3$ for the matrix entry. Similarly, integrals in the fifth entry of B cancel, leaving a zero entry. System equations are therefore

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5/12 & -1/4 & -1/6 \\ 0 & 0 & 0 & -1/4 & 3/4 & -1/2 \\ 0 & 0 & 0 & -1/6 & -1/2 & 2/3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_1 \\ a_3 \\ a_4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \int_{C_1^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_2^{(1)}} \left(\phi_3^{(1)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_4^{(2)}} \left(\phi_1^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{4}{3} \\ 0 \\ \int_{C_1^{(2)}} \left(\phi_4^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -11/10 \\ -8/15 \\ -43/15 \\ -7/10 \\ -9/5 \\ -1 \end{pmatrix}.$$

To eliminate the fourth and fifth equations, we first add the the fourth column of the stiffness matrix to the first, and the fifth column to the third,

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 5/12 & -1/4 & -1/6 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & -1/4 & 3/4 & -1/2 \\ 0 & 0 & 0 & 5/12 & -1/4 & -1/6 \\ 0 & 0 & 0 & -1/4 & 3/4 & -1/2 \\ 0 & 0 & 0 & -1/6 & -1/2 & 2/3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_1 \\ a_3 \\ a_4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \int_{C_1^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_2^{(1)}} \left(\phi_3^{(1)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_4^{(2)}} \left(\phi_1^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{4}{3} \\ 0 \\ \int_{C_1^{(2)}} \left(\phi_4^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -11/10 \\ -8/15 \\ -43/15 \\ -7/10 \\ -9/5 \\ -1 \end{pmatrix}.$$

awe now delete the fourth and fifth rows and columns of the stiffness matrix, and the fourth and fifth entries of the remaining matrices,

$$\begin{pmatrix} 1/2 & -1/2 & 0 & -1/6 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 2/3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \int_{C_1^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_2^{(1)}} \left(\phi_3^{(1)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_1^{(2)}} \left(\phi_4^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -11/10 \\ -8/15 \\ -43/15 \\ -1 \end{pmatrix}.$$

The Dirichlet boundary condition along $x = 2(3 - y)$ requires $a_3 = a_4 = 0$,

$$\begin{pmatrix} 1/2 & -1/2 & 0 & -1/6 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 2/3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \int_{C_1^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_4^{(2)}} \left(\phi_3^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \int_{C_2^{(1)}} \left(\phi_3^{(1)} \frac{\partial V_1}{\partial n}\right) ds \\ \int_{C_1^{(2)}} \left(\phi_4^{(2)} \frac{\partial V_1}{\partial n}\right) ds + \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -11/10 \\ -8/15 \\ -43/15 \\ -1 \end{pmatrix}.$$

The first two equations can be solved for $a_1 = -82/15$ and $a_2 = -23/5$. The two-element, linear approximation is therefore

$$V_2(x, y) = \begin{cases} -\dfrac{82}{15} \cdot \dfrac{1}{2}(2 - x) - \dfrac{23}{5} \cdot \dfrac{1}{2}(x - y), & (x, y) \text{ in } R_1 \\ -\dfrac{82}{15} \cdot \dfrac{1}{6}(6 - x - 2y), & (x, y) \text{ in } R_2 \end{cases}$$

$$= \begin{cases} \dfrac{1}{30}(-164 + 13x + 69y), & (x, y) \text{ in } R_1 \\ -\dfrac{41}{45}(6 - x - 2y), & (x, y) \text{ in } R_2. \end{cases}$$

EXERCISES 16.7

§16.8 Isoparametric Triangular Elements

In Section 16.7, we used linear basis functions on triangular elements to approximate solutions of boundary value problems. Basis functions were developed directly from the requirement that coefficients of basis functions be values of the polynomial approximation at the nodes (vertices) of the triangle. In this section, we continue with triangular elements, first showing that isoparametric elements can be used to give the same linear basis functions. We then use isoparametric elements to develop quadratic approximations for triangular elements.

Linear Approximations on Isoparametric Triangles

In the isoparametric approach, parental basis functions are established for a triangle in the $\xi\eta$-plane, and these are then mapped to basis functions for triangles in the xy-plane. The triangle chosen in the $\xi\eta$-plane is the right-angled one in Figure 16.17. Basis functions that satisfy requirement 16.41 are easily seen to be

$$\phi_1(\xi,\eta) = 1 - \xi - \eta, \qquad \phi_2(\xi,\eta) = \xi, \qquad \phi_3(\xi,\eta) = \eta. \tag{16.46}$$

The isoparametric transformation that maps nodes (vertices) of this triangle to the nodes of the generic triangle in Figure 16.18 is

$$x = \phi_1 x_1 + \phi_2 x_2 + \phi_3 x_3 = (1 - \xi - \eta)x_1 + \xi x_2 + \eta x_3, \tag{16.47a}$$

$$y = \phi_1 y_1 + \phi_2 y_2 + \phi_3 y_3 = (1 - \xi - \eta)y_1 + \xi y_2 + \eta y_3. \tag{16.47b}$$

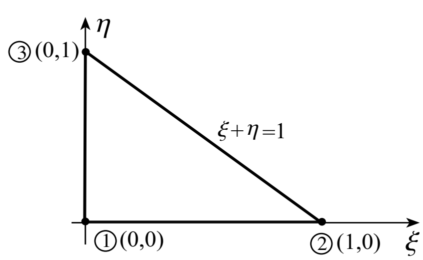


Figure 16.17

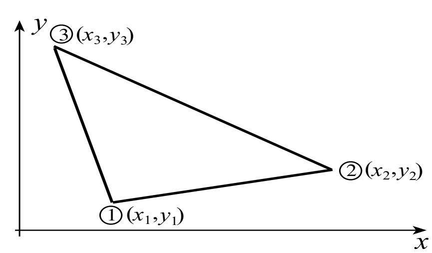


Figure 16.18

To find basis functions for the xy-triangle, we solve these equations for ξ and η in terms of x and y. The result is

$$\xi = \frac{(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y}{(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)},$$

$$\eta = \frac{(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y}{(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)}. \tag{16.48}$$

(See Exercise 1.) Substitution of these into equations 16.46 gives basis functions 16.39.

Problem 16.36 on the triangle of Figure 16.13 provides a simple illustration. The transformation that maps vertices of the triangle in Figure 16.17 to those in Figure 16.13 is

$$x = 2\xi, \quad y = 3\eta,$$

so that $\xi = x/2$ and $\eta = y/3$. Substitution of these into equations 16.46 gives basis functions for the triangle in Figure 16.13,

$$\phi_1(x,y) = 1 - \frac{x}{2} - \frac{y}{3}, \quad \phi_2(x,y) = \frac{x}{2}, \quad \phi_3(x,y) = \frac{y}{3}.$$

Quadratic Approximations on Isoparametric Triangles

We now develop quadratic approximations for isoparametric triangles. As in the linear case, the parent triangle is that in Figure 16.17, but additional nodes are needed. The complete second-order polynomial in ξ and η is

$$p(\xi,\eta) = a + b\xi + c\eta + d\xi^2 + f\xi\eta + g\eta^2. \tag{16.49}$$

(By complete, we mean that the polynomial contains all second order terms in ξ and η.) To determine the six coefficients, values of the polynomial at six points in the triangle are needed. We choose the three vertices and the midpoints of the sides (Figure 16.19).

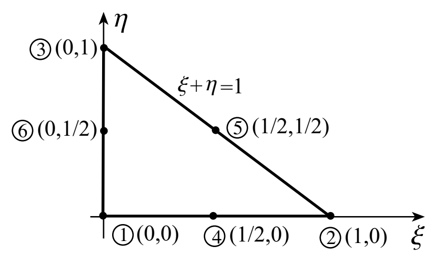


Figure 16.19

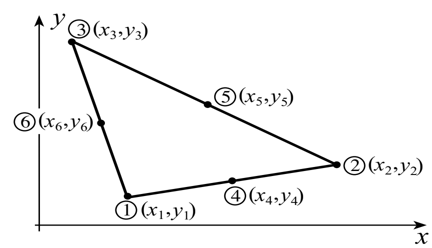


Figure 16.20

We could take the direct approach that we have used in the past to find quadratic basis functions by specifying values of $p(\xi,\eta)$ at each of the nodes, solving the equations for a, b, c, d, f, and g, substituting these into equation 16.49, and rearranging terms. Preferable is to use the fact that basis functions $\phi_i(\xi,\eta)$ shoud satisfy property 16.41. If we take the basis function $\phi_1(\xi,\eta)$ associated with node 1 in form 16.49, and invoke these conditions, we obtain the equations

$$\begin{aligned}
1 &= \phi_1(\xi_1,\eta_1) = \phi_1(0,0) = a,\\
0 &= \phi_1(\xi_2,\eta_2) = \phi_1(1,0) = a + b + d,\\
0 &= \phi_1(\xi_3,\eta_3) = \phi_1(0,1) = a + c + g,\\
0 &= \phi_1(\xi_4,\eta_4) = \phi_1(1/2,0) = a + b/2 + d/4,\\
0 &= \phi_1(\xi_5,\eta_5) = \phi_1(1/2,1/2) = a + b/2 + c/2 + d/4 + f/4 + g/4,\\
0 &= \phi_1(\xi_6,\eta_6) = \phi_1(0,1/2) = a + c/2 + g/4.
\end{aligned}$$

These are easily solved for $a = 1$, $b = -3$, $c = -3$, $d = 2$, $f = 4$, and $g = 2$, so that the first basis function is

$$\phi_1(\xi,\eta) = 1 - 3\xi - 3\eta + 2\xi^2 + 4\xi\eta + 2\eta^2 = (1-\xi-\eta)(1-2\xi-2\eta). \tag{16.50a}$$

Similar procedures give basis functions associated with the other five nodes

$$\phi_2(\xi,\eta) = \xi(2\xi-1), \tag{16.50b}$$
$$\phi_3(\xi,\eta) = \eta(2\eta-1), \tag{16.50c}$$
$$\phi_4(\xi,\eta) = 4\xi(1-\xi-\eta), \tag{16.50d}$$
$$\phi_5(\xi,\eta) = 4\xi\eta, \tag{16.50e}$$
$$\phi_6(\xi,\eta) = 4\eta(1-\xi-\eta). \tag{16.50f}$$

In practice, we can formulate the basis functions by inspection. For instance, in order that $\phi_1(\xi,\eta)$ vanish at nodes 2, 5, and 3, we include in it the factor $1-\xi-\eta$. To vanish at nodes 4 and 6, we include $2-\xi-\eta$. In other words, $\phi_1(\xi,\eta)$ must be of the form $\phi_1(\xi,\eta) = a(1-\xi-\eta)(2-\xi-\eta)$. The requirement that $\phi_1(0,0) = 1$ yields $a = 1$. This procedure can also be followed for the other five basis functions. We have shown $\phi_1(\xi,\eta)$ and $\phi_4(\xi,\eta)$ in Figures 16.21a,b.

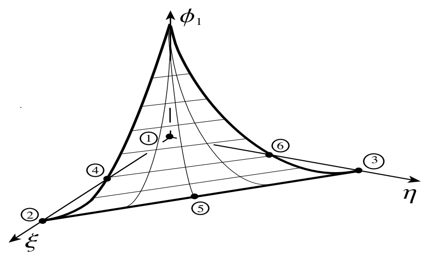


Figure 16.21a

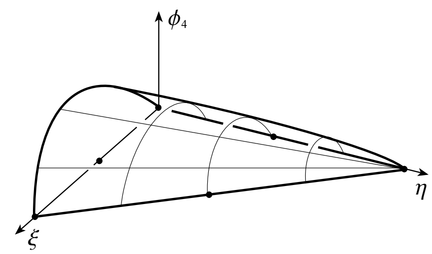


Figure 16.21b

Suppose now that we want basis functions for the triangle in Figure 16.20. We map the nodes in Figure 16.19 to those in Figure 16.20 and use this mapping to transform basis functions. The mapping to accomplish this is

$$x = \sum_{i=1}^{6} \phi_i(\xi,\eta)x_i, \qquad y = \sum_{i=1}^{6} \phi_i(\xi,\eta)y_i. \tag{16.51}$$

Depending on the positions of nodes along the sides of the triangle in Figure 16.20, basis functions may, or may not be, quadratic polynomials in x and y. However, the most common choice of side nodes in triangles is midpoints, just like in the parent element, in which case,

$$x_4 = \frac{x_1+x_2}{2}, \qquad x_5 = \frac{x_2+x_3}{2}, \qquad x_6 = \frac{x_1+x_3}{2}.$$

These will imply similar expressions for y_4, y_5, and y_6. Substitution of these into maping 16.51 gives

$$\begin{aligned} x &= (1-\xi-\eta)(1-2\xi-2\eta)x_1 + \xi(2\xi-1)x_2 + \eta(2\eta-1)x_3 \\ &\quad + 4\xi(1-\xi-\eta)\left(\frac{x_2+x_2}{2}\right) + 4\xi\eta\left(\frac{x_2+x_3}{2}\right) + 4\eta(1-\xi-\eta)\left(\frac{x_1+x_3}{2}\right) \\ &= x_1 + (x_2-x_1)\xi + (x_3-x_1)\eta, \end{aligned}$$

with an identical result for y,

$$y = y_1 + (y_2-y_1)\xi + (y_3-y_1)\eta.$$

This is linear mapping 16.47; when solved for ξ and η in terms of x and y, equations 16.48 result. When these are substituted into parent basis functions 16.50, basis functions for the triangle in Figure 16.20 with midpoint nodes are obtained. Because the mapping is linear, the xy-basis functions will remain quadratic. As an illustration, suppose the xy-triangle is that in Figure 16.13. The transformation that maps vertices of the triangle in Figure 16.17 to those in Figure 16.13 and preserves midpoints is $x = 2\xi$, $y = 3\eta$, so that $\xi = x/2$ and $\eta = y/3$. Substitution of these into equations 16.50 gives quadratic basis functions for the triangle in Figure 16.13 with midpoint nodes,

$$\phi_1(x,y) = \left(1 - \frac{x}{2} - \frac{y}{3}\right)\left(1 - x - \frac{2y}{3}\right) = \frac{1}{18}(6 - 3x - 2y)(3 - 3x - 2y),$$
$$\phi_2(x,y) = \frac{x}{2}(x-1),$$
$$\phi_3(x,y) = \frac{\eta}{3}\left(\frac{2\eta}{3} - 1\right) = \frac{1}{9}\eta(2\eta - 3),$$
$$\phi_4(x,y) = 4\left(\frac{x}{2}\right)\left(1 - \frac{x}{2} - \frac{y}{3}\right) = \frac{1}{3}x(6 - 3x - 2y),$$
$$\phi_5(x,y) = 4\left(\frac{x}{2}\right)\left(\frac{y}{3}\right) = \frac{2}{3}xy,$$
$$\phi_6(x,y) = 4\left(\frac{y}{3}\right)\left(1 - \frac{x}{2} - \frac{y}{3}\right) = \frac{2}{9}y(6 - 3x - 2y).$$

When a triangular element has two nodes on the boundary of a region (Figure 16.22), approximations are enhanced if quadratic basis functions are used, and one of the midpoint nodes is chosen on the boundary. This results in a triangle with two straight sides and one curved side (Figure 16.23). Transformation 16.51 still maps nodes of the parent element in Figure 16.19 to nodes of the curved-sided triangle in Figure 16.23, but it is no longer linear. For instance,

$$\begin{aligned} x &= (1 - \xi - \eta)(1 - 2\xi - 2\eta)x_1 + \xi(2\xi - 1)x_2 + \eta(2\eta - 1)x_3 \\ &\quad + 4\xi(1 - \xi - \eta)\left(\frac{x_1 + x_2}{2}\right) + 4\xi\eta x_5 + 4\eta(1 - \xi - \eta)\left(\frac{x_1 + x_3}{2}\right) \\ &= (1 - \xi - \eta)x_1 + \xi(1 - 2\eta)x_2 + \eta(1 - 2\xi)x_3 + 4\xi\eta x_5, \end{aligned}$$

with a similar equation for y. Basis functions for the curved triangle will not be quadratic.

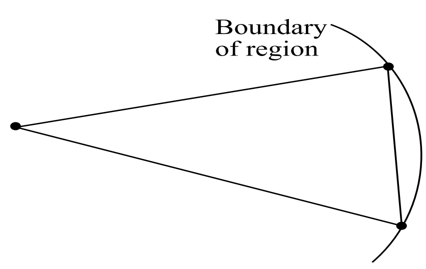


Figure 16.22

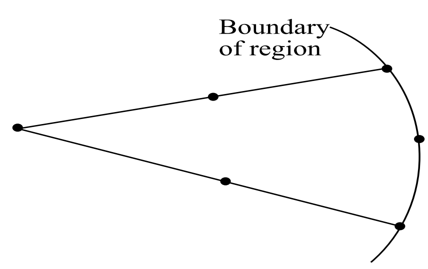


Figure 16.23

We close this section with a brief mention of cubic approximations. The complete cubic polynomial in ξ and η is

$$p(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3.$$

The parent triangle, which must have ten nodes, is usually chosen as that in Figure 16.24, nine boundary nodes and one interior node. Cubic basis functions for this triangle are developed in Exercise 2.

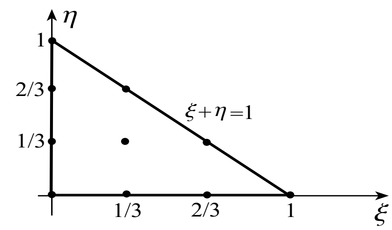


Figure 16.24

EXERCISES 16.8

1. Verify the results in equations 16.48.

2. Develop cubic basis functions for the ten-node parent triangle in Figure 16.24.

§16.9 Isoparametric Quadrilateral Elements

We continue with the isoparametric approach in developing basis functions for quadrilateral elements. The parent quadrilateral is taken to be the square in Figure 16.25. Recall that the number of nodes for the square dictates the order of the polynomial approximation, and conversely, the order of a polynomial determines the number of nodes that must be used. Usually the number of nodes is chosen, with due regard to symmetry, and terms in the polynomial are chosen correspondingly. We shall illustrate. There are two sets of polynomials in common use, depending on whether nodes interior to the square are used.

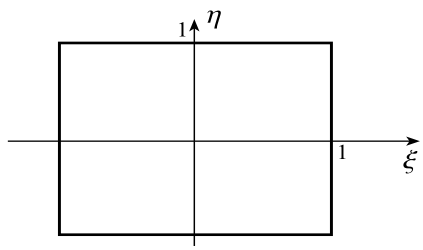


Figure 16.25

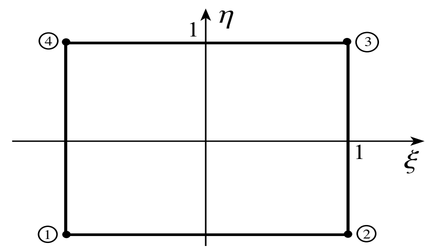


Figure 16.26

Lagrange Type Basis Functions

Two-dimensional Lagrange type basis functions use products of one-dimensional Lagrange interpolation formulas. For example, products of the two linear basis functions $(1-\xi)/2$ and $(1+\xi)/2$ in the ξ-coordinate by their counterparts in η yield what are called **bilinear** basis functions

$$\phi_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta), \quad \phi_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta),$$
$$\phi_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta), \quad \phi_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta).$$

These bilinear basis functions correspond to the minimum number of nodes for a square, its vertices (Figure 16.26). They continue to satisfy property 16.41. Approximations are ultimately polynomials in 1, ξ, η, and $\xi\eta$. As a result, bilinear basis functions constitute a complete set of linear functions, but an incomplete set of quadratics.

Two-dimensional biquadratic basis functions are products of the one-dimensional quadratic basis functions 16.23. Since they involve terms in

$$1,\ \xi,\ \eta,\ \xi^2,\ \xi\eta,\ \eta^2,\ \xi^2\eta,\ \xi\eta^2,\ \xi^2\eta^2,$$

they are a complete set of quadratic basis functions, but an incomplete set of cubics and an incomplete set of quartics. They use the nine nodes in Figure 16.27, eight boundary nodes and one interior.

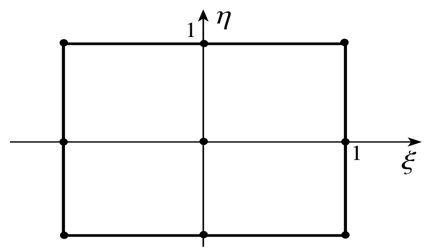


Figure 16.27

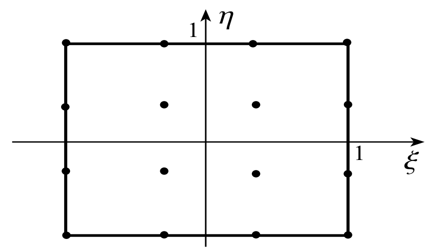


Figure 16.28

Two-dimensional bicubic basis functions are products of the one-dimensional cubic basis functions in Exercise 3 of Section 16.4. Since they involve terms in

$$1,\ \xi,\ \eta,\ \xi^2,\ \xi\eta,\ \eta^2,\ \xi^3,\ \xi^2\eta,\ \xi\eta^2,\ \eta^3,\ \xi^3\eta,\ \xi^2\eta^2,\ \xi\eta^3,\ \xi^3\eta^2,\ \xi^2\eta^3,\ \xi^3\eta^3,$$

they are a complete set of cubic basis functions, but an incomplete set of quartics, quintics, and sextics. They use the sixteen nodes in Figure 16.28, twelve boundary nodes and four interior.

Serendipity Basis Functions

Serendipity basis functions use only boundary nodes of the quadrilateral and in the process remove some of the incomplete terms in the Lagrange basis functions. The Lagrange bilinear basis functions use only the boundary nodes in Figure 16.26, and hence will remain as serendipity functions for the four-node quadrilateral. The bilinear quadratics, however, use a centre node (Figure 16.27), and will be replaced. Boundary nodes for the complete quadratic serendipity basis functions are shown in Figure 16.29. Because there are eight nodes, basis functions are chosen to involve the terms

$$1,\ \xi,\ \eta,\ \xi^2,\ \xi\eta,\ \eta^2,\ \xi^2\eta,\ \xi\eta^2.$$

They are to satisfy conditions 16.3, and we could use these conditions to generate them. For instance, if we take the basis function $\phi_1(\xi,\eta)$ associated with node 1 as

$$\phi_1(\xi,\eta) = a + b\xi + c\eta + d\xi^2 + e\xi\eta + f\eta^2 + g\xi^2\eta + h\xi\eta^2,$$

then substituting each of the nodes gives eight equations in the eight coefficients. Alternatively, we can obtain the basis functions by inspection. For $\phi_1(\xi,\eta)$ to vanish at nodes 2, 6, 3, 7, and 4, we include factors of $1-\xi$ and $1-\eta$. For vanishing at nodes 5 and 8, we include the factor $1+\xi+\eta$. In other words, $\phi(\xi,\eta)$ must be of the form

$$\phi(\xi,\eta) = a(1-\xi)(1-\eta)(1+\xi+\eta).$$

To meet the requirement $\phi(-1,-1)=1$, a must be $-1/4$, and therefore

$$\phi_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta).$$

A similar procedure gives the remaining seven serendipity basis functions for the eight-node quadrilateral,

$$\phi_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta), \tag{16.52a}$$

$$\phi_2(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta), \tag{16.52b}$$

$$\phi_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta), \tag{16.52c}$$

$$\phi_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta), \tag{16.52d}$$

$$\phi_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta), \qquad \phi_6(\xi,\eta) = \frac{1}{2}(1+\xi)(1-\eta^2), \tag{16.52e}$$

$$\phi_7(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta), \qquad \phi_8(\xi,\eta) = \frac{1}{2}(1-\xi)(1-\eta^2). \tag{16.52f}$$

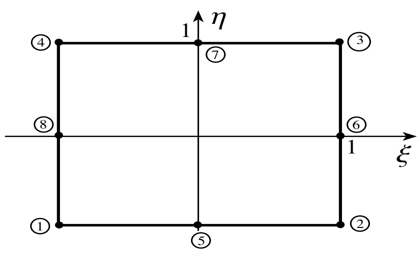


Figure 16.29

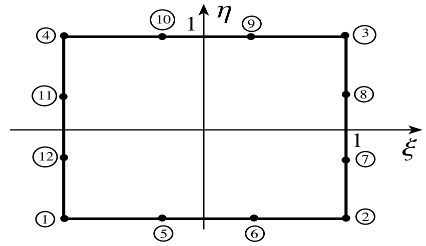


Figure 16.30

Serendipity functions for the quadrilateral in Figure 16.30, (the quadrilateral in Figure 16.28 stripped of interior nodes), are developed in Exercise 1.

The parent square in Figure 16.25 can be mapped to an arbitrary quadrilateral with an isoparametric mapping. For the four-node square in Figure 16.26 to be mapped to the four-node quadrilateral in Figure 16.31, we use

$$x = \phi_1(\xi,\eta)x_1 + \phi_2(\xi,\eta)x_2 + \phi_3(\xi,\eta)x_3 + \phi_4(\xi,\eta)x_4, \tag{16.53a}$$

$$y = \phi_1(\xi,\eta)y_1 + \phi_2(\xi,\eta)y_2 + \phi_3(\xi,\eta)y_3 + \phi_4(\xi,\eta)y_4. \tag{16.53b}$$

Additionally, by using more nodes, we can map the parent square to "quadrilaterals" with curved sides. In particular, the isoparametric mapping

$$x = \sum_{i=1}^{8} \phi_i(\xi,\eta)x_i, \qquad y = \sum_{i=1}^{8} \phi_i(\xi,\eta)y_i, \tag{16.54}$$

with basis functions 16.52, maps the eight-node square in Figure 16.29 to the "quadrilateral" in Figure 16.32.

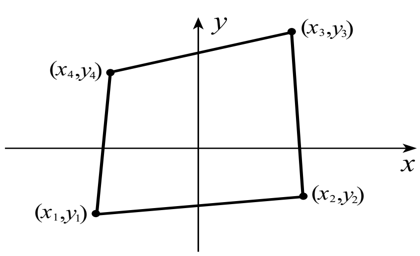


Figure 16.31

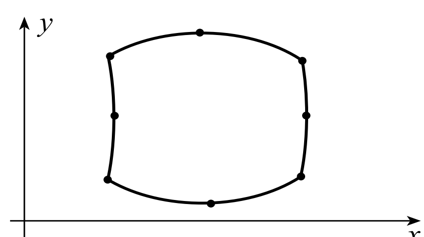


Figure 16.32

EXERCISES 16.9

1. Find the twelve serendipity basis functions for the quadrilateral in Figure 16.30.

§16.10 Finite Elements and Diffusion Problems

We use finite elements, in conjunction with separation of variables, to reduce an initial boundary value problem associated with the two-dimensional diffusion equation to an initial value problem for a system of ODEs. To illustrate, consider the general, one-dimensional, heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{kG(x,t)}{\kappa}, \quad 0 < x < L, \quad t > 0, \tag{16.55a}$$

$$U(0,t) = g(t), \quad t > 0, \tag{16.55b}$$

$$U_x(L,t) = h(t), \quad t > 0, \tag{16.55c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{16.55d}$$

Suppose we create a mesh of N elements $x_{j-1} \le x \le x_j$, $j = 1, \ldots, N$ for the interval $0 \le x \le L$ (with $x_0 = 0$ and $x_N = L$). We pick n basis functions $\phi_i^{(j)}(x)$, $i = 1, \ldots, n$, on the j^{th} element. We approximate the solution of problem 16.55 on the j^{th} element with separated functions

$$U_N^{(j)}(x,t) = \sum_{i=1}^{n} a_i^{(j)}(t)\phi_i^{(j)}(x). \tag{16.56}$$

To ease the notation, we temporarily drop the element designation and write

$$U_N(x,t) = \sum_{i=1}^{n} a_i(t)\phi_i(x). \tag{16.57}$$

Realize, however, that we are about to do is being done on every element. We will insert superscripts later. The residual of this approximation is

$$R = \frac{\partial U_N}{\partial t} - k\frac{\partial^2 U_N}{\partial x^2} - \frac{kG}{\kappa}.$$

Application of Galerkin's method requires

$$\int_{x_{j-1}}^{x_j} \left[\frac{\partial U_N}{\partial t} - k\frac{\partial^2 U_N}{\partial x^2} - \frac{kG}{\kappa}\right]\phi_m\,dx = 0, \quad m = 1, \ldots, n.$$

We apply integration by parts on the second derivative term,

$$0 = -k\left\{\frac{\partial U_N}{\partial x}\phi_m\right\}_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j}\left(k\frac{\partial U_N}{\partial x}\phi_m' + \frac{\partial U_N}{\partial t}\phi_m - \frac{kG}{\kappa}\phi_m\right)dx.$$

We now substitute for $U_N(x,t)$ in the integral,

$$0 = -k\left\{\frac{\partial U_N}{\partial x}\phi_m\right\}_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j}\left[\sum_{i=1}^{n}(ka_i\phi_i'\phi_m' + a_i'\phi_i\phi_m) - \frac{kG}{\kappa}\phi_m\right]dx,$$

or,

$$\sum_{i=1}^{n} a_i'\int_{x_{j-1}}^{x_j}\phi_i\phi_m\,dx + k\sum_{i=1}^{n} a_i\int_{x_{j-1}}^{x_j}\phi_i'\phi_m'\,dx = k\left\{\frac{\partial U_N}{\partial x}\phi_m\right\}_{x_{j-1}}^{x_j} + \frac{k}{\kappa}\int_{x_{j-1}}^{x_j} G\phi_m\,dx. \tag{16.58}$$

When the n basis functions ϕ_m are chosen, these are the element equations for the j^{th} element. They constitute a coupled system of n linear, first-order ODEs in the n coefficients $a_i(t)$. Perhaps it would be appropriate for us to now put on the designation for the j^{th} element,

$$\sum_{i=1}^{n} a_i^{(j)\prime} \int_{x_{j-1}}^{x_j} \phi_i^{(j)} \phi_m^{(j)}\, dx + k \sum_{i=1}^{n} a_i^{(j)} \int_{x_{j-1}}^{x_j} \phi_i^{(j)\prime} \phi_m^{(j)\prime}\, dx$$
$$= k \left\{ \frac{\partial U_N}{\partial x} \phi_m^{(j)} \right\}_{x_{j-1}}^{x_j} + \frac{k}{\kappa} \int_{x_{j-1}}^{x_j} G \phi_m^{(j)}\, dx, \quad j = 1, \ldots, N. \tag{16.59}$$

These element equations will be assembled into system equations for the N elements in the x-mesh. Boundary conditions 16.55b,c are incorporated as follows. The last element equation ($m = n$) of the last element ($j = N$) contains, on the left side,

$$k \left\{ \frac{\partial U_N}{\partial x} \phi_n^{(N)} \right\}_{x_{N-1}}^{x_N} = k \left[\frac{\partial U_N(L,t)}{\partial x} \phi_n^{(N)}(L) - \frac{\partial U_N(x_{N-1},t)}{\partial x} \phi_n^{(N)}(x_{N-1}) \right].$$

Since $\phi_n^{(N)}(x_{N-1}) = 0$ and $\phi_n^{(N)}(L) = 1$, this expression reduces to

$$k \frac{\partial U_N(L,t)}{\partial x}.$$

We substitute from boundary condition 16.55c to replace this term with $kh(t)$, and since this is on the right side of the equation, it is an implicit condition. Boundary condition 16.55b must be incorporated explicitly. On the first element ($j = 1$),

$$U_N(x,t) = \sum_{i=1}^{n} a_i^{(1)}(t) \phi_i^{(1)}(x).$$

Since $\phi_i^{(1)}(0) = 0$ when $i > 1$, and $\phi_1^{(1)}(0) = 1$, this reduces at $x = 0$ to

$$U_N(0,t) = a_1^{(1)}(t).$$

Boundary condition 16.55b is satisfied if we set $a_1^{(1)}(t) = g(t)$. Interelement boundary conditions must also be applied to the system equations. The residual at time $t = 0$ is

$$R_{|t=0} = \sum_{i=1}^{n} a_i(0) \phi_i(x) - f(x).$$

If we subject it to Galerkin's requirement, we obtain

$$0 = \int_{x_{j-1}}^{x_j} \left[\sum_{i=1}^{n} a_i(0) \phi_i(x) - f(x) \right] \phi_m(x)\, dx, \quad m = 1, \ldots, n.$$

This gives n linear equations that can be solved for initial values $a_i(0)$ for differential equations 16.58 in $a_i(t)$,

$$\sum_{i=1}^{n} a_i(0) \int_{x_{j-1}}^{x_j} \phi_i \phi_m\, dx = \int_{x_{j-1}}^{x_j} f \phi_m\, dx, \quad m = 1, \ldots, n. \tag{16.60}$$

In finite element programs, the system ODEs are solved numerically with one of the finite difference techniques discussed in Chapter 14.

§16.11 Finite Elements and the Wave Equation

Material in this section parallels that in Section 16.10 in that both deal with initial boundary value problems. The only difference here is that the wave equation contains a second-order time derivative as opposed to a first-order derivative in the diffusion equation. Consider then the one-dimensional vibration problem

$$\frac{\partial^2 Y}{\partial t^2} = c^2\frac{\partial^2 Y}{\partial x^2} + \frac{F(x,t)}{\rho}, \quad 0 < x < L, \quad t > 0, \tag{16.61a}$$

$$Y(0,t) = h_1(t), \quad t > 0, \tag{16.61b}$$

$$Y(L,t) = h_2(t), \quad t > 0, \tag{16.61c}$$

$$Y(x,0) = f(x), \quad 0 < x < L, \tag{16.61d}$$

$$Y_t(x,0) = g(x), \quad 0 < x < L. \tag{16.61c}$$

Suppose we create a mesh of N elements $x_{j-1} \le x \le x_j$, $j = 1,\ldots,N$ for the interval $0 \le x \le L$ (with $x_0 = 0$ and $x_N = L$). We pick n basis functions $\phi_i^{(j)}(x)$, $i = 1,\ldots,n$, on the j^{th} element. We approximate the solution of problem 16.61 on the j^{th} element with separated functions

$$Y_N^{(j)}(x,t) = \sum_{i=1}^{n} a_i^{(j)}(t)\phi_i^{(j)}(x). \tag{16.62}$$

To ease the notation, we temporarily drop the element designation and write

$$Y_N(x,t) = \sum_{i=1}^{n} a_i(t)\phi_i(x). \tag{16.63}$$

Realize, however, that we are about to do is being done on every element. We will insert superscripts later. The residual of this approximation is

$$R = \frac{\partial^2 Y_N}{\partial t^2} - c^2\frac{\partial^2 Y_N}{\partial x^2} - \frac{F}{\rho}.$$

Application of Galerkin's method requires

$$\int_{x_{j-1}}^{x_j}\left[\frac{\partial^2 Y_N}{\partial t^2} - c^2\frac{\partial^2 Y_N}{\partial x^2} - \frac{F}{\rho}\right]\phi_m\,dx = 0, \quad m = 1,\ldots,n.$$

We apply integration by parts on the second derivative term in x,

$$0 = -c^2\left\{\frac{\partial Y_N}{\partial x}\phi_m\right\}_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j}\left(c^2\frac{\partial Y_N}{\partial x}\phi_m' + \frac{\partial^2 Y_N}{\partial t^2}\phi_m - \frac{F}{\rho}\phi_m\right)dx.$$

We now substitute for $Y_N(x,t)$ in the integral,

$$0 = -c^2\left\{\frac{\partial Y_N}{\partial x}\phi_m\right\}_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j}\left[\sum_{i=1}^{n}(c^2 a_i\phi_i'\phi_m' + a_i''\phi_i\phi_m) - \frac{F}{\rho}\phi_m\right]dx,$$

or,

$$\sum_{i=1}^{n} a_i''\int_{x_{j-1}}^{x_j}\phi_i\phi_m\,dx + c^2\sum_{i=1}^{n} a_i\int_{x_{j-1}}^{x_j}\phi_i'\phi_m'\,dx = c^2\left\{\frac{\partial Y_N}{\partial x}\phi_m\right\}_{x_{j-1}}^{x_j} + \frac{1}{\rho}\int_{x_{j-1}}^{x_j}F\phi_m\,dx. \tag{16.64}$$

When the n basis functions ϕ_m are chosen, these are the element equations for the j^{th} element. Perhaps it would be appropriate for us to now put on the designation for the j^{th} element,

$$\sum_{i=1}^{n} a_i^{(j)\prime\prime} \int_{x_{j-1}}^{x_j} \phi_i^{(j)} \phi_m^{(j)}\, dx + c^2 \sum_{i=1}^{n} a_i^{(j)} \int_{x_{j-1}}^{x_j} \phi_i^{(j)\prime} \phi_m^{(j)\prime}\, dx$$
$$= c^2 \left\{ \frac{\partial Y_N}{\partial x} \phi_m^{(j)} \right\}_{x_{j-1}}^{x_j} + \frac{1}{\rho} \int_{x_{j-1}}^{x_j} F \phi_m^{(j)}\, dx, \quad j = 1, \ldots, N. \tag{16.65}$$

These element equations will be assembled into system equations for the N elements in the x-mesh. They constitute a system of coupled, second-order ODEs in the coefficients $a_i^{(j)}(t)$. Boundary conditions 16.61b,c, being Dirichlet, must be incorporated explicitly. On the first element ($j = 1$),

$$Y_N(x,t) = \sum_{i=1}^{n} a_i^{(1)}(t) \phi_i^{(1)}(x).$$

Since $\phi_i^{(1)}(0) = 0$ when $i > 1$, and $\phi_1^{(1)}(0) = 1$, this reduces at $x = 0$ to

$$Y_N(0,t) = a_1^{(1)}(t).$$

Boundary condition 16.61b is satisfied if we set $a_1^{(1)}(t) = h_1(t)$. On the last element ($j = N$),

$$Y_N(x,t) = \sum_{i=1}^{n} a_i^{(N)}(t) \phi_i^{(N)}(x).$$

Since $\phi_i^{(N)}(0) = 0$ when $i < n$, and $\phi_n^{(N)}(L) = 1$, this reduces at $x = L$ to

$$Y_N(L,t) = a_n^{(N)}(t).$$

Boundary condition 16.61c is satisfied if we set $a_n^{(N)}(t) = h_2(t)$. Interelement boundary conditions must also be applied to the system equations. The displacement residual at time $t = 0$ is

$$R_{|t=0} = \sum_{i=1}^{n} a_i(0) \phi_i(x) - f(x).$$

If we subject it to Galerkin's requirement, we obtain

$$0 = \int_{x_{j-1}}^{x_j} \left[\sum_{i=1}^{n} a_i(0) \phi_i(x) - f(x) \right] \phi_m(x)\, dx, \quad m = 1, \ldots, n.$$

This gives n linear equations that can be solved for initial values $a_i(0)$ for differential equations 16.65 in $a_i(t)$,

$$\sum_{i=1}^{n} a_i(0) \int_{x_{j-1}}^{x_j} \phi_i \phi_m\, dx = \int_{x_{j-1}}^{x_j} f \phi_m\, dx, \quad m = 1, \ldots, n. \tag{16.66}$$

The velocity residual at time $t = 0$ is

$$R_{|t=0} = \sum_{i=1}^{n} a_i'(0)\phi_i(x) - g(x).$$

If we subject it to Galerkin's requirement, we obtain

$$0 = \int_{x_{j-1}}^{x_j} \left[\sum_{i=1}^{n} a_i'(0)\phi_i(x) - g(x) \right] \phi_m(x)\, dx, \quad m = 1, \ldots, n.$$

This gives n linear equations that can be solved for initial values $a_i'(0)$ for differential equations 16.65,

$$\sum_{i=1}^{n} a_i'(0) \int_{x_{j-1}}^{x_j} \phi_i \phi_m \, dx = \int_{x_{j-1}}^{x_j} g\phi_m \, dx, \quad m = 1, \ldots, n. \tag{16.67}$$

In finite element programs, the system ODEs are solved numerically with one of the finite difference techniques discussed in Chapter 14.

APPENDIX A Convergence of Fourier Series

In order to establish convergence of a Fourier series to the function that it represents, we require a few preliminary results on trigonometric integrals. These results are formulated so as to make them useful for Fourier integrals in Appendix B as well.

Theorem A.1 (**Riemann's Theorem**) If $f(x)$ is piecewise continuous on $a \le x \le b$, then

$$\lim_{\lambda\to\infty} \int_a^b f(x)\sin\lambda x\, dx = 0 = \lim_{\lambda\to\infty} \int_a^b f(x)\cos\lambda x\, dx. \tag{A.1}$$

Proof The interval $a \le x \le b$ can be divided into a finite number of subintervals $p \le x \le q$ in each of which $f(x)$ is continuous even at the end points, provided we use the limits from the interior as values of $f(x)$ at the end points. The theorem then follows if we can show that

$$\lim_{\lambda\to\infty} \int_p^q f(x)\sin\lambda x\, dx = 0 = \lim_{\lambda\to\infty} \int_p^q f(x)\cos\lambda x\, dx$$

for continuous $f(x)$ on $p \le x \le q$. If we divide this interval into n equal parts by points $x_j = p + (q-p)j/n$, $j = 0, \ldots, n$, then

$$\begin{aligned}\int_p^q f(x)\sin\lambda x\, dx &= \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} f(x)\sin\lambda x\, dx \\ &= \sum_{j=0}^{n-1} \left[f(x_j) \int_{x_j}^{x_{j+1}} \sin\lambda x\, dx + \int_{x_j}^{x_{j+1}} [f(x) - f(x_j)]\sin\lambda x\, dx \right] \\ &= \sum_{j=0}^{n-1} f(x_j) \left(\frac{\cos\lambda x_j - \cos\lambda x_{j+1}}{\lambda} \right) + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} [f(x) - f(x_j)]\sin\lambda x\, dx.\end{aligned}$$

When we use the triangle inequality, $|a+b| \le |a| + |b|$, on each of these summations, and note that $|\sin\lambda x| \le 1$, we obtain

$$\left| \int_p^q f(x)\sin\lambda x\, dx \right| \le \sum_{j=0}^{n-1} |f(x_j)| \left| \frac{\cos\lambda x_j - \cos\lambda x_{j+1}}{\lambda} \right| + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} |f(x) - f(x_j)|\, dx.$$

Clearly, $|\cos\lambda x_j - \cos\lambda x_{j+1}| \le |\cos\lambda x_j| + |\cos\lambda x_{j+1}| \le 2$, and if we denote the maximum value of $|f(x)|$ on $p \le x \le q$ by M, then

$$\left| \int_p^q f(x)\sin\lambda x\, dx \right| \le \frac{2Mn}{\lambda} + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} |f(x) - f(x_j)|\, dx.$$

Because a continuous function $[f(x)]$ on a closed interval $[p \le x \le q]$ is uniformly continuous thereon, we can state that corresponding to any number $\epsilon > 0$, no matter how small, there exists an N large enough that when $n > N$ and $x_j \le x \le x_{j+1}$,

$$|f(x) - f(x_j)| < \frac{\epsilon}{2(q-p)}.$$

For $n > N$, then,

$$\left|\int_p^q f(x)\sin\lambda x\,dx\right| \le \frac{2Mn}{\lambda} + \sum_{j=0}^{n-1}\frac{\epsilon}{2(q-p)}(x_{j+1}-x_j) = \frac{2Mn}{\lambda} + \frac{\epsilon}{2}.$$

Finally, if λ is chosen so large that $2Mn/\lambda < \epsilon/2$, then

$$\left|\int_p^q f(x)\sin\lambda x\,dx\right| < \epsilon;$$

that is, λ can be chosen so large that the value of the integral can be made arbitrarily close to zero. This is tantamount to saying that

$$\lim_{\lambda\to\infty}\int_p^q f(x)\sin\lambda x\,dx = 0.$$

A similar proof yields the other limit. ∎

When λ is set equal to $n\pi/L$, we obtain the following corollary to Theorem A.1.

Corollary If $f(x)$ is piecewise continuous on $0 \le x \le 2L$, then

$$\lim_{n\to\infty}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx = 0 = \lim_{n\to\infty}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx. \tag{A.2}$$

Theorem A.2 If $f(x)$ is piecewise continuous on $0 \le x \le b$ and has a right derivative at $x = 0$, then

$$\lim_{\lambda\to\infty}\int_0^b f(x)\frac{\sin\lambda x}{x}dx = \frac{\pi}{2}f(0+). \tag{A.3}$$

Proof We begin by expressing the integral in the form

$$\int_0^b f(x)\frac{\sin\lambda x}{x}dx = \int_0^b\left[\frac{f(x)-f(0+)}{x}\right]\sin\lambda x\,dx + f(0+)\int_0^b\frac{\sin\lambda x}{x}dx. \, (A.4)$$

Now the function $[f(x) - f(0+)]/2$ is piecewise continuous on $0 \le x \le b$ (since $f(x)$ is), and provided we define the value at $x = 0$ by the limit that is the right derivative of $f(x)$ at $x = 0$). Hence, by Riemann's theorem

$$\lim_{\lambda\to\infty}\int_0^b\left[\frac{f(x)-f(0+)}{x}\right]\sin\lambda x\,dx = 0,$$

and the first integral on the right of A.4 vanishes in the limit as $\lambda \to \infty$. Further, by the change of variable $u = \lambda x$ in the second integral, we find that

$$\lim_{\lambda\to\infty}\int_0^b\frac{\sin\lambda x}{x}dx = \lim_{\lambda\to\infty}\int_0^{b\lambda}\frac{\sin u}{u}du = \int_0^\infty\frac{\sin u}{u}du = \frac{\pi}{2}^*.$$

Consequently, the limit of A.4 as $\lambda \to \infty$ yields A.3. ∎

*This integral is quoted in many sources. See, for example, any edition of *Standard Mathematical Tables* by Chemical Rubber Publishing Company.

Theorem A.3 If $f(x)$ is piecewise continuous on $a \leq x \leq b$, then at every x in $a < x < b$ at which $f(x)$ has a right and left derivative,

$$\lim_{\lambda\to\infty} \frac{1}{\pi}\int_a^b f(t)\frac{\sin\lambda(x-t)}{x-t}dt = \frac{f(x+)+f(x-)}{2}. \tag{A.5}$$

Proof We begin by subdividing the interval of integration,

$$\int_a^b f(t)\frac{\sin\lambda(x-t)}{x-t}dt = \int_a^x f(t)\frac{\sin\lambda(x-t)}{x-t}dt + \int_x^b f(t)\frac{\sin\lambda(x-t)}{x-t}dt,$$

and make the changes of variables $u = x - t$ and $u = t - x$, respectively,

$$\int_a^b f(t)\frac{\sin\lambda(x-t)}{x-t}dt = \int_0^{x-a} f(x-u)\frac{\sin\lambda u}{u}du + \int_0^{b-x} f(x+u)\frac{\sin\lambda u}{u}du.$$

For fixed x, $f(x-u)$ is piecewise continuous in u on $0 \leq u \leq x-a$ and has a right derivative at $u = 0$ (namely, the negative of the left derivative of $f(x)$ at x). It follows, then, from Theorem A.2 that

$$\lim_{\lambda\to\infty}\int_0^{x-a} f(x-u)\frac{\sin\lambda u}{u}du = \frac{\pi}{2}f(x-).$$

A similar discussion yields

$$\lim_{\lambda\to\infty}\int_0^{b-x} f(x+u)\frac{\sin\lambda u}{u}du = \frac{\pi}{2}f(x+),$$

and these two facts give the theorem. ∎

We are now prepared to prove Theorem 3.2 in Section 3.1.

Theorem A.4 If $f(x)$ is piecewise continuous and of period $2L$, then at every x at which $f(x)$ has a right and left derivative, the Fourier series of $f(x)$ converges to $[f(x+)+f(x-)]/2$.

Proof The n^{th} partial sum of the Fourier series of $f(x)$ is

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right).$$

Substitutions from definitions 3.12 in Section 3.1 for a_0, a_k, and b_k yield

$$\begin{aligned}
S_n(x) &= \frac{1}{2L}\int_0^{2L} f(t)\,dt + \sum_{k=1}^{n}\left[\cos\frac{k\pi x}{L}\frac{1}{L}\int_0^{2L} f(t)\cos\frac{k\pi t}{L}dt + \sin\frac{k\pi x}{L}\frac{1}{L}\int_0^{2L} f(t)\sin\frac{k\pi t}{L}dt\right]\\
&= \frac{1}{L}\int_0^{2L}\left[\frac{1}{2}f(t) + \sum_{k=1}^{n} f(t)\left(\cos\frac{k\pi x}{L}\cos\frac{k\pi t}{L} + \sin\frac{k\pi x}{L}\sin\frac{k\pi t}{L}\right)\right]dt\\
&= \frac{1}{L}\int_0^{2L} [f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos\frac{k\pi(x-t)}{L}\right]dt\\
&= \frac{1}{L}\int_0^{2L} f(t)\frac{\sin\dfrac{(n+1/2)\pi(x-t)}{L}}{2\sin\dfrac{\pi(x-t)}{2L}}dt.^*
\end{aligned}$$

Since the integrand is of period $2L$, we may integrate over any interval of length $2L$. We choose an interval beginning at a, where $a < x < a + 2L$, and rearrange the integrand into the following form,

$$S_n(x) = \frac{1}{L}\int_a^{a+2L}\left[f(t)\frac{x-t}{2\sin\dfrac{\pi(x-t)}{2L}}\right]\frac{\sin\dfrac{(n+1/2)\pi(x-t)}{L}}{x-t}\,dt.$$

In order to take limits as $n \to \infty$ and apply Theorem A.3, we require piecewise continuity of

$$F(t) = f(t)\frac{x-t}{2\sin\dfrac{\pi(x-t)}{2L}}$$

on $a \le t \le a + 2L$ and existence of both of its one-sided derivatives at $t = x$ (x fixed). This will follow if

$$\frac{x-t}{2\sin\dfrac{\pi(x-t)}{2L}}$$

has these properties (since $f(t)$ has, by assumption). Since $t = x$ is the only point in the interval $a \le t \le a + 2L$ at which the denominator of this function vanishes, it follows that it is indeed piecewise continuous thereon. Furthermore, it is easily shown that this function has a right and left derivative at $t = x$. By Theorem A.3, then,

$$\begin{aligned}\lim_{n\to\infty} S_n(x) &= \lim_{n\to\infty}\frac{1}{L}\int_a^{a+2L}\left[f(t)\frac{x-t}{2\sin\dfrac{\pi(x-t)}{2L}}\right]\frac{\sin\dfrac{(n+1/2)\pi(x-t)}{L}}{x-t}\,dt\\ &= \frac{\pi}{L}\lim_{n\to\infty}\frac{1}{\pi}\int_a^{a+2L}F(t)\frac{\sin\dfrac{(2n+1)\pi(x-t)}{2L}}{x-t}\,dt = \frac{\pi}{2L}[F(x+)+F(x-)].\end{aligned}$$

Since $F(x+) = \lim_{t\to x^+} F(t) = f(x+)(L/\pi)$, and similarly for $F(x-)$, it follows that

$$\lim_{n\to\infty} S_n(x) = \frac{\pi}{2L}\left[\frac{L}{\pi}f(x+) + \frac{L}{\pi}f(x-)\right] = \frac{f(x+)+f(x-)}{2}. \blacksquare$$

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*We have used the identity

$$\frac{1}{2} + \sum_{k=1}^{n}\cos k\theta = \frac{\sin{(n+1/2)\theta}}{2\sin{(\theta/2)}}.$$

This formula can be established by expressing $\cos k\theta$ as a complex exponential $(e^{ik\theta} + e^{-ik\theta})/2$ and summing the two resulting geometric series. The identity is regarded in the limit sense at angles for which $\sin{(\theta/2)} = 0$.

APPENDIX B Convergence of Fourier Integrals

In order to establish convergence of a Fourier integral to the function that it represents, we require some preliminary results on trigonometric integrals. They parallel and utilize analogous properties in Appendix A.

Theorem B.1 (**Riemann's Theorem**) If $f(x)$ is piecewise continuous on every finite interval and absolutely integrable on $-\infty < x < \infty$, then

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx = 0 = \lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx. \tag{B.1}$$

Proof Since

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx = \lim_{\lambda\to\infty}\left[\lim_{\substack{r\to\infty\\ s\to\infty}}\int_{-s}^{r} f(x)\sin\lambda x\,dx\right],$$

and the limit on r and s is absolutely and uniformly convergent with respect to λ, limits may be reversed,

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx = \lim_{\substack{r\to\infty\\ s\to\infty}}\left[\lim_{\lambda\to\infty}\int_{-s}^{r} f(x)\sin\lambda x\,dx\right].$$

But Riemann's theorem for finite intervals (Theorem A.1 in Appendix A) implies that the integral on the right converges for all r and s. ∎

Theorem B.2 If $f(x)$ is piecewise continuous on every finite interval and absolutely integrable on $-\infty < x < \infty$, then at every x at which $f(x)$ has a right and left derivative,

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\lambda(x-t)}{x-t}dt = \frac{f(x+)+f(x-)}{2}. \tag{B.2}$$

Proof For each fixed x, the function

$$F(t) = f(t)\frac{\sin\lambda(x-t)}{x-t}$$

is piecewise continuous in t on every finite interval (provided we define $F(x)$ by the limit as t approaches x). Further, since

$$|F(t)| = |\lambda||f(t)|\left|\frac{\sin\lambda(x-t)}{\lambda(x-t)}\right| \le |\lambda||f(t)|,$$

and $f(t)$ is absolutely integrable on $-\infty < t < \infty$, it follows that the improper integral

$$\int_{-\infty}^{\infty} F(t)\,dt = \int_{-\infty}^{\infty} f(t)\frac{\sin\lambda(x-t)}{x-t}dt$$

converges. If a and b are numbers such that $a < x < b$, then

$$\left|\int_{-\infty}^{\infty} F(t)\,dt - \pi\left[\frac{f(x+)+f(x-)}{2}\right]\right| \le \int_{-\infty}^{a} |F(t)|\,dt + \left|\int_{a}^{b} F(t)\,dt - \pi\left[\frac{f(x+)+f(x-)}{2}\right]\right| + \int_{b}^{\infty} |F(t)|\,dt.$$

Now,

$$\int_{-\infty}^{a} |F(t)|\, dt \leq \int_{-\infty}^{a} \frac{|f(t)|}{|x-t|} dt \leq \frac{1}{x-a} \int_{-\infty}^{a} |f(t)|\, dt.$$

Given any $\epsilon > 0$, there exists $a(\epsilon) < 0$, independent of λ, such that

$$\int_{-\infty}^{a} |F(t)|\, dt \leq \frac{1}{x-a} \int_{-\infty}^{a} |f(t)|\, dt < \frac{\epsilon}{3}.$$

Similarly, there exists $b(\epsilon) > 0$ such that

$$\int_{b}^{\infty} |F(t)|\, dt < \frac{\epsilon}{3}.$$

Since $f(t)$ is piecewise continuous on $a \leq t \leq b$ and has both one-sided derivatives at $t = x$, $a < x < b$, we have from Theorem A.3 in Appendix A that

$$\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{a}^{b} f(t) \frac{\sin \lambda(x-t)}{x-t} dt = \frac{f(x+) + f(x-)}{2};$$

that is, there exists $\lambda(\epsilon)$ such that whenever $\lambda > \lambda(\epsilon)$,

$$\left| \int_{a}^{b} f(t) \frac{\sin \lambda(x-t)}{x-t} dt - \pi \left[\frac{f(x+) + f(x-)}{2} \right] \right| < \frac{\epsilon}{3}.$$

Combining these three results, we have, for $\lambda > \lambda(\epsilon)$,

$$\left| \int_{-\infty}^{\infty} F(t)\, dt - \pi \left[\frac{f(x+) + f(x-)}{2} \right] \right| < \epsilon.$$

Since ϵ can be made arbitrarily small, it follows that

$$\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \lambda(x-t)}{x-t} dt = \frac{f(x+) + f(x-)}{2}. \blacksquare$$

We can now establish Theorem 11.1 in Section 11.2.

Theorem B.3 (**Fourier Integral Theorem**) If $f(x)$ is piecewise continuous on every finite interval and absolutely integrable on $-\infty < x < \infty$, then at every x at which $f(x)$ has a right- and left-derivative,

$$\frac{f(x+) + f(x-)}{2} = \int_{0}^{\infty} [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x]\, d\lambda \tag{B.3a}$$

when

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \lambda x\, dx, \qquad B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \lambda x\, dx. \tag{B.3b}$$

Proof By Theorem B.2, we may write

$$\frac{f(x+) + f(x-)}{2} = \lim_{\alpha \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \alpha(x-t)}{x-t} dt.$$

Since

$$\int_0^\alpha \cos\lambda(x-t)\,d\lambda = \left\{\frac{\sin\lambda(x-t)}{x-t}\right\}_0^\alpha = \frac{\sin\alpha(x-t)}{x-t},$$

it follows that

$$\begin{aligned}\frac{f(x+)+f(x-)}{2} &= \lim_{\alpha\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left[\int_0^\alpha \cos\lambda(x-t)\,d\lambda\right]dt\\ &= \lim_{\alpha\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^\alpha f(t)\cos\lambda(x-t)\,d\lambda\,dt.\end{aligned}$$

Since

$$\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\,dt$$

is uniformly convergent with respect to λ, we may interchange the order of integration and write

$$\begin{aligned}\frac{f(x+)+f(x-)}{2} &= \lim_{\alpha\to\infty}\frac{1}{\pi}\int_0^\alpha\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\,dt\,d\lambda\\ &= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\,dt\,d\lambda\\ &= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)[\cos\lambda x\cos\lambda t+\sin\lambda x\sin\lambda t]\,dt\,d\lambda.\end{aligned}$$

This is the result in equation B.3. ∎

APPENDIX C Vector Analysis

In this appendix we briefly mention the theorems from vector analysis that are used throughout the book.

When $f(x,y,z)$ is a scalar function with first partial derivatives in some region V of space, its gradient is a vector-valued function defined by

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}. \tag{C.1}$$

This a very important vector in applied mathematics, principally due to the properties stated in the following theorem.

Theorem C.1 The directional derivative of a function $f(x,y,z)$ in any direction is the component of ∇f in that direction. Furthermore, $f(x,y,z)$ increases most rapidly in the direction ∇f, and its rate of change in this direction is $|\nabla f|$.

When $\mathbf{F}(x,y,z) = P(x,y,z)\hat{\mathbf{i}}+Q(x,y,z)\hat{\mathbf{j}}+R(x,y,z)\hat{\mathbf{k}}$ is a vector function with first partial derivatives in some region V, its divergence and curl are defined as

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}, \tag{C.2}$$

and

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}. \tag{C.3}$$

The gradient, divergence, and curl are linear operators that satisfy the following identities:

$$\nabla(fg) = f\nabla g + g\nabla f, \tag{C.4a}$$

$$\nabla\cdot(f\mathbf{F}) = \nabla f\cdot\mathbf{F} + f\nabla\cdot\mathbf{F}, \tag{C.4b}$$

$$\nabla\times(f\mathbf{F}) = \nabla f\times\mathbf{F} + f(\nabla\times\mathbf{F}), \tag{C.4c}$$

$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G}), \tag{C.4d}$$

$$\nabla\times(\nabla f) = \mathbf{0}, \tag{C.4e}$$

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0, \tag{C.4f}$$

provided $f(x,y,z)$ and the components of vectors are sufficiently differentiable.

The line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

of a continuous vector function $\mathbf{F} = P\hat{\mathbf{i}}+Q\hat{\mathbf{j}}+R\hat{\mathbf{k}}$ along a smooth curve C can always be evaluated by substituting from parametric equations for C and evaluating the resulting definite integral. For example, the value of

$$\int_C y\,dx + x\,dy + z\,dz$$

along the curve C: $x = t^2$, $y = t+1$, $z = 3t$, $0 \le t \le 1$, can be calculated with the definite integral

$$\int_0^1 (t+1)(2t\,dt) + t^2\,dt + 3t(3\,dt) = \int_0^1 (3t^2 + 11t)\,dt = \frac{13}{2}.$$

In the event that a line integral is independent of path, and this occurs when $\mathbf{F}$ is the gradient of some scalar function $f(x,y,z)$, the value of the line integral is the difference in the values of $f(x,y,z)$ at terminal and initial points. The above line integral is independent of path since $\nabla(xy + z^2/2) = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and therefore

$$\int_C y\,dx + x\,dy + z\,dz = \left\{xy + \frac{z^2}{2}\right\}_{(0,1,0)}^{(1,2,3)} = \frac{13}{2}.$$

The surface integral

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

of the normal component of a vector function $\mathbf{F}(x,y,z)$ over a smooth surface S with unit normal vector $\hat{\mathbf{n}}$ is usualy evaluated by projecting the surface in a one-to-one fashion onto a coordinate plane, expressing $\mathbf{F}\cdot\hat{\mathbf{n}}$ and dS in terms of coordinates in this plane, and evaluating the resulting double integral. For example, when $\mathbf{F} = x^2y\hat{\mathbf{i}} + xz\hat{\mathbf{j}}$ and when $\hat{\mathbf{n}}$ is the upper normal to the surface S: $z = 4 - x^2 - y^2$, $z \ge 0$, it is appropriate to project S onto the xy-plane (Figure C.1). The unit upper normal to S is

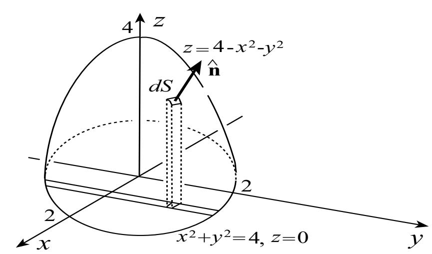


Figure C.1

$$\hat{\mathbf{n}} = \frac{\nabla(z - 4 + x^2 + y^2)}{|\nabla(z - 4 + x^2 + y^2)|} = \frac{(2x, 2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$

The relationship between a rectangular area $dy\,dx$ in the xy-plane, and its projection dS on S is

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dy\,dx = \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx.$$

Since S projects onto the circle $x^2 + y^2 \le 4$, the value of the surface integral of the normal component of $\mathbf{F}$ over S is

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2y, xz, 0)\cdot\frac{(2x,2y,1)}{\sqrt{1+4x^2+4y^2}}\sqrt{1+4x^2+4y^2}\,dy\,dx\\ &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x^3y + 2xyz)\,dy\,dx\\ &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [2x^3y + 2xy(4 - x^2 - y^2)]\,dy\,dx\end{aligned}$$

$$= 0.$$

When a surface does not project in a one-to-one fashion onto a coordinate plane (such would be the case, for example, if the surface were closed), it must divided into subsurfaces that do project one-to-one. Alternatively, if the surface is indeed closed, the surface integral can be replaced by a triple integral over its interior. This is the result of the following theorem.

Theorem C.2 (**Divergence Theorem**) Let S be a piecewise smooth surface enclosing a volume V. Let $\mathbf{F}(x, y, z)$ be a vector function whose components have continuous first partial derivatives in an open region containing S in its interior. If $\hat{\mathbf{n}}$ is the unit outward-pointing normal to S, then

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV. \tag{C.5}$$

For example, consider evaluating the surface integral of $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ over the surface S that encloses the volume described by $x^2 + y^2 \le 4$, $0 \le z \le 2$ (Figure C.2). To do so by surface integrals would require that the top and bottom of the cylinder be projected onto the xy-plane and the cylindrical side be divided into two parts, each of which projects one-to-one onto the xz-pane (or yz-plane). Alternatively, the divergence theorem yields

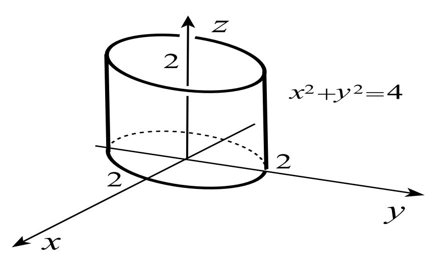


Figure C.2

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V (1 + 1 + 1)\, dV = 3 \iiint_V dV = 3(\text{volume of } V) = 24\pi.$$

If we set $\mathbf{F} = u\nabla v$ in equation C.5, where u and v are arbitrary functions of x, y, and z, and use identity C.4b, we immediately obtain

$$\oiint_S (u\nabla v) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (u\nabla^2 v + \nabla u \cdot \nabla v)\, dV. \tag{C.6}$$

This result is called **Green's first identity**. When u and v are interchanged in this equation and the equations are subtracted, the result is called **Green's second identity**,

$$\oiint_S (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (u\nabla^2 v - v\nabla^2 u)\, dV. \tag{C.7}$$

Stokes's theorem relates line integals around closed curves to surface integals over surfaces that have the curves as boundaries.

Theorem C.3 (**Stokes's Theorem**) Let C be a closed, piecewise smooth, non-self-intersecting curve, and let S be a piecewise smooth (orientable) surface with C as boundary (Figure C.3). Let $\mathbf{F}$ be a vector function whose components have continuous first partial derivatives in an open region that contains S and C in its interior. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS, \tag{C.8}$$

where $\hat{\mathbf{n}}$ is the unit normal to S chosen in the following way. If when moving along C, the surface S is on the left side, then $\hat{\mathbf{n}}$ must be chosen as the unit normal on that side. On the other hand, if when moving along C, the surface is on the right, then $\hat{\mathbf{n}}$ must be chosen on the opposite side of S.

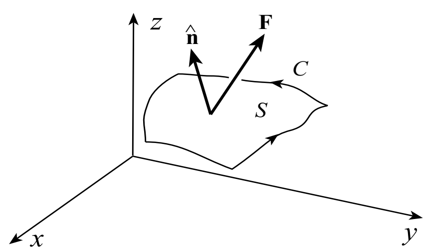


Figure C.3

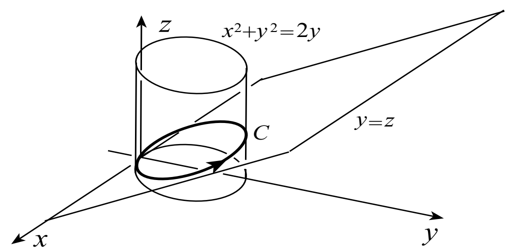


Figure C.4

For example, consider the integral

$$\oint_C y^2\,dx + xy\,dy + xz\,dz,$$

where C is the curve of intersection of the surfaces $x^2+y^2=2y$ and $y=z$, directed so that y increases when x is positive (Figure C.4). If we choose S as that part of the plane $y=z$ interior to C, then

$$\hat{\mathbf{n}} = \frac{\nabla(z-y)}{|\nabla(z-y)|} = \frac{(0,-1,1)}{\sqrt{2}}.$$

Since $\nabla\times\mathbf{F} = (0,-z,-y)$, it follows by Stokes's theorem that

$$\oint_C y^2\,dx + xy\,dy + xz\,dz = \iint_S (0,-z,-y)\cdot\frac{(0,-1,1)}{\sqrt{2}}dS = \iint_S \frac{z-y}{\sqrt{2}}dS = 0,$$

since $z=y$ at every point of S.

When C is a curve in the xy-plane (directed counterclockwise) and S is chosen as that part A of the xy-plane interior to C, we obtain Green's theorem as a special case of Stokes's theorem (Figure C.5),

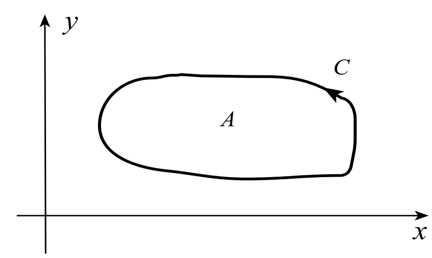


Figure C.5

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy\,dx. \tag{C.9}$$

The two-dimensional version of Green's first identity is obtained from this equation by setting $P=-u\partial v/\partial y$ and $Q=u\partial v/\partial x$, where u and v are functions of x and y,

$$\oint_C (u\nabla v)\cdot\hat{\mathbf{n}}\,ds = \iint_A (u\nabla^2 v + \nabla u\cdot\nabla v)\,dA, \tag{C.10}$$

provided $\hat{\mathbf{n}}$ is the unit outward pointing normal to C. When u and v are interchanged and the equations are subtracted, the result is **Green's second identity** in the plane,

$$\oint_C (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, ds = \iint_A (u\nabla^2 v - v\nabla^2 u)\, dA. \tag{C.11}$$

An alternative form of Green's theorem, which casts it as a two-dimensional version of the divergence theorem is

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_A \nabla \cdot \mathbf{F}\, dA, \tag{C.12}$$

where $\hat{\mathbf{n}}$ is the unit outward-pointing normal to C.

APPENDIX D Complex Functions

In this appendix we present those elements of the theory of functions of a complex variable that are necessary for thorough discussions of Laplace transforms (Chapter 10) and Fourier transforms (Chapter 11). No attempt is made at a complete treatment of the subject. An exhaustive development can be found in the author's text *Introduction to Complex Analysis and Its Applications.*

Complex numbers can be represented in three forms, Cartesian, polar, and exponential. The Cartesian form of a complex number is its representation $z = x+yi$ in terms of its real and imaginary parts x and y. The polar and exponential forms of a complex number express the number in terms of its modulus r and argument θ. They are essentially the same, the exponential form being more compact and operationally more suggestive. The polar form is $z = r(\cos\theta + \sin\theta\, i)$, and the exponential form is $z = re^{\theta i}$, replacing therefore $\cos\theta + \sin\theta\, i$ with $e^{\theta i}$.

A complex function f of the complex variable z is the assignment of a complex number $f(z)$ to each possible value of z. For example, the function might be a polynomial $f(z) = 2z^3-z^2+4$, or a rational function $f(z) = (2z^2-4)/(z^3+5z^2-2z)$. Since $f(z)$ is a complex number, it has real and imaginary parts that we denote by $f(z) = u(x,y)+v(x,y)i$, where $z = x+yi$. For example, when $f(z) = z^2 = (x+yi)^2$, we find that $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. Below is a list of the complex functions that are encountered in the text, identifying their real and imaginary parts:

$$e^z = e^x(\cos y + \sin y\, i), \tag{D.1a}$$

$$\sin z = \sin x \cosh y + \cos x \sinh y\, i, \tag{D.1b}$$

$$\cos z = \cos x \cosh y - \sin x \sinh y\, i, \tag{D.1c}$$

$$\sinh z = \sinh x \cos y + \cosh x \sin y\, i, \tag{D.1d}$$

$$\cosh z = \cosh x \cos y + \sinh x \sin y\, i, \tag{D.1e}$$

$$\sqrt{z} = \sqrt{r}e^{\theta i/2}, \tag{D.1f}$$

where θ is the value of the argument of z (satisfying $0 \le \theta < 2\pi$). This is one of branches of the square root function.

Complex functions can be differentiated and the usual power, product, quotient, and chain rules apply. Derivatives of the above functions are identical to what they are for their real counterparts. A function $f(z)$ is said to be analytic in an open set S if $f'(z)$ exists at each point of S; it is said to be entire, if it is analytic at every point in the complex plane. A complex function is said to be analytic at a point z_0 if it is analytic in some open set containing z_0. One way to verify that a function is analytic in an open set is to show that the first partial derivatives of its real and imaginary parts $u(x,y)$ and $v(x,y)$ are continuous and satisfy the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{D.2}$$

One of the features of complex functions that make them so different from real functions is that if a complex function has a derivative in an open set, then it has derivatives of all orders in that set. Existence of the first derivative of a real function does not guarantee existence of its second derivative, let alone derivatives of all orders. For example, the function $f(x) = x^{5/3}$ has a first derivative for all values of x, but it does not have a second derivative at $x = 0$. Not so for complex functions, existence of a first derivative guarantees existence of all derivatives.

Particularly important about complex functions are their zeros and singularities. A point z_0 is called a **zero** of order m of a complex function f if f is analytic at z_0, and

$$0 = f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0), \quad f^{(m)}(z_0) \neq 0. \tag{D.3}$$

The above representations immediately show that e^z has no zeros, $\sin z$ has zeros of order 1 (also called simple zeros) at $z = n\pi$, where n is an integer, $\cos z$ has simple zeros at $z = (2n+1)\pi/2$, $\sinh z$ has simple zeros at $z = n\pi i$, and $\cosh z$ has simple zeros at $z = (2n+1)\pi i/2$. The square root function $\sqrt{z}$ has no zeros. The rational function $f(z) = (z^3 + z^2)/(z-4)^3$ has a simple zero at $z = -1$ and a zero of order two at $z = 0$.

A point z_0 is called a **singularity** of a complex function f if f is not analytic at z_0, but every neighbourhood of z_0 contains at least one point at which f is analytic. The functions e^z, $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ have no singularities; they are entire functions. The square root function, however, has singularities at $z = 0$ and every point on the positive real axis, its branch cut. A singularity z_0 is said to be isolated, if there exists an open set containing z_0 in which it is the only singularity of the function. The singularities of $\sqrt{z}$ are not isolated. The rational function $f(z) = (z+1)/(z^3 + z)$ has isolated singularities at $z = 0$ and $z = \pm i$.

When a complex function $f(z)$ is analytic at a point z_0, it can be expanded in a Taylor series around z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \tag{D.4}$$

and the series converges to $f(z)$ in every circle that contains only points at which the function is analytic. When $z_0 = 0$, the series is called the Maclaurin series for the function. Below are some useful Maclaurin series together with their circles of convergence:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1, \tag{D.5a}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty, \tag{D.5b}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad |z| < \infty, \tag{D.5c}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \quad |z| < \infty, \tag{D.5d}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad |z| < \infty, \tag{D.5e}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad |z| < \infty. \tag{D.5f}$$

Taylor series can be used to verify the following useful result for determining the order of a zero of a function. A function f has a zero of order m at z_0 if and only if it can be written in the form

$$f(z) = (z - z_0)^m g(z), \tag{D.6}$$

valid in some circle $|z - z_0| < R$, where g is analytic at z_0 and $g(z_0) \neq 0$.

When a complex function $f(z)$ has an isolated singularity at a point z_0, the function can be expanded in a Laurent series valid in an open annulus $0 < |z - z_0| < R$ around the point,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots \tag{D.7}$$

Isolated singularities are classified according to the number of terms in the Laurent series that have negative powers of $z - z_0$:

(i) if $a_n = 0$ for all $n < 0$, z_0 is called a **removable** singularity;
(ii) if $a_n = 0$ for $n < -m$, m a fixed positive integer, but $a_{-m} \neq 0$, z_0 is called a **pole** of
order m;
(iii) if $a_n \neq 0$ for an infinity of negative integers n, z_0 is called an **essential** singularity.

Of particular importance is the coefficient a_{-1} in the Laurent series. It is called the residue of $f(z)$ at $z = z_0$. We can find it by writing out the Laurent series for the function around z_0. Alternatively, when it is known that z_0 is a pole of order m, the residue at z_0 is also given by the formula

$$\text{Res}\,[f(z), z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]. \tag{D.8}$$

To determine the order of the pole of a function, the following result is useful. A function f has a pole of order m at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m}, \tag{D.9}$$

valid in some annulus $0 < |z - z_0| < R$, where g is analytic at z_0 and $g(z_0) \neq 0$.

The counterpart of the definite integral for a real function $f(x)$ is the contour integral of a complex function $f(z)$. It is the integral of $f(z)$ from some point z_0 to another point z_1 along a curve C joining the points, denoted by

$$\int_C f(z)\,dz. \tag{D.10}$$

An indispensable property of contour integrals in applications is the fact that

$$\left| \int_C f(z)\,dz \right| le ML, \tag{D.11}$$

where M is the maximum value of $|f(z)|$ on C, and L is the length of C.

When $f(z)$ and dz are expressed in terms of their real and imaginary parts, this integral can be separated into two real line integrals,

$$\int_C f(z)\,dz = \int_C (u+vi)(dx+dy\,i) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \quad \text{(D.12)}$$

These line integrals can be evaluated by substituting from parametric equations for C; $x = x(t)$, $y = y(t)$, $t_0 \le t \le t_1$, but there are more efficient methods for evaluating contour integrals, one of which is to use Cauchy's residue theorem. It states that if a function f is analytic inside and on a simple, closed, piecewise smooth curve C, except at singularities $z_1, \ldots, z_n$ in its interior, then,

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}[f, z_j]. \quad \text{(D.13)}$$

Another feature of complex functions that make them so different from real functions is that if a function $f(z)$ is analytic in an open set containing a closed curve C, then values of $f(z)$ are completely determined by its values on C. This is a result called Cauchy's integral formula. It states that values of $f(z)$ at points z interior to C are given by the contour integral

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta. \quad \text{(D.14)}$$

Furthermore, values of all derivatives of $f(z)$ at z interior to C are given by the Cauchy's generalized integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta. \quad \text{(D.15)}$$

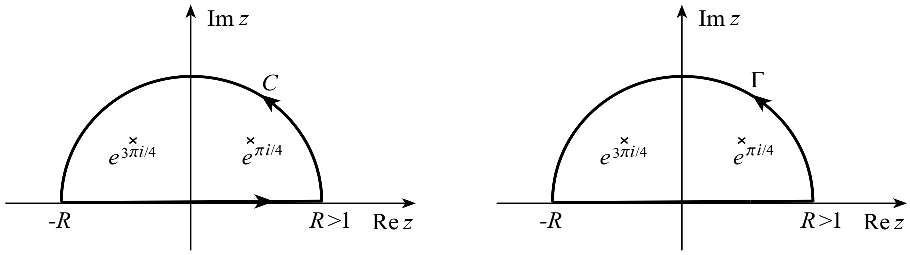
Contour integrals can be used to evaluate many real improper integrals with infinite limits. We illustrate with one example, and then state some useful shortcuts. Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx.$$

It is fairly clear that were we to evaluate the contour integral

$$\oint_C \frac{1}{1+z^4}\,dz$$

where C is the curve in the left figure below, and were we to let $R \to \infty$, then that part of the contour integral along the real axis would give rise to the required improper integral. Let us consider this contour integral then.



The integrand $(1+z^4)^{-1}$ has simple poles at the four fourth roots of -1,

$$e^{\pi i/4}, \qquad e^{3\pi i/4}, \qquad e^{5\pi i/4}, \qquad e^{7\pi i/4},$$

only the first two of which are interior to C. L'Hôpital's rule gives

$$\operatorname{Res}\left[\frac{1}{1+z^4}, e^{\pi i/4}\right] = \lim_{z\to e^{\pi i/4}} \frac{z-e^{\pi i/4}}{1+z^4} = \lim_{z\to e^{\pi i/4}} \frac{1}{4z^3} = \frac{1}{4e^{3\pi i/4}} = -\frac{\sqrt{2}}{8}(1+i).$$

Similarly, $\operatorname{Res}\left[\dfrac{1}{1+z^4}, e^{3\pi i/4}\right] = \dfrac{\sqrt{2}}{8}(1-i)$. By Cauchy's residue theorem then,

$$\oint_C \frac{1}{1+z^4}\,dz = 2\pi i\left[-\frac{\sqrt{2}}{8}(1+i) + \frac{\sqrt{2}}{8}(1-i)\right] = \frac{\pi}{\sqrt{2}}.$$

Suppose we now divide C into a semicircular part Γ and a straight line part (right figure above). Then

$$\frac{\pi}{\sqrt{2}} = \int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_\Gamma \frac{1}{1+z^4}\,dz.$$

If we set $z = Re^{\theta i}$, $0 \le \theta \le \pi$, on Γ, then on the semicircle,

$$\left|\frac{1}{1+z^4}\right| \le \frac{1}{|z^4|-1} = \frac{1}{R^4-1}.$$

Hence, we can say that

$$\left|\int_\Gamma \frac{1}{1+z^4}\,dz\right| \le \frac{1}{R^4-1}(\pi R).$$

It is clear that the limit of this expression is zero as $R\to\infty$, and therefore

$$\frac{\pi}{\sqrt{2}} = \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_\Gamma \frac{1}{1+z^4}\,dz\right) = \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx.$$

The following two results save most of the work in evaluating real improper integrals like these.

Suppose that $P(x)$ and $Q(x)$ are polynomials (of degrees m and n), and $Q(x) \neq 0$ for all real x. When $n \ge m+2$,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i\left\{\begin{array}{l}\text{sum of the residues of } P(z)/Q(z) \text{ at}\\ \text{its poles in the half-plane } \operatorname{Im} z > 0\end{array}\right\}. \qquad \text{(D.16)}$$

Suppose that $P(x)$ and $Q(x)$ are polynomials (of degrees m and n), and $Q(x) \neq 0$ for any real x. When $a > 0$ and $n \ge m+1$,

$$\int_{-\infty}^{\infty} \frac{P(x)\cos ax}{Q(x)}\,dx = -2\pi\,\mathrm{Im}\left\{\begin{array}{l}\text{sum of the residues of } P(z)e^{azi}/Q(z)\\ \text{at its poles in the half-plane } \mathrm{Im}\,z>0\end{array}\right\} \tag{D,17a}$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)\sin ax}{Q(x)}\,dx = 2\pi\,\mathrm{Re}\left\{\begin{array}{l}\text{sum of the residues of } P(z)e^{azi}/Q(z)\\ \text{at its poles in the half-plane } \mathrm{Im}\,z>0\end{array}\right\} \tag{D.17b}$$

APPENDIX E Numerical Answers to Exercises

Exercises 1.1

1. (a) $u = -2x/(x^2y + 6y + 8x)$ (b) $u = -2x/[x^2y + 2x\sin(y/x)]$ (c) $u = -2x/(x^2y + 2ye^{-2y/x})$
2. $u = x$ **3.** $u = 3 - 2/x - 2/y$ **4.** $u = -(1/3)(u - 4y)e^{4(3x+u-4y)/3}$
5. $u = (3/5)(u+y)e^{x-(u+y)/5}$ **6.** $y = 2 - \sqrt{4 + x^2 + y^2 - 2x}$ **7.** $u = (x^2+y^2+2)/(2-2xy)$
8. No solution **9.** (a) $u = -1 + \sqrt{3 + 2c + c^2 - 2xy}$ (b) $u = -1 + \sqrt{1 + 2c + c^2 - 2xy}$ (c) No
10. $u = 2 - 2\sqrt{1 - x + y}$, $y \geq x - 1$ **11.** $u = nx$, n a constant, along the initial curve
12. (a) No solution (b) $u = (a + \sqrt{2x^2 + 2y^2 - a^2})/2$
13. (a) $u = 4\ln(y - x) + x^2/(y - x)^2$ (b) No solution (c) Infinite number of solutions
14. $lx + my + nu = \alpha$, $x^2 + y^2 + u^2 = \beta$
16. $u = -2/(2 + x + y)$ for $y > x$ and $y > -2 - x$; $u = 2/(2 - x - y)$ for $y < x$ and $y < 2 - x$
17. $u = [\tan x + f(y - x)]/[1 - f(y - x)\tan x]$
19. $u = 1$ for $y > x$ and $x < 1$; $u = (1 - x)/(1 - y)$ for $y < x$ and $x < 1$; $u = 0$ for $y < x$ and $x > 1$
20. (a) $u = x$ (b) $u^2 = y + 2(u - x) + (u - x)^2 + (u - x)\sqrt{1 + 4(u - x)}$ when $x > 1/2$; $u^2 = y + 2(u - x) + (u - x)^2 - (u - x)\sqrt{1 + 4(u - x)}$ when $x < 1/2$
21. $y(x - u) = y^2 - \left(\dfrac{x + u}{y} - 1\right)^3 - 2\left(\dfrac{x + u}{y} - 1\right)$
24. No

Exercises 1.2

1. $u = x + 4y$ **2.** $u = -(x \pm \sqrt{2}y)^2/4$ **3.** $u = x^2 + (1 \pm y)^2$ **4.** $u = (y - 1)e^{-x} + e^{-2x}$
5. $u = 2xy - 3y^2/2$
6. (a) $u = 1$ (b) $u = x/(1 - y)$ (c) $x = u(1 + u^3 - y)$ (d) $u = [y - 1 + \sqrt{(y - 1)^2 + 4x}]/2$
8. $u = (x/2)\sqrt{4y + x^2}$
9. $(u - u_0)^2 = 4(x - x_0)(y - y_0)$
10. $u = u_0 + (x - x_0)^2/(2y - 2y_0) + \{u_0 - (x - x_0)^2/[4(y - y_0)^2]\}(y - y_0)$
11. (a) $(u - u_0)^2 = 4u_0(x - x_0)(y - y_0)$ (b)(i) No solution (ii) $u = -[1 \pm (x - y)/2]^2$ (iii) $u = x(y + 1)$
12. (a) $k < -a^2/2$ (b) $u = 2a(a - \sqrt{x^2 + y^2})$
13. (a) $k^2 < 1 - a^2$ (b) $u = (a - \sqrt{x^2 + y^2})/\sqrt{a^2 - 1}$

Exercises 1.3

2. $t = 1/m$
3. (a) $u = x/t - [1 - \sqrt{1 + 4t(t - x)}]/(2t^2)$, $t + 1/(4t)$ (b) Same solution as in part (a), 1/2
8. $t = 1/(2a)$ **9.** $u = g(t - x/c)$ valid for $x > ct$, and $u = f(x - ct)$ valid for $x < ct$
10. $u = f(x - ct)e^{-\beta t}$
11. (a) $u = f(x - ct) + \displaystyle\int_0^t F(x + c(v - t), v)\,dv$ (b) $f(x - ct) + xt + (1 - c)t^2/2$
13. $u = u_0 + n[(x - x_0)\cos\theta + (y - y_0)\sin\theta]$, where $\cos\theta + a\sin\theta = b/n$

Exercises 2.1

1. (a) $V(0,y) = f_1(y),\ 0 < y < L',$
$V(L,y) = f_2(y),\ 0 < y < L',$
$V(x,0) = f_3(x),\ 0 < x < L,$
$V(x,L') = f_4(x),\ 0 < x < L$

(b) $-V_x(0,y) = f_1(y),\ 0 < y < L',$
$V_x(L,y) = f_2(y),\ 0 < y < L',$
$-V_y(x,0) = f_3(x),\ 0 < x < L,$
$V_y(x,L') = f_4(x),\ 0 < x < L$

(c) $-l_1V_x(0,y) + h_1V(0,y) = f_1(y),\ 0 < y < L',$
$l_2V_x(L,y) + h_2V(L,y) = f_2(y),\ 0 < y < L',$
$-l_3V_y(x,0) + h_3V(x,0) = f_3(x),\ 0 < x < L,$
$l_4V_y(x,L') + h_4V(x,L') = f_4(x),\ 0 < x < L$

2. (a) $V(0,y,z) = f_1(y,z),\ y > 0,\ z > 0,$
$V(L,y,z) = f_2(y,z),\ y > 0,\ z > 0,$
$V(x,0,z) = f_3(x,z),\ 0 < x < L,\ z > 0,$
$V(x,y,0) = f_4(x,y),\ 0 < x < L,\ y > 0$

(b) $-V_x(0,y,z) = f_1(y,z),\ y > 0,\ z > 0,$
$V_x(L,y,z) = f_2(y,z),\ y > 0,\ z > 0,$
$-V_y(x,0,z) = f_3(x,z),\ 0 < x < L,\ z > 0$
$-V_z(x,y,0) = f_4(x,y),\ 0 < x < L,\ y > 0$

(c) $-l_1V_x(0,y,z) + h_1V(0,y,z) = f_1(y,z),\ y > 0,\ z > 0,$
$l_2V_x(L,y,z) + h_2V(L,y,z) = f_2(y,z),\ y > 0,\ z > 0,$
$-l_3V_y(x,0,z) + h_3V(x,0,z) = f_3(x,z),\ 0 < x < L,\ z > 0$
$-l_4V_z(x,y,0) + h_4V(x,y,0) = f_4(x,y),\ 0 < x < L,\ y > 0$

3. (a) $V(r_0,\theta) = f(\theta),\ -\pi < \theta \le \pi$ (b) $V_r(r_0,\theta) = f(\theta),\ -\pi < \theta \le \pi$
(c) $lV_r(r_0,\theta) + hV(r_0,\theta) = f(\theta),\ -\pi < \theta \le \pi$

4. (a) $V(r_0,\theta) = f_1(\theta),\ 0 < \theta < \pi,$
$V(r,0) = f_2(r),\ 0 < r < r_0,$
$V(r,\pi) = f_3(r),\ 0 < r < r_0$

(b) $V_r(r_0,\theta) = f_1(\theta),\ 0 < \theta < \pi,$
$-r^{-1}V_\theta(r,0) = f_2(r),\ 0 < r < r_0,$
$r^{-1}V_\theta(r,\pi) = f_3(r),\ 0 < r < r_0$

(c) $l_1V_r(r_0,\theta) + h_1V(r_0,\theta) = f_1(\theta),\ 0 < \theta < \pi,$
$-l_2r^{-1}V_\theta(r,0) + h_2V(r,0) = f_2(r),\ 0 < r < r_0,$
$l_3r^{-1}V_\theta(r,\pi) + h_3V(r,\pi) = f_3(r),\ 0 < r < r_0$

5. (a) $V(r_0,\theta,z) = f_1(\theta,z),\ -\pi < \theta \le \pi,\ z > 0,$
$V(r,\theta,0) = f_2(r,\theta),\ 0 \le r < r_0,\ -\pi < \theta \le \pi$
(b) $V_r(r_0,\theta,z) = f_1(\theta,z),\ -\pi < \theta \le \pi,\ z > 0,$
$-V_z(r,\theta,0) = f_2(r,\theta),\ 0 \le r < r_0,\ -\pi < \theta \le \pi$
(c) $l_1V_r(r_0,\theta,z) + h_1V(r_0,\theta,z) = f_1(\theta,z),\ -\pi < \theta \le \pi,\ z > 0,$
$-l_2V_z(r,\theta,0) + h_2V(r,\theta,0) = f_2(r,\theta),\ 0 \le r < r_0,\ -\pi < \theta \le \pi$

6. (a) $V(r_0,\phi,\theta) = f(\phi,\theta), 0 \le \phi \le \pi, -\pi < \theta \le \pi$
(b) $V_r(r_0,\phi,\theta) = f(\phi,\theta), 0 \le \phi \le \pi, -\pi < \theta \le \pi$
(c) $lV_r(r_0,\phi,\theta) + hV(r_0,\phi,\theta) = f(\phi,\theta), 0 \le \phi \le \pi, -\pi < \theta \le \pi$

7. (a) $V(r_0,\phi,\theta) = f_1(\phi,\theta),\ 0 \le \phi < \pi/2,\ -\pi < \theta \le \pi,$
$V(r,\pi/2,\theta) = f_2(r,\theta),\ 0 \le r < r_0,\ -\pi < \theta \le \pi$
(b) $V_r(r_0,\phi,\theta) = f_1(\phi,\theta),\ 0 \le \phi < \pi/2,\ -\pi < \theta \le \pi,$
$r^{-1}V_\phi(r,\pi/2,\theta) = f_2(r,\theta),\ 0 < r < r_0,\ -\pi < \theta \le \pi$
(c) $l_1V_r(r_0,\phi,\theta) + h_1V(r_0,\phi,\theta) = f_1(\phi,\theta),\ 0 \le \phi < \pi/2,\ -\pi < \theta \le \pi,$
$l_2r^{-1}V_\phi(r,\pi/2,\theta) + h_2V(r,\pi/2,\theta) = f_2(r,\theta),\ 0 < r < r_0,\ -\pi < \theta \le \pi$

Exercises 2.2

1. (b) $-l_1U_x(0,t)+h_1U(0,t)=f_1(t),\ t>0 \quad l_2U_x(L,t)+h_2U(L,t)=f_2(t),\ t>0$

2.
$$\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2},\ 0<x<L,\ t>0,$$
$$U(0,t)=100,\ t>0,$$
$$U(L,t)=100,\ t>0,$$
$$U(x,0)=f(x),\ 0<x<L$$

3.
$$\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2},\ 0<x<L,\ t>0,$$
$$U_x(0,t)=0,\ t>0,$$
$$U(L,t)=100,\ t>0,$$
$$U(x,0)=f(x),\ 0<x<L$$

4.
$$\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2},\ 0<x<L,\ t>0,$$
$$U(0,t)=100,\ t>0,$$
$$U(L,t)=\{100t/T,\ 0<t<T; 100,\ t>T\}$$
$$U(x,0)=f(x),\ 0<x<L$$

5.
$$\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2},\ 0<x<L,\ t>0,$$
$$-\kappa U_x(0,t)+\mu_0 U(0,t)=\mu_0 U_0,\ t>0,$$
$$\kappa U_x(L,t)+\mu_L U(L,t)=\mu_L U_L,\ t>0,$$
$$U(x,0)=f(x),\ 0<x<L$$

6.
$$\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2}+\frac{kg(x,t)}{\kappa},\ 0<x<L,\ t>0,$$
$$U_x(0,t)=0,\ t>0,$$
$$U_x(L,t)=0,\ t>0,$$
$$U(x,0)=f(x),\ 0<x<L$$
$$g(x,t)=\begin{cases}0, & 0<x<L/4,\\ 2.093\times 10^5 q/3, & L/4<x<3L/4,\\ 0, & 3L/4<x<L\end{cases}$$

7.
$$\frac{\partial U}{\partial t}=k\frac{\partial^2 U}{\partial x^2},\ 0<x<L,\ t>0,$$
$$-U_x(0,t)=Q_0/\kappa,\ t>0,$$
$$U_x(L,t)=-Q_L(t)/(\kappa A),\ t>0,$$
$$U(x,0)=f(x),\ 0<x<L$$

8.
$$\frac{\partial U}{\partial t}=k\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right),\ 0<x<L,\ 0<y<L',\ t>0,$$
$$U(0,y,t)=50,\ 0<y<L',\ t>0,$$
$$U_x(L,y,t)=0,\ 0<y<L',\ t>0,$$
$$U_y(x,0,t)=0,\ 0<x<L,\ t>0,$$
$$U(x,L',t)=50,\ 0<x<L,\ t>0,$$
$$U(x,y,0)=f(x,y),\ 0<x<L,\ 0<y<L'$$

9.
$$\frac{\partial U}{\partial t}=k\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{kg(t)}{\kappa},\ 0<x<L,\ 0<y<L',\ t>0,$$
$$U(0,y,t)=50,\ 0<y<L',\ t>0,$$
$$U_x(L,y,t)=0,\ 0<y<L',\ t>0,$$
$$-\kappa U_y(x,0,t)+\mu U(x,0,t)=\mu f_1(t),\ 0<x<L,\ t>0,$$
$$U(x,L',t)=50,\ 0<x<L,\ t>0,$$
$$U(x,y,0)=f(x,y),\ 0<x<L,\ 0<y<L'$$

where $g(t) = \begin{cases} e^{\alpha t}, & 0 < t < T \\ 0, & t > T. \end{cases}$

10. $$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right) + \frac{kg(r)}{\kappa},\ 0 < r < r_0,\ -\pi < \theta \le \pi,\ t > 0,$$

$$\kappa U_r(r_0, \theta, t) + \mu U(r_0, \theta, t) = 0,\ -\pi < \theta \le \pi,\ t > 0,$$

$$U(r, \theta, 0) = f(r, \theta),\ 0 \le r < r_0,\ -\pi < \theta \le \pi,$$

where $g(r) = \begin{cases} 0, & 0 < r < r_1 \\ q, & r_1 < r < r_2 \\ 0, & r_2 < r < r_0 \end{cases}$

11.

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right),\ 0 < r < r_0,\ -\pi < \theta \le \pi,\ t > 0,$$

$$U(r_0, \theta, t) = f_1(\theta, t),\ -\pi < \theta \le \pi,\ t > 0,$$

$$U(r, \theta, 0) = f(r, \theta),\ 0 \le r < r_0,\ -\pi < \theta \le \pi,$$

12.

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}\right),\ 0 < r < r_0,\ -\pi < \theta \le \pi,\ 0 < z < L,\ t > 0,$$

$$U_z(r, \theta, 0, t) = 0,\ 0 \le r < r_0,\ -\pi < \theta \le \pi,\ t > 0$$

$$U_z(r, \theta, L, t) = 0,\ 0 \le r < r_0,\ -\pi < \theta \le \pi,\ t > 0,$$

$$U(r_0, \theta, z, t) = f_1(\theta, t),\ -\pi < \theta \le \pi,\ 0 < z < L,\ t > 0,$$

$$U(r, \theta, z, 0) = f(r, \theta, z),\ 0 \le r < r_0,\ -\pi < \theta \le \pi,\ 0 < z < L$$

13.

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}\right),\ 0 < r < r_0,\ -\pi < \theta \le \pi,\ 0 < z < L,\ t > 0,$$

$$U(r, \theta, 0, t) = 100,\ 0 \le r < r_0,\ -\pi < \theta \le \pi,\ t > 0$$

$$U(r, \theta, L, t) = 100,\ 0 \le r < r_0,\ -\pi < \theta \le \pi,\ t > 0,$$

$$U_r(r_0, \theta, z, t) = 0,\ -\pi < \theta \le \pi,\ 0 < z < L,\ t > 0,$$

$$U(r, \theta, z, 0) = f(r, \theta),\ 0 \le r < r_0,\ -\pi < \theta \le \pi,\ 0 < z < L$$

14. $$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right),\ 0 < r < r_0,\ 0 < z < L,\ t > 0,$$

$$U_z(r, 0, t) = 0,\ 0 \le r < r_0,\ t > 0$$

$$\kappa U_z(r, L, t) + \mu U(r, L, t) = 20\mu,\ 0 \le r < r_0,\ t > 0,$$

$$\kappa U_r(r_0, z, t) + \mu U(r_0, z, t) = 20\mu,\ 0 < z < L,\ t > 0,$$

$$U(r, z, 0) = f(r),\ 0 \le r < r_0,\ 0 < z < L$$

15.

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right),\ 0 < r < r_0,\ t > 0,$$

$$U(r_0, t) = f_1(t),\ t > 0,$$

$$U(r, 0) = f(r),\ 0 \le r < r_0$$

16. $$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right),\ 0 < r < r_0,\ 0 < \theta < \pi,\ t > 0,$$
$$U_r(r_0,\theta,t) = 0,\ 0 < \theta < \pi,\ t > 0,$$
$$U_\theta(r,0,t) = -qr/\kappa,\ 0 < r < r_0,\ t > 0,$$
$$U_\theta(r,\pi,t) = qr/\kappa,\ 0 < r < r_0,\ t > 0,$$
$$U(r,\theta,0) = f(r,\theta),\ 0 < r < r_0,\ 0 < \theta < \pi$$

17. $$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right),\ 0 < r < r_0,\ 0 < \theta < \pi,\ t > 0,$$
$$U_r(r_0,\theta,t) = -q/\kappa,\ 0 < \theta < \pi,\ t > 0,$$
$$-\kappa U_\theta(r,0,t) + \mu r U(r,0,t) = \mu U_0 r,\ 0 < r < r_0,\ t > 0,$$
$$\kappa U_\theta(r,\pi,t) + \mu r U(r,\pi,t) = \mu U_0 r,\ 0 < r < r_0,\ t > 0,$$
$$U(r,\theta,0) = f(r,\theta),\ 0 < r < r_0,\ 0 < \theta < \pi$$

18. $$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right),\ 0 < r < r_0,\ t > 0,$$
$$\kappa U_r(r_0,t) + \mu U(r_0,t) = 10\mu,\ t > 0,$$
$$U(r,0) = 100,\ 0 \le r < r_0$$

19. $$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial U}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 U}{\partial \theta^2} = 0,$$
$$0 < r < r_0,\ 0 < \phi < \pi/2,\ -\pi < \theta \le \pi,$$
$$U_r(r_0,\phi,\theta) = 0,\ 0 \le \phi < \pi/2,\ 0 < \theta < \pi,$$
$$U_\phi(r,\pi/2,\theta) = rf(r,\theta)/\kappa,\ 0 \le r < r_0,\ -\pi < \theta \le \pi$$
No

20. $U(x) = (U_L - U_0)x/L + U_0$ **21.** $U(x) = -q_0 x/\kappa + B$, where B is arbitrary

22. (a) Heat must flow (b) Temperature need not vary with time

23. $U(x) = \dfrac{\mu_0\mu_L(U_L - U_0)x - \kappa\mu_L U_L - \mu_0 U_0(\kappa + L\mu_L)}{\kappa\mu_L + \mu_0(\kappa + L\mu_L)}$

24. (b) $\displaystyle\iint_{\beta(V)} -\kappa f(x,y,z)dS = \iiint_V g(x,y,z)\,dV$

25. $\displaystyle\kappa\int_0^L [f_3(x) - f_4(x)]\,dx + \kappa\int_0^{L'} [f_1(y) - f_2(y)]\,dy = \iint_R g(x,y)\,dA$

27. (a) $U = U_b + G(b^2 - r^2)/(6\kappa)$ (b) $D - Gr^2/(6\kappa)$, D arbitrary
(c) $U = U_m + Gb/(3\mu) + G(b^2 - r^2)/(6\kappa)$

28. $U(r) = \dfrac{aU_a(b-r) + bU_b(r-a)}{(b-a)r}$ **29.** $U(r) = U_a + \dfrac{b^2 Q}{\kappa}\left(\dfrac{1}{a} - \dfrac{1}{r}\right)$

30. $U(r) = \dfrac{\mu ab^2(U_a - U_m)}{[\kappa a + \mu b(b-a)]r} + \dfrac{\kappa a U_a + \mu b(bU_m - aU_a)}{\kappa a + \mu b(b-a)}$

31. $U(r) = \dfrac{a^2 Q}{\kappa r} + U_m + \dfrac{Qa^2(\kappa - \mu b)}{\mu\kappa b^2}$

34. (a) $5A\kappa(2e^{-t} - 2 - t) + AL^2t^2/4$ (b) No **35.** 0

36. (a) $-\kappa A(U_{\text{out}} - U_{\text{in}})/L$ W (b) 660 W

37. (a) $-\dfrac{\kappa}{r}\left[\dfrac{U_{\text{out}} - U_{\text{in}}}{\ln(r_{\text{out}}/r_{\text{in}})}\right]$ (b) 2.36×10^4 W (c) Yes

38. (b) $-(\kappa n\pi/L)e^{-n^2\pi^2kt/L^2}$, $-(\kappa n\pi/L)e^{-n^2\pi^2kt/L^2}\cos(n\pi/2)$, $(-)^{n+1}(\kappa n\pi/L)e^{-n^2\pi^2kt/L^2}$

(c) Limits as $t \to 0^+$ are $-\kappa n\pi/L$, $-(\kappa n\pi/L)\cos(n\pi/2)$, $(-1)^{n+1}\kappa n\pi/L$

Limits as $t \to \infty$ are zero.

39. (b) $U(0-) = U(0+), \quad -\kappa_1 \dfrac{\partial U(0-)}{\partial n} = -\kappa_2 \dfrac{\partial U(0+)}{\partial n}$

40. (a) $U(r) = U_m + QR/(3\mu) + Q(R^2 - r^2)/(6\kappa)$

(b)
$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < R, \quad t > 0,$$
$$\kappa U_r(R,t) + \mu U(R,t) = \mu U_m, \quad t > 0,$$
$$U(r,0) = U_m + QR/(3\mu) + Q(R^2 - r^2)/(6\kappa), \quad 0 \le r < R$$

43. (a) $U(r) = \dfrac{U_a[\kappa + \mu b \ln(b/r)] + \mu b U_m \ln(r/a)}{\kappa + \mu b \ln(b/a)}$

44.
$$U(r) = \begin{cases} \dfrac{-I^2 r^2}{4\pi^2 a^4 \sigma\kappa} + U_m + \dfrac{I^2}{4\pi^2 a^2 \sigma}\left[\dfrac{1}{\kappa} + \dfrac{2}{\kappa^*}\ln(b/a) + \dfrac{2}{b\mu^*}\right], & 0 \le r \le a \\ \dfrac{I^2}{2\pi^2 a^2 \sigma\kappa^*}\left[\ln(b/r) + \dfrac{\kappa^*}{b\mu^*}\right] + U_m, & a < r < b \end{cases}$$

45.
$$U(r) = \begin{cases} \dfrac{-I^2 r^2}{4\pi^2 a^4 \sigma\kappa} + U_m + \dfrac{I^2}{4\pi^2 a^2 \sigma}\left[\dfrac{1}{\kappa} + \dfrac{2}{\kappa^*}\ln(b/a) + 2\left(\dfrac{1}{a\mu} + \dfrac{1}{b\mu^*}\right)\right], & 0 \le r \le a \\ \dfrac{I^2}{2\pi^2 a^2 \sigma\kappa^*}\ln(b/r) + \dfrac{I^2}{2\pi^2 a^2 b\sigma\mu^*} + U_m, & a < r \le b \end{cases}$$

Exercises 2.3

1.
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$y(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = g(x), \quad 0 < x < L$$

2.
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$y_x(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = g(x), \quad 0 < x < L$$

3.
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\beta}{\rho}\frac{\partial y}{\partial t} - g, \quad 0 < x < L, \quad t > 0,$$
$$-\tau y_x(0,t) + k_1 y(0,t) = 0, \quad t > 0,$$
$$\tau y_x(L,t) + k_2 y(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = g(x), \quad 0 < x < L$$

4. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$y_x(0,t) = -\tau^{-1}\cos\omega t, \quad t > 0,$$
$$y(L,t) = 0, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

5. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\beta}{\rho}\frac{\partial y}{\partial t} - g, \quad 0 < x < L, \quad t > 0,$$
$$-\tau y_x(0,t) + k_1 y(0,t) = \cos\omega t, \quad t > 0,$$
$$\tau y_x(L,t) + k_2 y(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = g(x), \quad 0 < x < L$$

6. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$y_x(L,t) = F/E, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

7. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$y_x(L,t) = 0, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

8. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + g, \quad 0 < x < L, \quad t > 0,$$
$$-AEy_x(0,t) + ky(0,t) = 0, \quad t > 0,$$
$$y_x(L,t) = 0, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

9. $y(L,t)$ and $\partial y(L,t)/\partial x$ have opposite signs

10. $-AE\dfrac{\partial y(0,t)}{\partial x} + ky(0,t) = kg(t), \quad t > 0$

11. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$y(L,t) = 0, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

$$y(x) = \frac{gx(x-L)}{2c^2}$$

12. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\beta}{\rho}\frac{\partial y}{\partial t} - g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$y(L,t) = 0, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$
$$y(x) = \frac{gx(x-L)}{2c^2}$$

13. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$\tau y_x(L,t) = F_L, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$
$$y(x) = \frac{gx(x-2L)}{2c^2} + \frac{F_L x}{\tau}$$

14. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + g, \quad 0 < x < L, \quad t > 0,$$
$$y_x(0,t) = 0, \quad t > 0,$$
$$y(L,t) = 0, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = v, \quad 0 < x < L$$

15. $L + \dfrac{gL^2}{2c^2}$ **16.** $L + \dfrac{gL}{2c^2}\left(L + \dfrac{2AE}{k}\right)$

18. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$m\frac{\partial^2 y(L,t)}{\partial t^2} = -AE\frac{\partial y(L,t)}{\partial x} + F_0 \sin \omega t, \quad t > 0,$$
$$y(x,0) = 0, \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

21. (a) $-\rho g \displaystyle\int_0^L [y(x,t) - y(x,0)]\, dx$ **22.** (a) $\displaystyle\int_0^t \int_0^L -\beta \left(\frac{\partial y}{\partial t}\right)^2 dx\, dt$

23. (a) $-\dfrac{k}{2}\displaystyle\int_0^L \{[y(x,t)]^2 - [y(x,0)]^2\}\, dx$

25. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = 0, \quad t > 0,$$
$$y_x(L,t) = 0, \quad t > 0,$$
$$y(x,0) = Fx/(AE), \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

26. $y(L,t) = 0$

Exercises 2.4

1. $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right), \quad 0 < r < r_1, \quad -\pi < \theta \le \pi, \quad t > 0,$$

$$z(r_1, \theta, t) = 0, \quad -\pi < \theta \le \pi, \quad t > 0,$$
$$z(r, \theta, 0) = f(r, \theta), \quad 0 \le r < r_1, \quad -\pi < \theta \le \pi,$$
$$z_t(r, \theta, 0) = 0, \quad 0 \le r < r_1, \quad -\pi < \theta \le \pi$$

2. $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r}\right), \quad 0 < r < r_1, \quad t > 0,$$

$$z(r_1, t) = 0, \quad t > 0,$$
$$z(r, 0) = f(r), \quad 0 \le r < r_1,$$
$$z_t(r, 0) = 0, \quad 0 \le r < r_1$$

3. $$0 = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}, \quad 0 < r < r_1, \quad 0 < \theta < \alpha,$$

$$z(r_1, \theta) = f(\theta), \quad 0 < \theta < \alpha,$$
$$z(r, 0) = 0, \quad 0 < r < r_1,$$
$$z(r, \alpha) = 0, \quad 0 < r < r_1$$

4. $$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{\rho g}{\tau}, \quad 0 < r < r_1, \quad 0 < \theta < \alpha,$$

$$z(r_1, \theta) = f(\theta), \quad 0 < \theta < \alpha,$$
$$z(r, 0) = 0, \quad 0 < r < r_1,$$
$$z(r, \alpha) = 0, \quad 0 < r < r_1$$

5. $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right) - g - \frac{\beta}{\rho}\frac{\partial y}{\partial t}, \quad 0 < r < r_1, \quad -\pi < \theta \le \pi, \quad t > 0,$$

$$z(r_1, \theta, t) = 0, \quad -\pi < \theta \le \pi, \quad t > 0,$$
$$z(r, \theta, 0) = f(r, \theta), \quad 0 \le r < r_1, \quad -\pi < \theta \le \pi,$$
$$z_t(r, \theta, 0) = 0, \quad 0 \le r < r_1, \quad -\pi < \theta \le \pi$$

6. $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{\cos \omega t}{\rho}, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0$$

$$z(0, y, t) = 0, \quad 0 < y < L', \quad t > 0,$$
$$z(L, y, t) = 0, \quad 0 < y < L', \quad t > 0,$$
$$z(x, 0, t) = 0, \quad 0 < x < L, \quad t > 0,$$
$$z(x, L', t) = 0, \quad 0 < x < L, \quad t > 0,$$
$$z(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L',$$
$$z_t(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L'$$

7. $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{\cos \omega t}{\rho}, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0$$

$$-\tau z_x(0, y, t) + k_0 z(0, y, t) = 0, \quad 0 < y < L', \quad t > 0,$$
$$\tau z_x(L, y, t) + k_L z(L, y, t) = 0, \quad 0 < y < L', \quad t > 0,$$
$$z(x, 0, t) = f_1(x, t), \quad 0 < x < L, \quad t > 0,$$
$$z(x, L', t) = f_2(x, t), \quad 0 < x < L, \quad t > 0,$$
$$z(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L',$$
$$z_t(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L'$$

8. $z(r) = \frac{\rho g}{4\tau}(r^2 - r_2^2)$

9. (a) $\frac{d^2 z}{dr^2} + \frac{1}{r}\frac{dz}{dr} = -\frac{f(r)}{\tau}, \quad 0 < r < r_2,$
$z(r_2) = 0$

(f) $z(r) = \frac{k}{36\tau}(9r_2r^2 - 4r^3 - 5r_2^3)$

Exercises 2.5

1.
$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 y}{\partial x^4} = -g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = y_x(0,t) = 0, \quad t > 0,$$
$$y_{xx}(L,t) = y_{xxx}(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

2.
$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 y}{\partial x^4} = -g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = y_{xx}(0,t) = 0, \quad t > 0,$$
$$y(L,t) = y_{xx}(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

3.
$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho + \overline{\rho}}\frac{\partial^4 y}{\partial x^4} = -g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = y_x(0,t) = 0, \quad t > 0,$$
$$y(L,t) = y_x(L,t) = 0, \quad t > 0,$$
$$y(x,0) = f(x), \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

4.
$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 y}{\partial x^4} = -g, \quad 0 < x < L, \quad t > 0,$$
$$y(0,t) = y_x(0,t) = 0, \quad t > 0,$$
$$y_{xx}(L,t) = y_{xxx}(L,t) = 0, \quad t > 0,$$
$$y(x,0) = -\frac{\rho g x^4}{24EI} + \frac{x^3}{6EI}(F + \rho g L) - \frac{Lx^2}{4EI}(2F + \rho g L), \quad 0 < x < L,$$
$$y_t(x,0) = 0, \quad 0 < x < L$$

5. (a)
$$\frac{d^4 y}{dx^4} = F/(EI), \quad 0 < x < L,$$
$$y(0) = y''(0) = y(L) = y''(L) = 0,$$

(b) -9.2×10^{-9} m (c) -1.68×10^9 N/m

Exercises 2.6

1.
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L',$$
$$V(0,y) = 0, \quad 0 < y < L',$$
$$V(L,y) = 100, \quad 0 < y < L',$$
$$V(x,0) = 0, \quad 0 < x < L,$$
$$V(x,L') = 100, \quad 0 < x < L$$

2. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = -\dfrac{\sigma(x,y)}{\epsilon_0}, \quad 0 < x < L, \quad 0 < y < L',$

$$V(0,y) = 0, \quad 0 < y < L',$$
$$V(L,y) = 100, \quad 0 < y < L',$$
$$V(x,0) = 0, \quad 0 < x < L, \qquad \sigma(x,y) = \begin{cases} \sigma, & L/4 < x < 3L/4,\ L'/4 < y < 3L'/4 \\ 0, & \text{otherwise} \end{cases}$$
$$V(x,L') = 100, \quad 0 < x < L$$

Exercises 2.8

2. Hyperbolic for $|y| > \sqrt{5}$; Elliptic for $|y| < \sqrt{5}$; Parabolic for $|y| = \sqrt{5}$
3. Hyperbolic for $y < 0$, but $x \neq 0$; Elliptic for $y > 0$, but $x \neq 0$; Parabolic for $x = 0$ and for $y = 0$
4. Hyperbolic for $y > -1/4$, $y \neq 0$ and $x \neq 0$; Elliptic for $y < -1/4$; Parabolic for $x = 0$, $y = 0$, and
$y = -1/4$
5. Hyperbolic for $x < 4y$ and $x < 0$, and for $x > 4y$ and $x > 0$; Elliptic for $x > 4y$ and $x < 0$, and for
$x < 4y$ and $x > 0$; Parabolic for $x = 0$ and for $x = 4y$
6. Hyperbolic for $(2n-1)\pi/4 < x < (2n+1)\pi/4$, n an even integer; Elliptic for
$(2n-1)\pi/4 < x < (2n+1)\pi/4$, n an odd integer; Parabolic for $x = (2n+1)\pi/4$
7. Hyperbolic for $x < 0$ and $y > 1$, and for $x > 0$ and $y < 1$; Elliptic for $x < 0$ and $y < 1$, and for
$x > 0$ and $y > 1$; Parabolic for $x = 0$ and for $y = 1$
8. Hyperbolic for $y < x^2/4$; Elliptic for $y > x^2/4$; Parabolic for $y = x^2/4$
9. Parabolic: $w_{\nu\nu} = w_\nu - (\nu+1)w_\eta$
10. Elliptic; $w_{\nu\nu} + w_{\eta\eta} = \dfrac{1}{4}\left[-3w_\nu - 6w_\eta + \left(\dfrac{\eta}{2} - \nu\right)w\right]$ **11.** Hyperbolic: $w_{\nu\eta} = 0$
12. Hyperbolic; $w_{\nu\eta} = w[(3+2\sqrt{2})w_\nu + (3-2\sqrt{2})w_\eta]/8$
13. If we set $\nu = x$, then $w_{\nu\nu} = 1 + \dfrac{1}{\nu^2}(\nu + 2\eta)w_\eta$
14. (b) $w_{\nu\eta} = \dfrac{-1}{6(\nu+\eta)}(w_\nu + w_\eta)$ (c) $w_{\nu\nu} + w_{\eta\eta} = -w_\nu/(3\nu)$ (d) $u_{yy} = 0$
15. Parabolic when $x = 0$ with $u_{yy} = u/4$; Elliptic when $x \neq 0$ with $w_{\nu\nu} + w_{\eta\eta} = (w - 2w_\eta)/4$
16. $w_{\nu\nu} = -2\eta w_\eta/(\eta - \nu^2)$ **17.** $y = x^3/3 + x/2 + C_1$, $y = -x/2 + C_2$ **18.** $w_{\nu\eta} = w_\eta/\nu$
20. $z_{\nu\nu} + z_{\eta\eta} = 45z/64$ **21.** $z_{\nu\eta} = z/64$ **22.** $z_{\nu\nu} = -2z_\eta$

Exercises 2.9

1. $y(x,t) = (t - 1 + e^{-t})/\rho$ **2.** $y(x,t) = [2\sin x - \sin(x+ct) - \sin(x-ct)]/(2\rho c^2)$
3. $y(x,t) = 5 + x^2 t + c^2 t^3/3 + (e^{x+ct} + e^{x-ct} - 2e^x)/(2\rho c^2)$
4. When $x - ct > 0$, $y(x,t) = [e^{-(x+ct)} + e^{-(x-ct)} - 2e^{-x}]/(2\rho c^2)$.
When $x - ct < 0 < x$, $y(x,t) = [e^{x-ct} + e^{-(x+ct)} - 2e^{-x} - 2(x-ct)]/(2\rho c^2)$.
When $x - ct < x < 0 < x + ct$, $y(x,t) = [e^{x-ct} + e^{-(x+ct)} - 2e^x + 2(x+ct)]/(2\rho c^2)$.
When $x + ct < 0$, $y(x,t) = [e^{x+ct} + e^{x-ct} - 2e^x]/(2\rho c^2)$
5. $y(x,t) = \dfrac{1}{2\rho c^2}\Big\{\ln(1+x^2) - \dfrac{1}{2}\ln[1+(x+ct)^2] - \dfrac{1}{2}\ln[1+(x-ct)^2]$
$-2x\,\mathrm{Tan}^{-1}x + (x+ct)\mathrm{Tan}^{-1}(x+ct) + (x-ct)\mathrm{Tan}^{-1}(x-ct)\Big\}$
6. $y(x,t) = \dfrac{1}{2}[f(x-ct) + f(x+ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta - \dfrac{gt^2}{2}$ **7.** No **8.** No

Exercises 2.10

1. $y(x,t) = [f(x+ct) + f(x-ct)]/2 - gt^2/2$ **2.** No **3.** No

Exercises 2.11

6. For nonhomogeneous problem, solution is 2.175. Drop the double integral for the homogeneous problem.

7. $y(x,t) = k(t - x/c) + \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta + \frac{1}{2c}\iint_D \frac{F(x,t)}{\rho}dA$

For homogeneous problem, drop first and last terms.

8. For nonhomogeneous problem,

$$y(x,t) = k(t + (x-L)/c) + \frac{1}{2}[f(x-ct) + f(2L-x-ct)] + \frac{1}{2c}\int_{x-ct}^{2L-x-ct} g(\zeta)\,d\zeta + \frac{1}{2c}\iint_D \frac{F(x,t)}{\rho}dA.$$

For homogenous problem, drop first and last terms.

9. For nonhomogeneous problem

$$y(x,t) = k(t - x/c) + m(t + (x-L)/c) - \frac{1}{2}[f(ct-x) + f(2L-x-ct)] + \frac{1}{2c}\int_{ct-x}^{2L-x-ct} g(\zeta)\,d\zeta + \frac{1}{2c}\iint_D \frac{F(x,t)}{\rho}dA.$$

For homogeneous problem, drop the first two terms and the last term.

10. For nonhomogenous problem,

$$y(x,t) = k(t - x/c) + m(t + (x-L)/c) + m(t - (L+x)/c) + \frac{1}{2}[f(2L-x-ct) - f(2L+x-ct)] + \frac{1}{2c}\int_{2L-x-ct}^{2L+x-ct} g(\zeta)\,d\zeta + \frac{1}{2c}\iint_D \frac{F(x,t)}{\rho}dA.$$

For homogeneous problem, drop the first three terms and the last term.

Exercises 3.1

1. $8 - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$ **2.** $\frac{8L^2}{3} - 1 + \frac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos\frac{n\pi x}{L} - \frac{\pi}{n}\sin\frac{n\pi x}{L}\right)$

3. $\frac{2L^2}{3} - 1 + \frac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}$ **4.** $3L - \frac{6L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}$

5. $\frac{6L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$ **6.** $\frac{3L}{4} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{3[1+(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{L} + \frac{\pi}{n}\sin\frac{n\pi x}{L}\right\}$

7. $1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + (-1)^{n+1}\cos\frac{n\pi}{3}\right]\sin\frac{2n\pi x}{3}$

8. $\frac{4}{3} - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1 + (-1)^{n+1}\cos\frac{n\pi}{3}\right]\cos\frac{n\pi x}{3}$ **9.** $1 + \sin x - \cos 2x$

10. $2\cos x - 3\sin 10x + 4\cos 2x$ **11.** $1/2 + (1/2)\cos 4x$ **12.** $(3/2)(\sin 7x + \sin 3x)$

13. $\frac{e^4 - 1}{4} + (e^4 - 1)\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2 + 4}\left(2\cos\frac{n\pi x}{2} - n\pi\sin\frac{n\pi x}{2}\right)$

14. $\frac{3}{\pi} - \frac{1}{2}\sin x - \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2 - 1}\cos 2nx$ **16.** No

17. $8 - \frac{6i}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} e^{-n\pi xi/2}$ **18.** $1 + \frac{i}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} \left[1 + (-1)^{n+1} \cos \frac{n\pi}{3}\right] e^{-2n\pi xi/3}$

19. $\frac{2i}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2n-1} e^{-(2n-1)\pi xi/L}$ $\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{L}$

20. $\frac{L}{2} - \frac{2L}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)\pi xi/L}$ $\frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}$

21. $\frac{L^2}{3} + \frac{2L^2}{\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n^2} e^{-n\pi xi/L}$

22. $\frac{4}{3} + \frac{3}{2\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2}(-2 + e^{2n\pi xi/3} + e^{4n\pi xi/3}) e^{-n\pi xi/3}$

23. $\frac{8L^2}{3} - 1 + \frac{4L^2}{\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1 - n\pi i}{n^2} e^{-n\pi xi/L}$ **24.** $\left(\frac{e^4 - 1}{2}\right) \sum_{n=-\infty}^{\infty} \left(\frac{2 - n\pi i}{n^2\pi^2 + 4}\right) e^{-n\pi xi/2}$

25. $\frac{3L}{4} + \frac{L}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{\frac{i}{n} + \frac{3}{n^2\pi}[1 + (-1)^{n+1}]\right\} e^{-n\pi xi/L}$

26. $\frac{1}{\pi} + \frac{1}{2\pi} \left\{\frac{i\pi}{2} e^{-xi} - \frac{i\pi}{2} e^{xi} + \sum_{\substack{n=-\infty \\ n\neq 0,\pm 1}}^{\infty} \frac{1 + (-1)^n}{1 - n^2} e^{-nxi}\right\}$ **27.** Yes $d_n = (a_n - ib_n)/2$

Exercises 3.2

1. $2 \sin 4x - 3 \sin x$ **2.** $\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos (2n-1)x$

3. $\frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{8}$ **4.** $\frac{2L^2}{3} - 1 + \frac{8L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{L}$

5. $\frac{1}{\pi} + \frac{1}{2} \cos x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \cos 2nx$ **6.** $\frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{L}$

7. $\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos \frac{2n\pi}{3}\right) \sin \frac{2n\pi x}{L}$

8. $\frac{L}{\pi^2} \sum_{n=1}^{\infty} \left\{\frac{[1 + (-1)^{n+1}]}{2n^2} \left(n\pi + 4 \sin \frac{n\pi}{4}\right) - \frac{4}{n^2} \sin \frac{n\pi}{2}\right\} \sin \frac{n\pi x}{L}$

9. $\frac{8L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{L}$ **10.** $\frac{1}{2} \sin \frac{2\pi x}{L}$

11. $-\frac{L}{2} + \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}$ **12.** $\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos \frac{(2n-1)\pi x}{L}$

13. $\frac{L^2}{6} - \frac{L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi x}{L}$ **14.** 1 **15.** $-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - 4} \cos \frac{(2n-1)\pi x}{L}$

16. $\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx$ **17.** $\int_0^L f(x)\, dx = 0$ **18.** (c) Yes

19. (a) $\frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}$ (b) $\frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}$
(c) $L - \frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2L}$ (d) $\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2L}$

20. (d) No **21.** (d) No

Exercises 3.4

6. (b) No **8.** (a) Yes (b) No **9.** (a) No (b) No (c) No

Exercises 4.1

1. Linear and homogeneous **2.** Not linear **3.** Not linear **4.** Not linear
5. Linear and homogeneous **6.** Linear and nonhomogeneous
7. Linear and nonhomogeneous **8.** Not linear **9.** Linear and homogeneous
10. Linear and homogeneous
11.

$$\begin{aligned}
\nabla^2 V_1 &= F(x,y,z), \quad 0<x<L, \quad 0<y<L', \quad 0<z<L'',\\
V_1(0,y,z) &= 0, \quad 0<y<L', \quad 0<z<L'',\\
V_1(L,y,z) &= 0, \quad 0<y<L', \quad 0<z<L'',\\
V_1(x,0,z) &= 0, \quad 0<x<L, \quad 0<z<L'',\\
V_1(x,L',z) &= 0, \quad 0<x<L, \quad 0<z<L'',\\
V_1(x,y,0) &= 0, \quad 0<x<L, \quad 0<y<L',\\
V_1(x,y,L'') &= 0, \quad 0<x<L, \quad 0<y<L';
\end{aligned}$$

$$\begin{aligned}
\nabla^2 V_2 &= 0, \quad 0<x<L, \quad 0<y<L', \quad 0<z<L'',\\
V_2(0,y,z) &= f_1(y,z), \quad 0<y<L', \quad 0<z<L'',\\
V_2(L,y,z) &= f_2(y,z), \quad 0<y<L', \quad 0<z<L'',\\
V_2(x,0,z) &= 0, \quad 0<x<L, \quad 0<z<L'',\\
V_2(x,L',z) &= 0, \quad 0<x<L, \quad 0<z<L'',\\
V_2(x,y,0) &= 0, \quad 0<x<L, \quad 0<y<L',\\
V_2(x,y,L'') &= 0, \quad 0<x<L, \quad 0<y<L';
\end{aligned}$$

$$\begin{aligned}
\nabla^2 V_3 &= 0, \quad 0<x<L, \quad 0<y<L', \quad 0<z<L'',\\
V_3(0,y,z) &= 0, \quad 0<y<L', \quad 0<z<L'',\\
V_3(L,y,z) &= 0, \quad 0<y<L', \quad 0<z<L'',\\
V_3(x,0,z) &= g_1(x,z), \quad 0<x<L, \quad 0<z<L'',\\
V_3(x,L',z) &= g_2(x,z), \quad 0<x<L, \quad 0<z<L'',\\
V_3(x,y,0) &= 0, \quad 0<x<L, \quad 0<y<L',\\
V_3(x,y,L'') &= 0, \quad 0<x<L, \quad 0<y<L';
\end{aligned}$$

$$\begin{aligned}
\nabla^2 V_4 &= 0, \quad 0 < x < L, \quad 0 < y < L', \quad 0 < z < L'', \\
V_4(0, y, z) &= 0, \quad 0 < y < L', \quad 0 < z < L'', \\
V_4(L, y, z) &= 0, \quad 0 < y < L', \quad 0 < z < L'', \\
V_4(x, 0, z) &= 0, \quad 0 < x < L, \quad 0 < z < L'', \\
V_4(x, L', z) &= 0, \quad 0 < x < L, \quad 0 < z < L'', \\
V_4(x, y, 0) &= h_1(x, y), \quad 0 < x < L, \quad 0 < y < L', \\
V_4(x, y, L'') &= h_2(x, y), \quad 0 < x < L, \quad 0 < y < L'.
\end{aligned}$$

12. (b) No

Exercises 4.2

1. U_0 **2.** $U(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}$ where $b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$

3. (a) $U(x,t) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} e^{-(2n-1)^2\pi^2 kt/L^2} \sin \frac{(2n-1)\pi x}{L}$

(b) $\frac{4\kappa}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2 kt/L^2}$; 0; $-\frac{4\kappa}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2 kt/L^2}$

(c) $-\kappa$, 0, κ; 0, 0, 0

4. (a) $U(x,t) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-(2n-1)^2\pi^2 kt/L^2} \sin \frac{(2n-1)\pi x}{L}$

(b) $-\frac{40\kappa}{L} \sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 kt/L^2}$; 0; $\frac{40\kappa}{L} \sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 kt/L^2}$

(c) $-\infty$, 0, ∞; 0, 0, 0 (d) 10; 0

5. (a) $\frac{4\kappa}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2 kt/L^2}$ (b) $-\kappa$

6. $U(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 kt/L^2} \cos \frac{n\pi x}{L}$ where $a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L} dx$; $a_0/2$

7. $U(x,t) = \frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2\pi^2 kt/(4L^2)} \cos \frac{(2n-1)\pi x}{2L}$

8. $U(x,t) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2\pi^2 kt/(4L^2)} \cos \frac{(2n-1)\pi x}{2L}$ where $a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} dx$

9. $U(x,t) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{(2n-1)\pi + 2(-1)^n}{(2n-1)^2} e^{-(2n-1)^2\pi^2 kt/(4L^2)} \sin \frac{(2n-1)\pi x}{2L}$

10. $U(x,t) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2\pi^2 kt/(4L^2)} \sin \frac{(2n-1)\pi x}{2L}$ where $a_n = \frac{2}{L}\int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} dx$

11. (a) $U(x,t) = \frac{a_0}{2} e^{\kappa t} + \sum_{n=1}^{\infty} a_n e^{(\kappa - n^2\pi^2 k/L^2)t} \cos \frac{n\pi x}{L}$, where $a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$

(b) $U(x,t) = Ce^{\kappa t}$

(c) $U(x,t) = e^{\kappa t}\left[\frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2 kt/L^2}\cos\frac{(2n-1)\pi x}{L}\right]$

12. (b) $U(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/a^2}\sin\frac{n\pi r}{a}$ where $b_n = \frac{2}{a}\int_0^a rf(r)\sin\frac{n\pi r}{a}dr$

13. A Robin boundary condition results at $r = a$.

15. $U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[(U_1 - U_0)\cos\frac{n\pi}{2} + U_0 + (-1)^{n+1}U_1\right]e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}$

16. $U(x,t) = e^{kat/\kappa}\sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}$ where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad a \le n^2\pi^2/L^2$

17. $U(x,t) = e^{kat/\kappa}\left(\frac{b_0}{2} + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L}\right)$ where $b_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$
$a \le 0$

18. (a) $y(x,t) = \frac{4L^2}{5\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$

(b) $y(x,t) = [f(x+ct) + f(x-ct)]/2$

19. (a) $y(x,t) = \frac{8L^3}{\pi^4 c}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$

(b) $y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$

20. $$y(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\Big[\frac{(-1)^{n+1}}{5(2n-1)^2}\cos\frac{(2n-1)\pi ct}{L} + \frac{2L^2}{\pi^2 c(2n-1)^4}\sin\frac{(2n-1)\pi ct}{L}\Big]\sin\frac{(2n-1)\pi x}{L}$$

21. (a) $y(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\sqrt{\frac{n^2\pi^2c^2}{L^2} + \frac{k}{\rho}}t + b_n\sin\sqrt{\frac{n^2\pi^2c^2}{L^2} + \frac{k}{\rho}}t\right)\sin\frac{n\pi x}{L}$ where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \qquad b_n = \frac{2}{\sqrt{n^2\pi^2c^2 + \frac{kL^2}{\rho}}}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

(b) No

22. (a) $y(x,t) = e^{-\beta t/(2\rho)}\sum_{n=1}^{\infty}(a_n\cos\omega_n t + b_n\sin\omega_n t)\sin\frac{n\pi x}{L}$ where $\omega_n = \sqrt{\frac{n^2\pi^2c^2}{L^2} - \frac{\beta^2}{4\rho^2}}$,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad b_n = \frac{2}{\omega_n L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx + \frac{\beta}{L\rho\omega_n}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

(b) No

23. (a) $y(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\right)\cos\frac{n\pi x}{L}$ where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad b_0 = \frac{2}{L}\int_0^L g(x)\,dx, \quad b_n = \frac{2}{n\pi c}\int_0^L g(x)\cos\frac{n\pi x}{L}dx$$

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$$

24. $y(x,t)=\sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}$ where $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}dx$;
$y(x,t)=[f(x+ct)+f(x-ct)]/2$

25. (b) $W(r,t)=\frac{1}{r}\sum_{n=1}^{\infty}\left(F_n\cos\frac{n\pi ct}{a}+G_n\sin\frac{n\pi ct}{a}\right)\sin\frac{n\pi r}{a}$ where

$$F_n=\frac{2}{a}\int_0^a rf(r)\sin\frac{n\pi r}{a}\,dr, \qquad G_n=\frac{2}{n\pi c}\int_0^a rg(r)\sin\frac{n\pi r}{a}\,dr$$

26. A Robin boundary condition at $r=a$ is obtained.

27. (a) $y(x,t)=\frac{L^*-L}{2}+\frac{4(L-L^*)}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{L}\cos\frac{(2n-1)\pi x}{L}$
(b) $y(x,t)=[f(x+ct)+f(x-ct)]/2$ (c) $y_t(x,t)=c[f'(x+ct)+f'(x-ct)]/2$, No

28. (a) $y(x,t)=\frac{8(L^*-L)}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{2L}\sin\frac{(2n-1)\pi x}{2L}$
(b) $[f(x+ct)+f(x-ct)]/2$ (c) $y_t(x,t)=(c/2)[f'(x+ct)-f'(x-ct)]$ No

29. $V(x,t)=\sum_{n=1}^{\infty} a_n e^{-(RC+LG)t/(2LC)}\left(\cos\omega_n t+\frac{RC+LG}{2LC\omega_n}\sin\omega_n t\right)\sin\frac{n\pi x}{M}$ where

$$\omega_n=\frac{\sqrt{4n^2\pi^2LC/M^2-R^2C^2-L^2G^2}}{2LC}, \qquad a_n=\frac{2}{M}\int_0^M f(x)\sin\frac{n\pi x}{M}dx$$

30. $V(x,y)=\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\operatorname{csch}\frac{(2n-1)\pi L}{L'}\sinh\frac{(2n-1)\pi(L-x)}{L'}\sin\frac{(2n-1)\pi y}{L'}$

31.
$$V(x,y)=\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\operatorname{csch}\frac{(2n-1)\pi L}{L'}\sinh\frac{(2n-1)\pi(L-x)}{L'}\sin\frac{(2n-1)\pi y}{L'}$$
$$+\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\operatorname{csch}\frac{(2n-1)\pi L'}{L}\sinh\frac{(2n-1)\pi(L'-y)}{L}\sin\frac{(2n-1)\pi x}{L}$$

32.
$$V(x,y)=\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[\cosh\frac{(2n-1)\pi x}{L'}\right.$$
$$\left.+\frac{1-\cosh\frac{(2n-1)\pi L}{L'}}{\sinh\frac{(2n-1)\pi L}{L'}}\sinh\frac{(2n-1)\pi x}{L'}\right]\sin\frac{(2n-1)\pi y}{L'}$$

33. $V(x,y)=\frac{-400L'}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\operatorname{sech}\frac{(2n-1)\pi L}{L'}\sinh\frac{(2n-1)\pi(L-x)}{L'}\sin\frac{(2n-1)\pi y}{L'}$

34. $V(x,y)=100x+B$ where B is arbitrary; $V(x,y)=100x-50L$

35. No

36.
$$U(x,y)=\frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left\{\cosh\frac{(2n-1)\pi y}{L}\right.$$
$$\left.-\operatorname{csch}\frac{(2n-1)\pi L'}{L}\left[1+\cosh\frac{(2n-1)\pi L'}{L}\right]\sinh\frac{(2n-1)\pi y}{L}\right\}\sin\frac{(2n-1)\pi x}{L}$$

37. $U(x,y)=\frac{4qL}{\pi^2\kappa}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\operatorname{sech}\frac{(2n-1)\pi L'}{L}\sinh\frac{(2n-1)\pi(L'-y)}{L}\sin\frac{(2n-1)\pi x}{L}$

38. $U(x,y)=10+\frac{4qL}{\pi^2\kappa}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left\{\operatorname{csch}\frac{(2n-1)\pi L'}{L}\left[1+\cosh\frac{(2n-1)\pi L'}{L}\right]*\right.$

$$\cosh\frac{(2n-1)\pi y}{L} - \sinh\frac{(2n-1)\pi y}{L}\bigg\}\sin\frac{(2n-1)\pi x}{L}$$

39. $z(x,y) = \frac{-2kL}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\text{csch}\frac{n\pi L'}{L}\sinh\frac{n\pi(L'-y)}{L}\sin\frac{n\pi x}{L}$

$$-\frac{2kL}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\text{csch}\frac{n\pi L}{L'}\sinh\frac{n\pi(L-x)}{L'}\sin\frac{n\pi y}{L'}$$

40. (a) $V(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi(L'-y)}{L}$ where $b_n = \frac{2}{L}\text{csch}\frac{n\pi L'}{L}\int_0^L h_1(x)\sin\frac{n\pi x}{L}dx$

(b) $V(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$ where $b_n = \frac{2}{L}\text{csch}\frac{n\pi L'}{L}\int_0^L h_2(x)\sin\frac{n\pi x}{L}dx$

(c) $V(x,y) = \sum_{n=1}^{\infty}\left(a_n\cosh\frac{n\pi y}{L} + b_n\sinh\frac{n\pi y}{L}\right)\sin\frac{n\pi x}{L}$ where $a_n = \frac{2}{L}\int_0^L h_1(x)\sin\frac{n\pi x}{L}dx$

and $b_n = \text{csch}\frac{n\pi L'}{L}\left[\frac{2}{L}\int_0^L h_2(x)\sin\frac{n\pi x}{L}dx - \frac{2}{L}\cosh\frac{n\pi L'}{L}\int_0^L h_1(x)\sin\frac{n\pi x}{L}dx\right]$

(d) $V(x,y) = \sum_{n=1}^{\infty}\left(a_n\cosh\frac{n\pi x}{L'} + b_n\sinh\frac{n\pi x}{L'}\right)\sin\frac{n\pi y}{L'}$ where $a_n = \frac{2}{L'}\int_0^{L'} g_1(y)\sin\frac{n\pi y}{L'}dy$

and

$$b_n = \text{csch}\frac{n\pi L}{L'}\left[\frac{2}{L'}\int_0^{L'} g_2(y)\sin\frac{n\pi y}{L'}dy - \frac{2}{L'}\cosh\frac{n\pi L}{L'}\int_0^{L'} g_1(y)\sin\frac{n\pi y}{L'}dy\right]$$

41. $U(x,y) = \frac{16L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{2(-1)^n + (2n-1)\pi}{(2n-1)^3}\text{sech}\frac{(2n-1)\pi L'}{2L} *$

$$\cosh\frac{(2n-1)\pi(L'-y)}{2L}\cos\frac{(2n-1)\pi x}{2L}$$

44. No

Exercises 4.3

1. $U(x,t) = \frac{I^2x(L-x)}{2\kappa A^2\sigma} + \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{20}{2n-1} - \frac{I^2L^2}{\kappa A^2\sigma\pi^2(2n-1)^3}\right]e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}$

2. $U(x,t) = 100 + \frac{I^2x(L-x)}{2\kappa A^2\sigma} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{80}{2n-1} + \frac{I^2L^2}{\kappa A^2\sigma\pi^2(2n-1)^3}\right] *$

$$e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}$$

3. $U(x,t) = \frac{I^2x(L-x)}{2\kappa A^2\sigma} + \left(\frac{U_L - U_0}{L}\right)x + U_0 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{U_0 + U_L(-1)^{n+1}}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}$

$$+\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{20}{2n-1} - \frac{I^2L^2}{\kappa A^2\sigma\pi^2(2n-1)^3}\right]e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}$$

4. $U(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left\{\frac{20e^{-(2n-1)^2\pi^2kt/L^2}}{2n-1} + \frac{kL^2(e^{-2\alpha t} - e^{-(2n-1)^2\pi^2kt/L^2})}{\kappa A^2\sigma(2n-1)[(2n-1)^2\pi^2k - 2\alpha L^2]}\right\} *$

$$\sin\frac{(2n-1)\pi x}{L}$$

5. $U(x,t) = \dfrac{100xe^{-t}}{L} + \dfrac{200}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{L^2(-1)^{n+1}(e^{-t} - e^{-n^2\pi^2kt/L^2})}{n(n^2\pi^2k - L^2)} + \dfrac{e^{-n^2\pi^2kt/L^2}}{n}\right]\sin\dfrac{n\pi x}{L}$;

$$U(x,t) = \frac{100xe^{-t}}{L} + \frac{200}{m\pi}[1 + (-1)^{m+1}t]e^{-t}\sin\frac{m\pi x}{L} + \frac{200}{\pi}\sum_{\substack{n=1\\ n\neq m}}^{\infty}\left[\frac{L^2(-1)^{n+1}(e^{-t} - e^{-n^2\pi^2kt/L^2})}{n(n^2\pi^2k - L^2)} + \frac{e^{-n^2\pi^2kt/L^2}}{n}\right]\sin\frac{n\pi x}{L}$$

6. $U(x,t) = \dfrac{kI^2t}{\kappa A^2\sigma} + 20$ **7.** $U(x,t) = 20 + \dfrac{k}{2\alpha\kappa A^2\sigma}(1 - e^{-2\alpha t})$

8. (a) $U(x,t) = \dfrac{kL^2}{\kappa(m^2\pi^2k - \alpha L^2)}(e^{-\alpha t} - e^{-m^2\pi^2kt/L^2})\sin\dfrac{m\pi x}{L}$

(b) $U(x,t) = \dfrac{kt}{\kappa}e^{-m^2\pi^2kt/L^2}\sin\dfrac{m\pi x}{L}$

11. (b) $U(x,t) = \psi(x) - \dfrac{4L^2kI^2e^{-ht}}{\kappa A^2\sigma\pi}\sum_{n=1}^{\infty}\dfrac{e^{-(2n-1)^2\pi^2kt/L^2}}{(2n-1)[hL^2 + (2n-1)^2\pi^2k]}\sin\dfrac{(2n-1)\pi x}{L}$

13. $U(x,t) = -\dfrac{g}{\kappa}(x-a)h(x-a) + \dfrac{g}{\kappa L}(L-a)x - \dfrac{2gL}{\pi^2\kappa}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi a}{L}e^{-n^2\pi^2kt/L^2}\sin\dfrac{n\pi x}{L}$

14. $$U(x,t) = -\frac{g}{2\kappa(b-a)}[(x-a)^2h(x-a) - (x-b)^2h(x-b)] + \frac{g}{2\kappa L(b-a)}[(L-a)^2 - (L-b)^2]x + \frac{2gL^2}{\kappa\pi^3}\sum_{n=1}^{\infty}\frac{1}{(b-a)n^3}\left(\cos\frac{n\pi b}{L} - \cos\frac{n\pi a}{L}\right)e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}$$

15. $U(r,t) = \dfrac{1}{r}\left[\dfrac{gr(a^2-r^2)}{6\kappa} + \sum_{n=1}^{\infty}b_ne^{-n^2\pi^2kt/a^2}\sin\dfrac{n\pi r}{a}\right]$, where

$$b_n = \frac{2}{a}\int_0^a\left[rf(r) - \frac{gr(a^2-r^2)}{6\kappa}\right]\sin\frac{n\pi r}{a}\,dr$$

16. (a) $$U(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left\{\frac{10}{2n-1}e^{-(2n-1)^2\pi^2kt/L^2} + \frac{kL^2[e^{-\alpha t} - e^{-(2n-1)^2\pi^2kt/L^2}]}{\kappa(2n-1)[(2n-1)^2\pi^2k - \alpha L^2]}\right\} * \sin\frac{(2n-1)\pi x}{L}$$

(b) When m is an even integer, the solution is the same as that in part (a). When m is an odd integer,

$$U(x,t) = \frac{4}{\pi}\sum_{\substack{n=1\\ 2n-1\neq m}}^{\infty}\left\{\frac{10}{2n-1}e^{-(2n-1)^2\pi^2kt/L^2} + \frac{L^2[e^{-m^2\pi^2kt/L^2} - e^{-(2n-1)^2\pi^2kt/L^2}]}{\kappa\pi^2(2n-1)[(2n-1)^2 - m^2]}\right\} * \sin\frac{(2n-1)\pi x}{L} + \frac{4}{m\pi}\left(10 + \frac{kt}{\kappa}\right)e^{-m^2\pi^2kt/L^2}\sin\frac{m\pi x}{L}$$

17. (a) $y(x,t) = \psi(x) + \sum_{n=1}^{\infty}\left(a_n\cos\dfrac{n\pi ct}{L} + b_n\sin\dfrac{n\pi ct}{L}\right)\sin\dfrac{n\pi x}{L}$, where $\psi(x) = -\dfrac{kx(L-x)}{2\rho c^2}$

$$a_n = \frac{2}{L}\int_0^L[f(x) - \psi(x)]\sin\frac{n\pi x}{L}dx, \quad b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

(b) $y(x,t) = \psi(x) + \dfrac{1}{2}[f(x+ct) + f(x-ct)] - \dfrac{1}{2}[\psi(x+ct) + \psi(x-ct)] + \dfrac{1}{2c}\int_{x-ct}^{x+ct}g(u)\,du$

18. (a) $y(x,t) = \psi(x) + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$ where

$$\psi(x) = \frac{mg}{\rho c^2}(x-a)h(x-a) - \frac{mg(L-a)x}{\rho c^2 L} \text{ and } b_n = \frac{2}{L}\int_0^L f(x) - \psi(x)]\sin\frac{n\pi x}{L}dx$$

(b) $y(x,t) = \psi(x) + \frac{1}{2}[f(x+ct) + f(x-ct)] - \frac{1}{2}[\psi(x-ct) + \psi(x+ct)]$

19. (a) $y(x,t) = \psi(x) + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$, where $\psi(x) = y_L x/L$,

$$a_n = \frac{2}{L}\int_0^L [f(x) - \psi(x)]\sin\frac{n\pi x}{L}dx, \qquad b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

(b) $y(x,t) = \psi(x) + \frac{1}{2}[f(x+ct) + f(x-ct)] - \frac{1}{2}[\psi(x+ct) + \psi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$

20. (a) No (c) $y(x,t) = \frac{F}{E}\left[x + \frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2L}\cos\frac{(2n-1)\pi ct}{2L}\right]$

(d) $y(x,t) = \psi(x) - \frac{1}{2}[\psi(x+ct) + \psi(x-ct)]$ where $\psi(x) = Fx/E$

22. $y(x,t) = \frac{2F_0L^2}{\pi^2\rho c}\sum_{n=1}^{\infty}\frac{1}{n^2(n^2\pi^2c^2 - \omega^2L^2)}\left[\cos\frac{n\pi a}{L} - \cos\frac{n\pi b}{L}\right] *$

$$\left[c\pi n\sin\omega t - \omega L\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}$$

23. $y(x,t) = \frac{2F_0L}{\pi\rho c}\sum_{n=1}^{\infty}\frac{1}{n(n^2\pi^2c^2 - \omega^2L^2)}\sin\frac{n\pi x_0}{L}\left[c\pi n\sin\omega t - \omega L\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}$

24. $y(x,t) = -\frac{(\rho+k)gx}{24EI}(x^3 - 2Lx^2 + L^3)$

$$+\frac{4(\rho+k)gL^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin\frac{(2n-1)\pi x}{L}\cos\frac{(2n-1)^2\pi^2 ct}{L^2}$$

25. $y(x,t) = -\frac{(\rho+k)gx}{24EI}(x^3 - 2Lx^2 + L^3) + \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{L}\cos\frac{n^2\pi^2ct}{L^2}$

$$+\frac{4(\rho+k)gL^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin\frac{(2n-1)\pi x}{L}\cos\frac{(2n-1)^2\pi^2ct}{L^2}, \text{ where}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

27. $V(x,y) = \sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi y}{L} + D_n\sinh\frac{n\pi y}{L}\right)\sin\frac{n\pi x}{L}$

$$+\sum_{n=1}^{\infty}\left(B_n\cosh\frac{n\pi x}{L'} + C_n\sinh\frac{n\pi x}{L'}\right)\sin\frac{n\pi y}{L'}$$

$$A_n = \frac{2}{L}\int_0^L h_1(x)\sin\frac{n\pi x}{L}\,dx, \qquad B_n = \frac{2}{L'}\int_0^{L'} g_1(y)\sin\frac{n\pi y}{L'}\,dy,$$

$$C_n = \frac{2}{L'\sinh(n\pi L/L')}\int_0^{L'}\left[g_2(y) - g_1(y)\cosh\frac{n\pi L}{L'}\right]\sin\frac{n\pi y}{L'}\,dy,$$

$$D_n = \frac{2}{L\sinh(n\pi L'/L)}\int_0^L\left[h_2(x) - h_1(x)\cosh\frac{n\pi L'}{L}\right]\sin\frac{n\pi x}{L}\,dx$$

28. (a) $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=-\frac{\sigma(x,y)}{\epsilon_0},\quad 0<x<L,\quad 0<y<L',$$
$$V(0,y)=0,\quad 0<y<L',$$
$$V(L,y)=0,\quad 0<y<L',$$
$$V(x,0)=0,\quad 0<x<L,$$
$$V(x,L')=0,\quad 0<x<L$$

(b) $$V(x,y)=\frac{-4L^2\sigma}{\pi^3\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3\sinh[(2n-1)\pi L'/L]}\Big[\sinh\frac{(2n-1)\pi(L'-y)}{L}+\sinh\frac{(2n-1)\pi y}{L}-\sinh\frac{(2n-1)\pi L'}{L}\Big]\sin\frac{(2n-1)\pi x}{L}$$

$$V(x,y)=\frac{-4(L')^2\sigma}{\pi^3\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3\sinh[(2n-1)\pi L/L']}\Big[\sinh\frac{(2n-1)\pi(L-x)}{L'}+\sinh\frac{(2n-1)\pi x}{L'}-\sinh\frac{(2n-1)\pi L}{L'}\Big]\sin\frac{(2n-1)\pi y}{L'}$$

(c) $$V(x,y)=\frac{\sigma x(L-x)}{2\epsilon_0}-\frac{4\sigma L^2}{\pi^3\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\operatorname{csch}\frac{(2n-1)\pi L'}{L}\Big[\sinh\frac{(2n-1)\pi(L'-y)}{L}+\sinh\frac{(2n-1)\pi y}{L}\Big]\sin\frac{(2n-1)\pi x}{L}$$

(d) $$V(x,y)=\frac{L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sigma_n}{n^2}\left\{\frac{1}{\sinh(n\pi L'/L)}\left[\sinh\frac{n\pi(L'-y)}{L}+\sinh\frac{n\pi y}{L}-\sinh\frac{n\pi L'}{L}\right]\right\}*\sin\frac{n\pi x}{L},\qquad \text{where } \sigma_n=\frac{2}{L}\int_0^L-\frac{\sigma(x)}{\epsilon_0}\sin\frac{n\pi x}{L}\,dx$$

(e) $$V(x,y)=\frac{yx(L^2-x^2)}{6\epsilon_0}+\frac{2L^3L'}{\pi^2\epsilon_0}\sum_{n=1}^{\infty}\frac{(-1)^n\sinh(n\pi y/L)}{n^3\sinh(n\pi L'/L)}\sin\frac{n\pi x}{L}$$

29. $$U(x,y)=\frac{g(L^2-x^2)}{2\kappa}+\frac{16L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[\frac{g(-1)^n}{\kappa}+2(-1)^n+(2n-1)\pi\right]\operatorname{sech}\frac{(2n-1)\pi L'}{2L}\cosh\frac{(2n-1)\pi(L'-y)}{2L}\cos\frac{(2n-1)\pi x}{2L}$$

Exercises 5.1

2. $\lambda_n=n^2\pi^2/9$, $y_n(x)=\sqrt{2/3}\sin(n\pi x/3)$
3. $\lambda_n=n^2\pi^2/16$, $y_0(x)=1/2$, $y_n(x)=(1/\sqrt{2})\cos(n\pi x/4)$
4. $\lambda_n=(2n-1)^2\pi^2/324$, $y_n(x)=(\sqrt{2}/3)\sin[(2n-1)\pi x/18]$
5. $\lambda_n=(2n-1)^2\pi^2/4$, $y_n(x)=\sqrt{2}\cos[(2n-1)\pi x/2]$
6. $\lambda_n=(2n-1)^2\pi^2/(4L^2)$, $y_n(x)=\sqrt{2/L}\cos[(2n-1)\pi x/(2L)]$
7. $\lambda_n=n^2\pi^2/81$, $y_n(x)=(\sqrt{2}/3)\sin[n\pi(x-1)/9]$
8. $\lambda_n=1/4+n^2\pi^2$, $y_n(x)=\sqrt{2}e^{x/2}\sin n\pi x$
9. $\lambda_n=\dfrac{1}{4}+\dfrac{n^2\pi^2}{16}$, $y_0(x)=\dfrac{1}{\sqrt{e^5-e}}$,
$$y_n(x)=\sqrt{\frac{2}{n^2\pi^2+4}}e^{-x/2}\left[\frac{n\pi}{2}\cos\frac{n\pi(x-1)}{4}+\sin\frac{n\pi(x-1)}{4}\right]$$
10. $\lambda_n=n^2\pi^2$, $y_n(x)=\sqrt{2}\sin(n\pi\ln x)$
11. $\lambda_n=n^2\pi^2/L^2$, $y_0(x)=A$, $y_n(x)=A\cos(n\pi x/L)+B\sin(n\pi x/L)$
12. Yes **17.** Sometimes

Exercises 5.2

2.

$\sin\lambda_n L$	$\cos\lambda_n L$
$\dfrac{(-1)^{n+1}\lambda_n(h_1/l_1+h_2/l_2)}{\left[\left(\lambda_n^2+h_1^2/l_1^2\right)\left(\lambda_n^2+h_2^2/l_2^2\right)\right]^{1/2}}$	$\dfrac{(-1)^{n+1}\left(\lambda_n^2-h_1h_2/(l_1l_2)\right)}{\left[\left(\lambda_n^2+h_1^2/l_1^2\right)\left(\lambda_n^2+h_2^2/l_2^2\right)\right]^{1/2}}$
$\dfrac{(-1)^{n+1}(h_1/l_1)}{\sqrt{\lambda_n^2+(h_1/l_1)^2}}$	$\dfrac{(-1)^{n+1}\lambda_n}{\sqrt{\lambda_n^2+(h_1/l_1)^2}}$
$\dfrac{(-1)^{n+1}\lambda_n}{\sqrt{\lambda_n^2+(h_1/l_1)^2}}$	$\dfrac{(-1)^{n}(h_1/l_1)}{\sqrt{\lambda_n^2+(h_1/l_1)^2}}$
$\dfrac{(-1)^{n+1}(h_2/l_2)}{\sqrt{\lambda_n^2+(h_2/l_2)^2}}$	$\dfrac{(-1)^{n+1}\lambda_n}{\sqrt{\lambda_n^2+(h_2/l_2)^2}}$
0	$(-1)^n$
$(-1)^{n+1}$	0
$\dfrac{(-1)^{n+1}\lambda_n}{\sqrt{\lambda_n^2+(h_2/l_2)^2}}$	$\dfrac{(-1)^{n}(h_2/l_2)}{\sqrt{\lambda_n^2+(h_2/l_2)^2}}$
$(-1)^{n+1}$	0
0	$(-1)^n$

4. $\dfrac{\sqrt{2L^3}}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sqrt{\dfrac{2}{L}}\sin\dfrac{n\pi x}{L}$, $0\le x<L$

5. $\dfrac{L^{3/2}/2}{\sqrt{L}}-\dfrac{2\sqrt{2L^3}}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\sqrt{\dfrac{2}{L}}\cos\dfrac{(2n-1)\pi x}{L}$, $0\le x\le L$

6. $\dfrac{4\sqrt{2L^3}}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\sqrt{\dfrac{2}{L}}\sin\dfrac{(2n-1)\pi x}{2L}$, $0\le x\le L$

7. $\dfrac{\sqrt{8L^3}}{\pi^2}\sum_{n=1}^{\infty}\left[\dfrac{\pi(-1)^{n+1}}{2n-1}-\dfrac{2}{(2n-1)^2}\right]\sqrt{\dfrac{2}{L}}\cos\dfrac{(2n-1)\pi x}{2L}$, $0\le x\le L$

8. $\sum_{n=1}^{\infty}c_n\dfrac{1}{N}\cos\lambda_n x$, where $2N^2=L+\dfrac{h_2/l_2}{\lambda_n^2+(h_2/l_2)^2}$ and

$$c_n=\frac{1}{N}\left[\frac{L}{\lambda_n}\sin\lambda_n L+\frac{1}{\lambda_n^2}(\cos\lambda_n L-1)\right]$$

9. $\sum_{n=1}^{\infty}c_n\dfrac{1}{N}\sin\lambda_n x$, where $2N^2=L+\dfrac{h_2/l_2}{\lambda_n^2+(h_2/l_2)^2}$ and $c_n=\dfrac{1}{N}\left(-\dfrac{L}{\lambda_n}\cos\lambda_n L+\dfrac{1}{\lambda_n^2}\sin\lambda_n L\right)$

10. $\dfrac{8\sqrt{2}L^{5/2}}{\pi^3}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^{n+1}\pi}{(2n-1)^2}-\dfrac{2}{(2n-1)^3}\right]\sqrt{\dfrac{2}{L}}\sin\dfrac{(2n-1)\pi x}{2L}$, $0\le x\le L$

11. $\lambda_0^2=0$, $y_0(x)=\sqrt{\dfrac{2}{e^{2L}-1}}$;

$$\lambda_n^2=1+n^2\pi^2/L^2,\ y_n(x)=\frac{\sqrt{2L}}{\sqrt{n^2\pi^2+L^2}}e^{-x}\left(\frac{n\pi}{L}\cos\frac{n\pi x}{L}+\sin\frac{n\pi x}{L}\right)$$

12. $\lambda_n^2=\beta^2/4+n^2\pi^2/L^2$, $y_n(x)=\sqrt{\dfrac{2}{L}}e^{-\beta x/2}\sin\dfrac{n\pi x}{L}$

13. $\lambda_0^2 = 0,\ y_0(x) = \sqrt{\dfrac{\beta}{e^{\beta L} - 1}};\ \lambda_n^2 = \dfrac{\beta^2}{4} + \dfrac{n^2\pi^2}{L^2},$

$$y_n(x) = \frac{\beta\sqrt{2L}}{\sqrt{\beta^2L^2 + 4n^2\pi^2}} e^{-\beta x/2}\left(\frac{2n\pi}{\beta L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi x}{L}\right)$$

14. (c) $y_0(x) = \dfrac{1}{\sqrt{2L}},\ y_n(x) = \dfrac{1}{\sqrt{L}}\cos\dfrac{n\pi(x+L)}{2L}$ **15.** $y_n(x) = \sqrt{\dfrac{2}{\ln L}}\sin\dfrac{n\pi\ln x}{\ln L}$

16. (a) $y_n(x) = \sqrt{\dfrac{2}{\ln L}}\cos\dfrac{(2n-1)\pi\ln x}{2\ln L}$ (b) $y_n(x) = \sqrt{\dfrac{2}{\ln L}}\cos\dfrac{n\pi\ln x}{\ln L}$

17. $y_n(x) = \dfrac{1}{N}\sin\lambda_n x,\ \tan\lambda\ln L = -l\lambda/(hL),\ 2N^2 = \ln L + \dfrac{(h/l)L}{\lambda_n^2 + (h/l)^2L^2}$

18. $y_n(x) = \cos(\lambda_n \ln x) + \dfrac{h_1}{l_1\lambda_n}\sin(\lambda_n\ln x)$, where $\tan\lambda\ln L = \dfrac{\lambda(h_1/l_1 + h_2/l_2)}{\lambda^2 - h_1h_2/(l_1l_2)}$

19. (a) $y_n(x) = \sqrt{\dfrac{2}{\ln(b/a)}}\sin\dfrac{n\pi\ln(x/a)}{\ln(b/a)}$ (b) $y_n(x) = \sqrt{\dfrac{2}{\ln(b/a)}}\cos\dfrac{(2n-1)\pi\ln(x/a)}{2\ln(b/a)}$

(c) $y_0(x) = \dfrac{1}{\sqrt{\ln(b/a)}},\ y_n(x) = \sqrt{\dfrac{2}{\ln(b/a)}}\cos\dfrac{n\pi\ln(x/a)}{\ln(b/a)}$

20. $y_n(x) = N^{-1}\sin[\lambda_n\ln(x/a)]$ where $2N^2 = \ln(b/a) + \dfrac{hb/l}{\lambda_n^2 + h^2b^2/l^2}$ and

$$\cot[\lambda\ln(b/a)] = -\frac{hb}{l\lambda}$$

21. $\lambda_1^2 = 2.46494,\ \lambda_2^2 = 22.1844,\ \lambda_3^2 = 61.6234,\ \lambda_4^2 = 129.782$

22. $\lambda_1^2 = 9.84006,\ \lambda_2^2 = 39.3603,\ \lambda_3^2 = 88.5606,\ \lambda_4^2 = 157.441$

23. $\lambda_1^2 = 1.79160,\ \lambda_2^2 = 14.1550,\ \lambda_3^2 = 44.2273,\ \lambda_4^2 = 93.7048$

24. (a) $\dfrac{2\sqrt{2L}}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\sqrt{\frac{2}{L}}\cos\frac{(2n-1)\pi x}{L}$ (b) 0 (c) 1,−1

25. (a) $\dfrac{2\sqrt{2L}}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\sqrt{\frac{2}{L}}\sin\frac{2(2n-1)\pi x}{L}$ (b) 0 (c) 0,0

26. No **28.** Yes **29.** Yes **30.** $A\cosh\sqrt{-\lambda}x$

Exercises 5.3

1. (a) $\dfrac{4L}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}$ (b) $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{L}$

2. (a) $\dfrac{8L^2}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}$

3. (b) $\lambda_n = \displaystyle\int_a^b \{r(x)[y_n'(x)]^2 + q(x)[y_n(x)]^2\}\,dx$

Exercises 6.2

1. (a) $U(x,t) = \dfrac{8L}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\pi^2kt/(4L^2)}}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2L}$

(b) $\dfrac{4\kappa}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-(2n-1)^2\pi^2kt/(4L^2)}}{2n-1}$

2. $U(x,t) = \sum_{n=1}^{\infty} c_n e^{-(2n-1)^2\pi^2 kt/(4L^2)} \sqrt{\frac{2}{L}} \cos\frac{(2n-1)\pi x}{2L}$ where

$$c_n = \int_0^L f(x)\sqrt{\frac{2}{L}}\cos\frac{(2n-1)\pi x}{2L}dx$$

3. $U(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(x)$, where $X_n(x) = \frac{1}{N}\frac{\sin\lambda_n(L-x)}{\sin\lambda_n L}$,

$$2N^2 = L\left[1 + \left(\frac{\kappa}{\lambda_n \mu}\right)^2\right] + \frac{\kappa/\mu}{\lambda_n^2}, \quad \cot\lambda L = -\frac{\kappa}{\lambda\mu}$$

4. (a) $U(x,t) = e^{-ht}\left(\frac{c_0}{\sqrt{L}} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 kt/L^2}\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\right)$ where $c_0 = \int_0^L \frac{f(x)}{\sqrt{L}}dx$

and $c_n = \int_0^L f(x)\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}dx$ (b) Drop the e^{-ht} factor

6. (a) $y(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$ where

$$A_n = \int_0^L f(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx \text{ and } B_n = \frac{L}{n\pi c}\int_0^L g(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx$$

(b) $\sqrt{A_1^2 + B_1^2}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$, $\sqrt{A_2^2 + B_2^2}\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$, $\sqrt{A_3^2 + B_3^2}\sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}$

(c) No nodes; $x = L/2$; and $x = L/3$ and $x = 2L/3$

7. (a) $y(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\frac{(2n-1)\pi ct}{2L} + B_n\sin\frac{(2n-1)\pi ct}{2L}\right]\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}$ where

$$A_n = \int_0^L f(x)\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}dx, \quad B_n = \frac{2L}{(2n-1)\pi c}\int_0^L g(x)\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}dx$$

(b) $y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$

8. (a) $y(x,t) = \sum_{n=1}^{\infty}(A_n\cos\omega_n t + B_n\sin\omega_n t)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$ where $\omega_n = \sqrt{\frac{n^2\pi^2c^2}{L^2} + \frac{k}{\rho}}$ and

$$A_n = \int_0^L f(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx, \quad B_n = \frac{1}{\omega_n}\int_0^L g(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx$$

9. $y(x,t) = \sum_{n=1}^{\infty} e^{-\beta t/(2\rho)}(A_n\cos\omega_n t + B_n\sin\omega_n t)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$ where $\omega_n = \frac{1}{2}\sqrt{\frac{4n^2\pi^2c^2}{L^2} - \frac{\beta^2}{\rho^2}}$

and $A_n = \int_0^L f(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx$, $B_n = \frac{1}{\omega_n}\int_0^L\left[g(x) + \frac{\beta}{2\rho}f(x)\right]\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx$

11. $y(x,t) = \frac{kL + AE}{50}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sqrt{A^2E^2\lambda_n^2 + k^2}}{\lambda_n[L(A^2E^2\lambda_n^2 + k^2) + kAE]}\cos c\lambda_n t\sin\lambda_n x$, $\cot\lambda L = -\frac{k}{AE\lambda}$

Exercises 6.3

1. (a) $V(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh[(2n-1)\pi L/L']}\sinh\frac{(2n-1)\pi(L-x)}{L'}\sin\frac{(2n-1)\pi y}{L'}$

(b) 25

2. 50°C **3.** $U_0/4$

4. (a) $U(x,y) = C + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi y}{L} \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}$ where

$$A_n = \frac{L}{n\pi \sinh n\pi} \int_0^L f(x) \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L} dx$$

Function $f(x)$ must satisfy the condition $\int_0^L f(x)\, dx = 0$.

(b) $U(x,y) = 50 + \frac{4L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\operatorname{csch}(2n-1)\pi}{(2n-1)^3} \cosh \frac{(2n-1)\pi y}{L} \cos \frac{(2n-1)\pi x}{L}$

(c) No solution

5. (a) $U(x,y) = \frac{Ax+B}{\sqrt{L'}} + \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi x}{L'} + B_n \sinh \frac{n\pi x}{L'}\right) \sqrt{\frac{2}{L'}} \cos \frac{n\pi y}{L'}$, where

$$A = \int_0^{L'} f_1(y) \frac{1}{\sqrt{L'}} dy, \quad A_n = \frac{1}{\cosh(n\pi L/L')} \left[\int_0^{L'} f_2(y) Y_n(y)\, dy - B_n \sinh \frac{n\pi L}{L'}\right],$$

$$B = \int_0^{L'} \frac{f_2(y)}{\sqrt{L'}} dy - AL, \quad B_n = \frac{L'}{n\pi} \int_0^{L'} f_1(y) Y_n(y)\, dy$$

(b) $f_1(x-L) + f_2$

6. $z(x,y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi y}{L} + B_n \sinh \frac{n\pi y}{L}\right) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$

$$+ \sum_{n=1}^{\infty} \left(C_n \cosh \frac{n\pi x}{L'} + D_n \sinh \frac{n\pi x}{L'}\right) \sqrt{\frac{2}{L'}} \sin \frac{n\pi y}{L'} \text{ where}$$

$$A_n = \int_0^L g_1(x) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx, \quad C_n = \int_0^{L'} f_1(y) \sqrt{\frac{2}{L'}} \sin \frac{n\pi y}{L'} dy,$$

$$B_n = \frac{1}{\sinh(n\pi L'/L)} \left[\int_0^L g_2(x) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx - A_n \cosh \frac{n\pi L'}{L}\right],$$

$$D_n = \frac{1}{\sinh(n\pi L/L')} \left[\int_0^{L'} f_2(y) \sqrt{\frac{2}{L'}} \sin \frac{n\pi y}{L'} dy - C_n \cosh \frac{n\pi L}{L'}\right]$$

7. $V(x,y) = \sum_{n=1}^{\infty} \left[A_n \cosh \frac{(2n-1)\pi x}{2L'} + B_n \sinh \frac{(2n-1)\pi x}{2L'}\right] \sqrt{\frac{2}{L'}} \cos \frac{(2n-1)\pi y}{2L'}$

$$+ \sum_{n=1}^{\infty} C_n \cosh \frac{(2n-1)\pi y}{2L} \sqrt{\frac{2}{L}} \sin \frac{(2n-1)\pi x}{2L}, \text{ where}$$

$$A_n = \int_0^{L'} f_1(y) \sqrt{\frac{2}{L'}} \cos \frac{(2n-1)\pi y}{2L'}\, dy,$$

$$B_n = \frac{2L'}{(2n-1)\pi \cosh \frac{(2n-1)\pi L}{2L'}} \left[\int_0^{L'} f_2(y) \sqrt{\frac{2}{L'}} \cos \frac{(2n-1)\pi y}{2L'}\, dy - \frac{(2n-1)\pi A_n}{2L'} \sinh \frac{(2n-1)\pi L}{2L'}\right],$$

$$C_n = \frac{1}{\cosh[(2n-1)\pi L'/(2L)]} \int_0^L g(x) \sqrt{\frac{2}{L}} \sin \frac{(2n-1)\pi x}{2L}\, dx$$

8. (a) $V(r,\theta)=\sum_{n=1}^{\infty}A_n r^n\sqrt{\frac{2}{\pi}}\sin n\theta$ where $A_n=a^{-n}\int_0^{\pi}f(\theta)\sqrt{\frac{2}{\pi}}\sin n\theta\,d\theta$

(b) $V(r,\theta)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta,\quad V(r,\pi/2)=\frac{4}{\pi}\mathrm{Tan}^{-1}\left(\frac{r}{a}\right)$

9. (b) 1/2

10. $U(r,\theta)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\left(\frac{r}{a}\right)^{2n}\cos 2n\theta$

11. (a) $V(r,\theta)=\frac{A_0}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}r^n\left(A_n\frac{\cos n\theta}{\sqrt{\pi}}+B_n\frac{\sin n\theta}{\sqrt{\pi}}\right)$, where A_0 is arbitrary and

$$A_n=\frac{1}{na^{n-1}}\int_{-\pi}^{\pi}f(\theta)\frac{\cos n\theta}{\sqrt{\pi}}\,d\theta,\quad B_n=\frac{1}{na^{n-1}}\int_{-\pi}^{\pi}f(\theta)\frac{\sin n\theta}{\sqrt{\pi}}\,d\theta$$

12. $V(r,\theta)=\frac{A_0}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}r^n\left(A_n\frac{\cos n\theta}{\sqrt{\pi}}+B_n\frac{\sin n\theta}{\sqrt{\pi}}\right)$ where $A_0=\frac{1}{h}\int_{-\pi}^{\pi}\frac{f(\theta)}{\sqrt{2\pi}}d\theta$ and

$$A_n=\frac{1}{a^{n-1}(ha+nl)}\int_{-\pi}^{\pi}f(\theta)\frac{\cos n\theta}{\sqrt{\pi}}d\theta,\ B_n=\frac{1}{a^{n-1}(ha+nl)}\int_{-\pi}^{\pi}f(\theta)\frac{\sin n\theta}{\sqrt{\pi}}d\theta$$

13. (b) No **14.** (b) Yes

15. (a) $V(r,\theta)=\frac{A_0}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}r^{-n}\left(A_n\frac{\cos n\theta}{\sqrt{\pi}}+B_n\frac{\sin n\theta}{\sqrt{\pi}}\right)$, where $A_0=\frac{1}{h}\int_{-\pi}^{\pi}\frac{f(\theta)}{\sqrt{2\pi}}\,d\theta$,

$$A_n=\frac{a^{n+1}}{ha+ln}\int_{-\pi}^{\pi}f(\theta)\frac{\cos n\theta}{\sqrt{\pi}}\,d\theta,\quad B_n=\frac{a^{n+1}}{ha+ln}\int_{-\pi}^{\pi}f(\theta)\frac{\sin n\theta}{\sqrt{\pi}}\,d\theta$$

(b) A_0 must be zero

16. $V(r,\theta)=\frac{A_0+B_0\ln r}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\left[(A_nr^n+B_nr^{-n})\frac{\cos n\theta}{\sqrt{\pi}}+(C_nr^n+D_nr^{-n})\frac{\sin n\theta}{\sqrt{\pi}}\right]$ where

$$A_0=\frac{1}{\ln(R/a)}\int_{-\pi}^{\pi}\frac{f_1(\theta)\ln R-f_2(\theta)\ln a}{\sqrt{2\pi}}d\theta,\quad B_0=\frac{1}{\ln(R/a)}\int_{-\pi}^{\pi}\frac{f_2(\theta)-f_1(\theta)}{\sqrt{2\pi}}d\theta,$$

$$A_n=\frac{1}{R^{2n}-a^{2n}}\int_{-\pi}^{\pi}[R^nf_2(\theta)-a^nf_1(\theta)]\frac{\cos n\theta}{\sqrt{\pi}}d\theta,$$

$$B_n=\frac{a^nR^n}{R^{2n}-a^{2n}}\int_{-\pi}^{\pi}[R^nf_1(\theta)-a^nf_2(\theta)]\frac{\cos n\theta}{\sqrt{\pi}}d\theta$$

$$C_n=\frac{1}{R^{2n}-a^{2n}}\int_{-\pi}^{\pi}[R^nf_2(\theta)-a^nf_1(\theta)]\frac{\sin n\theta}{\sqrt{\pi}}d\theta,$$

$$D_n=\frac{a^nR^n}{R^{2n}-a^{2n}}\int_{-\pi}^{\pi}[R^nf_1(\theta)-a^nf_2(\theta)]\frac{\sin n\theta}{\sqrt{\pi}}d\theta$$

17. $V(r,\theta)=\frac{A_0+B_0\ln r}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\left[\left(A_nr^n+B_nr^{-n}\right)\frac{\cos n\theta}{\sqrt{\pi}}+\left(C_nr^n+D_nr^{-n}\right)\frac{\sin n\theta}{\sqrt{\pi}}\right]$,

where A_0 is arbitrary, and

$$A_n=\frac{1}{n\left(R^{2n}-a^{2n}\right)}\int_{-\pi}^{\pi}\left[a^{n+1}f_1(\theta)+R^{n+1}f_2(\theta)\right]\frac{\cos n\theta}{\sqrt{\pi}}\,d\theta$$

$$B_n=\frac{(Ra)^{n+1}}{n\left(R^{2n}-a^{2n}\right)}\int_{-\pi}^{\pi}\left[a^{n-1}f_2(\theta)+R^{n-1}f_1(\theta)\right]\frac{\cos n\theta}{\sqrt{\pi}}\,d\theta$$

$$C_n=\frac{1}{n\left(R^{2n}-a^{2n}\right)}\int_{-\pi}^{\pi}\left[a^{n+1}f_1(\theta)+R^{n+1}f_2(\theta)\right]\frac{\sin n\theta}{\sqrt{\pi}}\,d\theta$$

$$D_n=\frac{(Ra)^{n+1}}{n\left(R^{2n}-a^{2n}\right)}\int_{-\pi}^{\pi}\left[a^{n-1}f_2(\theta)+R^{n-1}f_1(\theta)\right]\frac{\sin n\theta}{\sqrt{\pi}}\,d\theta$$

18. $V(r,\theta) = \dfrac{A_0 + B_0 \ln r}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left[(A_n r^n + B_n r^{-n})\dfrac{\cos n\theta}{\sqrt{\pi}} + (C_n r^n + D_n r^{-n})\dfrac{\sin n\theta}{\sqrt{\pi}}\right]$ where

$$A_0 = \frac{1}{M}\int_{-\pi}^{\pi} \frac{a(Rh_2 \ln R + l_2)f_1(\theta) + R(l_1 - ah_1 \ln a)f_2(\theta)}{\sqrt{2\pi}} d\theta,$$
$$B_0 = \frac{Ra}{M}\int_{-\pi}^{\pi} \frac{h_1 f_2(\theta) - h_2 f_1(\theta)}{\sqrt{2\pi}} d\theta,$$
$$A_n = \frac{1}{G}\int_{-\pi}^{\pi} [(h_2 R^{-n} - nl_2 R^{-n-1})f_1(\theta) - (h_1 a^{-n} + nl_1 a^{-n-1})f_2(\theta)]\frac{\cos n\theta}{\sqrt{\pi}} d\theta,$$
$$B_n = \frac{1}{G}\int_{-\pi}^{\pi} [(h_1 a^n - nl_1 a^{n-1})f_2(\theta) - (h_2 R^n + nl_2 R^{n-1})f_1(\theta)]\frac{\cos n\theta}{\sqrt{\pi}} d\theta,$$
$$C_n = \frac{1}{G}\int_{-\pi}^{\pi} [(h_2 R^{-n} - nl_2 R^{-n-1})f_1(\theta) - (h_1 a^{-n} + nl_1 a^{-n-1})f_2(\theta)]\frac{\sin n\theta}{\sqrt{\pi}} d\theta,$$
$$D_n = \frac{1}{G}\int_{-\pi}^{\pi} [(h_1 a^n - nl_1 a^{n-1})f_2(\theta) - (h_2 R^n + nl_2 R^{n-1})f_1(\theta)]\frac{\sin n\theta}{\sqrt{\pi}} d\theta,$$
$$M = h_1 h_2 aR \ln(R/a) + ah_1 l_2 + Rh_2 l_1,$$
$$G = (h_2 R^{-n} - nl_2 R^{-n-1})(h_1 a^n - nl_1 a^{n-1}) - (h_1 a^{-n} + al_1 a^{-n-1})(h_2 R^n + nl_2 R^{n-1})$$

19. (a) $U(r,\theta) = A_0 H_0(\theta) + \sum_{n=1}^{\infty} A_n r^{n\pi/\alpha}\sqrt{\dfrac{2}{\alpha}}\cos\dfrac{n\pi\theta}{\alpha}$, where $A_n = a^{-n\pi/\alpha}\int_0^{\alpha} f(\theta)H_n(\theta)\, d\theta$

(b) $U(r,\theta) = \dfrac{\alpha}{2} - \dfrac{4\alpha}{\pi^2}\sum_{n=1}^{\infty}\dfrac{r^{(2n-1)\pi/\alpha}}{(2n-1)^2}\cos\dfrac{(2n-1)\pi\theta}{\alpha}$

20. Sides must have constant temperatures k_1 and k_2, in which case $U(\theta) = k_1 + (k_2 - k_1)\theta/\alpha$.

21. (a) $z(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\alpha}\sqrt{\dfrac{2}{\alpha}}\sin\dfrac{n\pi\theta}{\alpha}$, where $A_n = a^{-n\pi/\alpha}\int_0^{\alpha} f(\theta)\sqrt{\dfrac{2}{\alpha}}\sin\dfrac{n\pi\theta}{\alpha}\, d\theta$

(b) $z(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n/2}\sqrt{\dfrac{2}{\alpha}}\sin\dfrac{n\pi\theta}{\alpha}$, where $A_n = a^{-n/2}\int_0^{2\pi} f(\theta)\sqrt{\dfrac{2}{\alpha}}\sin\dfrac{n\pi\theta}{\alpha}\, d\theta$

(c) $z(r,\theta) = \sqrt{\dfrac{r}{a}}\sin\left(\dfrac{\theta}{2}\right)$

22. (a) $V(r,\theta) = \sum_{n=1}^{\infty} A_n\left[\left(\dfrac{r}{a}\right)^n - \left(\dfrac{a}{r}\right)^n\right]\sqrt{\dfrac{2}{\pi}}\sin n\theta$ where

$$A_n = \left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right]^{-1}\int_0^{\pi} f(\theta)\sqrt{\frac{2}{\pi}}\sin n\theta\, d\theta$$

(b) $V(r,\theta) = \dfrac{4V_0}{\pi}\sum_{n=1}^{\infty}\left[\left(\dfrac{b}{a}\right)^{2n-1} - \left(\dfrac{a}{b}\right)^{2n-1}\right]^{-1}\dfrac{1}{2n-1}\left[\left(\dfrac{r}{a}\right)^{2n-1} - \left(\dfrac{a}{r}\right)^{2n-1}\right]\sin(2n-1)\theta$

23. (a) $V(r,\theta) = \sum_{n=1}^{\infty} A_n \sinh\dfrac{n\pi\theta}{\ln(b/a)}\sqrt{\dfrac{2}{\ln(b/a)}}\sin\dfrac{n\pi\ln(r/a)}{\ln(b/a)}$, where

$$A_n = \operatorname{csch}\frac{n\pi^2}{\ln(b/a)}\int_a^b \frac{1}{r}f(r)\sqrt{\frac{2}{\ln(b/a)}}\sin\frac{n\pi\ln(r/a)}{\ln(b/a)}\, dr$$

(b) $V(r,\theta) = \dfrac{4V_0}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\operatorname{csch}\dfrac{(2n-1)\pi^2}{\ln(b/a)}\sinh\dfrac{(2n-1)\pi\theta}{\ln(b/a)}\sin\dfrac{(2n-1)\pi\ln(r/a)}{\ln(b/a)}$

24. $V(r,\theta) = \dfrac{A_0 + B_0\theta}{\sqrt{\ln(b/a)}} + \displaystyle\sum_{n=1}^{\infty}\left[A_n \cosh\frac{n\pi\theta}{\ln(b/a)} + B_n \sinh\frac{n\pi\theta}{\ln(b/a)}\right]\sqrt{\frac{2}{\ln(b/a)}}\cos\frac{n\pi\ln(r/a)}{\ln(b/a)}$,

where $A_0 = \displaystyle\int_a^b \frac{1}{r}g(r)\frac{1}{\sqrt{\ln(b/a)}}\,dr, \quad A_n = \int_a^b \frac{1}{r}g(r)\sqrt{\frac{2}{\ln(b/a)}}\cos\frac{n\pi\ln(r/a)}{\ln(b/a)}\,dr,$

$B_0 = \dfrac{1}{\pi}\displaystyle\int_a^b \frac{1}{r}[f(r) - g(r)]\frac{1}{\sqrt{\ln(b/a)}}\,dr,$

$B_n = \operatorname{csch}\dfrac{n\pi^2}{\ln(b/a)}\displaystyle\int_a^b \frac{1}{r}\left[f(r) - \cosh\frac{n\pi^2}{\ln(b/a)}g(r)\right]\sqrt{\frac{2}{\ln(b/a)}}\cos\frac{n\pi\ln(r/a)}{\ln(b/a)}\,dr$

25. $V(r,\theta) = \displaystyle\sum_{n=1}^{\infty}(A_n\cosh\lambda_n\theta + B_n\sinh\lambda_n\theta)R_n(r)$, where $R_n(r) = \dfrac{1}{N}\sin[\lambda_n\ln(r/a)]$,

$2N^2 = \ln\left(\dfrac{b}{a}\right) + \dfrac{hb/l}{\lambda_n^2 + h^2b^2/l^2}$, $\cot[\lambda\ln(b/a)] = -hb/(l\lambda)$

$A_n = \displaystyle\int_a^b \frac{1}{r}g(r)R_n(r)\,dr, \quad B_n = \operatorname{csch}\lambda_n\pi\int_a^b \frac{1}{r}[f(r) - \cosh\lambda_n\pi g(r)]R_n(r)\,dr$

26. $V(r,\theta) = (A_0\ln r + B_0)\sqrt{\dfrac{2}{\pi}} + \displaystyle\sum_{n=1}^{\infty}\left(A_n r^{2n} + \frac{B_n}{r^{2n}}\right)\frac{2}{\sqrt{\pi}}\cos 2n\theta$, where

$A_0 = \dfrac{1}{\ln(b/a)}\displaystyle\int_0^{\pi/2}[g(\theta) - f(\theta)]\sqrt{\frac{2}{\pi}}\,d\theta,$

$B_0 = \displaystyle\int_0^{\pi/2}\left[f(\theta - \frac{\ln a}{\ln(b/a)}[g(\theta) - f(\theta)]\right]\sqrt{\frac{2}{\pi}}\,d\theta,$

$A_n = \dfrac{1}{(b/a)^{2n} - (a/b)^{2n}}\displaystyle\int_0^{\pi/2}\left[\frac{1}{a^{2n}}f(\theta) - \frac{1}{b^{2n}}g(\theta)\right]\frac{2}{\sqrt{\pi}}\cos 2n\theta\,d\theta,$

$B_n = \dfrac{1}{(b/a)^{2n} - (a/b)^{2n}}\displaystyle\int_0^{\pi/2}\left[b^{2n}f(\theta) - a^{2n}g(\theta)\right]\frac{2}{\sqrt{\pi}}\cos 2n\theta\,d\theta$

27. $V(r,\theta) = (A_0\theta + B_0)R_0(r) + \displaystyle\sum_{n=1}^{\infty}\left[A_n\cosh\frac{n\pi\theta}{\ln(b/a)} + B_n\sinh\frac{n\pi\theta}{\ln(b/a)}\right]R_n(r)$, where

$R_0(r) = \dfrac{1}{\sqrt{\ln(b/a)}}, \quad R_n(r) = \sqrt{\dfrac{2}{\ln(b/a)}}\cos\dfrac{n\pi\ln(r/a)}{\ln(b/a)}, \quad B_0 = \displaystyle\int_a^b \frac{1}{r}f(r)R_0(r)\,dr,$

$A_0 = \dfrac{2}{\pi}\displaystyle\int_a^b \frac{1}{r}[g(r) - f(r)]R_0(r)\,dr, \quad A_n = \int_a^b \frac{1}{r}f(r)R_n(r)\,dr,$

$B_n = \operatorname{csch}\dfrac{n\pi^2}{2\ln(b/a)}\displaystyle\int_a^b \frac{1}{r}\left[g(r) - \cosh\frac{n\pi^2}{2\ln(b/a)}f(r)\right]R_n(r)\,dr$

29. (b) 1/2, 1/2 (d) $r = a\left\{\dfrac{-\sin\theta \pm \sqrt{\sin^2\theta + \tan^2[(2V-1)\pi/2]}}{\tan[(2V-1)\pi/2]}\right\}$

30. $V(r,\theta) = \dfrac{1}{2}(V_1 + V_2) + \dfrac{1}{\pi}(V_1 - V_2)\operatorname{Tan}^{-1}\left(\dfrac{2ar\sin\theta}{a^2 - r^2}\right)$

31. $$V(r,\theta) = \begin{cases} 1 + \dfrac{1}{\pi}\left\{\operatorname{Tan}^{-1}\left[\dfrac{a+r}{a-r}\tan\left(\dfrac{\theta}{2}\right)\right] - \operatorname{Tan}^{-1}\left[\dfrac{a+r}{a-r}\tan\left(\dfrac{2\theta-\pi}{4}\right)\right]\right\}, & -\pi < \theta < -\dfrac{\pi}{2} \\ \dfrac{1}{\pi}\left\{\operatorname{Tan}^{-1}\left[\dfrac{a+r}{a-r}\tan\left(\dfrac{\theta}{2}\right)\right] - \operatorname{Tan}^{-1}\left[\dfrac{a+r}{a-r}\tan\left(\dfrac{2\theta-\pi}{4}\right)\right]\right\}, & -\dfrac{\pi}{2} \le \theta \le \pi \end{cases}$$

Exercises 6.4

1. $U(x,y,t) = \frac{2}{\sqrt{LL'}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-(n^2/L^2+m^2/L'^2)\pi^2 kt} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'}$, where

$$C_{mn} = \frac{2}{\sqrt{LL'}} \int_0^{L'} \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'} dx\, dy$$

2. $U(x,y,t) = \frac{2}{\sqrt{LL'}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-[(2n-1)^2/(4L^2)+(2m-1)^2/(4L'^2)]\pi^2 kt} *$

$$\cos\frac{(2n-1)\pi x}{2L} \sin\frac{(2m-1)\pi y}{2L'},$$

where $C_{mn} = \frac{2}{\sqrt{LL'}} \int_0^{L'} \int_0^L f(x,y) \cos\frac{(2n-1)\pi x}{2L} \sin\frac{(2m-1)\pi y}{2L'}\, dx\, dy$

3. (a) $U(x,y,t) = \sum_{n=1}^{\infty} C_{0n} e^{-n^2\pi^2 kt/L^2} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \frac{1}{\sqrt{L'}}$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-(n^2/L^2+m^2/L'^2)\pi^2 kt} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \sqrt{\frac{2}{L'}} \cos\frac{m\pi y}{L'},$$

where $C_{0n} = \sqrt{\frac{2}{LL'}} \int_0^{L'} \int_0^L f(x,y) \sin\frac{n\pi x}{L}\, dx\, dy,$

$$C_{mn} = \frac{2}{\sqrt{LL'}} \int_0^{L'} \int_0^L f(x,y) \sin\frac{n\pi x}{L} \cos\frac{m\pi y}{L'}\, dx\, dy$$

(b) $U(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}$, where $C_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$

4. $U(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-k(\lambda_n^2+\mu_m^2)t} X_n(x) Y_m(y)$, where

$$C_{mn} = \int_0^{L'} \int_0^L f(x,y) X_n(x) Y_m(y)\, dx\, dy,$$

$$X_n(x) = N^{-1} \sin\lambda_n x, \quad 2N^2 = L + \frac{h/l}{\lambda_n^2 + (h/l)^2}, \quad Y_m(y) = \sqrt{\frac{2}{L'}} \sin\frac{m\pi y}{L'}$$

5. $U(x,z,t) = \frac{16U_0}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-k\pi^2[(2n-1)^2/L^2+(2m-1)^2/(4L''^2)]t}}{(2m-1)(2n-1)} \sin\frac{(2n-1)\pi x}{L} \sin\frac{(2m-1)\pi z}{2L''}$

6. $U(x,z,t) = \frac{4U_0}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\mu_m L''}{(2n-1)\mu_m N^2} e^{-k[(2n-1)^2\pi^2/L^2+\mu_m^2]t} \sin\frac{(2n-1)\pi x}{L} \cos\mu_m z,$

where $2N^2 = L'' + \frac{h/l}{\mu_m^2 + (h/l)^2}, \quad \tan\mu L'' = \frac{h}{\mu l}$

7. $U(x,t) = \frac{4U_0}{\pi} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2\pi^2 kt/L^2}}{2n-1} \sin\frac{(2n-1)\pi x}{L}$

9. $\frac{64U_0}{\pi^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{m+1}}{(2n-1)(2m-1)(2j-1)} e^{-k\pi^2[(2n-1)^2/L^2+(2m-1)^2/(4L'^2)+(2j-1)^2/(4L''^2)]t} *$

$$\sin\frac{(2n-1)\pi x}{L} \cos\frac{(2m-1)\pi y}{2L'} \sin\frac{(2j-1)\pi z}{2L''}$$

10. (a) $$U(x,y,t) = \frac{4LL'}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}e^{-(2m-1)^2\pi^2kt/(4L'^2)}\cos\frac{(2m-1)\pi y}{2L'}$$
$$-\frac{32LL'}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{-[(2n-1)^2/L^2+(2m-1)^2/(4L'^2)]\pi^2kt}}{(2n-1)^2(2m-1)^2}\cos\frac{(2n-1)\pi x}{L}\cos\frac{(2m-1)\pi y}{2L'}$$

11. (a) $$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos\frac{\sqrt{n^2+m^2}\pi ct}{L}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L}}\sin\frac{m\pi y}{L}$$, where
$$A_{mn} = \frac{2}{L}\int_0^L\int_0^L f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L}\,dx\,dy$$
(b) $$\frac{2L}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+m}}{(2n-1)^2(2m-1)^2}\cos\frac{c\pi\sqrt{(2n-1)^2+(2m-1)^2}t}{L} *$$
$$\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2m-1)\pi y}{L}$$

12. (a) $$z(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos c\pi\sqrt{n^2/L^2+m^2/L'^2}tX_n(x)Y_m(y),$$ where
$$A_{mn} = \int_0^{L'}\int_0^L f(x,y)X_n(x)Y_m(y)\,dx\,dy, \quad X_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \quad Y_0(y) = \frac{1}{\sqrt{L'}},$$
$$Y_m(y) = \sqrt{\frac{2}{L'}}\cos\frac{m\pi y}{L'}$$
(b) $$z(x,y,t) = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$$

13. $x = pL/n$, $p = 1,\ldots,n-1$ and $y = qL'/m$, $q = 1,\ldots,m-1$

14. No

15. $$V(x,y,z) = \frac{2}{\sqrt{LL'}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\cosh\sqrt{\frac{n^2}{L^2}+\frac{m^2}{L'^2}}\pi z + B_{mn}\sinh\sqrt{\frac{n^2}{L^2}+\frac{m^2}{L'^2}}\pi z\right) *$$
$$\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'},$$
where $$A_{mn} = \frac{2}{\sqrt{LL'}}\int_0^{L'}\int_0^L f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}\,dx\,dy,$$
$$B_{mn} = \frac{2/\sqrt{LL'}}{\sinh\sqrt{\frac{n^2}{L^2}+\frac{m^2}{L'^2}}\pi L''}\int_0^{L'}\int_0^L\left[g(x,y)-f(x,y)\cosh\sqrt{\frac{n^2}{L^2}+\frac{m^2}{L'^2}}\pi L''\right] *$$
$$\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}\,dx\,dy$$

16. $$V(x,y,z) = \frac{2}{\sqrt{LL'}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\cosh\sqrt{\frac{n^2}{L^2}+\frac{m^2}{L'^2}}\pi z + B_{mn}\sinh\sqrt{\frac{n^2}{L^2}+\frac{m^2}{L'^2}}\pi z\right) *$$
$$\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}$$
$$+\frac{2}{\sqrt{L'L''}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(C_{mn}\cosh\sqrt{\frac{n^2}{L'^2}+\frac{m^2}{L''^2}}\pi x + D_{mn}\sinh\sqrt{\frac{n^2}{L'^2}+\frac{m^2}{L''^2}}\pi x\right) *$$
$$\sin\frac{n\pi y}{L'}\sin\frac{m\pi z}{L''}$$
where $$A_{mn} = \frac{2}{\sqrt{LL'}}\int_0^{L'}\int_0^L f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}\,dx\,dy,$$

$$B_{mn} = \frac{2/\sqrt{LL'}}{\sinh\sqrt{\frac{n^2}{L^2}+\frac{m^2}{L'^2}}\pi L''}\int_0^{L'}\int_0^{L}\left[g(x,y) - f(x,y)\cosh\sqrt{\frac{n^2}{L^2}+\frac{m^2}{L'^2}}\pi L''\right] *$$
$$\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}\,dx\,dy$$

$$C_{mn} = \frac{2}{\sqrt{L'L''}}\int_0^{L'}\int_0^{L''} h(y,z)\sin\frac{n\pi y}{L'}\sin\frac{m\pi z}{L''}\,dz\,dy,$$

$$D_{mn} = \frac{2/\sqrt{L'L''}}{\sinh\sqrt{\frac{n^2}{L'^2}+\frac{m^2}{L''^2}}\pi L}\int_0^{L'}\int_0^{L''}\left[k(y,z) - h(y,z)\cosh\sqrt{\frac{n^2}{L'^2}+\frac{m^2}{L''^2}}\pi L\right] *$$
$$\sin\frac{n\pi y}{L'}\sin\frac{m\pi z}{L''}\,dz\,dy$$

17. $U(x,y,z) = \frac{4L}{\pi^2\kappa}\sum_{n=1}^{\infty}\frac{\operatorname{csch}(2n-1)\pi}{(2n-1)^2}\left[Q\cosh\frac{(2n-1)\pi(L-z)}{L} + q\cosh\frac{(2n-1)\pi x}{L}\right] *$
$$\sin\frac{(2n-1)\pi y}{L}$$

Exercises 6.5

3. a constant

4. $\lambda_{mn}^2 = \frac{(2n-1)^2\pi^2}{4L^2} + \frac{m^2\pi^2}{L'^2}, \qquad W_{mn}(x,y) = \frac{2}{\sqrt{LL'}}\sin\frac{(2n-1)\pi x}{2L}\sin\frac{m\pi y}{L'}$

5. $\lambda_{mn}^2 = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{L'^2} \quad , \quad W_{mn}(x,y) = \begin{cases} \sqrt{\frac{2}{LL'}}\sin\frac{n\pi x}{L}, & m = 0 \\ \frac{2}{\sqrt{LL'}}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{L'}, & m > 0. \end{cases}$

6. $\lambda_{mn}^2 = \frac{(2n-1)^2\pi^2}{4L^2} + \frac{(2m-1)^2\pi^2}{4L'^2}, \qquad W_{mn}(x,y) = \frac{2}{\sqrt{LL'}}\cos\frac{(2n-1)\pi x}{2L}\sin\frac{(2m-1)\pi y}{2L'}$

7. $\lambda_{mn}^2 = \frac{n^2\pi^2}{L^2} + \nu_m^2$, where $\tan\nu L' = \frac{h}{\nu l}$;

$$W_{mn}(x,y) = \frac{\sqrt{2}}{N\sqrt{L}}\sin\frac{n\pi x}{L}\cos\nu_m y, \text{ where } 2N^2 = L' + \frac{h/l}{\nu_m^2 + (h/l)^2}$$

8. $\lambda_{mn}^2 = \mu_n^2 + \nu_m^2, \quad W_{mn}(x,y) = X_n(x)Y_m(y)$, where μ_n, ν_m, $X_n(x)$, and $Y_m(y)$ are given in line 1 of
Table 5.1

12. (b) The line integral vanishes

Exercises 6.6

4. No

Exercises 6.8

1. (a) $y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

2. (a) $y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$

Exercises 7.1

1. $\dfrac{8\sqrt{2L^3}}{(2n-1)^2\pi^2}\left[(-1)^{n+1}(L-1)-\dfrac{2L}{(2n-1)\pi}\right]$ **2.** $\dfrac{5\sqrt{2L}[1+(-1)^{n+1}]}{n\pi}$

3. $\tilde{f}(0)=5\sqrt{L},\ \tilde{f}(\lambda_n)=0$ for $n>0$

4. $-\dfrac{L}{N\lambda_n}\cos\lambda_n L+\dfrac{1}{N\lambda_n^2}\sin\lambda_n L$, where $2N^2=L+\dfrac{h_2/l_2}{\lambda_n^2+(h_2/l_2)^2}$, $\cot\lambda L=-\dfrac{h_2}{\lambda l_2}$

5. $\dfrac{1}{N\lambda_n^2}(1-\cos\lambda_n L)$, where $2N^2=L+\dfrac{h_2/l_2}{\lambda_n^2+(h_2/l_2)^2}$, $\tan\lambda L=\dfrac{h_2}{\lambda l_2}$

6. $\dfrac{2\sqrt{2L}}{(2n-1)^2\pi^2-4L^2}\left[(-1)^{n+1}(2n-1)\pi\sin L-2L\right]$

7. $\tilde{f}(0)=\dfrac{1}{\sqrt{L}}(e^L-1)$, $\tilde{f}(\lambda_n)=\dfrac{-\sqrt{2L^3}}{n^2\pi^2+L^2}\left[1+(-1)^{n+1}e^L\right]$

8. $\dfrac{\sqrt{L^5/2}}{(2n-1)\pi}\left\{\dfrac{32}{(2n-1)^2\pi^2}\left[\cos\dfrac{(2n-1)\pi}{4}-1\right]+\dfrac{8}{(2n-1)\pi}\sin\dfrac{(2n-1)\pi}{4}-\cos\dfrac{(2n-1)\pi}{4}\right\}$

9. $\tilde{f}(\lambda_2)=\sqrt{L/8}$, $\tilde{f}(\lambda_n)=0$ for $n\neq 2$

10. $\tilde{f}(0)=\sqrt{\dfrac{e^{2L}-1}{2}}$, $\tilde{f}(\lambda_n)=0$ for $n>0$ **11.** $2x$ **12.** $-3x^2$ **13.** $2\sqrt{2}$

14. $2x-1$

Exercises 7.2

1. $U(x,t)=\dfrac{U_L x}{L}+\displaystyle\sum_{n=1}^{\infty}c_n e^{-n^2\pi^2kt/L^2}\sin\dfrac{n\pi x}{L}$, where $c_n=\dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx+\dfrac{2(-1)^n U_L}{n\pi}$

5. $U(x,t)=U_0+\dfrac{8U_0}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n e^{-(2n-1)^2\pi^2kt/(4L^2)}}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{2L}$

6. $U(x,t)=\displaystyle\sum_{n=1}^{\infty}\left[\tilde{f}(\lambda_n)e^{-k\lambda_n^2 t}+\dfrac{k}{\kappa}\int_0^t\tilde{g}(\lambda_n,u)e^{k\lambda_n^2(u-t)}\,du\right]\sqrt{\dfrac{2}{L}}\sin\dfrac{n\pi x}{L}$, where

$$\tilde{f}(\lambda_n)=\int_0^L f(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx,\quad \tilde{g}(\lambda_n,t)=\int_0^L g(x,t)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}dx$$

7. $U(x,t)=\displaystyle\sum_{n=0}^{\infty}\left[\tilde{f}(\lambda_n)e^{-k\lambda_n^2 t}+\dfrac{k}{\kappa}\int_0^t\tilde{g}(\lambda_n,u)e^{k\lambda_n^2(u-t)}\,du\right]X_n(x)$, where $X_0(x)=\dfrac{1}{\sqrt{L}}$,

$$X_n(x)=\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L},\quad \tilde{f}(\lambda_n)=\int_0^L f(x)X_n(x)\,dx,\quad \tilde{g}(\lambda_n,t)=\int_0^L g(x,t)X_n(x)\,dx$$

8. (b) $U(x,t)=\dfrac{100x}{L}e^{-t}+\dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\dfrac{1}{n}e^{-n^2\pi^2kt/L^2}+\dfrac{(-1)^n L^2(e^{-n^2\pi^2kt/L^2}-e^{-t})}{n(n^2\pi^2k-L^2)}\right]\sin\dfrac{n\pi x}{L}$

(c) $U(x,t)=\dfrac{100x}{L}e^{-t}+\dfrac{200}{\pi}\displaystyle\sum_{\substack{n=1\\n\neq m}}^{\infty}\left[\dfrac{1}{n}e^{-n^2\pi^2kt/L^2}+\dfrac{(-1)^n L^2(e^{-n^2\pi^2kt/L^2}-e^{-t})}{n(n^2\pi^2k-L^2)}\right]\sin\dfrac{n\pi x}{L}$

$$+\frac{200}{m\pi}[1+(-1)^{m+1}t]e^{-t}\sin\frac{m\pi x}{L}$$

9. $U(x,t)=\dfrac{2gL}{\kappa\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{1-e^{-n^2\pi^2kt/L^2}}{n^2}\sin\dfrac{n\pi b}{L}\sin\dfrac{n\pi x}{L}$

12. (a) $$U(x,t) = 4U_0 \sum_{n=1}^{\infty} \left\{ \left[\frac{1}{(2n-1)\pi} + \frac{2(-1)^n}{(2n-1)^2\pi^2} \right] e^{-(2n-1)^2\pi^2kt/(4L^2)} + \frac{(2n-1)k\pi}{(2n-1)^2\pi^2k - 4\alpha L^2} [e^{-\alpha t} - e^{-(2n-1)^2\pi^2kt/(4L^2)}] \right\} \sin \frac{(2n-1)\pi x}{2L}$$

(b) $$U(x,t) = 4U_0 \sum_{\substack{n=1\\ n\neq m}}^{\infty} \left\{ \left[\frac{1}{(2n-1)\pi} + \frac{2(-1)^n}{(2n-1)^2\pi^2} \right] e^{-(2n-1)^2\pi^2kt/(4L^2)} + \frac{(2n-1)k\pi}{(2n-1)^2\pi^2k - 4\alpha L^2} [e^{-\alpha t} - e^{-(2n-1)^2\pi^2kt/(4L^2)}] \right\} \sin \frac{(2n-1)\pi x}{2L} + 2U_0 \left[\frac{2}{(2m-1)\pi} + \frac{4(-1)^m}{(2m-1)^2\pi^2} + \frac{(2m-1)\pi kt}{2L^2} \right] e^{-\alpha t} \sin \frac{(2m-1)\pi x}{2L}$$

14. $$U(x,t) = \frac{qx}{\kappa} + U_0 + \frac{8(U_0\kappa + qL)}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} e^{-(2n-1)^2\pi^2kt/(4L^2)} \sin \frac{(2n-1)\pi x}{2L}$$

15. (a) $$U(x,t) = U_0 + \frac{q}{\kappa}(L-x) - \frac{8qL}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2\pi^2kt/(4L^2)} \cos \frac{(2n-1)\pi x}{2L}$$

(b) $$U(x,t) = U_0 + \frac{8qL}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left[e^{-(2n-1)^2\pi^2k(t-t_0)/(4L^2)} - e^{-(2n-1)^2\pi^2kt/(4L^2)} \right] * \cos \frac{(2n-1)\pi x}{2L}$$

(c) $U = U_0$

16. (a) $$U(x,t) = U_0 + \frac{kqt}{\kappa L} + \frac{q}{6\kappa L}(3x^2 - 6Lx + 2L^2) - \frac{2qL}{\pi^2\kappa} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2kt/L^2} \cos \frac{n\pi x}{L}$$

(b) $$U(x,t) = U_0 + \frac{kqt_0}{\kappa L} + \frac{2qL}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[e^{-n^2\pi^2k(t-t_0)/L^2} - e^{-n^2\pi^2kt/L^2} \right] \cos \frac{n\pi x}{L}$$

(c) $U = U_0 + \dfrac{kqt_0}{\kappa L}$

19. (a) $$y(x,t) = \frac{kx(x-L)}{2\rho c^2} + \sum_{n=1}^{\infty} \left\{ \left[\tilde{f}(\lambda_n) + \frac{k\tilde{1}_n}{\rho c^2\lambda_n^2} \right] \cos \frac{n\pi ct}{L} + \frac{L\tilde{g}(\lambda_n)}{n\pi c} \sin \frac{n\pi ct}{L} \right\} * \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \qquad \text{where } \tilde{1}_n = \frac{\sqrt{2L}[1 + (-1)^{n+1}]}{n\pi},$$

$$\tilde{f}(\lambda_n) = \int_0^L f(x)\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}\, dx, \quad \tilde{g}(\lambda_n) = \int_0^L g(x)\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}\, dx$$

(b) $$y(x,t) = \psi(x) + \frac{1}{2}[f(x+ct) + f(x-ct)] - \frac{1}{2}[\psi(x+ct) + \psi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du, \text{ where } \psi(x) = \frac{kx(x-L)}{2\rho c^2}$$

20. (a) $$y(x,t) = \frac{y_L x}{L} + \sum_{n=1}^{\infty} \left\{ \left[\tilde{f}(\lambda_n) + \frac{\sqrt{2L}(-1)^n y_L}{n\pi} \right] \cos \frac{n\pi ct}{L} + \frac{L\tilde{g}(\lambda_n)}{n\pi c} \sin \frac{n\pi ct}{L} \right\} * \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L},$$

$$\text{where } \tilde{f}(\lambda_n) = \int_0^L f(x)\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx, \quad \tilde{g}(\lambda_n) = \int_0^L g(x)\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx$$

(b) $y(x,t) = \psi(x) + \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] - \frac{1}{2}\left[\psi(x+ct) + \psi(x-ct)\right]$
$+\frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$, where $\psi(x) = y_L x/L$

21. (a) $y(x,t) = \psi(x) + \sum_{n=1}^{\infty}\left[\tilde{f}(\lambda_n) + \frac{mg}{\rho c^2\lambda_n^2}X_n(a)\right]\cos\frac{n\pi ct}{L}X_n(x)$, where

$$X_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \quad \psi(x) = \frac{mg}{\rho c^2}(x-a)h(x-a) - \frac{mg(L-a)x}{\rho c^2 L}$$

(b) $y(x,t) = \psi(x) + \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] - \frac{1}{2}\left[\psi(x+ct) + \psi(x-ct)\right]$

22. $y(x,t) = \frac{2L^2F_0}{\pi^2\rho c}\sum_{n=1}^{\infty}\frac{1}{n^2(n^2\pi^2c^2 - \omega^2L^2)}\left[\cos\frac{n\pi a}{L} - \cos\frac{n\pi b}{L}\right] *$
$\left[n\pi c\sin\omega t - \omega L\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}$

23. $y(x,t) = \frac{2LF_0}{\rho\pi c}\sum_{n=1}^{\infty}\frac{1}{n(n^2\pi^2c^2 - \omega^2L^2)}\sin\frac{n\pi x_0}{L}\left[n\pi c\sin\omega t - \omega L\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}$

24. (a) $y(x,t) = \frac{Fx}{E} - \frac{8LF}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{2L}\sin\frac{(2n-1)\pi x}{2L}$

(b) $y(x,t) = \psi(x) - \frac{1}{2}\left[\psi(x+ct) + \psi(x-ct)\right]$, where $\psi(x) = Fx/E$

(c) $y(L,t) = \frac{FL}{E} - \psi(L+ct)$

25. $y(x,t) = \frac{kL}{2} + \frac{c^2Ft^2}{2EL} + \frac{F}{6EL}(3x^2 - 6Lx + 2L^2)$
$-\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{F}{E} + k[1 + (-1)^{n+1}]\right]\cos\frac{n\pi ct}{L}\cos\frac{n\pi x}{L}$

26. (a) $y(x,t) = -\frac{4\omega F_0L^3}{\rho\pi^2 c}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2[(2n-1)^2\pi^2c^2 - \omega^2L^2]}\sin\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$
$+\frac{2F_0}{\rho\omega^2}\psi(x)\sin\omega t$ where $\psi(x) = \sec\frac{\omega L}{2c}\sin\frac{\omega x}{2c}\sin\frac{\omega(L-x)}{2c}$

(b) $y(x,t) = \frac{2F_0L}{m^2\pi^2\rho c}\left(\frac{L}{m\pi c}\sin\frac{m\pi ct}{L} - t\cos\frac{m\pi ct}{L}\right)\sin\frac{m\pi x}{L}$
$+\frac{4L^2F_0}{\rho c^2\pi^3}\sum_{\substack{n=1\\2n-1\neq m}}^{\infty}\frac{1}{(2n-1)[(2n-1)^2 - m^2]}\left[\sin\frac{m\pi ct}{L} - \frac{m}{2n-1}\sin\frac{(2n-1)\pi ct}{L}\right] *$
$\sin\frac{(2n-1)\pi x}{L}$

27. (a) $y(x,t) = \frac{8cLF_0}{E\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2\pi^2c^2 - 4\omega^2L^2}\left[\frac{2\omega L}{2n-1}\sin\frac{(2n-1)\pi ct}{2L} - c\pi\sin\omega t\right] *$
$\sin\frac{(2n-1)\pi x}{2L}$

(b) $y(x,t) = \frac{2F_0(-1)^{m+1}}{(2m-1)\pi E}\left[\frac{2L}{(2m-1)\pi}\sin\frac{(2m-1)\pi ct}{2L} - ct\cos\frac{(2m-1)\pi ct}{2L}\right] *$
$\sin\frac{(2m-1)\pi x}{2L}$

$$+\frac{8LF_0}{E\pi^2}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{(-1)^n}{(2n-1)^2-(2m-1)^2}\left[\frac{2m-1}{2n-1}\sin\frac{(2m-1)\pi ct}{2L}-\sin\frac{(2n-1)\pi ct}{2L}\right]*$$

$$\sin\frac{(2n-1)\pi x}{2L}$$

28. (a) $y(x,t)=2\pi Ac^2\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^2\pi^2c^2-\omega^2L^2}\left(\frac{\omega L}{n\pi c}\sin\frac{n\pi ct}{L}-\sin\omega t\right)\sin\frac{n\pi x}{L}$

(b) $y(x,t)=\frac{A(-1)^m}{L}\left(ct\cos\frac{m\pi ct}{L}-\frac{L}{m\pi}\sin\frac{m\pi ct}{L}\right)\sin\frac{m\pi x}{L}$

$$+\frac{2A}{\pi}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{n(-1)^n}{n^2-m^2}\left(\frac{m}{n}\sin\frac{n\pi ct}{L}-\sin\frac{m\pi ct}{L}\right)\sin\frac{n\pi x}{L}$$

29. $\omega=(2n-1)\pi c/(2L)$ **30.** $y(x,t)=\frac{F_0}{\rho\omega^2}(\omega t-\sin\omega t)$ **31.** $\omega=(2n-1)\pi c/(2L)$

32. $\omega=n\pi c/L$ **33.** $\omega=n\pi c/L$ **34.** $\omega=(2n-1)\pi c/(2L)$

35. $\omega=n\pi c/L$ or $\phi=n\pi c/L$. If $\omega=\phi$ and $A_0=B_0$, then $\omega=(2n-1)\pi c/L$. If $\omega=\phi$ and $A_0=-B_0$, then $\omega=2n\pi c/L$.

36. $\omega=n\pi c/L$ or $\phi=n\pi c/L$. If $\omega=\phi$ and $F_0=G_0$, then $\omega=2n\pi c/L$. If $\omega=\phi$ and $F_0=-G_0$, then $\omega=(2n-1)\pi c/L$.

37. (b) $y(x,t)=\psi(x)-\frac{1}{2}\psi(x+ct)-\frac{1}{2}\psi(x-ct)$, where $\psi(x)=\frac{gx(2L-x)}{2c^2}$ (c) Yes

38. (a) $y(x,t)=f(x)-\frac{gk}{AEc^2}\sum_{n=1}^{\infty}\frac{1}{N\lambda_n^4}\cos c\lambda_n tX_n(x)$, $f(x)=\frac{g}{2c^2}\left(-x^2+2Lx+\frac{2LAE}{k}\right)$,

$X_n(x)=\frac{1}{N\cos\lambda_n L}\cos\lambda_n(L-x)$, and $2N^2=L\left[1+\left(\frac{k}{AE\lambda_n}\right)^2\right]+\frac{k}{AE\lambda_n^2}$ (b) Yes

39. $y(x,t)=2A\pi c^2\sum_{n=1}^{\infty}n(-1)^n\left\{e^{-\beta t/(2\rho)}\left[\frac{-\beta\omega\rho L^2}{\rho^2(n^2\pi^2c^2-\omega^2L^2)^2+\beta^2\omega^2L^4}\cos\omega_n t\right.\right.$

$$\left.+\frac{2\omega\rho^2(n^2\pi^2c^2-\omega^2L^2)-\beta^2\omega L^2}{2\omega_n[\rho^2(n^2\pi^2c^2-\omega^2L^2)^2+\beta^2\omega^2L^4]}\sin\omega_n t\right]$$

$$\left.+\frac{\rho}{\rho^2(n^2\pi^2c^2-\omega^2L^2)^2+\beta^2\omega^2L^4}[\rho(\omega^2L^2-n^2\pi^2c^2)\sin\omega t+\beta\omega L^2\cos\omega t]\right\}\sin\frac{n\pi x}{L}$$

40. (a) $y(x,t)=-\frac{4L^2g}{c^2\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1-\cos\frac{(2n-1)\pi ct}{L}\right]\sin\frac{(2n-1)\pi x}{L}$

41. (a) $y(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\psi(x)-\frac{1}{2}[\psi(x+ct)+\psi(x-ct)]$, where $\psi(x)=\frac{g}{2c^2}(x^2-Lx)$

42. $y(x,t)=\frac{t^2}{2}\left(\frac{c^2F_0}{\tau L}-g\right)+\psi(x)-\frac{1}{2}[\psi(x+ct)+\psi(x-ct)]$, where $\psi(x)=\frac{F_0x^2}{2L\tau}$

43. $y(x,t)=\frac{2AL^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}\left[1-\cos\frac{(2n-1)^2\pi^2ct}{L^2}\right]\sin\frac{(2n-1)\pi x}{L}$

45. (b) No **46.** (b) No **47.** (b) No

48. (a) $V(x,y)=-\frac{4\sigma L^2}{\epsilon_0\pi^3}\sum_{n=1}^{\infty}\frac{\sinh[(2n-1)\pi y/L]+\sinh[(2n-1)\pi(L'-y)/L]}{(2n-1)^3\sinh[(2n-1)\pi L'/L]}\sin\frac{(2n-1)\pi x}{L}$

$$+\frac{\sigma x(L-x)}{2\epsilon_0}$$

$$V(x,y)=-\frac{4\sigma L'^2}{\epsilon_0\pi^3}\sum_{n=1}^{\infty}\frac{\sinh\left[(2n-1)\pi x/L'\right]+\sinh\left[(2n-1)\pi(L-x)/L'\right]}{(2n-1)^3\sinh\left[(2n-1)\pi L/L'\right]}\sin\frac{(2n-1)\pi y}{L'}$$

$$+\frac{\sigma y(L'-y)}{2\epsilon_0}$$

(b) $V(x,y)=\dfrac{\sqrt{2L^3}}{\epsilon_0\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{\tilde{\sigma}(\lambda_n)}{n^2}\left[1-\frac{\sinh\left(n\pi y/L\right)+\sinh\left[n\pi(L'-y)/L\right]}{\sinh\left(n\pi L'/L\right)}\right]\sin\frac{n\pi x}{L}$, where

$$\tilde{\sigma}(\lambda_n)=\int_0^L\sigma(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx$$

(c) $V(x,y)=\dfrac{2L^3}{\pi^3\epsilon_0}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left[\frac{L'\sinh\left(n\pi y/L\right)}{\sinh\left(n\pi L'/L\right)}-y\right]\sin\frac{n\pi x}{L}$

49. $V(x,y)=\sqrt{\dfrac{2}{L'}}\displaystyle\sum_{n=1}^{\infty}\frac{\tilde{f}(\lambda_n)}{\sinh\left(n\pi L/L'\right)}\sinh\frac{n\pi x}{L'}\sin\frac{n\pi y}{L'}+\frac{\sigma y(L'-y)}{2\epsilon_0}$

$$-\frac{4\sigma L'^2}{\epsilon_0\pi^3}\sum_{n=1}^{\infty}\frac{\sinh\left[(2n-1)\pi x/L'\right]+\sinh\left[(2n-1)\pi(L-x)/L'\right]}{(2n-1)^3\sinh\left[(2n-1)\pi L/L'\right]}\sin\frac{(2n-1)\pi y}{L'},$$

where $\tilde{f}(\lambda_n)=\displaystyle\int_0^{L'}f(y)Y_n(y)\,dy$

50. $V(x,y)=\sqrt{\dfrac{2}{L}}\displaystyle\sum_{n=1}^{\infty}\frac{\tilde{g}(\lambda_n)}{\sinh\left(n\pi L'/L\right)}\sinh\frac{n\pi(L'-y)}{L}\sin\frac{n\pi x}{L}+\frac{\sigma x(L-x)}{2\epsilon_0}$

$-\dfrac{4\sigma L^2}{\epsilon_0\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{\sinh\left[(2n-1)\pi y/L\right]+\sinh\left[(2n-1)\pi(L'-y)/L\right]}{(2n-1)^3\sinh\left[(2n-1)\pi L'/L\right]}\sin\frac{(2n-1)\pi x}{L}$, where

$$\tilde{g}(\lambda_n)=\int_0^L g(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx$$

51. $V(x,y)=\dfrac{\sigma x(L-x)}{2\epsilon_0}-\dfrac{4\sigma L^2}{\epsilon_0\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{\sinh\left[(2n-1)\pi y/L\right]+\sinh\left[(2n-1)\pi(L'-y)/L\right]}{(2n-1)^3\sinh\left[(2n-1)\pi L'/L\right]}*$

$$\sin\frac{(2n-1)\pi x}{L}$$

$+\displaystyle\sum_{n=1}^{\infty}b_n\sinh\frac{n\pi(L'-y)}{L}\sin\frac{n\pi x}{L}+\sum_{n=1}^{\infty}C_n\sinh\frac{n\pi(L-x)}{L'}\sin\frac{n\pi y}{L'}$, where

$$b_n=\frac{2}{L\sinh\left(n\pi L'/L\right)}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx,\quad C_n=\frac{2}{L'\sinh\left(n\pi L/L'\right)}\int_0^{L'}f(y)\sin\frac{n\pi y}{L'}\,dy$$

52. $U(x,y)=\dfrac{C}{2}(L^2-x^2)+\dfrac{16CL^2}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\cosh\dfrac{(2n-1)\pi y}{2L}}{(2n-1)^3\cosh\dfrac{(2n-1)\pi L'}{2L}}\cos\frac{(2n-1)\pi x}{2L}$

54. (a) $r^2\dfrac{d^2\tilde{U}}{dr^2}+r\dfrac{d\tilde{U}}{dr}-\lambda_n^2\tilde{U}=g(r)H_n'(\alpha)-f(r)H_n'(0),\quad \tilde{U}'(a,\lambda_n)=0$

(b) $U(\theta)=k_1+(k_2-k_1)\dfrac{\theta}{\alpha}$

(c) $U(r,\theta)=2a\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2-\alpha^2}\left[n\pi\left(\frac{r}{a}\right)-\alpha\left(\frac{r}{a}\right)^{n\pi/\alpha}\right]\sin\frac{n\pi\theta}{\alpha}$

Exercises 7.3

1. $$U(x,y,t) = \frac{2}{\sqrt{LL'}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \tilde{\tilde{f}}(\lambda_n, \mu_m) e^{-k\pi^2(n^2/L^2+m^2/L'^2)t} + \frac{A(\lambda_n, \mu_m)}{k\pi^2(n^2/L^2+m^2/L'^2)} [1 - e^{-k\pi^2(n^2/L^2+m^2/L'^2)t}] \right\} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'},$$ where
$$\tilde{\tilde{f}}(\lambda_n, \mu_m) = \frac{2}{\sqrt{LL'}} \int_0^{L'} \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'} dx\,dy,$$
$$A(\lambda_n, \mu_m) = \frac{2k}{\sqrt{L^3 L'^3}} \left\{ \frac{L'^2 n}{m} [U_1 + U_2(-1)^{n+1}][1 + (-1)^{m+1}] + \frac{L^2 m}{n} [U_3 + U_4(-1)^{m+1}][1 + (-1)^{n+1}] \right\}$$

2. (a) $$U(x,y,t) = \frac{2}{L'\pi^3} \sum_{n=1}^{\infty} \frac{\pi^2 L' n^2 [U_1 + U_2(-1)^{n+1}] - L^2(\kappa_2^{-1}\phi_2 + \kappa_1^{-1}\phi_1)[1 + (-1)^{n+1}]}{n^3} * \left[1 - e^{-n^2\pi^2 kt/L^2}\right] \sin\frac{n\pi x}{L} + \frac{8L^2 L'}{\pi^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\kappa_2^{-1}\phi_2(-1)^{m+1} - \kappa_1^{-1}\phi_1}{(2n-1)[(2n-1)^2 L'^2 + m^2 L^2]} \left[1 - e^{-[(2n-1)^2/L^2 + m^2/L'^2]\pi^2 kt}\right] * \sin\frac{(2n-1)\pi x}{L} \cos\frac{m\pi y}{L'}$$

(b) $$U(x,t) = U_1 + \frac{(U_2 - U_1)x}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{U_1 + U_2(-1)^{n+1}}{n} e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}$$

5. $$z(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{-g\tilde{\tilde{1}}_{nm}}{c^2(\lambda_n^2 + \mu_m^2)} (1 - \cos c\sqrt{\lambda_n^2 + \mu_m^2}t) + \tilde{\tilde{f}}(\lambda_n, \mu_m) \cos c\sqrt{\lambda_n^2 + \mu_m^2}t \right] * X_n(x) Y_m(y),$$

where $\lambda_n = n\pi/L$, $\mu_m = m\pi/L'$, $X_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$, $Y_m(y) = \sqrt{\frac{2}{L'}} \sin\frac{m\pi y}{L'}$,

$$\tilde{\tilde{1}}_{nm} = \frac{2\sqrt{LL'}[1 + (-1)^{n+1}][1 + (-1)^{m+1}]}{mn\pi^2},$$

$$\tilde{\tilde{f}}(\lambda_n, \mu_m) = \frac{2}{\sqrt{LL'}} \int_0^{L'} \int_0^L f(x,y) X_n(x) Y_m(y)\, dx\, dy$$

6. $$z(x,y,t) = \frac{16AL^2}{\rho\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos\omega t - \cos[c\pi\sqrt{(2n-1)^2 + (2m-1)^2}t/L]}{(2m-1)(2n-1)\{c^2\pi^2[(2n-1)^2 + (2m-1)^2] - \omega^2 L^2\}} * \sin\frac{(2n-1)\pi x}{L} \sin\frac{(2m-1)\pi y}{L}$$

7. $$z(x,y,t) = \frac{4\sqrt{2}AL}{\rho\pi^3 c} t \sin\frac{\sqrt{2}\pi ct}{L} \sin\frac{\pi x}{L} \sin\frac{\pi y}{L} + \frac{16AL^2}{\rho\pi^2} \sum_{\substack{m=1 \\ nm\neq 1}}^{\infty} \sum_{n=1}^{\infty} \frac{\cos\omega t - \cos[c\pi\sqrt{(2n-1)^2 + (2m-1)^2}t/L]}{(2n-1)(2m-1)\{c^2\pi^2[(2n-1)^2 + (2m-1)^2] - \omega^2 L^2\}} * \sin\frac{(2n-1)\pi x}{L} \sin\frac{(2m-1)\pi y}{L}$$

8. $$z(x,y,t) = \frac{16AL^2}{\rho\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos\omega t - \cos[c\pi\sqrt{(2n-1)^2 + (2m-1)^2}t/L]}{(2n-1)(2m-1)\{c^2\pi^2[(2n-1)^2 + (2m-1)^2] - \omega^2 L^2\}} *$$

$$\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2m-1)\pi y}{L}$$

9. $$z(x,y,t)=\frac{16AL^2}{\rho\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\cos\omega t-\cos\left[c\pi\sqrt{(2n-1)^2+(2m-1)^2}\,t/L\right]}{(2n-1)(2m-1)\{c^2\pi^2[(2n-1)^2+(2m-1)^2]-\omega^2L^2\}}*$$
$$\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2m-1)\pi y}{L}$$

10. $$z(x,y,t)=\frac{8AL}{3\sqrt{10}\rho\pi^3c}t\sin\frac{\sqrt{10}\pi ct}{L}\left(\sin\frac{\pi x}{L}\sin\frac{3\pi y}{L}+\sin\frac{3\pi x}{L}\sin\frac{\pi y}{L}\right)$$
$$+\frac{16AL^2}{\rho\pi^2}\sum_{\substack{m=1\\(2n-1)(2m-1)\neq 3}}^{\infty}\sum_{n=1}^{\infty}\frac{\cos\omega t-\cos\left[c\pi\sqrt{(2n-1)^2+(2m-1)^2}\,t/L\right]}{(2n-1)(2m-1)\{c^2\pi^2[(2n-1)^2+(2m-1)^2]-\omega^2L^2\}}*$$
$$\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2m-1)\pi y}{L}$$

11. $$z(x,y,t)=\frac{8AL}{\sqrt{130}\rho\pi^3c}t\sin\frac{\sqrt{130}\pi ct}{L}\left[\frac{1}{63}\left(\sin\frac{7\pi x}{L}\sin\frac{9\pi y}{L}+\sin\frac{9\pi x}{L}\sin\frac{7\pi y}{L}\right)\right.$$
$$\left.+\frac{1}{33}\left(\sin\frac{3\pi x}{L}\sin\frac{11\pi y}{L}+\sin\frac{11\pi x}{L}\sin\frac{3\pi y}{L}\right)\right]$$
$$+\frac{16AL^2}{\rho\pi^2}\sum_{\substack{m=1\\(2n-1)(2m-1)\neq(7,9),(9,7),(3,11),(11,3)}}^{\infty}\sum_{n=1}^{\infty}\frac{\cos\omega t-\cos\left[c\pi\sqrt{(2n-1)^2+(2m-1)^2}\,t/L\right]}{(2n-1)(2m-1)\{c^2\pi^2[(2n-1)^2+(2m-1)^2]-\omega^2L^2]\}}*$$
$$\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2m-1)\pi y}{L}$$

Exercises 8.2

1. $U(x,t)=\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\frac{e^{-k\lambda^2t}}{\lambda^3}[2(1-\cos\lambda L)-\lambda L\sin\lambda L]\sin\lambda x\,d\lambda$

2. $U(x,t)=\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\frac{e^{-k\lambda^2t}}{\lambda^3}[2\sin\lambda L-\lambda L(1+\cos\lambda L)]\cos\lambda x\,d\lambda$

3. $U(x,t)=\dfrac{1}{\sqrt{a\pi}}\displaystyle\int_0^{\infty}e^{-\lambda^2[kt+1/(4a)]}\cos\lambda x\,d\lambda$

5. (a) $U(x,t)=\dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_0^{L}u(L-u)e^{-(u-x)^2/(4kt)}du$

6. (a) $U(x,t)=\dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_0^{\infty}f(u)\left[e^{-(u-x)^2/(4kt)}-e^{-(u+x)^2/(4kt)}\right]du$

7. (a) $U(x,t)=\dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_0^{\infty}f(u)\left[e^{-(u-x)^2/(4kt)}+e^{-(u+x)^2/(4kt)}\right]du$ (b) U_0

9. (a) $y(0,t)=f(ct)$ (b) $y(x_0,t)=\dfrac{1}{2}[f(x_0+ct)+f(x_0-ct)]$

10. (a) $y(x,t)=\dfrac{1}{2}[f(x+ct)+f(x-ct)]$ (b) $y(x_0,t)=\dfrac{1}{2}[f(x_0+ct)+f(x_0-ct)]$

11. $y(x,t)=\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct}g(u)\,du$

12. $V(x,y)=\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\frac{\cos\lambda y}{\sinh\lambda L}\left\{\int_0^{\infty}[f_1(u)\sinh\lambda(L-x)+f_2(u)\sinh\lambda x]\cos\lambda u\,du\right\}d\lambda$

13. (b) $V(x,y)=\dfrac{2V_0}{\pi}\mathrm{Tan}^{-1}\left[\dfrac{\sin(\pi x/L)}{\sinh(\pi y/L)}\right]$ **14.** $\dfrac{2a}{\pi}\displaystyle\int_0^{\infty}\frac{\cos\lambda x}{a^2+\lambda^2}\,d\lambda$

15. $\frac{2}{\pi}\int_0^\infty \frac{1}{\lambda}\cos\frac{\lambda(a+b-2x)}{2}\,\sin\frac{\lambda(b-a)}{2}\,d\lambda$ **16.** $\frac{4b}{\pi a}\int_0^\infty \frac{1}{\lambda^2}\sin^2\left(\frac{\lambda a}{2}\right)\cos\lambda x\,d\lambda$

17. $\frac{4b}{\pi a^2}\int_0^\infty \frac{1}{\lambda^3}(\sin\lambda a - a\lambda\cos\lambda a)\cos\lambda x\,d\lambda$ **18.** $\frac{1}{\pi}\int_0^\infty \left(\frac{a\cos\lambda x+\lambda\sin\lambda x}{a^2+\lambda^2}\right)d\lambda$

19. $\int_0^\infty \frac{e^{-\lambda^2/(4k)}}{\sqrt{k\pi}}\cos\lambda x\,d\lambda$

20. $\frac{2}{\pi}\int_0^\infty \frac{1}{\lambda}(\cos\lambda a-\cos\lambda b)\sin\lambda x\,d\lambda,\quad \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda}(\sin\lambda b-\sin\lambda a)\cos\lambda x\,d\lambda;\quad 0{,}0$

21. $\frac{8b}{\pi a}\int_0^\infty \frac{1}{\lambda^2}\sin^2\left(\frac{\lambda a}{2}\right)\sin\lambda c\sin\lambda x\,d\lambda,\quad \frac{8b}{\pi a}\int_0^\infty \frac{1}{\lambda^2}\sin^2\left(\frac{\lambda a}{2}\right)\cos\lambda c\cos\lambda x\,d\lambda;\quad 0{,}0$

22. $\frac{1}{\pi}\int_0^\infty \left[\frac{\lambda+b}{a^2+(b+\lambda)^2}+\frac{\lambda-b}{a^2+(b-\lambda)^2}\right]\sin\lambda x\,d\lambda,$

$\frac{a}{\pi}\int_0^\infty \left[\frac{1}{a^2+(b+\lambda)^2}+\frac{1}{a^2+(b-\lambda)^2}\right]\cos\lambda x\,d\lambda;\quad 0{,}1$

23. $\frac{a}{\pi}\int_0^\infty \left[\frac{-1}{a^2+(b+\lambda)^2}+\frac{1}{a^2+(b-\lambda)^2}\right]\sin\lambda x\,d\lambda,$

$\frac{1}{\pi}\int_0^\infty \left[\frac{b+\lambda}{a^2+(b+\lambda)^2}+\frac{b-\lambda}{a^2+(b-\lambda)^2}\right]\cos\lambda x\,d\lambda;\quad 0{,}0$

24. (b) $I=\sqrt{\frac{\pi}{k}}e^{-\lambda^2/(4k)}$

Exercises 8.3

4. (a) $\mathcal{F}\{f^{(n)}(x)\}=(i\omega)^n\mathcal{F}\{f(x)\},\qquad \mathcal{F}^{-1}\{\omega^n\tilde f(\omega)\}=i^{-n}\frac{d^n}{dx^n}\mathcal{F}^{-1}\{\tilde f(\omega)\}$

5. $\mathcal{F}_S\{xf(x)\}=-\frac{d}{d\omega}\mathcal{F}_C\{f(x)\},\quad \mathcal{F}_S\{x^2f(x)\}=-\frac{d^2}{d\omega^2}\mathcal{F}_S\{f(x)\}$

$\mathcal{F}_C\{xf(x)\}=\frac{d}{d\omega}\mathcal{F}_S\{f(x)\},\quad \mathcal{F}_C\{x^2f(x)\}=-\frac{d^2}{d\omega^2}\mathcal{F}_C\{f(x)\}$

9. (b) $\mathcal{F}_C\{\tilde f(x)\}=\frac{\pi}{2}f(\omega),\quad \mathcal{F}_S\{\tilde f(x)\}=\frac{\pi}{2}f(\omega)$

10. (b) $\mathcal{F}_C\left\{\int_0^x f(u)\,du\right\}=-\frac{1}{\omega}\mathcal{F}_S\{f(x)\},\quad \mathcal{F}_S\left\{\int_0^x f(u)\,du\right\}=\frac{1}{\omega}\mathcal{F}_C\{f(x)\}$

11. $\frac{2a}{\omega^2+a^2}$ **12.** $\frac{n!}{(a+i\omega)^{n+1}}$ **13.** $\frac{2}{\omega}e^{-i\omega(a+b)/2}\sin\frac{\omega(b-a)}{2}$ **14.** $\pi[h(\omega+a)-h(\omega-a)]$

15. $\frac{4b}{a\omega^2}\sin^2\left(\frac{a\omega}{2}\right)$ **16.** $\frac{4b}{a^2\omega^3}\sin a\omega-\frac{4b}{a\omega^2}\cos a\omega$

17. $e^{-\omega^2/(4a)}\int_0^{\omega/(2a)} e^{ax^2}\,dx,\quad \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\omega^2/(4a)}$

18. $\frac{1}{\omega}(\cos a\omega-\cos b\omega),\quad \frac{1}{\omega}(\sin b\omega-\sin a\omega)$

19. $\frac{4b}{a\omega^2}\sin c\omega\sin^2\left(\frac{a\omega}{2}\right),\quad \frac{4b}{a\omega^2}\cos c\omega\sin^2\left(\frac{a\omega}{2}\right)$

23. (a) $\frac{\omega(h+al)}{(a^2+\omega^2)\sqrt{h^2+\omega^2l^2}}$ (b) $\frac{1}{\omega\sqrt{h^2+\omega^2l^2}}\,[\omega l(\sin\omega b+\sin\omega a)-h(\cos\omega a-\cos\omega b]$

24. (b) $\frac{2}{\omega}e^{-i\omega(a+b)/2}\sin\frac{\omega(b-a)}{2},\quad \frac{n!}{(a+i\omega)^{n+1}}$

(c) $\frac{1}{\omega}(\cos a\omega-\cos b\omega),\quad \frac{1}{\omega}(\sin b\omega-\sin a\omega);$

$$\frac{4b}{a\omega^2}\sin c\omega \sin^2\left(\frac{a\omega}{2}\right), \quad \frac{4b}{a\omega^2}\cos c\omega \sin^2\left(\frac{a\omega}{2}\right)$$

25. (a)(i) $\begin{cases} (x^2/2)e^{-8x}, & x>0 \\ 0, & x<0 \end{cases}$ (ii) $\begin{cases} \dfrac{b}{a}(x-a)[h(x-a)-1], & x>0 \\ 0, & x<0 \end{cases}$ (b) No

26. (b)(i) $-\dfrac{L}{\omega^2}(1+e^{i\omega L}) - \dfrac{2i}{\omega^3}(e^{i\omega L}-1)$ (ii) $\dfrac{e^{b(c-i\omega)} - e^{a(c-i\omega)}}{c-i\omega}$

27. (b)(i) $\dfrac{4b}{a\omega^2}\sin^2\left(\dfrac{a\omega}{2}\right)$ (ii) $\dfrac{2ia}{a^2-\omega^2}\sin\left(\dfrac{2\pi n\omega}{a}\right)$

Exercises 8.4

1. (a) $U(x,t) = \dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/(4kt)}\,du$

$$+\frac{k}{2\pi\kappa}\int_{-\infty}^{\infty}\left[\int_0^t \tilde{g}(\omega,u)e^{-k\omega^2(t-u)}\,du\right]e^{i\omega x}\,d\omega$$

(b)(i) $U(x,t) = \dfrac{1}{2}\operatorname{erf}\left(\dfrac{x+a}{2\sqrt{kt}}\right) - \dfrac{1}{2}\operatorname{erf}\left(\dfrac{x-a}{2\sqrt{kt}}\right)$

(ii) $U(x,t) = 1 - \dfrac{1}{2}\operatorname{erf}\left(\dfrac{x+a}{2\sqrt{kt}}\right) + \dfrac{1}{2}\operatorname{erf}\left(\dfrac{x-a}{2\sqrt{kt}}\right)$

2. (a) $\overline{U}\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)$ (b) No

3. (a) $U(x,t) = \dfrac{Q_0}{\kappa}\left[-x\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right) + 2\sqrt{\dfrac{kt}{\pi}}e^{-x^2/(4kt)}\right]$ (b) $\left(\dfrac{2Q_0\sqrt{k}}{\kappa\sqrt{\pi}}\right)\sqrt{t}$

4. (a) $U(x,t) = \dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_0^{\infty} f(u)\left[e^{-(x-u)^2/(4kt)} - e^{-(x+u)^2/(4kt)}\right]du$

$$+\frac{2}{\pi}\int_0^{\infty}\sin\omega x\left\{\int_0^t e^{-k\omega^2(t-u)}\left[k\omega f_1(u) + \frac{k}{\kappa}\tilde{g}(\omega,u)\right]du\right\}d\omega$$

(b) $U_0\operatorname{erf}\left(\dfrac{x}{2\sqrt{kt}}\right)$ (c) $\overline{U}\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)$

5. (a) $U(x,t) = \dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_0^{\infty} f(u)\left[e^{-(x-u)^2/(4kt)} + e^{-(x+u)^2/(4kt)}\right]du$

$$+\frac{2k}{\kappa\pi}\int_0^{\infty}\cos\omega x\left\{\int_0^t e^{-k\omega^2(t-u)}\left[f_1(u)+\tilde{g}(\omega,u)\right]du\right\}d\omega$$

(b) U_0 (c) $\dfrac{Q_0}{\kappa}\left[-x\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right) + \dfrac{2\sqrt{kt}}{\sqrt{\pi}}e^{-x^2/(4kt)}\right]$

6. $U(x,t) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\tilde{U}(\omega,t)X_\omega(x)\,d\omega$, where $\tilde{U}(\omega,t) = \dfrac{\mu U_m X_\omega(0)}{\kappa\omega^2}(1-e^{-k\omega^2 t})$

7. $U(x,t) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\tilde{U}(\omega,t)X_\omega(x)\,d\omega$, where

$$\tilde{U}(\omega,t) = \tilde{f}(\omega)e^{-k\omega^2 t} + \frac{k}{\kappa}\int_0^t e^{-k\omega^2(t-u)}\left[\tilde{g}(\omega,u) + X_\omega(0)f_1(u)\right]du$$

8. $U(x,t) = \displaystyle\int_{-\infty}^{\infty} f(u)\frac{1}{\sqrt{4k\pi t}}e^{-(x-u+\alpha t)^2/(4kt)}du$

9. $y(x,t) = \dfrac{1}{2}[f(x+ct)+f(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(v)\,dv + \dfrac{c}{\tau}F_1(t-x/c)h(t-x/c),$

where $F_1(t) = \int_0^t f_1(u)\,du$

10. $y(x,t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\left[\tilde{f}(\omega)\cos c\omega t + \dfrac{\tilde{g}(\omega)}{c\omega}\sin c\omega t\right]e^{i\omega x}d\omega$

$y(x,t) = \dfrac{1}{2}[f(x+ct)+f(x-ct)] + \dfrac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$

11. $y(x,t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\left[\tilde{f}(\omega)\cos\sqrt{k+c^2\omega^2}t + \dfrac{\tilde{g}(\omega)}{c\omega}\sin\sqrt{k+c^2\omega^2}t\right]e^{i\omega x}d\omega$ No

12. $y(x,t) = \dfrac{1}{2\pi}\int_{-\infty}^{-\beta/(2c)} e^{-\beta t}\left[\tilde{f}(\omega)\cos\sqrt{4c^2\omega^2-\beta^2}t + \dfrac{\tilde{g}(\omega)+\beta\tilde{f}(\omega)}{\sqrt{4c^2\omega^2-\beta^2}}\sin\sqrt{4c^2\omega^2-\beta^2}t\right]e^{i\omega x}d\omega$

$+\dfrac{1}{2\pi}\int_{-\beta/(2c)}^{\beta/(2c)} e^{-\beta t}\left[\tilde{f}(\omega)\cosh\sqrt{\beta^2-4c^2\omega^2}t + \dfrac{\tilde{g}(\omega)+\beta\tilde{f}(\omega)}{\sqrt{\beta^2-4c^2\omega^2}}\sinh\sqrt{\beta^2-4c^2\omega^2}t\right]e^{i\omega x}d\omega$

$+\dfrac{1}{2\pi}\int_{\beta/(2c)}^{\infty} e^{-\beta t}\left[\tilde{f}(\omega)\cos\sqrt{4c^2\omega^2-\beta^2}t + \dfrac{\tilde{g}(\omega)+\beta\tilde{f}(\omega)}{\sqrt{4c^2\omega^2-\beta^2}}\sin\sqrt{4c^2\omega^2-\beta^2}t\right]e^{i\omega x}d\omega$

No d'Alembert solution

13. $V(x,y) = \dfrac{2}{\pi}\int_0^{\infty}\left[\dfrac{\tilde{f}_1(\omega)\sinh\omega(L-x)}{\sinh\omega L} + \dfrac{\tilde{f}_2(\omega)\sinh\omega x}{\sinh\omega L}\right]\sin\omega y\,d\omega$

14. $U(x,y) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\left[\dfrac{\tilde{f}(\omega)\sinh\omega(L'-y)}{\sinh\omega L'} + \dfrac{\tilde{g}(\omega)\sinh\omega y}{\sinh\omega L'}\right]e^{i\omega x}\,d\omega$

15. $U(x,t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\left[\dfrac{\tilde{g}(\omega)\cosh\omega y}{\cosh\omega L'} - \dfrac{\tilde{f}(\omega)\sinh\omega(L'-y)}{\omega\cosh\omega L'}\right]e^{i\omega x}\,d\omega$

16. (a)(i) $V(x,y) = \sum_{n=1}^{\infty} B_n e^{-n\pi x/L'}\sqrt{\dfrac{2}{L'}}\sin\dfrac{n\pi y}{L'}$, where $B_n = \int_0^{L'} f(y)\sqrt{\dfrac{2}{L'}}\sin\dfrac{n\pi y}{L'}\,dy$

$V(x,y) = \dfrac{4k}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}e^{-(2n-1)\pi x/L'}\sin\dfrac{(2n-1)\pi y}{L'}$

(ii) $V(x,y) = \dfrac{2}{\pi}\int_0^{\infty}\dfrac{\tilde{g}(\omega)}{\sinh\omega L'}\sinh\omega y\,\sin\omega x\,d\omega$

(iii) $V(x,y) = \dfrac{2}{\pi}\int_0^{\infty}\dfrac{\tilde{g}(\omega)}{\sinh\omega L'}\sinh\omega(L'-y)\sin\omega x\,d\omega$

(iv) $V(x,y) = \sum_{n=1}^{\infty}\left[\int_0^{L'} f(u)\sqrt{\dfrac{2}{L'}}\sin\dfrac{n\pi y}{L'}\,du\right]e^{-n\pi x/L'}\sqrt{\dfrac{2}{L'}}\sin\dfrac{n\pi y}{L'}$

$+\dfrac{2}{\pi}\int_0^{\infty}\dfrac{\tilde{g}_1(\omega)}{\sinh\omega L'}\sinh\omega(L'-y)\sin\omega x\,d\omega + \dfrac{2}{\pi}\int_0^{\infty}\dfrac{\tilde{g}_2(\omega)}{\sinh\omega L'}\sinh\omega y\,\sin\omega x\,d\omega$

(b)(i) $V(x,y) = \dfrac{2}{\pi}\int_0^{\infty}\left\{\tilde{g}_1(\omega)\cosh\omega y + \dfrac{\sinh\omega y}{\sinh\omega L'}\left[\tilde{g}_2(\omega) - \tilde{g}_1(\omega)\cosh\omega L'\right.\right.$

$\left.\left. + \int_0^{L'} f(u)\sinh\omega(L'-u)\,du\right] - \int_0^{y} f(u)\sinh\omega(y-u)\,du\right\}\sin\omega x\,d\omega$

(ii) Fails

17. (a) $U(x,y) = \dfrac{4U_0}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}e^{-(2n-1)\pi x/(2L')}\sin\dfrac{(2n-1)\pi y}{2L'}$

(b) $U(x,y) = \dfrac{8L'Q_0}{\pi^2\kappa} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left[1 - 2\cos\frac{(2n-1)\pi}{4}\right] e^{-(2n-1)\pi x/(2L')} \sin\frac{(2n-1)\pi y}{2L'}$

(c) $U(x,y) = \dfrac{8\mu L' U_m}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)[2\mu L' + (2n-1)\pi\kappa]} e^{-(2n-1)\pi x/(2L')} \sin\frac{(2n-1)\pi y}{2L'}$

18. (a) $U(x,y) = U_0$ (b) $U(x,y) = \dfrac{B_0}{\sqrt{L'}} + \dfrac{4L'Q_0}{\pi^2\kappa} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} e^{-(2n-1)\pi x/L'} \cos\frac{(2n-1)\pi y}{L'}$

(c) $U(x,y) = U_m$

19. Yes It is unbounded.

20. (a) $U(x,y) = \displaystyle\sum_{n=1}^{\infty} \left[\tilde{f}(\lambda_n) e^{-\lambda_n x} + \frac{\mu U_m \sin\lambda_n L'}{N\kappa\lambda_n^2}(1 - e^{-\lambda_n x})\right] \frac{1}{N} \sin\lambda_n y,$ where

$$\tilde{f}(\lambda_n) = \int_0^{L'} f(y)\frac{1}{N}\sin\lambda_n y\,dy, \quad 2N^2 = L' + \frac{\kappa/\mu}{\lambda_n^2 + \kappa^2/\mu^2}, \quad \cot\lambda L' = -\frac{\kappa}{\lambda\mu}$$

(b) $U(x,y) = 2U_0 \displaystyle\sum_{n=1}^{\infty} \frac{\kappa^2 + \mu^2\lambda_n^2}{\lambda_n[\kappa\mu + L'(\mu^2\lambda_n^2 + \kappa^2)]} \left[1 + \frac{(-1)^{n+1}\kappa}{\sqrt{\mu^2\lambda_n^2 + \kappa^2}}\right] e^{-\lambda_n x} \sin\lambda_n y$

21. (a) $U(x,y) = \displaystyle\sum_{n=1}^{\infty} \left[\tilde{f}(\lambda_n) e^{-\lambda_n x} + \frac{\mu U_m \cos\lambda_n L'}{N\kappa\lambda_n^2}(1 - e^{-\lambda_n x})\right] \frac{1}{N} \cos\lambda_n y,$ where

$$\tilde{f}(\lambda_n) = \int_0^{L'} f(y)\frac{1}{N}\cos\lambda_n y\,dy, \quad 2N^2 = L' + \frac{\kappa/\mu}{\lambda_n^2 + \kappa^2/\mu^2}, \quad \tan\lambda L' = \frac{\kappa}{\lambda\mu}$$

(b) $U(x,y) = 2lU_0 \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{\mu^2\lambda_n^2 + \kappa^2}}{L'(\mu^2\lambda_n^2 + \kappa^2) + \kappa\mu} e^{-\lambda_n x} \cos\lambda_n y$

22. (a)(i) $V(x,y) = \dfrac{\sigma x(L-x)}{2\epsilon} + \dfrac{V_L x}{L} + \dfrac{2L}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \left\{-\frac{\sigma L[1 + (-1)^{n+1}]}{n^3\pi\epsilon} + \frac{(-1)^n \pi V_L}{nL}\right\} *$

$$e^{-n\pi y/L} \sin\frac{n\pi x}{L}$$

(ii) $-\dfrac{\sigma x^2}{2\epsilon} + \left(\dfrac{V_L}{L} + \dfrac{\sigma L}{2\epsilon}\right) x + \dfrac{2}{\pi} \displaystyle\sum_{n=1}^{\infty} \left\{\frac{(-1)^n V_L}{n} - \frac{\sigma L^2[1 + (-1)^{n+1}]}{n^3\pi^2\epsilon}\right\} e^{-n\pi y/L} \sin\frac{n\pi x}{L}$

(b) No

23. $V(x,y) = \dfrac{V_L x}{L} + \dfrac{e^{-y}}{\epsilon}\left[\dfrac{\sin(L-x) + \sin x}{\sin L} - 1\right]$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{V_L(-1)^n}{n} - \frac{L^2[1 + (-1)^{n+1}]}{n\epsilon(n^2\pi^2 - L^2)}\right] e^{-n\pi y/L} \sin\frac{n\pi x}{L}$$

24. (a) $V(x,y) = \dfrac{e^{-y}}{\epsilon}\left[\dfrac{\sin(L-x) + \sin x}{\sin L} - 1\right]$

$$-\frac{4L^2}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)\pi y/L}}{(2n-1)[(2n-1)^2\pi^2 - L^2]} \sin\frac{(2n-1)\pi x}{L}$$

(b) $V(x,y) = \dfrac{2}{\pi\epsilon}\displaystyle\int_0^{\infty} \frac{1}{\omega(1+\omega^2)}\left[\frac{\sin\omega L - \sin\omega x - \sin\omega(L-x)}{\sin\omega L}\right] \sin\omega x\,d\omega$

25. (d) $V(x,y) = \dfrac{2}{\pi}\mathrm{Tan}^{-1}\left(\dfrac{x}{y}\right)$

(e) $V(x,y) = \dfrac{y}{\pi}\displaystyle\int_0^{\infty} f(u)\left[\frac{1}{(x-u)^2 + y^2} - \frac{1}{(x+u)^2 + y^2}\right] du$

$$+\frac{x}{\pi}\int_0^{\infty} g(u)\left[\frac{1}{x^2 + (y-u)^2} - \frac{1}{x^2 + (y+u)^2}\right] du$$

26. (b) $V(x,y) = \frac{1}{2} + \frac{1}{\pi}\text{Tan}^{-1}\left(\frac{x}{y}\right)$ **27.** $V(x,y) = \frac{2}{\pi}\text{Tan}^{-1}\left(\frac{\sin\frac{\pi y}{L'}}{\sinh\frac{\pi x}{L'}}\right)$

Exercises 9.2

1. (a) 120 (b) 2.9812 (c) 7.3619 (d) −5.7386 (e) 0.6891 (f) −1.0276

Exercises 9.3

3. (a) 0.9604 (b) 0.6201 (c) 0.3688 (d) 0.0955 (e) 0.4448 (f) −0.2769 (g) 0.4333 (h) 0.1190

4. (a) 0.4448 (b) −0.2769 (c) 0.4333 (d) 0.1190

Exercises 9.4

2. (a) $2a^{\nu-1}\sum_{n=1}^{\infty}\frac{J_\nu(\lambda_{\nu n}r)}{\lambda_{\nu n}J_{\nu+1}(\lambda_{\nu n}a)}$ (b) $2\nu a^{\nu}\sum_{n=1}^{\infty}\frac{J_\nu(\lambda_{\nu n}r)}{[\lambda_{\nu n}^2a^2-\nu^2]J_\nu(\lambda_{\nu n}a)}$

3. When $hl/\neq 0$, $\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_1(\lambda_{0n}a)J_0(\lambda_{0n}r)}{\lambda_{0n}\left[1+\left(\frac{h}{\lambda_{0n}l}\right)^2\right][J_0(\lambda_{0n}a)]^2}$;

when $l=0$, $\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(\lambda_{0n}r)}{\lambda_{0n}J_1(\lambda_{0n}a)}$; when $h=0$, $\frac{a}{\sqrt{2}}R_{00}(r)$

Exercises 9.5

1. $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$, $P_3(x)=\frac{1}{2}(5x^3-3x)$,

$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$, $P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$,

$P_6(x)=\frac{1}{16}(231x^6-315x^4+105x^2-5)$

10. (b) $Q_2(x)=\left(\frac{3x^2}{2}-\frac{1}{2}\right)Q_0-\frac{3x}{2}$, $Q_3(x)=\left(\frac{5x^3}{2}-\frac{3x}{2}\right)Q_0-\frac{5x^2}{2}+\frac{2}{3}$

$Q_4(x)=\left(\frac{35x^4}{8}-\frac{15x^2}{4}+\frac{3}{8}\right)Q_0-\frac{35x^3}{8}+\frac{55x}{24}$

(c) $Q_2(x)=P_2(x)Q_0(x)-\frac{3x}{2}$, $Q_3(x)=P_3(x)Q_0(x)-\frac{5x^2}{2}+\frac{2}{3}$

$Q_4(x)=P_4(x)Q_0(x)-\frac{35x^3}{8}+\frac{55x}{24}$

Exercises 9.6

1. $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-2)!(4n-1)}{2^{2n}n!(n-1)!}P_{2n-1}(\cos\phi)$

2. $\frac{\sqrt{2}}{5}\left(\frac{1}{\sqrt{2}}\right)+\frac{4\sqrt{10}}{35}\left[\sqrt{\frac{5}{2}}P_2(\cos\phi)\right]+\frac{8\sqrt{2}}{105}\left[\frac{3}{\sqrt{2}}P_4(\cos\phi)\right]$

3. $\frac{1}{4}+\frac{1}{2}\cos\phi+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-2)!(4n+1)}{2^{2n+1}(n-1)!(n+1)!}P_{2n}(\cos\phi)$

4. $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(4n+1)(2n-2)!}{2^{2n}(n-1)!(n+1)!}P_{2n}(\cos\phi)$

5. $\lambda_n = 2n(2n-1)$, $n \geq 1$; $\Phi_n(\phi) = \sqrt{4n-1}P_{2n-1}(\cos\phi)$
6. $\lambda_n = 2n(2n+1)$, $n \geq 0$; $\Phi_n(\phi) = \sqrt{4n+1}P_{2n}(\cos\phi)$

Exercises 10.1

1. (a) $U(r,t) = \sum_{n=1}^{\infty} C_n e^{-k\lambda_n^2 t} \frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}$, where $C_n = \int_0^a rf(r)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}\,dr$

(b) $U(r,t) = \frac{2U_0}{a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1(\lambda_n a)}e^{-k\lambda_n^2 t}J_0(\lambda_n r)$ (c) $U(r,t) = \frac{8}{a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}e^{-k\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

2. (a) $U(r,t) = \sum_{n=0}^{\infty} C_n e^{-k\lambda_n^2 t} \frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}$, where $C_n = \int_0^a rf(r)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}\,dr$

(b) $\frac{1}{\pi a^2}\int_{-\pi}^{\pi}\int_0^a rf(r)\,dr\,d\theta$

3.
$$U(r,\theta,t) = \sum_{n=1}^{\infty} A_{0n}\frac{1}{\sqrt{2\pi}}R_{0n}(r)e^{-k\lambda_{0n}^2 t} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{mn}(r)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right)e^{-k\lambda_{mn}^2 t},$$
where $R_{mn}(r) = \frac{\sqrt{2}J_m(\lambda_{mn}r)}{aJ_{m+1}(\lambda_{mn}a)}$, $A_{0n} = \int_{-\pi}^{\pi}\int_0^a rf(r,\theta)\frac{R_{0n}(r)}{\sqrt{2\pi}}\,dr\,d\theta$,
$$A_{mn} = \int_{-\pi}^{\pi}\int_0^a rf(r,\theta)R_{mn}(r)\frac{\cos m\theta}{\sqrt{\pi}}\,dr\,d\theta,\quad B_{mn} = \int_{-\pi}^{\pi}\int_0^a rf(r,\theta)R_{mn}(r)\frac{\sin m\theta}{\sqrt{\pi}}\,dr\,d\theta$$

5. $U(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-k\lambda_{mn}^2 t}R_{mn}(r)H_m(\theta)$, where
$$C_{mn} = \int_0^{\pi/2}\int_0^a rf(r,\theta)R_{mn}(r)H_m(\theta)\,dr\,d\theta,\quad H_m(\theta) = \frac{2}{\sqrt{\pi}}\sin 2m\theta,$$
$$R_{mn}(r) = \frac{\sqrt{2}J_{2m}(\lambda_{mn}r)}{J_{2m+1}(\lambda_{mn}a)}$$

6. $U(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-k\lambda_{mn}^2 t}R_{mn}(r)H_m(\theta)$, where
$$C_{mn} = \int_0^{\pi/2}\int_0^a rf(r,\theta)R_{mn}(r)H_m(\theta)\,dr\,d\theta,\quad H_0(\theta) = \sqrt{\frac{2}{\pi}},$$
$$H_m(\theta) = \frac{2}{\sqrt{\pi}}\cos 2m\theta,\quad R_{mn}(r) = \frac{\sqrt{2}J_{2m}(\lambda_{mn}r)}{J_{2m+1}(\lambda_{mn}a)}$$

7. $U(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-k\lambda_{mn}^2 t}R_{mn}(r)H_m(\theta)$, where
$$C_{mn} = \int_0^{\pi/2}\int_0^a rf(r,\theta)R_{mn}(r)H_m(\theta)\,dr\,d\theta,$$
$$R_{mn}(r) = R_{mn}(r) = N^{-1}J_{2m}(\lambda_{mn}r),\quad 2N^2 = a^2\left[1 - \left(\frac{2m}{\lambda_{mn}a}\right)^2\right][J_{2m}(\lambda_{mn}a)]^2$$

8. $U(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}e^{-k\lambda_{mn}^2 t}R_{mn}(r)H_m(\theta)$, where
$$C_{mn} = \int_0^{\pi/2}\int_0^a rf(r,\theta)R_{mn}(r)H_m(\theta)\,dr\,d\theta,\quad H_0(\theta) = \sqrt{\frac{2}{\pi}},$$

$$H_m(\theta) = \frac{2}{\sqrt{\pi}} \cos 2m\theta, \quad R_{mn}(r) = N^{-1} J_{2m}(\lambda_{mn} r),$$

$$2N^2 = a^2 \left[1 - \left(\frac{2m}{\lambda_{mn} a}\right)^2\right] [J_{2m}(\lambda_{mn} a)]^2$$

9. $U(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} e^{-k\lambda_{mn}^2 t} R_{mn}(r) H_m(\theta)$, where $C_{mn} = \frac{2\sqrt{2\alpha}\bar{U}}{(2m-1)\pi} \int_0^1 r^2 R_{mn}(r)\, dr$,

$$H_m(\theta) = \sqrt{\frac{2}{\alpha}} \sin \frac{(2m-1)\pi\theta}{\alpha}, \quad R_{mn}(r) = N^{-1} J_{(2m-1)\pi/\alpha}(\lambda_{mn} r),$$

$$2N^2 = \left[1 - \left(\frac{(2m-1)\pi}{\alpha\lambda_{mn}}\right)^2\right] \left[J_{(2m-1)\pi/\alpha}(\lambda_{mn})\right]^2$$

10. $U(r,t) = \sum_{n=1}^{\infty} C_n e^{-k\lambda_n^2 t} \frac{\sqrt{2} J_0(\lambda_n r)}{J_1(\lambda_n)}$, where $C_n = \int_0^1 r f(r) \frac{\sqrt{2} J_0(\lambda_n r)}{J_1(\lambda_n)}\, dr$

11. $U(r,\theta,t) = \sum_{n=1}^{\infty} C_{0n} e^{-k\lambda_{0n}^2 t} \frac{1}{\sqrt{\alpha}} R_{0n}(r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} e^{-k\lambda_{mn}^2 t} \sqrt{\frac{2}{\alpha}} \cos \frac{m\pi\theta}{\alpha} R_{mn}(r)$, where

$$C_{0n} = \int_0^{\alpha}\int_0^1 f(r,\theta) \frac{r}{\sqrt{\alpha}} R_{0n}(r)\, dr\, d\theta,$$

$$C_{mn} = \int_0^{\alpha}\int_0^1 f(r,\theta) \sqrt{\frac{2}{\alpha}} \cos \frac{m\pi\theta}{\alpha} R_{mn}(r) r\, dr\, d\theta,$$

$$H_0(\theta) = \frac{1}{\sqrt{\alpha}}, \quad H_m(\theta) = \sqrt{\frac{2}{\alpha}} \cos \frac{m\pi\theta}{\alpha}, \quad R_{mn}(r) = \frac{\sqrt{2} J_{m\pi/\alpha}(\lambda_{mn} r)}{J_{m\pi/\alpha+1}(\lambda_{mn})}$$

12. $U(r,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} e^{-k(\nu_m^2+\lambda_n^2)t} R_n(r) Z_m(z)$, where

$$C_{mn} = \int_0^L \int_0^a r f(r,z) R_n(r) Z_m(z)\, dr\, dz,$$

$$Z_m(z) = \sqrt{\frac{2}{L}} \cos \frac{(2m-1)\pi z}{2L}, \quad \nu_m = \frac{(2m-1)\pi}{2L}, \quad R_n(r) = \frac{\sqrt{2}}{a} \frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$$

13. (a) $U(r,t) = \sum_{n=1}^{\infty} C_n e^{-k\lambda_n^2 t} R_n(r)$, where $C_n = \int_0^a r f(r) R_n(r)\, dr$,

$$R_n(r) = \frac{1}{N} J_0(\lambda_n r), \quad \frac{1}{N} = \frac{\sqrt{2}}{a\left[1 + \left(\frac{h}{\lambda_n l}\right)^2\right]^{1/2} J_0(\lambda_n a)}$$

(b) $U(r,t) = \frac{2U_0 hl}{a} \sum_{n=1}^{\infty} \frac{1}{(h^2 + l^2\lambda_n^2)} \frac{J_0(\lambda_n r)}{J_0(\lambda_n a)} e^{-k\lambda_n^2 t}$

14. (a) $U(r,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 kt/a^2} \sqrt{\frac{2}{a}} r^{-1} \sin \frac{n\pi r}{a}$, where $C_n = \int_0^a r^2 f(r) \sqrt{\frac{2}{a}} r^{-1} \sin \frac{n\pi r}{a}\, dr$

(b) $U(r,t) = \frac{2U_0 a}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2\pi^2 kt/a^2} \sin \frac{n\pi r}{a}$ (c) $0.318U_0$ (d) U_0

15. (a) $U(r,t) = \sum_{n=0}^{\infty} C_n e^{-k\lambda_n^2 t} R_n(r)$, where $C_n = \int_0^a r^2 f(r) R_n(r)\, dr$,

$$R_0(r) = \frac{\sqrt{3}}{a^{3/2}}, \qquad R_n(r) = \frac{\sqrt{2}\sqrt{1 + \lambda_n^2 a^2}}{\lambda_n a^{3/2}} \frac{\sin \lambda_n r}{r},$$

$$\lim_{t\to\infty} U(r,t) = \frac{1}{(4/3)\pi a^3}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^a f(r)\,r^2\sin\phi\,dr\,d\phi\,d\theta \qquad \text{(b) } U_0$$

16. (a) $U(r,t) = \sum_{n=1}^{\infty} C_n e^{-k\lambda_n^2 t} R_n(r), \quad \text{where } C_n = \int_0^a r^2 f(r) R_n(r)\,dr,$

$$R_n(r) = \frac{1}{Nr}\sin\lambda_n r, \qquad N^2 = \frac{a}{2}\left[1 + \frac{\mu a/\kappa - 1}{\lambda_n^2 a^2 + \left(1 - \frac{\mu a}{\kappa}\right)^2}\right]$$

(b) $U(r,t) = \frac{2U_0\mu a}{\kappa r}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sqrt{\lambda_n^2 a^2 + (1-\mu a/\kappa)^2}}{\lambda_n\left[\lambda_n^2 a^2 + \frac{\mu a}{\kappa}\left(\frac{\mu a}{\kappa} - 1\right)\right]} e^{-k\lambda_n^2 t}\sin\lambda_n r$

17. (a) $U(r,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn} e^{-k\lambda_{mn}^2 t}\Phi_m(\phi) R_{mn}(r), \quad \text{where}$

$$C_{mn} = \int_0^a\int_0^{\pi} r^2\sin\phi\, f(r,\phi)\Phi_m(\phi) R_{mn}(r)\,d\phi\,dr,$$

$$\Phi_m(\phi) = \sqrt{\frac{2m+1}{2}}P_m(\cos\phi), \qquad R_{mn}(r) = \frac{\sqrt{2}J_{m+1/2}(\lambda_{mn} r)}{a\sqrt{r}J_{m+3/2}(\lambda_{mn} a)}$$

18. (a) $U(r,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} e^{-k\lambda_{mn}^2 t}\Phi_m(\phi) R_{mn}(r), \quad \text{where}$

$$C_{mn} = \int_0^a\int_0^{\pi} r^2\sin\phi\, f(r,\phi)\Phi_m(\phi) R_{mn}(r)\,d\phi\,dr, \qquad \Phi_m(\phi) = \sqrt{\frac{2m+1}{2}}P_m(\cos\phi),$$

$$R_{mn}(r) = \frac{1}{N\sqrt{r}}J_{m+1/2}(\lambda_{mn} r),$$

$$2N^2 = a^2\left[1 - \left(\frac{m+1/2}{\lambda_{mn} a}\right)^2 + \left(\frac{1}{2\lambda_{mn} a}\right)^2\right][J_{m+1/2}(\lambda_{mn} a)]^2$$

(b) $\frac{1}{(4/3)\pi a^3}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^a f(r,\phi) r^2\sin\phi\,dr\,d\phi\,d\theta$

20. $U(r,z,t) = \frac{4La^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}e^{-(2m-1)^2\pi^2 kt/(4L^2)}\cos\frac{(2m-1)\pi z}{2L}$

$$-\frac{32L}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2\lambda_n^2}e^{-k[\lambda_n^2 + (2m-1)^2\pi^2/(4L^2)]t}\frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}\cos\frac{(2m-1)\pi z}{2L}$$

21. (a) $A_n = \int_0^a r f(r)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}\,dr$ (b) No nodal curves (c) Nodal curve $r = 0.4356$

(d) No, yes

22. $z(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos c\lambda_{mn} t H_m(\theta) R_{mn}(r), \quad \text{where}$

$$A_{mn} = \int_0^{2\pi}\int_0^a f(r,\theta) H_m(\theta) R_{mn}(r) r\,dr\,d\theta$$

$$H_m(\theta) = \frac{1}{\sqrt{\pi}}\sin\frac{m\theta}{2}, \qquad R_{mn}(r) = \frac{\sqrt{2}J_{m/2}(\lambda_{mn} r)}{aJ_{m/2+1}(\lambda_{mn} a)}$$

23. $z(r,t) = \frac{8}{a}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n^3 J_1(\lambda_n a)}\cos c\lambda_n t$

24. $z(r,t) = -\frac{2v_0}{ca}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n^2 J_1(\lambda_n a)}\sin c\lambda_n t$

25. (b) $r = 2.4048a/5.5201$ (c) $r = 2.4048a/8.6537$, $r = 5.5201a/8.6537$ (e) y axis; x axis
(f) y axis and $r = 3.8317a/7.0156$; x axis and $r = 3.8317a/7.0156$
(g) $y = \pm x$, $r = 5.1356a/11.620$, $r = 8.4172a/11.620$; x and y axes, $r = 5.1356a/11.620$, $r = 8.4172a/11.620$

26. (b) $$y(x,t) = \frac{1}{L}\sum_{n=1}^{\infty}\left\{\frac{J_0(\alpha_n\sqrt{x/L})\cos[\alpha_n\sqrt{g/(4L)}t]}{[J_1(\alpha_n)]^2}\int_0^L f(u)J_0(\alpha_n\sqrt{u/L})\,du + \frac{2\sqrt{L/g}J_0(\alpha_n\sqrt{x/L})\sin[\alpha_n\sqrt{g/(4L)}t]}{\alpha_n[J_1(\alpha_n)]^2}\int_0^L h(u)J_0(\alpha_n\sqrt{u/L})\,du\right\}$$ where $J_0(\alpha_n) = 0$

27. (a) $\frac{1}{\sqrt{2\pi}}R_{0n}(r)$, $\quad \frac{1}{\sqrt{\pi}}R_{mn}(r)\cos m\theta$, $\quad \frac{1}{\sqrt{\pi}}R_{mn}(r)\sin m\theta$, where
$$R_{mn}(r) = \frac{\sqrt{2}J_m(\lambda_{mn}r)}{aJ_{m+1}(\lambda_{mn}a)}$$
(b) $$z(r,\theta,t) = \sum_{n=1}^{\infty} A_{0n}\frac{R_{0n}(r)}{\sqrt{2\pi}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{mn}(r)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right)\cos c\lambda_{mn}t,$$
where $$A_{0n} = \int_{-\pi}^{\pi}\int_0^a f(r,\theta)\frac{R_{0n}(r)}{\sqrt{2\pi}}\,r\,dr\,d\theta, \quad A_{mn} = \int_{-\pi}^{\pi}\int_0^a f(r,\theta)\frac{R_{mn}(r)}{\sqrt{\pi}}\cos m\theta\,r\,dr\,d\theta,$$
$$B_{mn} = \int_{-\pi}^{\pi}\int_0^a f(r,\theta)\frac{R_{mn}(r)}{\sqrt{\pi}}\sin m\theta\,r\,dr\,d\theta$$

28. (a) $$V(r,\theta) = \frac{A_0J_0(kr)}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} J_n(kr)\left(A_n\frac{\cos n\theta}{\sqrt{\pi}} + B_n\frac{\sin n\theta}{\sqrt{\pi}}\right),$$ where
$$A_0 = \frac{1}{J_0(ka)}\int_{-\pi}^{\pi}\frac{f(\theta)}{\sqrt{2\pi}}\,d\theta, \quad A_n = \frac{1}{J_n(ka)}\int_{-\pi}^{\pi} f(\theta)\frac{\cos n\theta}{\sqrt{\pi}}\,d\theta,$$
$$B_n = \frac{1}{J_n(ka)}\int_{-\pi}^{\pi} f(\theta)\frac{\sin n\theta}{\sqrt{\pi}}\,d\theta$$
(b) $1/J_0(ka)$ times the average value (c) $V(r,\theta) = \dfrac{J_0(kr)}{J_0(ka)}$

29. $$V(r,z) = \sum_{n=1}^{\infty} A_n\sinh\lambda_n z\,\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}, \quad \text{where } A_n = \frac{\sqrt{2}}{aJ_1(\lambda_n a)\sinh\lambda_n L}\int_0^a rf(r)J_0(\lambda_n r)\,dr$$

31. (a) $$U(r,z) = U(r,z) = \sum_{n=1}^{\infty} C_n\sinh\lambda_n(L-z)\,R_n(r),$$ where
$$C_n = \frac{1}{\sinh\lambda_n L}\int_0^a rf(r)R_n(r)\,dr, \quad R_n(r) = \frac{1}{N}J_0(\lambda_n r),$$
$$\frac{1}{N} = \frac{\sqrt{2}}{a\left[1+\left(\frac{\mu}{\lambda_n\kappa}\right)^2\right]^{1/2} J_0(\lambda_n a)}$$
(b) $$U(r,z) = \frac{2U_0\mu\kappa}{a}\sum_{n=1}^{\infty}\frac{1}{(\mu^2+\lambda_n^2\kappa^2)\sinh\lambda_n L}\sinh\lambda_n(L-z)\frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}$$

32. $$U(r,z) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n z}\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}, \quad \text{where } C_n = \int_0^a rf(r)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}dr$$

33. $$U(r,z) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n z}\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}, \quad \text{where } C_n = \int_0^a rf(r)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}dr$$

34. $V(r,\phi)=\sum_{n=0}^{\infty}A_n r^n\sqrt{\frac{2n+1}{2}}P_n(\cos\phi)$, where

$$A_n=\frac{1}{a^n}\int_0^{\pi}f(\phi)\sin\phi\sqrt{\frac{2n+1}{2}}P_n(\cos\phi)\,d\phi \qquad \text{When } f(\phi)=k,\ V(r,\phi)=k.$$

36. $V(r,\phi,\theta)=\sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi}}A_{0n}r^n\Phi_{0n}(\phi)+\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}r^n\Phi_{mn}(\phi)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}}+B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right)$,

where $\Phi_{mn}(\phi)=\sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}P_{mn}(\cos\phi)$,

$$A_{0n}=\frac{1}{na^{n-1}}\int_{-\pi}^{\pi}\int_0^{\pi}f(\phi,\theta)\frac{1}{\sqrt{2\pi}}\sin\phi\,\Phi_{0n}(\phi)\,d\phi\,d\theta, \quad n>0,$$

$$A_{mn}=\frac{1}{na^{n-1}}\int_{-\pi}^{\pi}\int_0^{\pi}f(\phi,\theta)\frac{\cos m\theta}{\sqrt{\pi}}\sin\phi\,\Phi_{mn}(\phi)\,d\phi\,d\theta,$$

$$B_{mn}=\frac{1}{na^{n-1}}\int_{-\pi}^{\pi}\int_0^{\pi}f(\phi,\theta)\frac{\sin m\theta}{\sqrt{\pi}}\sin\phi\,\Phi_{mn}(\phi)\,d\phi\,d\theta$$

37. $V(r,\phi,\theta)=\sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi}}A_{0n}r^n\Phi_{0n}(\phi)+\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}r^n\Phi_{mn}(\phi)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}}+B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right)$,

where $\Phi_{mn}(\phi)=\sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}P_{mn}(\cos\phi)$,

$$A_{00}=\frac{1}{h}\int_{-\pi}^{\pi}f(\phi,\theta)\frac{1}{\sqrt{2\pi}}\sin\phi\,\Phi_{00}(\phi)\,d\phi,$$

$$A_{0n}=\frac{1}{a^{n-1}(ln+ha)}\int_{-\pi}^{\pi}\int_0^{\pi}f(\phi,\theta)\frac{1}{\sqrt{2\pi}}\sin\phi\,\Phi_{0n}(\phi)\,d\phi\,d\theta, \quad n>0,$$

$$A_{mn}=\frac{1}{a^{n-1}(ln+ha)}\int_{-\pi}^{\pi}\int_0^{\pi}f(\phi,\theta)\frac{\cos m\theta}{\sqrt{\pi}}\sin\phi\,\Phi_{mn}(\phi)\,d\phi\,d\theta,$$

$$B_{mn}=\frac{1}{a^{n-1}(ln+ha)}\int_{-\pi}^{\pi}\int_0^{\pi}f(\phi,\theta)\frac{\sin m\theta}{\sqrt{\pi}}\sin\phi\,\Phi_{mn}(\phi)\,d\phi\,d\theta$$

38. $U(r,\phi)=\sum_{n=1}^{\infty}A_n r^{2n-1}\sqrt{4n-1}P_{2n-1}(\cos\phi)$, where

$$A_n=\frac{1}{a^{2n-1}}\int_0^{\pi/2}f(\phi)\sin\phi\sqrt{4n-1}P_{2n-1}(\cos\phi)\,d\phi$$

When $f(\phi)=k$, $U(r,\phi)=k\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(4n-1)(2n-2)!}{2^{2n-1}n!(n-1)!}\left(\frac{r}{a}\right)^{2n-1}P_{2n-1}(\cos\phi)$

39. $U(r,\phi)=\sum_{n=1}^{\infty}A_n r^{2n}\sqrt{4n+1}P_{2n}(\cos\phi)$, where

$$A_n=\frac{1}{a^{2n}}\int_0^{\pi/2}f(\phi)\sin\phi\sqrt{4n+1}P_{2n}(\cos\phi)\,d\phi \qquad \text{When } f(\phi)=k,\ U(r,\phi)=k$$

40. $U(r,\phi)=\sum_{n=1}^{\infty}\frac{A_n}{r^{2n}}\sqrt{4n-1}P_{2n-1}(\cos\phi)$, where

$$A_n=a^{2n}\int_0^{\pi/2}f(\phi)\sin\phi\sqrt{4n-1}P_{2n-1}(\cos\phi)\,d\phi$$

41. $V(r,\phi)=V_0\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(4n-1)(2n-2)!}{2^{2n}n!\,(n-1)!}\left(\frac{r}{a}\right)^{2n-1}P_{2n-1}(\cos\phi)\right]$

42. $V(r,\phi) = \frac{1}{2}(V_0 + V_1) + (V_0 - V_1)\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(4n-1)(2n-2)!}{2^{2n} n!\,(n-1)!} \left(\frac{r}{a}\right)^{2n-1} P_{2n-1}(\cos\phi)$

45. (a) V_0 (b) aV_0/r

46. $V(r,\phi,\theta) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{A_{0n}}{r^{n+1}} \Phi_{0n}(\phi) + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \frac{1}{r^{n+1}} \Phi_{mn}(\phi) \left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right),$

where $\Phi_{mn}(\phi) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} P_{mn}(\cos\phi),$

$$A_{0n} = \frac{a^{n+2}}{n+1} \int_{-\pi}^{\pi}\int_{0}^{\pi} f(\phi,\theta) \frac{1}{\sqrt{2\pi}} \sin\phi\, \Phi_{0n}(\phi)\, d\phi\, d\theta,$$

$$A_{mn} = \frac{a^{n+2}}{n+1} \int_{-\pi}^{\pi}\int_{0}^{\pi} f(\phi,\theta) \frac{\cos m\theta}{\sqrt{\pi}} \sin\phi\, \Phi_{mn}(\phi)\, d\phi\, d\theta,$$

$$B_{mn} = \frac{a^{n+2}}{n+1} \int_{-\pi}^{\pi}\int_{0}^{\pi} f(\phi,\theta) \frac{\sin m\theta}{\sqrt{\pi}} \sin\phi\, \Phi_{mn}(\phi)\, d\phi\, d\theta$$

47. $V(r,\phi,\theta) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{A_{0n}}{r^{n+1}} \Phi_{0n}(\phi) + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \frac{1}{r^{n+1}} \Phi_{mn}(\phi) \left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right),$

where $\Phi_{mn}(\phi) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} P_{mn}(\cos\phi),$

$$A_{00} = \frac{a}{h} \int_{-\pi}^{\pi}\int_{0}^{\pi} f(\phi,\theta) \frac{1}{\sqrt{2\pi}} \sin\phi\, \Phi_{00}(\phi)\, d\phi\, d\theta,$$

$$A_{0n} = \frac{a^{n+2}}{l(n+1)+ha} \int_{-\pi}^{\pi}\int_{0}^{\pi} f(\phi,\theta) \frac{1}{\sqrt{2\pi}} \sin\phi\, \Phi_{0n}(\phi)\, d\phi\, d\theta,$$

$$A_{mn} = \frac{a^{n+2}}{l(n+1)+ha} \int_{-\pi}^{\pi}\int_{0}^{\pi} f(\phi,\theta) \frac{\cos m\theta}{\sqrt{\pi}} \sin\phi\, \Phi_{mn}(\phi)\, d\phi\, d\theta,$$

$$B_{mn} = \frac{a^{n+2}}{l(n+1)+ha} \int_{-\pi}^{\pi}\int_{0}^{\pi} f(\phi,\theta) \frac{\sin m\theta}{\sqrt{\pi}} \sin\phi\, \Phi_{mn}(\phi)\, d\phi\, d\theta$$

48. (c) $U(r,z) = \sum_{n=1}^{\infty} A_n I_0(\lambda_n r) Z_n(z), \quad \text{where } A_n = \frac{1}{I_0(\lambda_n a)} \int_0^L f(z) Z_n(z)\, dz$

(d) $\frac{4U_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{I_0[(2n-1)\pi r/L]}{I_0[(2n-1)\pi a/L]} \sin\frac{(2n-1)\pi z}{L}$

49. $U(r,z) = \sum_{n=0}^{\infty} A_n I_0(\lambda_n r) Z_n(z), \quad \text{where } A_n = \frac{1}{I_0(\lambda_n a)} \int_0^L f(z) Z_n(z)\, dz$

When $f(z) = U_0$, $U(r,z) = U_0$.

50. (b) $V(r,0) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) \sqrt{\frac{2n+1}{2}}$

(c) For $r < a$, $V(r,\phi) = \frac{Q}{4\pi\epsilon_0 a} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\phi)$

For $r > a$, $V(r,\phi) = \frac{Q}{4\pi\epsilon_0 r} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\phi)$

51. For $r < a$, $V(r,\phi) = \frac{Q}{2\pi\epsilon_0 a} \left[1 - \left(\frac{r}{a}\right)\cos\phi + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-2)!}{2^{2n-1} n!\,(n-1)!} \left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\phi)\right]$

For $r > a$, $V(r,\phi) = \dfrac{Q}{4\pi\epsilon_0 a}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}n!\,(n+1)!}\left(\frac{a}{r}\right)^{2n+1}P_{2n}(\cos\phi)$

Exercises 10.2

1. $U(r,t) = \dfrac{2k}{a}\displaystyle\sum_{n=1}^{\infty}A_n\lambda_n e^{-k\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$, where $A_n = \displaystyle\int_0^t f(u)e^{k\lambda_n^2 u}\,du$

2. (a) $U(r,t) = \dfrac{2k}{\kappa a}\displaystyle\sum_{n=0}^{\infty}A_n e^{-k\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}$, where $A_n = \displaystyle\int_0^t f_1(u)e^{k\lambda_n^2 u}\,du$

(b) $U(t,r) = \dfrac{Q}{4\kappa a}(2r^2 - a^2 + 8kt) - \dfrac{2Q}{\kappa a}\displaystyle\sum_{n=1}^{\infty}\frac{e^{-k\lambda_n^2 t}}{\lambda_n^2}\frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}$ (c) No

3. (a) $U(r,t) = \dfrac{\sqrt{2}}{a}\displaystyle\sum_{n=1}^{\infty}\left[\tilde{f}(\lambda_n)e^{-k\lambda_n^2 t} + \frac{k}{\kappa}\int_0^t \tilde{g}(\lambda_n,u)e^{-k\lambda_n^2(t-u)}\,du\right]\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$, where

$$\tilde{f}(\lambda_n) = \int_0^a rf(r)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}\,dr, \quad \tilde{g}(\lambda_n,t) = \int_0^a rg(r,t)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}\,dr$$

(b) $U(r,t) = \dfrac{2g}{\kappa a}\displaystyle\sum_{n=1}^{\infty}\frac{1-e^{-k\lambda_n^2 t}}{\lambda_n^3}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

4. (a) $\dfrac{\sqrt{2}}{a}\displaystyle\sum_{n=0}^{\infty}\left[\tilde{f}(\lambda_n)e^{-k\lambda_n^2 t} + \frac{k}{\kappa}\int_0^t \tilde{g}(\lambda_n,u)e^{-k\lambda_n^2(t-u)}\,du\right]\frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}$, where

$$\tilde{f}(\lambda_n) = \int_0^a rf(r)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}\,dr, \quad \tilde{g}(\lambda_n,t) = \int_0^a rg(r,t)\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}\,dr$$

(b) $U(r,t) = \dfrac{kgt}{\kappa}$

5. (b) $U(r,t) = \dfrac{2\mu}{a}\displaystyle\sum_{n=1}^{\infty}\left(\frac{g+\kappa U_m\lambda_n^2}{\mu^2+\kappa^2\lambda_n^2}\right)\frac{1-e^{-k\lambda_n^2 t}}{\lambda_n^2}\frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}$

6. (a) $U(r,t) = \dfrac{\sqrt{2/a}}{r}\displaystyle\sum_{n=1}^{\infty}\left\{\tilde{f}(\lambda_n)e^{-k\lambda_n^2 t} + k\int_0^t\left[\frac{\tilde{g}(\lambda_n,u)}{\kappa} - a^2 f_1(u)R_n'(a)\right]e^{-k\lambda_n^2(t-u)}\,du\right\} *$

$$\sin\frac{n\pi r}{a},$$

where $R_n(r) = \sqrt{\dfrac{2}{a}}r^{-1}\sin\lambda_n r$, $\quad \tilde{f}(\lambda_n) = \displaystyle\int_0^a r^2 f(r)R_n(r)\,dr$,

$\tilde{g}(\lambda_n,t) = \displaystyle\int_0^a r^2 g(r,t)R_n(r)\,dr$

(b) $U(r,t) = \dfrac{2a^3 g}{\pi^3\kappa r}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}(1-e^{-n^2\pi^2 kt/a^2})\sin\frac{n\pi r}{a}$

(c) $U(r,t) = \dfrac{g}{6\kappa}(a^2-r^2) + \dfrac{2a^3 g}{\pi^3\kappa r}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}e^{-n^2\pi^2 kt/a^2}\sin\frac{n\pi r}{a}$

(d) $U(r,t) = \dfrac{2af_1}{\pi r}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(1-e^{-n^2\pi^2 kt/a^2})\sin\frac{n\pi r}{a}$

(e) $U(r,t) = f_1\left[1 + \dfrac{2a}{\pi r}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 kt/a^2}\sin\frac{n\pi r}{a}\right]$

7. (a) $U(r,t) = \sum_{n=0}^{\infty} \left[\tilde{f}(\lambda_n) e^{-k\lambda_n^2 t} + \frac{k}{\kappa} \int_0^t [\tilde{g}(\lambda_n, u) + a^2 R_n(a) f_1(u)] e^{-k\lambda_n^2 (t-u)}\, du \right] R_n(r),$

where $R_0(r) = \frac{\sqrt{3}}{a^{3/2}}, \quad R_n(r) = \frac{1}{rN} \sin \lambda_n r, \quad \frac{1}{N} = \frac{\sqrt{2 + 2\lambda_n^2 a^2}}{\lambda_n a^{3/2}},$

$\tilde{f}(\lambda_n) = \int_0^a r^2 f(r) R_n(r)\, dr, \quad \tilde{g}(\lambda_n, t) = \int_0^a r^2 g(r,t) R_n(r)\, dr$

(b) $U(r,t) = \frac{3kf_1 t}{\kappa a} + \frac{f_1}{10\kappa a}(5r^2 - 3a^2) + \frac{2f_1}{\kappa a r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\lambda_n^3} \sqrt{1 + \lambda_n^2 a^2}\, e^{-k\lambda_n^2 t} \sin \lambda_n r$

(c) Yes

8. $U(r,z,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \int_0^t k \left[-f_1(u) Z_m'(L) \frac{\sqrt{2}}{\lambda_n} - a f_2(u) R_n'(a) \frac{\sqrt{2/L}(-1)^{m+1}}{\mu_m} \right] * \right.$

$\left. e^{k(\lambda_n^2 + \mu_m^2)(u-t)} du + C_{mn} e^{-k(\lambda_n^2 + \mu_m^2)t} \right\} R_n(r) Z_m(z)$, where $R_n(r) = \frac{\sqrt{2} J_0(\lambda_n r)}{a J_1(\lambda_n a)}$

$C_{mn} = \int_0^a \int_0^L f(r,z) R_n(r) Z_m(z)\, dz\, dr, \quad Z_m(z) = \sqrt{\frac{2}{L}} \cos \frac{(2m-1)\pi z}{2L},$

9. $U(r,\phi,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[-ka^2 \frac{\sqrt{4m-1}(-1)^{m-1}(2m-2)!}{2^{2m-1}\, m!\, (m-1)!} R_{mn}'(a) \int_0^t f_1(u) e^{k\lambda_{mn}^2 (u-t)} du \right] *$

$R_{mn}(r) \Phi_m(\phi),$

where $\Phi_m(\phi) = \sqrt{4m-1} P_{2m-1}(\cos\phi), \quad R_{mn}(r) = \frac{1}{N\sqrt{r}} J_{(4m-1)/2}(\lambda_{mn} r),$

$2N^2 = a^2 [J_{(4m+1)/2}(\lambda_{mn} a)]^2$

10. $U(r,\theta,t) = -\frac{2\sin\theta}{a} \sum_{n=1}^{\infty} \frac{1 - e^{-k\lambda_n^2 t}}{\lambda_n} \frac{J_1(\lambda_n r)}{J_0(\lambda_n a)}$

11. (a) $\frac{1}{2\pi} \sum_{n=1}^{\infty} \left[\tilde{\tilde{f}}(\lambda_{0n}, 0) e^{-k\lambda_{0n}^2 t} + \frac{2\pi a \overline{U} R_{0n}'(a)}{\lambda_{0n}^2} (e^{-k\lambda_{0n}^2 t} - 1) \right] R_{0n}(r)$

$+ \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathrm{Re} \left[\tilde{\tilde{f}}(\lambda_{mn}, m) e^{im\theta} \right] e^{-k\lambda_{mn}^2 t} R_{mn}(r)$, where

$\tilde{\tilde{f}}(\lambda_{mn}, m) = \int_{-\pi}^{\pi} \int_0^a f(r,\theta) e^{-im\theta} R_{mn}(r)\, dr\, d\theta, \quad R_{mn}(r) = \frac{\sqrt{2} J_m(\lambda_{mn} r)}{a J_{m+1}(\lambda_{mn} a)},$

(b) Yes

12. (a) $\frac{1}{2\pi} \sum_{n=0}^{\infty} \left[\tilde{\tilde{f}}(\lambda_{0n}, 0) e^{-k\lambda_{0n}^2 t} + \frac{2\pi a k R_{0n}(a)}{\kappa} \int_0^t f_1(u) e^{-k\lambda_{0n}^2 (u-t)}\, du \right] R_{0n}(r)$

$+ \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathrm{Re} \left[\tilde{\tilde{f}}(\lambda_{mn}, m) e^{im\theta} \right] e^{-k\lambda_{mn}^2 t} R_{mn}(r)$, where

$\tilde{\tilde{f}}(\lambda_{mn}, m) = \int_{-\pi}^{\pi} \int_0^a f(r,\theta) e^{-im\theta} R_{mn}(r)\, dr\, d\theta, \quad R_{mn}(r) = N^{-1} J_m(\lambda_{mn} r),$

$2N^2 = a^2 \left[1 - \left(\frac{m}{\lambda_{mn} a} \right)^2 \right] [J_m(\lambda_{mn} a)]^2$

(b) $U(r,\theta,t) = \frac{Q}{4\kappa a}(2r^2 - a^2 + 8kt) - \frac{2Q}{\kappa a} \sum_{n=1}^{\infty} \frac{e^{-k\lambda_{0n}^2 t}}{\lambda_{0n}^2} \frac{J_0(\lambda_{0n} r)}{J_0(\lambda_{0n} a)}$

$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\tilde{\tilde{f}}(\lambda_{0n},0)e^{-k\lambda_{0n}^2 t}R_{0n}(r)+\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\text{Re}\,[\tilde{\tilde{f}}(\lambda_{mn},m)e^{im\theta}]e^{-k\lambda_{mn}^2 t}R_{mn}(r)$$

13. $$U(r,\theta,t)=\frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\tilde{\tilde{f}}(\lambda_{0n},0)e^{-k\lambda_{0n}^2 t}+\frac{2\pi a\mu U_e R_{0n}(a)}{\kappa\lambda_{0n}^2}(1-e^{-k\lambda_{0n}^2 t})\right]R_{0n}(r)$$
$$+\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\text{Re}\left[\tilde{\tilde{f}}(\lambda_{mn},m)e^{im\theta}\right]e^{-k\lambda_{mn}^2 t}R_{mn}(r),\text{ where}$$
$$\tilde{\tilde{f}}(\lambda_{mn},m)=\int_{-\pi}^{\pi}\int_0^a f(r,\theta)e^{-im\theta}R_{mn}(r)\,dr\,d\theta,\qquad R_{mn}(r)=\frac{1}{N}J_m(\lambda_{mn}r),$$
$$2N^2=a^2\left[1-\left(\frac{m}{\lambda_{mn}a}\right)^2+\left(\frac{\mu}{\lambda_{mn}\kappa}\right)^2\right][J_m(\lambda_{mn}a)]^2$$

14. $$U(r,t)=U_o+\frac{2(U_i-U_o)r_1}{r_2^2}\sum_{n=1}^{\infty}\frac{J_1(\lambda_n r_1)}{\lambda_n[J_1(\lambda_n r_2)]^2}e^{-k\lambda_n^2 t}J_0(\lambda_n r)$$

15. (a) $$U(r,t)=\sqrt{\frac{2}{r_2-r_1}}\frac{1}{r}\sum_{n=1}^{\infty}\left\{\tilde{f}(\lambda_n)e^{-k\lambda_n^2 t}+k\int_0^t\left[\frac{\tilde{g}(\lambda_n,u)}{\kappa}+[r_1^2f_1(u)R_n'(r_1)\right.\right.$$
$$\left.\left.-r_2^2f_2(u)R_n'(r_2)]\right]e^{-k\lambda_n^2(t-u)}\,du\right\}\sin\frac{n\pi(r-r_1)}{r_2-r_1},$$
where $\tilde{f}(\lambda_n)=\int_{r_1}^{r_2}r^2f(r)R_n(r)\,dr,\quad \tilde{g}(\lambda_n,t)=\int_{r_1}^{r_2}r^2g(r,t)R_n(r)\,dr$
$$R_n(r)=\sqrt{\frac{2}{r_2-r_1}}\frac{1}{r}\sin\lambda_n(r-r_1)$$

(b) $$U(r,t)=\frac{2(r_2-r_1)^2g}{\pi^3\kappa r}\sum_{n=1}^{\infty}\frac{[r_1+(-1)^{n+1}r_2]}{n^3}[1-e^{-n^2\pi^2kt/(r_2-r_1)^2}]\sin\frac{n\pi(r-r_1)}{r_2-r_1}$$

(c) $$U(r,t)=\frac{g}{6\kappa}\left[r_1^2+r_1r_2+r_2^2)-\frac{r_1r_2(r_1+r_2)}{r}-r^2\right]$$
$$-\frac{2(r_2-r_1)^2g}{\pi^3\kappa r}\sum_{n=1}^{\infty}\frac{[r_1+(-1)^{n+1}r_2]}{n^3}e^{-n^2\pi^2kt/(r_2-r_1)^2}\sin\frac{n\pi(r-r_1)}{r_2-r_1}$$

(d) $$U(r,t)=\frac{2}{\pi r}\sum_{n=1}^{\infty}\frac{[r_1f_1+(-1)^{n+1}r_2f_2]}{n}[1-e^{-n^2\pi^2kt/(r_2-r_1)^2}]\sin\frac{n\pi(r-r_1)}{r_2-r_1}$$

(e) $$U(r,t)=\frac{r_1r_2(f_1-f_2)}{(r_2-r_1)r}+\frac{r_2f_2-r_1f_1}{r_2-r_1}$$
$$-\frac{2}{\pi r}\sum_{n=1}^{\infty}\frac{[r_1f_1+(-1)^{n+1}r_2f_2]}{n}e^{-n^2\pi^2kt/(r_2-r_1)^2}\sin\frac{n\pi(r-r_1)}{r_2-r_1}$$

16. (a) $$U(r,t)=\sum_{n=1}^{\infty}\left\{\tilde{f}(\lambda_n)e^{-k\lambda_n^2 t}+k\int_0^t\left[\frac{\tilde{g}(\lambda_n,u)}{\kappa}+[\kappa^{-1}r_1^2R_n(r_1)Q(u)\right.\right.$$
$$\left.\left.-r_2^2f_2(u)R_n'(r_2)]\right]e^{-k\lambda_n^2(t-u)}\,du\right\}\frac{1}{Nr}\sin\lambda_n(r_2-r),\text{ where}$$
$$\tilde{f}(\lambda_n)=\int_{r_1}^{r_2}r^2f(r)R_n(r)\,dr,\qquad R_n(r)=\frac{1}{Nr}\sin\lambda_n(r_2-r),$$
$$2N^2=r_2-r_1+\frac{r_1}{1+\lambda_n^2r_1^2}$$

(b) $$U(r,t)=\frac{r_1^2Q}{\kappa}\left(\frac{1}{r}-\frac{1}{r_2}\right)+\frac{2r_1^2Q}{\kappa r}\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{1+\lambda_n^2r_1^2}}{\lambda_n[r_2+(r_2-r_1)r_1^2\lambda_n^2]}e^{-k\lambda_n^2 t}\sin\lambda_n(r_2-r)$$

17. $z(r,t) = \dfrac{F}{4\rho c^2}(a^2 - r^2) + \sum_{n=1}^{\infty} B_n \cos c\lambda_n t \, \dfrac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)} - \dfrac{2F}{\rho c^2 a}\sum_{n=1}^{\infty} \dfrac{J_0(\lambda_n r)\cos c\lambda_n t}{\lambda_n^3 J_1(\lambda_n a)}$,

where $B_n = \int_0^a r f(r) \dfrac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}\, dr$

18. $z(r,t) = \dfrac{k}{16\rho c^2}(a^4 - r^4) + \sum_{n=1}^{\infty} B_n \cos c\lambda_n t \dfrac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}$

$$+\frac{2k}{\rho c^2 a}\sum_{n=1}^{\infty} \frac{(\lambda_n^2 a^2 - 1)J_0(\lambda_n r)\cos c\lambda_n t}{\lambda_n^5 J_1(\lambda_n a)},$$

where $B_n = \int_0^a r f(r) \dfrac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)} dr$

19. $z(r,t) = \dfrac{g}{4c^2}(r^2 - a^2) - \dfrac{2v_0}{ca}\sum_{n=1}^{\infty} \dfrac{J_0(\lambda_n r)\sin c\lambda_n t}{\lambda_n^2 J_1(\lambda_n a)}$

20. (a) $z(r,\theta,t) = \dfrac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left\{\left[\tilde{\tilde{f}}(\lambda_{mn}, m) + \dfrac{a\tilde{f}_1(m)R'_{mn}(a)}{\lambda_{mn}^2}\right]\cos c\lambda_{mn} t\right.$

$$\left. - \frac{a\tilde{f}_1(m)R'_{mn}(a)}{\lambda_{mn}^2}\right\} e^{im\theta} R_{mn}(r), \text{ where } \tilde{f}_1(m) = \int_{-\pi}^{\pi} f_1(\theta)e^{-im\theta} d\theta,$$

$\tilde{\tilde{f}}(\lambda_{mn}, m) = \int_{-\pi}^{\pi}\int_0^a f(r,\theta)e^{-im\theta} R_{mn}(r)\, r\, dr\, d\theta, \qquad \dfrac{\sqrt{2}J_m(\lambda_{mn} r)}{aJ_{m+1}(\lambda_{mn} a)}$

(b) $z(r,t) = \sum_{n=1}^{\infty}\left[\left(\tilde{f}(\lambda_n) + \dfrac{af_1 R'_n(a)}{\lambda_n^2}\right)\cos c\lambda_n t - \dfrac{af_1 R'_n(a)}{\lambda_n^2}\right] R_n(r)$, where

$R_n(r) = \dfrac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}$

(c) $z(r,t) = f_1 + \sum_{n=1}^{\infty}\left[a_n - \dfrac{\sqrt{2}f_1}{\lambda_n}\right]\cos c\lambda_n t \dfrac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)}$, where $a_n = \int_0^a r f(r)\dfrac{\sqrt{2}J_0(\lambda_n r)}{aJ_1(\lambda_n a)} dr$

21. $y(x,t) = \sum_{n=1}^{\infty}\left[\tilde{f}(\lambda_n)\cos\lambda_n t + \dfrac{\tilde{h}(\lambda_n)}{\lambda_n}\sin\lambda_n t + \dfrac{1}{\rho}\int_0^t \tilde{F}(\lambda_n, u)\sin\lambda_n(t-u)\, du\right] Z_n(\sqrt{4x/g})$,

where $Z_n(z) = \dfrac{\sqrt{2}J_0(\lambda_n z)}{MJ_1(\lambda_n M)}, \quad \tilde{f}(\lambda_n) = \int_0^M z f(z) Z_n(z)\, dz,$

$\tilde{F}(\lambda_n, t) = \int_0^M zF(z,t)Z_n(z)\, dz$

22. $z(r,t) = \dfrac{-2Ac}{a}\sum_{n=1}^{\infty} \dfrac{c\lambda_n \sin\omega t - \omega\sin c\lambda_n t}{(\omega^2 - c^2\lambda_n^2)J_1(\lambda_n a)} J_0(\lambda_n r)$ When $\omega = c\lambda_m$,

$$z(r,t) = \frac{A(\sin c\lambda_m t - c\lambda_m t\cos c\lambda_m t)}{a\lambda_m J_1(\lambda_m a)} J_0(\lambda_m r)$$

$$-\frac{2A}{a}\sum_{\substack{n=1\\ n\neq m}}^{\infty} \frac{\lambda_n \sin c\lambda_m t - \lambda_m \sin c\lambda_n t}{(\lambda_m^2 - \lambda_n^2)J_1(\lambda_n a)} J_0(\lambda_n r)$$

23. (a) $z(r,t) = \dfrac{2F_0}{\rho ca}\sum_{n=1}^{\infty} \dfrac{1}{\lambda_n^2(c^2\lambda_n^2 - \omega^2)}(c\lambda_n \sin\omega t - \omega\sin c\lambda_n t)\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

(b) $z(r,t) = \dfrac{F_0 J_0(\lambda_m r)}{\rho ac^2\lambda_m^3 J_1(\lambda_m a)}(\sin c\lambda_m t - c\lambda_m t\cos c\lambda_m t)$

$$+\frac{2F_0}{\rho c^2 a}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{1}{\lambda_n^2(\lambda_n^2-\lambda_m^2)}(\lambda_n\sin c\lambda_m t-\lambda_m\sin c\lambda_n t)\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$$

24. $$z(r,\theta,t)=\frac{1}{2\pi}\left[\tilde{\tilde{f}}(\lambda_{0k},0)\cos c\lambda_{0k}t-\frac{\pi aAR'_{0k}(a)}{\lambda_{0k}^2}\sin c\lambda_{0k}t+\frac{\pi caAR'_{0k}(a)}{\lambda_{0k}}t\cos c\lambda_{0k}t\right]R_{0k}(r)$$

$$+\frac{1}{2\pi}\sum_{\substack{n=1\\n\neq k}}^{\infty}\left[\tilde{\tilde{f}}(\lambda_{0n},0)\cos c\lambda_{0n}t+\frac{2\pi AaR'_{0n}(a)}{\lambda_{0n}(\lambda_{0k}^2-\lambda_{0n}^2)}(\lambda_{0n}\sin c\lambda_{0k}t-\lambda_{0k}\sin c\lambda_{0n}t)\right]R_{0n}(r)$$

$$+\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathrm{Re}[\tilde{\tilde{f}}(\lambda_{mn},m)e^{im\theta}]\cos c\lambda_{mn}tR_{mn}(r)$$

25. Yes

26. $$V(r,\theta)=\frac{4\sigma\beta^2}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{r^2}{(2n-1)[(2n-1)^2\pi^2-4\beta^2]}\left[1-\left(\frac{r}{a}\right)^{(2n-1)\pi/\beta-2}\right]\sin\frac{(2n-1)\pi\theta}{\beta}$$

(a) $$V(r,\theta)=\frac{\sigma r^2}{\epsilon\pi}\ln\left(\frac{a}{r}\right)\sin 2\theta+\frac{\sigma}{\epsilon\pi}\sum_{n=2}^{\infty}\frac{r^2}{(2n-1)[(2n-1)^2-1]}\left[1-\left(\frac{r}{a}\right)^{4n-4}\right]\sin 2(2n-1)\theta$$

(b) $$V(r,\theta)=\frac{4\sigma}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{r^2}{(2n-1)[(2n-1)^2-4]}\left[1-\left(\frac{r}{a}\right)^{2n-3}\right]\sin(2n-1)\theta$$

(c) $$V(r,\theta)=\frac{9\sigma r^2}{8\epsilon\pi}\left[\left(\frac{a}{r}\right)^{4/3}-1\right]\sin\frac{2\theta}{3}+\frac{\sigma r^2}{3\pi\epsilon}\ln\left(\frac{a}{r}\right)\sin 2\theta+\frac{9\sigma}{\pi\epsilon}\sum_{n=3}^{\infty}\frac{r^2}{(2n-1)[(2n-1)^2-9]}\left[1-\left(\frac{r}{a}\right)^{4(n-2)/3}\right]\sin\frac{2(2n-1)\theta}{3}$$

27. $$V(r,\phi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\,d\theta+\frac{k(a^{n+2}-r^{n+2})}{\epsilon(n+2)^2}+\frac{1}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^m\int_{-\pi}^{\pi}f(u)\cos m(\theta-u)\,du$$

28. $$V(r,\phi)=\sum_{n=1}^{\infty}\left\{\left[V_1\tilde{1}_n-\frac{(\sigma/\epsilon)\tilde{1}_n a^2}{2(2n^2-n+1)}+\frac{V_2\Phi'_n(\pi/2)}{2n(2n-1)}\right]\left(\frac{r}{a}\right)^{2n-1}+\frac{(\sigma/\epsilon)\tilde{1}_n r^2}{2(2n^2-n+1)}-\frac{V_2\Phi'_n(\pi/2)}{2n(2n-1)}\right\}\Phi_n(\phi),\ \text{where } \tilde{1}_n=\frac{\sqrt{4n-1}(-1)^{n-1}(2n-2)!}{2^{2n-1}\,n!\,(n-1)!},$$

$$\Phi_n(\phi)=\sqrt{4n-3}P_{2n-1}(\cos\phi)$$

29. (a) $$\frac{\partial^2U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2U}{\partial\theta^2}=-\frac{g}{\kappa},\quad 0<r<a,\quad 0<\theta<\beta,$$

$$U_r(a,\theta)=-Q/\kappa,\quad 0<\theta<\beta,\qquad ga\beta=2q+2\beta Q$$

$$U_\theta(r,0)=qr/\kappa,\quad 0<r<a,$$

$$U_\theta(r,\beta)=0,\quad 0<r<a.$$

(b) $$U(r,\theta)=\frac{A_0}{\sqrt{\beta}}-\frac{gr^2}{4\kappa}+\frac{qr\cos(\beta-\theta)}{\kappa\sin\beta}+\frac{2q\beta^2a}{\kappa\pi}\sum_{n=1}^{\infty}\frac{(r/a)^{n\pi/\beta}}{n(n^2\pi^2-\beta^2)}\cos\frac{n\pi\theta}{\beta}$$

30. (a) $$U(r,z)=\sum_{n=1}^{\infty}\left\{\left[\tilde{f}(\lambda_n)-\frac{\tilde{g}(\lambda_n)}{\kappa\lambda_n^2}\right]\frac{I_0(\lambda_n r)}{I_0(\lambda_n a)}+\frac{\tilde{g}(\lambda_n)}{\kappa\lambda_n^2}\right\}\sqrt{\frac{2}{L}}\sin\frac{n\pi z}{L},$$

where $\tilde{f}(\lambda_n)=\int_0^L f(z)\sqrt{\frac{2}{L}}\sin\frac{n\pi z}{L}dz,\quad \tilde{g}(\lambda_n)=\int_0^L g(z)\sqrt{\frac{2}{L}}\sin\frac{n\pi z}{L}dz$

(b) $U(r,z) = \dfrac{Gz(L-z)}{2\kappa}$
$$+\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{U_0}{2n-1} - \frac{L^2G}{(2n-1)^3\pi^2\kappa}\right]\frac{I_0[(2n-1)\pi r/L]}{I_0[(2n-1)\pi a/L]}\sin\frac{(2n-1)\pi z}{L}$$

Exercises 10.3

1. $U(r,t) = \dfrac{ag}{\kappa}\displaystyle\int_0^\infty \frac{1}{\lambda^2}\left(1-e^{-k\lambda^2 t}\right)J_1(\lambda a)J_0(\lambda r)\,d\lambda$

2. $U(r,\theta,t) = \overline{U} - \dfrac{4\overline{U}}{\alpha}\displaystyle\sum_{n=1}^{\infty}\left[\int_0^\infty \frac{1}{\lambda}e^{-k\lambda^2 t}J_{(2n-1)\pi/\alpha}(\lambda r)\,d\lambda\right]\sin\frac{(2n-1)\pi\theta}{\alpha}$

3. (a) $z(r,t) = \displaystyle\int_0^\infty \lambda\tilde{f}(\lambda)\cos c\lambda t J_0(\lambda r)\,d\lambda$ **4.** $z(r,t) = \dfrac{1}{c}\displaystyle\int_0^\infty \tilde{f}(\lambda)\sin c\lambda t J_0(\lambda r)\,d\lambda$

5. $U(r,z) = \dfrac{aQ}{\kappa}\displaystyle\int_0^\infty \frac{1}{\lambda}e^{-\lambda z}J_1(\lambda a)J_0(\lambda r)\,d\lambda$ **6.** $U(r,z) = a\overline{U}\displaystyle\int_0^\infty e^{-\lambda z}J_1(\lambda a)J_0(\lambda r)\,d\lambda$

7. $U(r,z) = \dfrac{2\overline{U}}{\pi}\displaystyle\int_0^\infty \frac{1}{\lambda}e^{-\lambda z}J_0(\lambda r)\sin\lambda a\,d\lambda$ **8.** $U(r,z) = \dfrac{2\overline{U}}{\pi}\displaystyle\int_0^\infty \frac{1}{\lambda}e^{-\lambda|z|}J_0(\lambda r)\sin\lambda a\,d\lambda$

Exercises 11.1

1. (b)(i) $\dfrac{6}{(s+5)^4}$ (ii) $\dfrac{s+1}{(s+1)^2+4} + \dfrac{2}{(s-3)^2+4}$ (iii) $\dfrac{s-a}{(s-a)^2-16} - \dfrac{4}{(s+a)^2-16}$

(c)(i) $\dfrac{1}{2}e^t\sin 2t$ (ii) $\dfrac{1}{\sqrt{\pi t}}e^{-3t}$ (c) $e^{-2t}\left(\cosh\sqrt{3}t - \dfrac{2}{\sqrt{3}}\sinh\sqrt{3}t\right)$

2. (b)(i) $e^{-3s}\left(\dfrac{1}{s^2}+\dfrac{1}{s}\right)$ (ii) $\dfrac{e^{-as}}{s}$ (iii) $\dfrac{1}{s}\left(1-e^{-as}\right)$ (iv) $\dfrac{1}{s}\left(e^{-as}-e^{-bs}\right)$

(c)(i) $(t-2)h(t-2)$ (ii) $\sin(t-3)h(t-3)$ (iii) $\cosh\sqrt{2}(t-5)h(t-5)$

3. (b)(i) $\dfrac{1}{1-e^{-as}}\left[\dfrac{1}{s^2} - e^{-as}\left(\dfrac{1}{s^2}+\dfrac{a}{s}\right)\right]$ (ii) $\dfrac{1-e^{-as}}{s(1+e^{-as})}$ (iii) $\dfrac{a\left(1+e^{-\pi s/a}\right)}{(s^2+a^2)\left(1-e^{-\pi s/a}\right)}$

5. $1-e^{-t}$ **6.** $\dfrac{1}{6}(-\sin 2t + 2\sin t)$ **7.** $-\dfrac{2}{7}e^{-4t} + \dfrac{2}{7}\cosh\sqrt{2}t - \dfrac{\sqrt{2}}{14}\sinh\sqrt{2}t$

8. $\dfrac{1}{5}(\cosh 3t - \cosh 2t)$ **9.** $\dfrac{2}{s^2} - e^{-s}\left(\dfrac{1}{s}+\dfrac{1}{s^2}\right)$ **10.** $e^{-s}\left(\dfrac{1}{s}-\dfrac{2}{s^3}\right)+\dfrac{2}{s^3}$

11. $\dfrac{1-e^{-as}}{s^2(1+e^{-as})}$ **12.** $\dfrac{1}{s(1+e^{-as})}$ **13.** $\dfrac{e^{-as}}{s}$ **14.** $\dfrac{(1-e^{-s})e^{-as}}{s}$ **15.** $2e^{2t}-e^t$

16. $-1+e^{-t}-e^{-t/2}+e^{t/2}$ **17.** $e^{-5(t-3)}h(t-3)$ **18.** $[-e^{-2(t-2)}+e^{-(t-2)}]h(t-2)$

19. $\dfrac{1}{3}e^{-t}+\dfrac{1}{3}e^{t/2}\left(-\cos\dfrac{\sqrt{3}t}{2}+\sqrt{3}\sin\dfrac{\sqrt{3}t}{2}\right)$ **20.** $\dfrac{1}{3}e^{-2t/3}\left(5\cos\dfrac{2\sqrt{5}t}{3}-\dfrac{8\sqrt{5}}{5}\sin\dfrac{2\sqrt{5}t}{3}\right)$

21. $[1-\cos(t-1)]h(t-1)-[1-\cos(t-2)]h(t-2)$ **22.** $e^{-t}\left(\dfrac{t^3}{6}-\dfrac{t^4}{24}\right)$

23. $\dfrac{1}{3}e^{-t}(\sin t+\sin 2t)$ **24.** $\dfrac{1}{4}(\sinh 2t + 2t\cosh 2t)$

25. $y(t) = \dfrac{1}{2}e^t+\dfrac{1}{2}e^{-t}\left(\cosh\sqrt{2}t+\dfrac{4}{\sqrt{2}}\sinh\sqrt{2}t\right)$ **26.** $y(t) = e^{-t}-\cos t+\sin t$

27. $y(t) = -2+t+2e^{-t}+2te^{-t}$ **28.** $y(t) = \dfrac{t^5}{60}e^t+e^t-te^t-\dfrac{t^2}{2}e^t$

29. $y(t) = \dfrac{1}{5}\cos 2t+\dfrac{4}{5}\cos 3t+\dfrac{4}{5}\sin 3t$ **30.** $y(t) = \left(\dfrac{t^5}{60}+C_1+C_2t+C_4t^2\right)e^t$

31. $y(t) = \dfrac{1}{a}\displaystyle\int_0^t f(u)\sinh a(t-u)\,du + A\cosh at + \dfrac{B}{a}\sinh at$

34. (b) $\mathcal{L}\{f'(t)\} = s\tilde{f}(s) - f(0+)$

Exercises 11.2

3. (a) $U(x,t) = U_0\left[1 - \operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)\right] + \dfrac{x}{2\sqrt{k\pi}}\displaystyle\int_0^t f_1(t-u)u^{-3/2}e^{-x^2/(4ku)}\,du$

(b) $U_0 + (\overline{U} - U_0)\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)$

4. (a) $U(x,t) = U_0 + \dfrac{\sqrt{k}}{\kappa}\displaystyle\int_0^t f_1(t-u)\dfrac{1}{\sqrt{\pi u}}e^{-x^2/(4ku)}\,du$

(b) $U_0 + \dfrac{Q_0}{\kappa}\left[2\sqrt{\dfrac{kt}{\pi}}e^{-x^2/(4kt)} - x\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)\right]$

5. (a) $U(x,t) = \dfrac{\mu}{\kappa}\displaystyle\int_0^t f_1(t-u)\left[\sqrt{\dfrac{k}{\pi u}}e^{-x^2/(4ku)} - \dfrac{\mu k}{\kappa}e^{\mu x/\kappa + \mu^2 ku/\kappa^2}\operatorname{erfc}\left(\dfrac{x}{2\sqrt{ku}} + \dfrac{\mu\sqrt{ku}}{\kappa}\right)\right]du$

(b) $U(x,t) = U_m\left[\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right) - e^{\mu x/\kappa + k\mu^2 t/\kappa^2}\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}} + \dfrac{\mu\sqrt{kt}}{\kappa}\right)\right]$

6. (c) $U(x,t) = \dfrac{U_0}{2}\left[1 + \operatorname{erf}\left(\dfrac{x}{2\sqrt{kt}}\right)\right]$

8. $y(x,t) = f_1(t - x/c)h(t - x/c)$

9. $y(x,t) = \dfrac{c}{\tau}F_1(t - x/c)h(t - x/c)$, where $F_1(t) = \displaystyle\int_0^t f_1(u)\,du$

Exercises 11.3

1. $\dfrac{t}{2}(t+2)e^t$ **2.** $\dfrac{t}{4}\sin 2t$ **3.** $\dfrac{3t-1}{9} + \dfrac{1}{9}e^{-3t}$

4. $-\dfrac{1}{256}(1+12t)e^{-t} + \dfrac{1}{256}(1+8t+88t^2)e^{3t}$ **5.** $-\dfrac{1}{3}\sin t + \dfrac{2}{3}\sin 2t$ **6.** $\cosh t$

7. $\dfrac{t^2}{8}\cosh 2t + \dfrac{3t}{16}\sinh 2t$ **8.** $\dfrac{e^t}{2}(\sin t - t\cos t)$ **9.** $\dfrac{t}{2}e^t\sin t$ **10.** $(t+1)e^t\sin t$

11. $\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}e^{-n^2\pi^2 t}\sin n\pi x$ **12.** $\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}\sin n\pi t\sin n\pi x\sin n\pi u$

13. $\dfrac{8}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3}\cos(2n-1)\pi t\sin(2n-1)\pi x$

14. $t^2 + \left(\dfrac{x^2}{2} - \dfrac{1}{6}\right) + \dfrac{2}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^2}\cos n\pi t\cos n\pi x$

15. $\dfrac{1}{2\pi}\sin\dfrac{\pi t}{2}\sin\dfrac{\pi x}{2} + \dfrac{2}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{4n^2-1}\sin n\pi t\sin n\pi x$

16. $\dfrac{1}{2\pi^2}\left[-2\pi t\cos\pi t\sin\pi x + (\sin\pi x - 2\pi x\cos\pi x)\sin\pi t\right] + \dfrac{2}{\pi^2}\displaystyle\sum_{n=2}^{\infty}\dfrac{(-1)^n}{n^2-1}\sin n\pi t\sin n\pi x$

17. $\dfrac{e^t}{2\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{(1-y^2)\cos yt + 2y\sin yt}{(1+y^2)^2}\,dt + \dfrac{ie^t}{2\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{(-2y\cos yt + (1-y^2)\sin yt}{(1+y^2)^2}\,dt$

Exercises 11.4

1. $U(x,t) = e^{-m^2\pi^2 kt/L^2} \sin \frac{m\pi x}{L}$ **2.** $U(x,t) = U_0$

3. $U(x,t) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-(2n-1)^2\pi^2 kt/L^2} \sin \frac{(2n-1)\pi x}{L}$

4. (a) $U(x,t) = \frac{kL^2}{\kappa(m^2\pi^2 k - \alpha L^2)} \left(e^{-\alpha t} - e^{-m^2\pi^2 kt/L^2}\right) \sin \frac{m\pi x}{L}$

(b) $U(x,t) = \frac{kt}{\kappa} e^{-m^2\pi^2 kt/L^2} \sin \frac{m\pi x}{L}$

5. $U(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(u) \sin \frac{n\pi u}{L} du\right] e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}$

6. $U(x,t) = \frac{1}{L} \int_0^L f(u)\, du + \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L f(u) \cos \frac{n\pi u}{L}\, du\right] e^{-n^2\pi^2 kt/L^2} \cos \frac{n\pi x}{L}$

7. $U(x,t) = U_L \left[\frac{x}{L} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}\right]$

$U(x,t) = U_L \sum_{n=0}^{\infty} \left\{\text{erf}\left[\frac{(2n+1)L+x}{2\sqrt{kt}}\right] - \text{erf}\left[\frac{(2n+1)L-x}{2\sqrt{kt}}\right]\right\}$

8. $U(x,t) = U_0 + \frac{Q(3x^2 - L^2)}{6L\kappa} + \frac{Qkt}{L\kappa} + \frac{2LQ}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} e^{-n^2\pi^2 kt/L^2} \cos \frac{n\pi x}{L}$

10. $U(x,t) = \frac{Qx}{\kappa} + \frac{8QL}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} e^{-(2n-1)^2\pi^2 kt/(4L^2)} \sin \frac{(2n-1)\pi x}{2L}$

$$U(x,t) = \frac{\sqrt{k}Q}{\kappa} \sum_{n=0}^{\infty} (-1)^n \left\{4\sqrt{\frac{t}{\pi}}\, e^{-[(2n+1)^2L^2 + x^2]/(4kt)} \sinh \frac{(2n+1)Lx}{2kt}\right.$$
$$+ \frac{(2n+1)L}{\sqrt{k}} \left[\text{erf}\left(\frac{(2n+1)L-x}{2\sqrt{kt}}\right) - \text{erf}\left(\frac{(2n+1)L+x}{2\sqrt{kt}}\right)\right]$$
$$\left. - \frac{x}{\sqrt{k}} \left[\text{erf}\left(\frac{(2n+1)L-x}{2\sqrt{kt}}\right) + \text{erf}\left(\frac{(2n+1)L+x}{2\sqrt{kt}}\right)\right] + \frac{2x}{\sqrt{k}}\right\}$$

13. $U(x,t) = U_L x/L$

14. $U(x,t) = U_0 + \frac{(U_L - U_0)x}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[U_0 + (-1)^{n+1}(U_L - aL)]}{n} e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}$

15. $U(x,t) = 100e^{-t} \frac{\sin(x/\sqrt{k})}{\sin(L/\sqrt{k})} + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 + \frac{(-1)^{n+1}L^2}{L^2 - n^2\pi^2 k}\right] e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}$

16. (a) $$U(x,t) = \frac{ke^{-\alpha t}}{\kappa\alpha} \left[-1 + \frac{\cos\sqrt{\frac{\alpha}{k}}\left(\frac{L}{2} - x\right)}{\cos\sqrt{\frac{\alpha}{k}}\frac{L}{2}}\right]$$
$$+ \frac{4kL^2}{\kappa\pi} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2\pi^2 kt/L^2}}{(2n-1)[\alpha L^2 - (2n-1)^2\pi^2 k]} \sin \frac{(2n-1)\pi x}{L}$$

(b) $$U(x,t) = \frac{1}{\kappa} e^{-m^2\pi^2 kt/L^2} \left[-\frac{L^2}{m^2\pi^2} + \frac{L}{m^2\pi^2}(L - 2x) \cos \frac{m\pi x}{L}\right.$$
$$\left. + \left(\frac{4kt}{m\pi} + \frac{3L^2}{m^3\pi^3}\right) \sin \frac{m\pi x}{L}\right] + \frac{4L^2}{\kappa\pi^3} \sum_{\substack{n=1 \\ 2n-1 \neq m}}^{\infty} \frac{e^{-(2n-1)^2\pi^2 kt/L^2}}{(2n-1)[m^2 - (2n-1)^2]} \sin \frac{(2n-1)\pi x}{L}$$

17. $y(x,t) = \dfrac{8kL^2}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \cos \dfrac{(2n-1)\pi ct}{L} \sin \dfrac{(2n-1)\pi x}{L}$

18. $y(x,t) = \dfrac{8L^3}{\pi^4 c} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^4} \sin \dfrac{(2n-1)\pi ct}{L} \sin \dfrac{(2n-1)\pi x}{L}$

19. $y(x,t) = \dfrac{2}{c\pi} \sum_{n=1}^{\infty} \left[\dfrac{1}{n} \int_0^L g(u) \sin \dfrac{n\pi u}{L} du \right] \sin \dfrac{n\pi x}{L} \sin \dfrac{n\pi ct}{L}$

20. $y(x,t) = \dfrac{gx(2L-x)}{2c^2} - \dfrac{16L^2 g}{\pi^3 c^2} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \cos \dfrac{(2n-1)\pi ct}{2L} \sin \dfrac{(2n-1)\pi x}{2L}$

21. $y(x,t) = \dfrac{2F_0}{\rho\omega^2} \left[\dfrac{\sin \dfrac{\omega(L-x)}{2c} \sin \dfrac{\omega x}{2c}}{\cos \dfrac{\omega L}{2c}} \right] \sin \omega t$

$$+ \frac{4F_0 \omega L^3}{\rho c \pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2[\omega^2 L^2 - (2n-1)^2 \pi^2 c^2]} \sin \frac{(2n-1)\pi ct}{L} \sin \frac{(2n-1)\pi x}{L}$$

22. $\dfrac{2}{L} \sum_{n=1}^{\infty} \left\{ \int_0^L \left[f(u) \cos \sqrt{\dfrac{n^2\pi^2c^2}{L^2} + \dfrac{k}{\rho}}\, t + \dfrac{g(u)}{\sqrt{\frac{n^2\pi^2c^2}{L^2} + \frac{k}{\rho}}} \sin \sqrt{\dfrac{n^2\pi^2c^2}{L^2} + \dfrac{k}{\rho}}\, t \right] * \right.$

$$\left. \sin \frac{n\pi u}{L}\, du \right\} \sin \frac{n\pi x}{L}$$

29. $y(x,t) = \dfrac{gx(x-L)}{2c^2} + \dfrac{4gL^2}{\pi^3 c^2} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \cos \dfrac{(2n-1)\pi ct}{L} \sin \dfrac{(2n-1)\pi x}{L}$

$$+ \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(u) \sin \frac{n\pi u}{L}\, du \right] \cos \frac{n\pi ct}{L} \sin \frac{n\pi x}{L}$$

30. (a) $y(x,t) = A \dfrac{\sin(\omega x/c) \sin \omega t}{\sin(\omega L/c)} + 2A\omega Lc \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2\pi^2c^2 - \omega^2 L^2} \sin \dfrac{n\pi ct}{L} \sin \dfrac{n\pi x}{L}$

(b) $y(x,t) = \dfrac{A(-1)^m}{2m\pi L} \left(2m\pi ct \sin \dfrac{m\pi x}{L} \cos \dfrac{m\pi ct}{L} - L \sin \dfrac{m\pi x}{L} \sin \dfrac{m\pi ct}{L} \right.$

$$\left. + 2m\pi x \cos \frac{m\pi x}{L} \sin \frac{m\pi ct}{L} \right) + \frac{2Am}{\pi} \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{(-1)^n}{n^2 - m^2} \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L}$$

32. (a) $\dfrac{4cF_0}{AE\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{2n-1} \sin \dfrac{(2n-1)\pi ct}{2L} \sin \dfrac{(2n-1)\pi x}{2L}$

33. $y(x,t) = \psi(x) - \dfrac{1}{2}[\psi(x+ct) + \psi(x-ct)] + \dfrac{t^2}{2} \left(\dfrac{c^2 F_0}{\tau L} - g \right)$, where $\psi(x) = \dfrac{F_0 x^2}{2\tau L}$

34. $y(x,t) = \dfrac{2c}{L} \sum_{n=1}^{\infty} C_n(t) \sin \dfrac{n\pi x}{L}$, where

$$C_n(t) = \int_0^t \left[f_1(u) + (-1)^{n+1} f_2(u) \right] \sin \frac{n\pi c(t-u)}{L} du$$

36. $y(x,t) = \sum_{n=1}^{\infty} \left[\dfrac{2}{L} \int_0^L f(u) \sin \dfrac{n\pi u}{L}\, du \right] \cos \dfrac{n^2\pi^2 ct}{L^2} \sin \dfrac{n\pi x}{L}$

37. (b) $-\dfrac{g}{2c^2} \left(x^2 - 2Lx - \dfrac{2AEL}{k} \right)$

(d) $y(x,t) = -\dfrac{g}{2c^2}\left(x^2 - 2Lx - \dfrac{2AEL}{k}\right) - \dfrac{kg}{AEc^2}\sum_{n=1}^{\infty}\dfrac{\cos c\lambda_n t\cos\lambda_n(L-x)}{N^2\lambda_n^4\cos\lambda_n L}$, where

$$2N^2 = L\left[1 + \left(\frac{k}{AE\lambda_n}\right)^2\right] + \frac{k}{AE\lambda_n^2}$$

39. When $v > 2Lg/c$, $t = 2L/c$; when $v < 2Lg/c$, $t = 4L/c - v/g$. 9.6×10^{-5} s

Exercises 11.5

1. $U(r,t) = \dfrac{2U_0}{a}\sum_{n=1}^{\infty}\dfrac{e^{-k\lambda_n^2 t}}{\lambda_n}e^{-k\lambda_n^2 t}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$ **2.** $U(r,t) = \dfrac{8}{a}\sum_{n=1}^{\infty}\dfrac{1}{\lambda_n^3}e^{-k\lambda_n^2 t}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

5. (a) $U(r,t) = \dfrac{2k}{a}\sum_{n=1}^{\infty}A_n\lambda_n e^{-k\lambda_n^2 t}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$, where $A_n = \int_0^t f(u)e^{k\lambda_n^2 u}\,du$

(b) $U(r,t) = \overline{U} - \dfrac{2\overline{U}}{a}\sum_{n=1}^{\infty}\dfrac{1}{\lambda_n}e^{-k\lambda_n^2 t}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

6. $U(r,t) = \dfrac{Q}{4\kappa a}(2r^2 - a^2 + 8kt) - \dfrac{2Q}{\kappa a}\sum_{n=1}^{\infty}\dfrac{e^{-k\lambda_n^2 t}}{\lambda_n^2}\dfrac{J_0(\lambda_n r)}{J_0(\lambda_n a)}$

7. (f) $U(r,t) = \dfrac{8U_0}{a\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{1}{(2m-1)\lambda_n}e^{-k[\lambda_n^2 + (2m-1)^2\pi^2/L^2]t}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}\sin\dfrac{(2m-1)\pi z}{L}$

8. $z(r,t) = \dfrac{8}{a}\sum_{n=1}^{\infty}\dfrac{1}{\lambda_n^3}\cos c\lambda_n t\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

9. $z(r,t) = \dfrac{2v_0}{ca}\sum_{n=1}^{\infty}\dfrac{1}{\lambda_n^2}\sin c\lambda_n t\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

10. $z(r,t) = \dfrac{F_0}{\rho\omega^2}\left[\dfrac{J_0(\omega r/c)}{J_0(\omega a/c)} - 1\right] - \dfrac{2F_0\omega}{\rho ac}\sum_{n=1}^{\infty}\dfrac{\sin c\lambda_n t}{\lambda_n^2(c^2\lambda_n^2 - \omega^2)}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

Exercises 12.1

1. 3 **2.** $\sin 1$ **3.** $e^{-3} + 9$ **4.** 0 **5.** 100 **6.** $-39 - \cos 10$

7. $y(x) = \dfrac{1}{\tau}\begin{cases} x/2, & 0 \le x \le L/2 - \epsilon \\ \dfrac{1}{6\epsilon^2}(-x + L/2 - \epsilon)^3 + x/2, & L/2 - \epsilon \le x \le L/2 \\ \dfrac{1}{6\epsilon^2}(x - L/2 - \epsilon)^3 - x/2 + L/2, & L/2 \le x \le L/2 + \epsilon \\ -x/2 + L/2, & L/2 + \epsilon \le x \le L \end{cases}$

9. (a) $y(x) = \begin{cases} -gx/\tau, & 0 \le x \le L/3 \\ -gL/(3\tau), & L/3 \le x \le 2L/3 \\ g(x-L)/\tau, & 2L/3 \le x \le L \end{cases}$

(b) $y(x) = \dfrac{g}{\tau}[(x - L/3)h(x - L/3) + (x - 2L/3)h(x - 2L/3)] - \dfrac{gx}{\tau}$

10. (a) $y(x) = \dfrac{1}{EI}\begin{cases} \dfrac{x^3}{6} - \dfrac{Lx^2}{4}, & 0 \le x \le L/2 \\ -\dfrac{L^2 x}{8} + \dfrac{L^3}{48}, & L/2 \le x \le L. \end{cases}$

(b) $y(x) = \dfrac{1}{EI}\left[-\dfrac{1}{6}(x - L/2)^3 h(x - L/2) + \dfrac{x^3}{6} - \dfrac{Lx^2}{4}\right]$

11. (a) $y(x) = \dfrac{1}{EI}\left(\dfrac{x^3}{6} - \dfrac{Lx^2}{2}\right)$ (b) $y(x) = \dfrac{1}{EI}\left(\dfrac{x^3}{6} - \dfrac{Lx^2}{2}\right)$

12. (a),(b) $y(t) = \begin{cases} 0, & 0 \le t \le T \\ (1/\sqrt{kM}) \sin \sqrt{k/M}(t-T), & t \ge T. \end{cases}$

(c) $y(t) = \dfrac{1}{\sqrt{kM}} \sin \sqrt{k/M}(t-T)h(t-T)$

14. (a) $z(x,y) = \displaystyle\sum_{n=1}^{\infty} a_n(y)\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$, where

$$a_n(y) = \frac{L^2 F_n}{n^2\pi^2\tau}\begin{cases} \dfrac{1}{\cosh(n\pi/2)} \sinh \dfrac{n\pi\epsilon}{2L} \sinh \dfrac{n\pi y}{L}, & 0 \le y \le \dfrac{L-\epsilon}{2}, \\ \cosh \dfrac{n\pi(L-\epsilon)}{2L}\left[-\cosh \dfrac{n\pi y}{L} + \dfrac{\cosh n\pi - 1}{\sinh n\pi} \sinh \dfrac{n\pi y}{L}\right] + 1, & \dfrac{L-\epsilon}{2} \le y \le \dfrac{L+\epsilon}{2}, \\ \dfrac{1}{\cosh(n\pi/2)} \sinh \dfrac{n\pi\epsilon}{2L} \sinh \dfrac{n\pi(L-y)}{L}, & \dfrac{L+\epsilon}{2} \le y \le L, \end{cases}$$

(b) $$z(x,y) = -\frac{1}{\pi\tau}\begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n\cosh(n\pi/2)} \sinh \frac{n\pi y}{L} \sin \frac{n\pi x}{L}, & 0 \le y \le L/2 \\ \displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n\cosh(n\pi/2)} \sinh \frac{n\pi(L-y)}{L} \sin \frac{n\pi x}{L}, & L/2 \le y \le L \end{cases}$$

(d) No

15. (a) $z(r) = \begin{cases} \dfrac{r^2}{4\pi\epsilon^2\tau} - \dfrac{[1 + 2\ln(R/\epsilon)]}{4\pi\tau}, & 0 \le r \le \epsilon \\ \dfrac{\ln r}{2\pi\tau} - \dfrac{\ln R}{2\pi\tau}, & \epsilon \le r \le R \end{cases}$

(b) $z(r) = \dfrac{1}{2\pi\tau} \ln\left(\dfrac{r}{R}\right)$ (c) $z(r) = \dfrac{1}{2\pi\tau} \ln\left(\dfrac{r}{R}\right)$

Exercises 12.3

1. $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) + 3y = F(x)$ **2.** $\dfrac{d}{dx}\left(e^x \dfrac{dy}{dx}\right) - 2e^x y = e^x F(x)$

3. $\dfrac{d}{dx}\left(x^2 \dfrac{dy}{dx}\right) - (x+1)y = F(x)$ **4.** $\dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{dy}{dx}\right) - \dfrac{x+1}{x^3}y = \dfrac{F(x)}{x^3}$

5. $\dfrac{d}{dx}\left(e^{4x}\dfrac{dy}{dx}\right) = e^{4x}F(x)$ **6.** $g(x;X) = -X\,h(x-X) - x\,h(X-x)$

7. $g(x;X) = \dfrac{-1}{\cos L}[\sin x \cos(L-X)h(X-x) + \sin X \cos(L-x)h(x-X)]$

8. $g(x;X) = \dfrac{-1}{k \sin k\pi}[\sin kx \sin k(\pi - X)h(X-x) + \sin kX \sin k(\pi - x)h(x-X)]$

9. $g(x;X) = -1 - x\,h(X-x) - X\,h(x-X)$

10. $$g(x;X) = \frac{1}{5(1+4e^{5L})}\Big[\left(e^{-x} - e^{4x}\right)\left(4e^{5L-X} + e^{4X}\right)h(X-x) + \left(e^{-X} - e^{4X}\right)\left(4e^{5L-x} + 4e^{4x}\right)h(x-X)\Big]$$

11. $$g(x;X) = \frac{1}{4}e^{-(x+X)}[(2\cos 2x + \sin 2x)\sin 2X\,h(X-x)\ (2\cos 2X + \sin 2X)\sin 2x\,h(x-X)]$$

12. $g(x;X) = -\left(\dfrac{9X^5 + 86}{530X^3}\right)x^2 - \left(\dfrac{3X^5 + 64}{265X^3}\right)\dfrac{1}{x^3} - \dfrac{1}{5}\left[\dfrac{x^2}{X^3}h(X-x) + \dfrac{X^2}{x^3}h(x-X)\right]$

13. $g(x;X) = -\dfrac{1}{k \sin kL}[\sin kx \sin k(L-X)h(X-x) + \sin kX \sin k(L-x)h(x-X)]$, provided $k \neq n\pi/L$ for any positive integer n

14. $g(x;X) = -\dfrac{1}{k\cos kL}[\sin kx \cos k(L-X)h(X-x) + \sin kX \cos k(L-x)h(x-X)]$,
provided $k \neq (2n-1)\pi/(2L)$ for any positive n

15. $$g(x;X) = \frac{1}{2k[1-\cos k(\beta-\alpha)]}\{[\sin k(\beta-\alpha-X+x) + \sin k(X-x)]h(X-x) + [\sin k(\beta-\alpha-x+X) + \sin k(x-X)]h(x-X)\},$$
provided $k(\beta-\alpha) \neq 2n\pi$ for any positive n

16. $g(x;X) = \dfrac{\pi}{2[J_0(\alpha)Y_0(\beta) - J_0(\beta)Y_0(\alpha)]}[u(x)v(X)h(X-x) + u(X)v(x)h(x-X)]$

17. $g(x;X) = \dfrac{xX}{L + l_2/h_2} - x\,h(X-x) - X\,h(x-X)$

18. (a) $g(x;X) = \dfrac{x(X + l_1/h_1)}{L + l_1/h_1 + l_2/h_2} + \dfrac{(l_1/h_1)(X - L - l_2/h_2)}{L + l_1/h_1 + l_2/h_2} - xh(X-x) - Xh(x-X)$

(b) $$g(x;X) = \frac{1}{L + l_1/h_1 + l_2/h_2}[(x + l_1/h_1)(X - L - l_2/h_2)h(X-x) + (X + l_1/h_1)(x - L - l_2/h_2)h(x-X)]$$

20. $g(x;X) = \dfrac{1}{6EI}(x-X)^3h(x-X) - \dfrac{x^3}{6EI} + \dfrac{Xx^2}{2EI}$

21. $g(x;X) = \dfrac{1}{6EI}(x-X)^3h(x-X) - \dfrac{x^3(L-X)}{6EIL} + \dfrac{x(-3LX^2 + 2L^2X + X^3)}{6EIL}$

22. $$g(x;X) = \frac{1}{6EI}(x-X)^3h(x-X) + \frac{x^3}{6EIL^3}(-L^3 + 3LX^2 - 2X^3) + \frac{x^2}{2EIL^2}(X^3 - 2LX^2 + L^2X)$$

23. $$g(x;X) = \frac{1}{6EI}(x-X)^3h(x-X) + \frac{x^3}{12EIL^3}(3LX^2 - X^3 - 2L^3) + \frac{x^2}{4EIL^2}(X^3 - 3LX^2 + 2L^2X)$$

26. $g(x;X) = \dfrac{-2L}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}$

Exercises 12.4

1. $g(x;X) = \dfrac{1}{\kappa L}[x(L-X)h(X-x) + X(L-x)h(x-X)]$

2. $U(x) = \displaystyle\int_\alpha^\beta g(x;X)F(X)\,dX$, where

$$g(x;X) = \frac{1}{\displaystyle\int_\alpha^\beta \frac{1}{\kappa(\tau)}d\tau}\left\{\left[\int_X^\beta \frac{1}{\kappa(\tau)}d\tau\right]\left[\int_\alpha^x \frac{1}{\kappa(\tau)}d\tau\right]h(X-x) + \left[\int_\alpha^X \frac{1}{\kappa(\tau)}d\tau\right]\left[\int_x^\beta \frac{1}{\kappa(\tau)}d\tau\right]h(x-X)\right\}$$

3. $$g(x;X) = \frac{1}{\kappa_1 L_2 + \kappa_2 L_1}\begin{cases} X(L_1 + L_2 - x), & 0 \le X \le L_1 \le x \\ \dfrac{[\kappa_2 L_1 + \kappa_1(X - L_1)](L_1 + L_2 - x)}{\kappa_2}, & L_1 \le X \le x \\ \dfrac{(L_1 + L_2 - X)[\kappa_2 L_1 + \kappa_1(x - L_1)]}{\kappa_2}, & x \le X \le L_1 + L_2 \end{cases}$$

4. $y(x) = \frac{k}{2\tau}x(x-L)$ **5.** $y(x) = \frac{kx}{24\tau}(4x^2 - 3L^2)$

6. $y(x) = \begin{cases} -\frac{kLx}{4\tau}, & 0 \le x \le L/4 \\ \frac{k}{32\tau}(L^2 - 16Lx + 16x^2), & L/4 \le x \le 3L/4 \\ \frac{kL}{4\tau}(x-L), & 3L/4 \le x \le L \end{cases}$

7. $y(x) = \frac{-\bar{k}}{\tau}\begin{cases} x, & 0 < x \le L/4 \\ L/4, & L/4 \le x \le 3L/4 \\ L-x, & 3L/4 \le x < L. \end{cases}$

8. $y(x) = \frac{1}{4\tau}\begin{cases} 2kx(x-L) - 4\bar{k}x, & 0 < x \le L/4 \\ 2kx(x-L) - \bar{k}L, & L/4 \le x \le 3L/4 \\ 2kx(x-L) - 4\bar{k}(L-x), & 3L/4 \le x < L \end{cases}$

9. $y(x) = -\frac{k}{4\tau}(L-x)(2x-L)$ **10.** $L + \frac{gL^2}{2c^2}$ **11.** $L + \frac{gL^2}{2c^2} + \frac{mgL}{2AE}$

12. $L + \frac{gL^2}{2c^2} + \frac{MgL}{AE}$

13. $y(x) = \begin{cases} \frac{gx(2L-x)}{2c^2} + \frac{mg}{AE}\left(x + \frac{AE}{k}\right) + \frac{Mgx}{AE} + \frac{Mg}{k} + \frac{AELg}{kc^2}, & 0 \le x \le L/2 \\ \frac{gx(2L-x)}{2c^2} + \frac{mg}{AE}\left(\frac{L}{2} + \frac{AE}{k}\right) + \frac{Mgx}{AE} + \frac{Mg}{k} + \frac{AELg}{kc^2}, & L/2 \le x \le L \end{cases}$

14. $y(x) = \frac{1}{48EI}\begin{cases} 8x^3 - 12Lx^2, & 0 \le x \le L/2 \\ L^3 - 6L^2x, & L/2 \le x \le L, \end{cases}$ **15.** $y(x) = \frac{x^3}{6EI} - \frac{Lx^2}{2EI}$

16. $y(x) = \frac{wx(L-x)(x^2 - Lx - L^2)}{24EI}$

17. $y(x) = -\frac{wx^2(L-x)^2}{24EI} - \frac{k}{48EI}\begin{cases} 3Lx^2 - 4x^3, & 0 \le x \le L/2, \\ -(L-x)^2(L-4x), & L/2 \le x \le L \end{cases}$

18. $y(x) = \frac{w}{24EI}(4Lx^3 - 6L^2x^2 - x^4)$

$$+\frac{W}{24EI}\begin{cases} Lx^2(2x-3L), & 0 \le x \le L/4 \\ -x^4 + 3Lx^3 - 27L^2x^2/8 + L^3x/16 - L^4/256, & L/4 \le x \le 3L/4 \\ 5L^4/16 - 13L^3x/8, & 3L/4 \le x \le L. \end{cases}$$

19. $y(x) = -\frac{k(x-L/4)^3}{6EI} + \frac{3kx^2(13x-7L)}{256EI} + \frac{W(2L^4 - 16L^3x + 21L^2x^2 - 7Lx^3 - 32x^4)}{768EI}$

20. $y(x) = (x-2)\int_1^x F(X)\,dX + \int_x^2 (X-2)F(X)\,dX + m_1(x-2) + m_2$

$y(x) = (x-2)e^x + m_1(x-2) + m_2$

21. $y(x) = -\frac{\cos(1-x)}{\cos 1}\int_0^x F(X)\sin X\,dX - \frac{\sin x}{\cos 1}\int_x^1 F(X)\cos(1-X)\,dX$

$$+\frac{m_2\sin x + m_1\cos(1-x)}{\cos 1}$$

$y(x) = \frac{1}{2}x\sin x - \frac{\sin x}{2\cos 1}(\sin 1 + \cos 1) + \frac{m_2\sin x + m_1\cos(1-x)}{\cos 1}$

22. $y(x) = \frac{-\cos k(\beta - x)}{k\cos k(\beta-\alpha)}\int_\alpha^x \sin k(X-\alpha)F(X)\,dX$

$$-\frac{\sin k(x-\alpha)}{k\cos k(\beta-\alpha)}\int_x^\beta \cos k(\beta - X)F(X)\,dX + \frac{\sin k(x-\alpha)}{k\cos k(\beta-\alpha)}, \quad k(\beta-\alpha) \ne n\pi - \pi/2$$

$$y(x) = \frac{1}{k^2} + \frac{k \sin k(x-\alpha) - \cos k(\beta - x)}{k^2 \cos k(\beta - \alpha)}$$

23. $y(x) = \displaystyle\int_\alpha^\beta \frac{1}{2k[1-\cos k(\beta-\alpha)]}\{[\sin k(\beta-\alpha-X+x) + \sin k(X-x)]h(X-x)$
$\quad + [\sin k(\beta-\alpha-x+X) + \sin k(x-X)]h(x-X)\}F(X)\,dX$

$$y(x) = \frac{x}{k^2} + \frac{(\beta-\alpha)[\cos k(x-\alpha) - \cos k(\beta-x)]}{2k^2[1-\cos k(\beta-\alpha)]}$$

24. $y(x) = \dfrac{\ln(x+1) - \ln 2}{\ln 2}\displaystyle\int_0^x \ln(X+1)F(X)\,dX + \dfrac{\ln(x+1)}{\ln 2}\int_x^1 \ln\left(\frac{X+1}{2}\right)F(X)\,dX$

$$y(x) = \frac{x^2}{4} - \frac{x}{2} + \frac{\ln(x+1)}{4\ln 2}$$

25. $y(x) = \dfrac{e^{2x}}{2(e^{2\pi}-1)}\displaystyle\int_0^x \left[e^{2(\pi-X)}(\sin 2x \cos 2X - \sin 2x \sin 2X)\right.$
$\quad \left. + e^{-2X}\sin 2(X-x)\right]F(X)\,dX$
$\quad + \dfrac{e^{2(x+\pi)}}{2(e^{2\pi}-1)}\displaystyle\int_x^\pi \left[e^{-2X}(\cos 2x - \sin 2x)\sin 2X\right]F(X)\,dX$

$$y(x) = \frac{1}{4}e^{2x}(1-\cos 2x)$$

Exercises 12.5

1. $y(x) = D + \dfrac{1}{\kappa}\displaystyle\int_x^L (x-X)F(X)\,dX \qquad y(x) = E + \dfrac{L^2}{\kappa\pi^2}\cos\dfrac{\pi x}{L}$

3. (a) $y(x) = \left(D + \dfrac{x}{4}\right)\sin 2x \qquad$ (b) $y(x) = \left(D + \dfrac{x}{4}\right)\sin 2x + m_1 \cos 2x, \quad m_1 = m_2$

4. $y(x) = D\sin\dfrac{n\pi x}{L} + \dfrac{L}{n\pi}\displaystyle\int_x^L \sin\dfrac{n\pi(X-x)}{L}F(X)\,dX$

5. (a) $y(x) = D\sin\dfrac{3\pi x}{L} - m_2\cos\dfrac{3\pi x}{L} + \dfrac{L}{3\pi}\displaystyle\int_x^L F(X)\sin\dfrac{3\pi(X-x)}{L}\,dX$

(b) $y(x) = E\sin\dfrac{3\pi x}{L} + \left(-m_2 + \dfrac{L^3}{9\pi^2}\right)\cos\dfrac{3\pi x}{L} + \dfrac{L^2 x}{9\pi^2}, \quad m_1 + m_2 = \dfrac{L^3}{9\pi^2}$

6. $y(x) = D\sin\dfrac{(2n-1)\pi x}{2L} + \dfrac{2L}{(2n-1)\pi}\displaystyle\int_x^L F(X)\sin\dfrac{(2n-1)\pi(X-x)}{2L}dX$

7. (a) $y(x) = E\sin\dfrac{5\pi x}{2L} - \dfrac{2Lm_2}{5\pi}\cos\dfrac{5\pi x}{2L} + \dfrac{2L}{5\pi}\displaystyle\int_x^L F(X)\sin\dfrac{5\pi(X-x)}{2L}\,dX$

(b) $y(x) = F\sin\dfrac{5\pi x}{2L} + \left(\dfrac{16L^4}{125\pi^3} - \dfrac{2Lm_2}{5\pi}\right)\cos\dfrac{5\pi x}{2L} + \dfrac{4L^2x^2}{25\pi^2} - \dfrac{32L^4}{625\pi^4},$

$$\frac{5\pi m_1}{2L} + m_2 = \frac{8L^3}{25\pi^2} - \frac{16L^3}{125\pi^3}$$

8. (c) $U(x) = \dfrac{L^2}{k\pi^2}\cos\dfrac{\pi x}{L} + E \qquad$ **9.** (b) Yes

10. $\overline{g}_s(x;X) = \begin{cases} \dfrac{1}{2kL}(x^2+X^2) - \dfrac{X}{k} + \dfrac{L}{3k}, & 0 \le x \le X \\ \dfrac{1}{2kL}(x^2+X^2) - \dfrac{x}{k} + \dfrac{L}{3k}, & X < x \le L \end{cases}$

11. $\overline{g}_s(x;X) = \dfrac{1}{8\pi}\begin{cases} (4x\cos 2x - \sin 2x)\sin 2X + 4(X-\pi)\sin 2x \cos 2X, & 0 \le x \le X \\ (4X\cos 2X - \sin 2X)\sin 2x + 4(x-\pi)\sin 2X\cos 2x, & X < x \le \pi \end{cases}$

12. (b) Yes

13. $$\overline{g}_s(x;X) = \frac{X-x}{2\pi}\sin(x-X) + C\sin x\sin X + D\cos x\cos X + \begin{cases} -\dfrac{1}{2}\sin(x-X), & 0 \le x \le X \\ \dfrac{1}{2}\sin(x-X), & X < x \le 2\pi \end{cases}$$

Exercises 12.6

1. $$x(t) = \frac{1}{M}\int_0^t (t-T)F(T)\,dT + x_0 + v_0 t$$

2. $$x(t) = \frac{1}{M\omega}\int_0^t e^{-\beta(t-T)/(2M)}\sin\omega(t-T)\,F(T)\,dT + e^{-\beta t/(2M)}\left[x_0\cos\omega t + \left(\frac{v_0}{\omega} + \frac{\beta x_0}{2M\omega}\right)\sin\omega t\right]$$

Exercises 13.2

3. $$G(r,\theta;R,\Theta) = -\frac{1}{\pi a^2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_{0n}r)J_0(\lambda_{0n}R)}{\lambda_{0n}^2[J_1(\lambda_{0n}a)]^2} - \frac{2}{\pi a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_m(\lambda_{mn}r)J_m(\lambda_{mn}R)\cos m(\theta-\Theta)}{\lambda_{mn}^2[J_{m+1}(\lambda_{mn}a)]^2}$$

4. $$G(r,\theta;R,\Theta) = \frac{-4}{\pi a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_m(\lambda_{mn}r)J_m(\lambda_{mn}R)\sin m\theta\sin m\Theta}{\lambda_{mn}^2[J_{m+1}(\lambda_{mn}a)]^2}$$

5. $$G(r,\theta;R,\Theta) = \frac{-4}{a^2 L}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_\nu(\lambda_{mn}r)J_{\nu_m}(\lambda_{mn}R)\sin\dfrac{m\pi\theta}{L}\sin\dfrac{m\pi\Theta}{L}}{\lambda_{mn}^2[J_{\nu_m+1}(\lambda_{mn}a)]^2}$$

6. $$G(x,y,z;X,Y,Z) = \frac{-8}{LL'L''}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\dfrac{n\pi X}{L}\sin\dfrac{j\pi Y}{L'}\sin\dfrac{m\pi Z}{L''}\sin\dfrac{n\pi x}{L}\sin\dfrac{j\pi y}{L'}\sin\dfrac{m\pi z}{L''}}{\dfrac{n^2\pi^2}{L^2} + \dfrac{j^2\pi^2}{L'^2} + \dfrac{m^2\pi^2}{L''^2}}$$

7. $$G(r,\theta,z;R,\Theta,Z) = -\frac{2}{\pi a^2 L}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{J_0(\mu_{0n}r)J_0(\mu_{0n}R)}{(\mu_{0n}^2 + j^2\pi^2/L^2)[J_1(\mu_{0n}a)]^2}\sin\frac{j\pi z}{L}\sin\frac{j\pi Z}{L}$$
$$-\frac{4}{\pi a^2 L}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{J_m(\mu_{mn}r)J_m(\mu_{mn}R)}{(\mu_{mn}^2 + j^2\pi^2/L^2)[J_{m+1}(\mu_{mn}a)]^2}\sin\frac{j\pi z}{L}\sin\frac{j\pi Z}{L}\cos m(\theta-\Theta)$$

8. $$G(r,\theta,\phi;R,\Theta,\Phi) = -\frac{1}{2\pi a^2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{J_{m+1/2}(\lambda_{mn}r)J_{m+1/2}(\lambda_{mn}R)}{\lambda_{mn}^3 r[J_{m+3/2}(\lambda_{mn}a)]^2}(2m+1)* P_{0m}(\cos\phi)P_{0m}(\cos\Phi)$$
$$-\frac{1}{\pi a^2}\sum_{j=1}^{\infty}\sum_{m\ge j}^{\infty}\sum_{n=1}^{\infty}\frac{J_{m+1/2}(\lambda_{mn}r)J_{m+1/2}(\lambda_{mn}R)}{\lambda_{mn}^3 r[J_{m+3/2}(\lambda_{mn}a)]^2}\frac{(2m+1)(m-j)!}{(m+j)!}* P_{jm}(\cos\phi)P_{jm}(\cos\Phi)\cos j(\theta-\Theta)$$

9. $$G(r,\theta,\phi;R,\Theta,\Phi) = \frac{-1}{4\pi\sqrt{r^2+R^2-2Rr[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}} + \frac{a}{4\pi\sqrt{R^2r^2+a^4-2a^2Rr[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}} + \frac{1}{4\pi\sqrt{r^2+R^2-2Rr[-\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}}$$

$$-\frac{a}{4\pi\sqrt{R^2r^2+a^4-2a^2Rr[-\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}}$$

10. $G(r,\theta;R,\Theta)=\dfrac{1}{4\pi}\ln\left\{a^2\left[\dfrac{r^2+R^2-2rR\cos(\theta-\Theta)}{a^4+r^2R^2-2a^2rR\cos(\theta-\Theta)}\right]\right\}$

11. $G(r,\theta;R,\Theta)=\dfrac{1}{4\pi}\ln\dfrac{[a^4+r^2R^2-2a^2rR\cos(\theta+\Theta)][R^2+r^2-2rR\cos(\theta-\Theta)]}{[a^4+r^2R^2-2a^2rR\cos(\theta-\Theta)][R^2+r^2-2rR\cos(\theta+\Theta)]}$

12. $G(x,y;X,Y)=\displaystyle\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+m+1}}{4\pi}\ln[(x-x_n)^2+(y-y_m)^2]$

13. (d) $G(r,\theta;R,\Theta)=\dfrac{1}{2\pi}\begin{cases}\ln(R/a)+\displaystyle\sum_{n=1}^{\infty}\left\{\left[\left(\frac{rR}{a^2}\right)^n-\left(\frac{r}{R}\right)^n\right]\frac{\cos n(\theta-\Theta)}{n}\right\}, & 0\le r\le R\\ \ln(R/a)+\displaystyle\sum_{n=1}^{\infty}\left\{\left[\left(\frac{rR}{a^2}\right)^n-\left(\frac{R}{r}\right)^n\right]\frac{\cos n(\theta-\Theta)}{n}\right\}, & 0\le r\le R\end{cases}$

14. $G(r,\theta;R,\Theta)=\dfrac{1}{2\pi}\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{rR}{a^2}\right)^n-\left(\frac{r}{R}\right)^n\right][\cos n(\theta-\Theta)-\cos n(\theta+\Theta)], & 0\le r\le R\\ \displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{rR}{a^2}\right)^n-\left(\frac{R}{r}\right)^n\right][\cos n(\theta-\Theta)-\cos n(\theta+\Theta)], & R<r\le a\end{cases}$

15. $G(r,\theta;R,\Theta)=\dfrac{1}{4L}\ln\left[\dfrac{\left(a^4+r^2R^2-2a^2rR\cos\frac{\pi(\theta+\Theta)}{L}\right)\left(R^2+r^2-2rR\cos\frac{\pi(\theta-\Theta)}{L}\right)}{\left(a^4+r^2R^2-2a^2rR\cos\frac{\pi(\theta-\Theta)}{L}\right)\left(R^2+r^2-2rR\cos\frac{\pi(\theta+\Theta)}{L}\right)}\right]$

16. $G(x,y,z;X,Y,Z)=\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{-\lambda_{mn}\sinh\lambda_{mn}L''}[\sinh\lambda_{mn}z\sinh\lambda_{mn}(L''-Z)h(Z-z)$

$$-\sinh\lambda_{mn}Z\sinh\lambda_{mn}(L''-z)h(z-Z)]\frac{2}{\sqrt{LL'}}\sin\frac{n\pi X}{L}\sin\frac{m\pi Y}{L'},$$

where $\lambda_{mn}^2=\dfrac{n^2\pi^2}{L^2}+\dfrac{m^2\pi^2}{L'^2}$

18. $G(x,y;X,Y)=\dfrac{4}{LL'}\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\frac{n\pi X}{L}\sin\frac{m\pi Y}{L'}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}}{k^2-\frac{n^2\pi^2}{L^2}-\frac{m^2\pi^2}{L'^2}}$

19. $G(r,\theta;R,\Theta)=\dfrac{1}{\pi a^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{k^2-\lambda_{0n}^2}\frac{J_0(\lambda_{0n}R)J_0(\lambda_{0n}r)}{[J_1(\lambda_{0n}a)]^2}$

$$+\frac{2}{\pi a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^2-\lambda_{mn}^2}\frac{J_m(\lambda_{mn}R)J_m(\lambda_{mn}r)\cos m(\theta-\Theta)}{J_{m+1}(\lambda_{mn}a)]^2}$$

20. $G(r,\theta;R,\Theta)=\dfrac{4}{\pi a^2}\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_m(\lambda_{mn}R)J_m(\lambda_{mn}r)\sin m\Theta\sin m\theta}{(k^2-\lambda_{mn}^2)[J_{m+1}(\lambda_{mn}a)]^2}$

21. $G(r,\theta;R,\Theta)=\dfrac{4}{a^2L}\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^2-\lambda_{mn}^2}\frac{J_\nu(\lambda_{mn}R)J_\nu(\lambda_{mn}r)}{[J_{\nu+1}(\lambda_{mn}a)]^2}\sin\frac{m\pi\Theta}{L}\sin\frac{m\pi\theta}{L}$

22. $G(x,y,z;X,Y,Z)$

$$=\frac{8}{LL'L''}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\frac{n\pi X}{L}\sin\frac{j\pi Y}{L'}\sin\frac{m\pi Z}{L''}\sin\frac{n\pi x}{L}\sin\frac{j\pi y}{L'}\sin\frac{m\pi z}{L''}}{k^2-\frac{n^2\pi^2}{L^2}-\frac{j^2\pi^2}{L'^2}-\frac{m^2\pi^2}{L''^2}}$$

23. $G(r,\theta,z;R,\Theta,Z) = \dfrac{2}{\pi a^2 L}\displaystyle\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k^2-\mu_{0n}^2-j^2\pi^2/L^2}\frac{J_0(\mu_{0n}R)J_0(\mu_{0n}r)}{[J_1(\mu_{0n}a)]^2}\sin\frac{j\pi Z}{L}\sin\frac{j\pi z}{L}$

$$+\frac{4}{\pi a^2 L}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k^2-\mu_{mn}^2-j^2\pi^2/L^2}\frac{J_m(\mu_{mn}R)J_m(\mu_{mn}r)}{[J_{m+1}(\mu_{mn}a)]^2}*$$

$$\sin\frac{j\pi Z}{L}\sin\frac{j\pi z}{L}\cos m(\theta-\Theta)$$

24. $G(r,\theta,\phi;R,\Theta,\Phi) =$

$$\frac{1}{2\pi a^2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^2-\lambda_{mn}^2}\frac{J_{m+1/2}(\lambda_{mn}R)J_{m+1/2}(\lambda_{mn}r)(2m+1)P_{0m}(\cos\Phi)P_{0m}(\cos\phi)}{\lambda_{mn}r[J_{m+3/2}(\lambda_{mn}a)]^2}$$

$$+\frac{1}{\pi a^2}\sum_{j=1}^{\infty}\sum_{m\geq j}^{\infty}\sum_{n=1}^{\infty}\frac{J_{m+1/2}(\lambda_{mn}R)J_{m+1/2}(\lambda_{mn}r)(2m+1)(m-j)!}{(k^2-\lambda_{mn}^2)(m+j)!\,\lambda_{mn}r[J_{m+3/2}(\lambda_{mn}a)]^2}*$$

$$P_{jm}(\cos\Phi)P_{jm}(\cos\phi)\cos j(\theta-\Theta)$$

26. (d) $G(r;0) = \dfrac{1}{2\pi r} + D$

Exercises 13.3

3. $u(x,y) = \displaystyle\int_0^{L'}\int_0^{L} G(x,y;X,Y)F(X,Y)\,dX\,dY$

$$+\frac{2}{L'}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi y}{L'}\sinh\frac{n\pi x}{L'}}{\sinh(n\pi L/L')}\int_0^{L'} g(Y)\sin\frac{n\pi Y}{L'}dY$$

$$+\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}}{\sinh(n\pi L'/L)}\int_0^{L} f(X)\sin\frac{n\pi X}{L}dX$$

4. $u(r,\theta) = \displaystyle\int_0^{\pi}\int_0^{a} F(R,\Theta)G(r,\theta;R,\Theta)R\,dR\,d\Theta$

$$+\frac{ar(a^2-r^2)}{\pi}\int_0^{\pi} f(\Theta)\left\{\frac{\cos(\theta-\Theta)-\cos(\theta+\Theta)}{[a^2+r^2-2ar\cos(\theta+\Theta)][a^2+r^2-2ar\cos(\theta-\Theta)]}\right\}d\Theta$$

$$+\frac{r(a^2-r^2)\sin\theta}{\pi}\int_0^{a}[g_1(R)+g_2(R)]\left\{\frac{a^2-R^2}{[a^4+r^2R^2-2a^2rR\cos\theta][r^2+R^2-2rR\cos\theta]}\right\}dR$$

6. $u(x,y,z) = \displaystyle\iiint_V G(x,y,z;X,Y,Z)F(X,Y,Z)\,dV + \oiint_{\beta(V)} K(X,Y,Z)\frac{\partial G(x,y,z;X,Y,Z)}{\partial N}dS$

7. (d) $\overline{G}(x,y;X,Y) = \Big[\dfrac{\sqrt{2}y}{\pi\sqrt{L}}\cos\dfrac{\pi y}{L}\sin\dfrac{\pi X}{L}\sin\dfrac{\pi Y}{L} + A(X)\sin\pi y$

$$-\frac{\sqrt{2L}}{\pi}\sin\frac{\pi X}{L}\sin\frac{\pi(y-Y)}{L}h(y-Y)\Big]\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$$

$$-\sum_{n=2}^{\infty}\frac{\sqrt{2L}\sin(n\pi x/L)}{\pi\sqrt{n^2-2}\sinh\sqrt{n^2-2}\pi}\left[\sinh\frac{\sqrt{n^2-2}\pi y}{L}\sinh\frac{\sqrt{n^2-2}\pi(L-Y)}{L}h(Y-y)\right.$$

$$\left.+\sinh\frac{\sqrt{n^2-2}\pi Y}{L}\sinh\frac{\sqrt{n^2-2}\pi(L-y)}{L}h(y-Y)\right]\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

(e) $u(x,y) = \displaystyle\iint_A \overline{G}(X,Y;x,y)F(X,Y)\,dA + Cw(x,y)$

Exercises 13.4

2. (a) $V(x,y) = D - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{\cosh \dfrac{n\pi(L-x)}{L'}}{n\sinh(n\pi L/L')}\left[\int_0^{L'} f(Y)\cos\frac{n\pi y}{L'}\,dY\right]\cos\frac{n\pi y}{L'}$

(b) $V(x,y) = -\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{\cosh \dfrac{n\pi(L-x)}{L'}}{n\sinh(n\pi L/L')}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)\cos\frac{n\pi y}{L'} \qquad V(x, L'/2) = 0$

3. $V(x,y) = D - \dfrac{2y+L'}{2L}\displaystyle\int_0^L f(X)\,dX$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cosh\dfrac{n\pi y}{L}}{n\sinh(n\pi L'/L)}\int_0^L \cos\frac{n\pi X}{L} f(X)\,dX\right]\cos\frac{n\pi x}{L}$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cosh n\pi(L'-y)}{n\sinh(n\pi L'/L)}\left[\int_0^L \cos\frac{n\pi X}{L} g(X)\,dX\right]\cos\frac{n\pi x}{L}$$

4. $U(x,y) = E + \dfrac{L}{4\kappa}(x+y) - \dfrac{y^2}{2\kappa}$, or $U(x,y) = F + \dfrac{L}{4\kappa}(x+y) - \dfrac{x^2}{2\kappa}$, or better

$$U(x,y) = D + \frac{L}{4\kappa}(x+y) - \frac{1}{4\kappa}(x^2+y^2)$$

5. (e) $u(r,\theta) = C - \dfrac{a}{2\pi}\displaystyle\int_{-\pi}^{\pi} K(\Theta)\ln\left\{\frac{[r^2+a^2-2ra\cos(\theta-\Theta)]}{ar}\right\}d\Theta$

$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^a RF(R,\Theta)\ln[H(r,\theta;R,\Theta)]\,dR\,d\Theta, \text{ where}$$

$$H(r,\theta;R,\Theta) = \frac{[r^2+R^2-2rR\cos(\theta-\Theta)][a^4+r^2R^2-2ra^2R\cos(\theta-\Theta)]}{a^4r^2}$$

6. (b) $u(x,y) = \displaystyle\iint_A \overline{N}(x,y;X,Y)F(X,Y)\,dA - \oint_{\beta(A)} \overline{N}(x,y;X,Y)K(X,Y)\,ds + C$

7. $u(r,\theta) = C - \dfrac{a}{2\pi}\displaystyle\int_{-\pi}^{\pi} \ln\left[\frac{r^2+a^2-2ra\cos(\theta-\Theta)}{ar}\right]K(\Theta)\,d\Theta$

$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^a RF(R,\Theta)\ln[H(r,\theta;R,\Theta)]\,dR\,d\Theta$$

$$H(r,\theta;R,\Theta) = \frac{[r^2+R^2-2rR\cos(\theta-\Theta)][a^4+r^2R^2-2ra^2R\cos(\theta-\Theta)]}{a^4r^2}$$

8. (b) $u(x,y,z) = \displaystyle\iiint_V N(x,y,x;X,Y,Z)F(X,Y,Z)\,dV$

$$-\oiint_{\beta(V)} N(x,y,z;X,Y,Z)K(X,Y,Z)\,dS + C$$

9. (a) $\nabla^2\overline{N} = \delta(x-X, y-Y, z-Z), \quad (x,y,z)$ in V,

$$\frac{\partial\overline{N}}{\partial n} = \frac{1}{\text{area}\,\beta(V)}, \quad (x,y,z) \text{ on } \beta(V)$$

(b) $u(x,y,z) = \displaystyle\iiint_V \overline{N}(x,y,z;X,Y,Z)F(X,Y,Z)\,dV$

$$-\oiint_{\beta(V)} \overline{N}(x,y,z;X,Y,Z)K(X,Y,Z)\,dS + C$$

Exercises 13.5

3. $u(x,y) = \iint_A G(X,Y;x,y)F(X,Y)\,dA + \sum_{n=1}^{\infty} a_n \sinh\frac{(2n-1)\pi(L-x)}{2L}\cos\frac{(2n-1)\pi y}{2L},$

$$\text{where } a_n = \frac{2}{L\sinh[(2n-1)\pi/2]}\int_0^L f(y)\cos\frac{(2n-1)\pi y}{2L}dy$$

Exercises 13.6

2. $u(x,y) = \int_{-\infty}^{\infty}\int_0^{\infty} G(x,y;X,Y)F(X,Y)\,dY\,dX + \frac{1}{2\pi}\int_{-\infty}^{\infty} K(X)\ln[(x-X)^2+y^2]\,dX$

3. $u(r,\theta) = \iint_A G(r,\theta;R,\Theta)F(R,\Theta)\,dA - \frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{K(\Theta)}{a^2+r^2-2ar\cos(\theta-\Theta)}\,d\Theta$ (b) Yes

4. (a) $u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \frac{4xy}{\pi}\int_0^{\infty}\frac{X\,K(X)}{[(x-X)^2+y^2][(x+X)^2+y^2]}dX$

$$+\frac{4xy}{\pi}\int_0^{\infty}\frac{Y\,H(Y)}{[x^2+(y-Y)^2][x^2+(y+Y)^2]}dY$$ (b) Yes

5. (a) $u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \frac{1}{2\pi}\int_0^{\infty} K(X)\ln\left[\frac{(x-X)^2+y^2}{(x+X)^2+y^2}\right]dX$

$$+\frac{x}{\pi}\int_0^{\infty} H(Y)\left[\frac{1}{x^2+(y-Y)^2}+\frac{1}{x^2+(y+Y)^2}\right]dY$$

(b) The first term is simpler.

6. $u(r,\phi,\theta) = \iiint_V G(r,\phi,\theta;R,\Phi,\Theta)F(R,\Phi,\Theta)\,dV$

$$-\frac{a(a^2-r^2)}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi}\frac{K(\Phi,\Theta)\sin\Phi}{\{R^2+a^2-2aR[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]\}^{3/2}}d\Phi\,d\Theta$$

Yes

7. $u(x,y,z) = \iiint_V G(x,y,z;X,Y,Z)F(X,Y,Z)\,dV$

$$+\frac{1}{2\pi}\iint_A \frac{K(X,Y)}{[(x-X)^2+(y-Y)^2+z^2]^{3/2}}dA$$

8. $u(x,y,z) = \iiint_V G(x,y,z;X,Y,Z)F(X,Y,Z)\,dV - \frac{1}{2\pi}\iint_A \frac{K(X,Y)}{\sqrt{(x-X)^2+(y-Y)^2+z^2}}dA$

9. $u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA$

$$-\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{K(X)[y(4n^2L'^2-y^2)-y(x-X)^2]}{[(x-X)^2+(Y-y-2nL')^2][(x-X)^2+(y+Y-2nL')^2]}dX$$

$$+\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(X)[y(L'^2-y^2+4n^2L'^2-4nL'^2)-y(x-X)^2]}{[(x-X)^2+(Y-y-2nL')^2][(x-X)^2+(y+Y-2nL')^2]}dX$$

10. $u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA$

$$+\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} K(X)\ln[(x-X)^2+(y+2nL')^2][(x-X)^2+(y-2nL')^2]\,dX$$

$$-\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} H(X)\ln[(x-X)^2+(L'-y-2nL')^2][(x-X)^2+(L'+y-2nL')^2]\,dX$$

11. (a) $G(x,y;X,Y) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left\{\frac{[(x-X)^2+(y-Y-2nL')^2][(x+X)^2+(y+Y-2nL')^2]}{[(x-X)^2+(y+Y-2nL')^2][(x+X)^2+(y-Y-2nL')^2]}\right\}$

(b) $G(x,y;X,Y) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left\{\frac{[(x-X)^2+(y-Y-2nL')^2][(x+X)^2+(y-Y-2nL')^2]}{[(x-X)^2+(y+Y-2nL')^2][(x+X)^2+(y+Y-2nL')^2]}\right\}$

(c) $G(x,y;X,Y) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left\{\frac{[(x-X)^2+(y-Y-2nL')^2][(x-X)^2+(y+Y-2nL')^2]}{[(x+X)^2+(y-Y-2nL')^2][(x+X)^2+(y+Y-2nL')^2]}\right\}$

12. (a) $$G(x,y,z;X,Y,Z) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left[\frac{-1}{\sqrt{(x-X)^2+(y-Y)^2+(z-Z-2nL'')^2}} + \frac{1}{\sqrt{(x-X)^2+(y-Y)^2+(z+Z-2nL'')^2}}\right]$$

(b) $$G(x,y,z;X,Y,Z) = -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left[\frac{1}{\sqrt{(x-X)^2+(y-Y)^2+(z-Z-2nL'')^2}} + \frac{1}{\sqrt{(x-X)^2+(y-Y)^2+(z+Z-2nL'')^2}}\right]$$

Exercises 13.7

1. $G(x,t;X,T) = \frac{2k}{\kappa L}\sum_{n=1}^{\infty} e^{-n^2\pi^2 k(t-T)/L^2}\sin\frac{n\pi x}{L}\sin\frac{n\pi x}{L}$

2. $G(x,t;X,T) = \frac{k}{\kappa}\left[\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty} e^{-n^2\pi^2 k(t-T)/L^2}\cos\frac{n\pi X}{L}\cos\frac{n\pi x}{L}\right]$

3. $G(x,t;X,T) = \frac{2k}{\kappa L}\sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)}\cos\frac{(2n-1)\pi X}{L}\cos\frac{(2n-1)\pi x}{L}$

4. $G(x,t;X,T) = \frac{k}{\kappa}\sum_{n=1}^{\infty}\frac{1}{N^2}e^{-k\lambda_n^2(t-T)}\sin\lambda_n X\sin\lambda_n x$, where

$$2N^2 = L + \frac{h_2/l_2}{\lambda_n^2+(h_2/l_2)^2} \text{ and eigenvalues satisfy } \cot\lambda L = -\frac{h_2}{l_2\lambda}$$

5. (a) $U(x,t) = \frac{kL^2}{\kappa(m^2\pi^2k-\alpha L^2)}\left(e^{-\alpha t}-e^{-m^2\pi^2kt/L^2}\right)\sin\frac{m\pi X}{L}$

(b) $U(x,t) = \frac{kt}{\kappa}e^{-m^2\pi^2kt/L^2}\sin\frac{m\pi X}{L}$

6. $$U(x,t) = \frac{4kL^2}{\kappa\pi}\sum_{n=1}^{\infty}\frac{e^{-\alpha t}-e^{-(2n-1)^2\pi^2kt/L^2}}{(2n-1)[(2n-1)^2\pi^2k-\alpha L^2]}\sin\frac{(2n-1)\pi x}{L} + \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\pi^2kt/L^2}}{2n-1}\sin\frac{(2n-1)\pi x}{L}$$

7. $U(x,t) = \frac{U_L x}{L} + \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}$, where

$$C_n = \frac{2}{L}\int_0^L f(X)\sin\frac{n\pi x}{L}\,dX + \frac{2(-1)^n U_L}{n\pi}$$

8. $$U(x,t) = \frac{1}{L}\int_0^L f(X)\,dX + \frac{k}{\kappa L}\int_0^t\int_0^L g(X,T)\,dX\,dT + \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(X)\cos\frac{n\pi X}{L}dX\right]e^{-n^2\pi^2kt/L^2}\cos\frac{n\pi x}{L}$$

$$+\frac{2k}{\kappa L}\sum_{n=1}^{\infty}\left[\int_0^t\int_0^L e^{-n^2\pi^2k(t-T)/L^2}g(X,T)\cos\frac{n\pi X}{L}\,dX\right]\cos\frac{n\pi x}{L}$$

9. (a) $U(x,t) = U_0 + \frac{q(L-x)}{\kappa} - \frac{8qL}{\pi^2\kappa}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\pi^2kt/(4L^2)}}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2L}$

(b) $U(x,t) = U_0 + \frac{8qL}{\pi^2\kappa}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[e^{-(2n-1)^2\pi^2k(t-t_0)/(4L^2)} - e^{-(2n-1)^2\pi^2kt/(4L^2)}\right]*$

$$\cos\frac{(2n-1)\pi x}{2L}$$

(c) $U = U_0$

12. $U(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\int_0^{L'}\int_0^L f(X,Y)f_n(X)g_m(Y)\,dX\,dY\right]e^{-(n^2/L^2+m^2/L'^2)\pi^2kt}f_n(x)g_m(y)$

$$+\frac{2\sqrt{LL'}}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1-e^{-(n^2/L^2+m^2/L'^2)\pi^2kt}}{n^2L'^2+m^2L^2}\left\{\frac{L'^2n}{m}\left[U_1+U_2(-1)^{n+1}\right]\left[1+(-1)^{m+1}\right]\right.$$

$$\left.+\frac{L^2m}{n}\left[U_3+U_4(-1)^{m+1}\right]\left[1+(-1)^{n+1}\right]\right\}f_n(x)g_m(y)$$

where $f_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$ and $g_m(x) = \sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}$

13. $U(x,y,t) = \frac{2}{\pi^3L'}\sum_{n=1}^{\infty}\left\{\frac{\sqrt{L'}\pi^2n^2[U_1+U_2(-1)^{n+1}] - L^2\left(\frac{\phi_1}{\kappa_1}+\frac{\phi_2}{\kappa_2}\right)[1+(-1)^{n+1}]}{n^3}\right\}*$

$$(1-e^{-n^2\pi^2kt/L^2})\sin\frac{n\pi x}{L}$$

$$-\frac{8L^2L'}{\pi^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left[\frac{\phi_1}{\kappa_1}+\frac{\phi_2}{\kappa_2}(-1)^m\right]\left[1-e^{-[(2n-1)^2/L^2+m^2/L'^2]\pi^2kt}\right]}{(2n-1)[(2n-1)^2L'^2+m^2L^2]}*$$

$$\sin\frac{(2n-1)\pi x}{L}\cos\frac{m\pi y}{L'}$$

14. $U(r,t) = \frac{2k}{\kappa a}\sum_{n=0}^{\infty}\left[\int_0^t e^{-k\lambda_n^2(t-T)}f_1(T)\,dT\right]\frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}$

15. (a) $U(r,t) = \frac{k}{\kappa}\sum_{n=1}^{\infty}\left[\int_0^t\int_0^a Rg(R,T)f_n(R)e^{-k\lambda_n^2(t-T)}dR\,dT\right]f_n(r)$

$$+\sum_{n=1}^{\infty}\left[\int_0^a Rf(R)f_n(R)\,dR\right]e^{-k\lambda_n^2t}f_n(r)$$

(b) $U(r,t) = \frac{2g}{\kappa a}\sum_{n=1}^{\infty}\frac{1-e^{-k\lambda_n^2t}}{\lambda_n^3}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$

Exercises 13.8

1. $G(x,t;X,T) = \frac{1}{\rho}\left[\frac{t-T}{L} + \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi c(t-T)}{L}\cos\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\right]$

2. $G(x,t;X,T) = \frac{4}{\rho\pi c}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi c(t-T)}{2L}\cos\frac{(2n-1)\pi X}{2L}\cos\frac{(2n-1)\pi x}{2L}$

3. $G(x,t;X,T) = \frac{4}{\rho\pi c}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi c(t-T)}{2L}\sin\frac{(2n-1)\pi X}{2L}\sin\frac{(2n-1)\pi x}{2L}$

4. $y(x,t) = -\frac{2kL^2}{\rho\pi^3c^2}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1-\cos\frac{n\pi ct}{L}\right)[1+(-1)^{n+1}]\sin\frac{n\pi x}{L}$

$$+\frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi ct}{L}\left[\int_0^L g(X)\sin\frac{n\pi X}{L}\,dX\right]\sin\frac{n\pi x}{L}$$

$$+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\left[\int_0^L f(X)\sin\frac{n\pi X}{L}\,dX\right]\sin\frac{n\pi x}{L}$$

5. $y(x,t) = \frac{8LF}{\pi^2E}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\left(1-\cos\frac{(2n-1)\pi ct}{2L}\right)\sin\frac{(2n-1)\pi x}{2L}$

6. $y(x,t) = \frac{kL}{2}+\frac{c^2Ft^2}{2LE}+\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\frac{F}{E}\left(1-\cos\frac{n\pi ct}{L}\right)-k[1+(-1)^{n+1}]\cos\frac{n\pi ct}{L}\right\}\cos\frac{n\pi x}{L}$

7. (a) $y(x,t) = \frac{2}{\rho\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left[\int_0^t\int_0^L F(X,T)\sin\frac{n\pi c(t-T)}{L}\sin\frac{n\pi x}{L}\,dX\,dT\right]\sin\frac{n\pi x}{L}$

(b) $y(x,t) = \frac{2F_0L}{\rho\pi^2c^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\cos\frac{n\pi ct}{L}\right)\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}$

(c) $y(x,t) = \frac{2F_0L}{\rho\pi^2c^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\left(1-\cos\frac{(2n-1)\pi ct}{L}\right)\sin\frac{(2n-1)\pi x}{L}$

(d) $y(x,t) = \frac{2F_0L}{\rho\pi^2c^2}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{1}{n^2}\left(1-\cos\frac{n\pi ct}{L}\right)\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}$

9. $z(x,y,t) = \frac{16AL^2}{\rho\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos\omega t-\cos c\pi\sqrt{(2n-1)^2+(2m-1)^2}t}{(2n-1)(2m-1)\{c^2\pi^2[(2n-1)^2+(2m-1)^2]-\omega^2L^2\}}*$

$$\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2m-1)\pi y}{L}$$

10. $z(r,t) = \frac{2Ac}{a}\sum_{n=1}^{\infty}\frac{c\lambda_n\sin\omega t-\omega\sin c\lambda_n t}{c^2\lambda_n^2-\omega^2}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$ When $\omega = c\lambda_m$ for some m,

$$z(r,t) = \frac{A}{a\lambda_m}(-c\lambda_m t\cos c\lambda_m t+\sin c\lambda_m t)\frac{J_0(\lambda_m r)}{J_1(\lambda_m a)}$$

$$+\frac{2A}{a}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{-\lambda_m\sin c\lambda_n t+\lambda_n\sin c\lambda_m t}{\lambda_n^2-\lambda_m^2}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}$$

Exercises 14.2

1. $2(3^{n-1})$, no limit **2.** $3(-2/3)^n$, 0 **5.** $8/3+11(-1)^n/[3(2^{n-1})]$, 8/3
6. $5/2+(9/2)(-1)^n$, no limit **7.** $(1/\sqrt{3})[(1+\sqrt{3})^{n-1}-(1-\sqrt{3})^{n-1}]$, no limit
8. $(2n-7)/5](-5/3)^{n-1}$, no limit **9.** $(1/\sqrt{5})\{[(1+\sqrt{5})/2]^n-[(1-\sqrt{5})/2]^n\}$

Exercises 14.8

1. $U_{n,m,p+1} = \left(1-\frac{2ks}{h_1^2}-\frac{2ks}{h_2^2}\right)U_{n,m,p}+\frac{ks}{h_1^2}(U_{n+1,m,p}+U_{n-1,m,p})$

$+\dfrac{ks}{h_2^2}(U_{n,m+1,p}+U_{n,m-1,p})$ where $h_1 = L/N$ and $h_2 = L'/N'$.

$$\Delta t \le \frac{1}{2k\left[\dfrac{1}{(\Delta x)^2}+\dfrac{1}{(\Delta y)^2}\right]}$$

Exercises 14.10

1. $y_{n,m+1} = 2\left(1-\dfrac{c^2s^2}{r^2}\right)y_{n,m} - y_{n,m-1} + \dfrac{c^2s^2}{r^2}\left(y_{n+1,m}+y_{n-1,m}\right) - gs^2$

2. $y_{n,m+1} = \left(1+\dfrac{\beta s}{2\rho}\right)^{-1}\left[2\left(1-\dfrac{c^2s^2}{r^2}\right)y_{n,m} - \left(1-\dfrac{\beta s}{2\rho}\right)y_{n,m-1} + \dfrac{c^2s^2}{r^2}\left(y_{n+1,m}+y_{n-1,m}\right)\right]$

3. $y_{n,m+1} = 2\left(1-\dfrac{k}{2\rho}-\dfrac{c^2s^2}{r^2}\right)y_{n,m} - y_{n,m-1} + \dfrac{c^2s^2}{r^2}\left(y_{n+1,m}+y_{n-1,m}\right)$

Exercises 14.11

1. $V_{n,m} = \dfrac{1}{4}\left(V_{n+1,m}+V_{n-1,m}+V_{n,m+1}+V_{n,m-1}-r^2f_{n,m}\right)$

2. $V_{n,m} = \dfrac{r^2(V_{n,m+1}+V_{n,m-1})+s^2(V_{n+1,m}+V_{n-1,m})-r^2s^2f_{n,m}}{2(r^2+s^2)}$

Exercises 15.3

1. (a) $Y_N(x) = 2+x+\displaystyle\sum_{n=1}^{N} b_n x[x^n-(n+1)L^n]$

(b) $Y_N(x) = c_0+\displaystyle\sum_{n=1}^{N-1} b_n x^2[2x^n-(n+2)L^n]$

(c) $Y_N(x) = \dfrac{3}{L+1}(1+x)+\displaystyle\sum_{n=2}^{N+1} b_n[(L+1)x^n-L^n(1+x)]$

(d) $Y_N(x) = 1-\dfrac{hx}{1+hL}+\displaystyle\sum_{n=1}^{N} b_n x[(1+hL)x^n-L^n(n+1+hL)$

3. (c)(i) $U_1(x) = x+(3-\sqrt{11})(x^2-x)$ (ii) $U_1(x) = x-(1/3)(x^2-x)$

(iii) $U_1(x) = x+\left(\dfrac{15-7\sqrt{5}}{2}\right)(x^2-x)$

(d)(i) $U_2(x) = x-0.599208(x^2-x)+0.191623(x^3-x)$

(ii) $U_2(x) = x-0.740482(x^2-x)+0.244289(x^3-x)$

(iii) $U_2(x) = x-\frac{3}{4}(x^2-x)+\frac{1}{4}(x^3-x)$

(iv) $U_2(x) = x-0.643726(x^2-x)+0.207311(x^3-x)$

4. (a) $U_2(x) = x+0.407585x(1-x)-0.191623x^2(1-x)$

(b) $U_2(x) = x+0.496193x(1-x)-0.244289x^2(1-x)$

(c) $U_2(x) = x+\frac{1}{2}x(1-x)-\frac{1}{4}x^2(1-x)$

(d) $U_2(x) = x+0.440592x(1-x)-0.215178x^2(1-x)$

5. (a) $20+80e^{-0.03t}$

(b) $100-2.25974t+0.0233766t^2$, $100-2.29325t+0.0237232t^2$, $100-2.2245t+0.0216672t^2$

6. (a) $U(x) = \sin x-(3+\sin 1)x+1$

(c) $U_1(x) = 1-3x+0.233773x(1-x)$, $U_2(x) = 1-3x+0.161938x(1-x)+0.14367x^2(1-x)$

$U_3(x) = 1-3x+0.157998x(1-x)+0.16337x^2(1-x)-0.0197002x^2(1-x)$

(d) 1.08091, 0.955701, 0.955701
(e) −0.007848, −0.007848, −0.000032

7. (b) $Y_1(x) = -x(1-x)/6, \quad Y_2(x) = -0.10687x(1-x) - 0.0381679x(1-x^2)$
(c) $Y_1(x) = -x(1-x)/6, \quad Y_2(x) = -0.145038x(1-x) - 0.0381679x^2(1-x)$

8. (b)(i) $-(3/4)(r-1)(2-r), \quad -(7/9)(r-1)(2-r), \quad -(23/30)(r-1)(2-r)$

9. $R_1(r) = (252/311)(r-1)(2-r) + r$

10. (b)(i) $V_2(x) = \frac{1}{4}(9-x) + 2.209925(x-1)(x-3) - 0.35600(x-1)(x^2+x-11)$
(ii) $V_2(x) = \frac{1}{4}(9-x) + 2.54167(x-1)(x-3) - 0.425926(x-1)(x^2+x-11)$
(iii) $V_2(x) = \frac{1}{4}(9-x) + 2.55723(x-1)(x-3) - 0.429385(x-1)(x^2+x-11)$
(iv) $V_2(x) = \frac{1}{4}(9-x) + 2.13777(x-1)(x-3) - 0.347697(x-1)(x^2+x-11)$
(c) $V(x) = 2/x + (1/2)\ln x$

11. (a) $R = \sum_{n=0}^{N-1} c_n(x\phi_n'' + \phi_n') - \frac{2}{x^2}$
(e) $V_3(x) = 2 - 2.21937(x-1) + 1.64097x(x-1) - 0.342326x^2(x-1)$

12. (a) $Y_2(x) = \frac{x}{3} + 0.0515489x\left(x - \frac{4}{3}\right) - 0.0650936x\left(x^2 - \frac{5}{3}\right)$
(b) $Y_2(x) = \frac{x}{3} - 0.0727162x\left(x - \frac{4}{3}\right) - 0.0392097x(1-x)^2$
(f) $Y_2(x) = 0.0759494x + 0.256329x^2$

13. (b) $Y_2(x) = (1+\sin 1)x - 0.525022x(1-x) + 0.508138x(1-x^2)$
(c) $Y_2(x) = Z_2(x) + (1+\sin 1)x = (1+\sin 1)x - 0.0234727x(1-x) + 0.521078x^2(1-x)$

16. (b) $T_1(r) = \frac{Da^2(28+5ah)}{21(5+ah)}\left[\left(1 + \frac{2}{ah}\right) - \frac{r^2}{a^2}\right]$
(c) $T_3(r) = \frac{Da^2}{60}\left[\left(13 + \frac{32}{ah}\right) - \frac{10r^2}{a^2} - \frac{3r^4}{a^4}\right]$

17. (b) $Y_2(x) = 2 - x + 0.8068(x-1)(x-2) - 0.2120x(x-1)x - 2)$
(d) $Y_2(x) = 2 - x + 0.7858(x-1)(x-2) - 0.1992x(x-1)x - 2)$

21. (c) $Y_N(x) = \frac{4w}{\pi k}\sum_{n=1}^{N} \frac{1}{(2n-1)\left[1 + \frac{(2n-1)^4\pi^4 EI}{kL^4}\right]} \sin\frac{(2n-1)\pi x}{L}$

22. $3\epsilon c^3 + 4(1-\omega^2)c - 4 = 0$

Exercises 15.4

1. 5.161 **2.** 13.76, 13.49 **3.** (a) 5.239 (b) 5.294, 5.253
4. (a) $5.609/a^2$ (b) $6.790/a^2$ (c) $5.830/a^2$
5. (a) $4/a^2$, $6/a^2$, $6/a^2$ (b) $9/(2a^2)$, $6/a^2$, $5.858/a^2$
6. (b) $17.14/a^2$ (c) $14.73/a^2$, $64.73/a^2$ **7.** (b) 2.667 (c) 2.562, 21.17 **8.** 9.26, 9.12

Exercises 15.5

1. (a) $\frac{-20}{6L^2 + 5L'^2}$ (b) $\frac{\sinh[\sqrt{10}(L'-y)/L]}{\sinh(\sqrt{10}L'/L)}$ (c) $\frac{4L^2[1+(-1)^{n+1}]}{\pi^3 n^3 \sinh(n\pi L'/L)} \sinh\frac{n\pi(L'-y)}{L}$ Yes

2. (a) $\frac{A}{\sinh(\sqrt{10}L'/L)} \sinh\frac{\sqrt{10}(L'-y)}{L}$ where $A = \frac{30}{L^5}\int_0^L f(x)x(L-x)\,dx$

3. (a) $A\cosh\frac{\sqrt{10}y}{L} + B\sinh\frac{\sqrt{10}y}{L}$ where $A = \frac{30}{L^5}\int_0^L f(x)x(L-x)\,dx$ and

$$B = \frac{30}{L^5 \sinh(\sqrt{10}L'/L)}\left[\int_0^L g(x)x(L-x)\,dx - \cosh\frac{\sqrt{10}L'}{L}\int_0^L f(x)x(L-x)\,dx\right]$$

4. (a) No (b) $-0.2812kL^2$

5. (a) $c_{nm} = \dfrac{-4LL'}{\pi^2(n^2L'^2+m^2L^2)}\displaystyle\int_0^L\int_0^{L'} F(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}\,dy\,dx$

6. $\dfrac{2}{3L^2(1+\epsilon^2)}$ $\quad\dfrac{5}{8L^2(1+\epsilon^2)}$

7. $A\cosh\dfrac{\sqrt{10}y}{L} + B\sinh\dfrac{\sqrt{10}y}{L} - \dfrac{Ly}{4}$ where where $A = \dfrac{30}{L^5}\displaystyle\int_0^L g(x)x(L-x)\,dx$ and

$$B = \frac{30}{L^5}\left[\operatorname{csch}\frac{\sqrt{10}L'}{L}\int_0^L h(x)x(L-x)\,dx - \coth\frac{\sqrt{10}L'}{L}\int_0^L g(x)x(L-x)\,dx\right]$$

8. $V_1(x,y) = g(x)\left(1 + \dfrac{L-x-y}{L}\right) + c_{11}xy(L-x-y)$ where

$$c_{11} = \frac{30}{7L^6}\int_0^L g(x)(x-L)(x^2-8Lx+4L^2)\,dx - \frac{90}{7L^5}\int_0^L\int_0^{L-x} F(x,y)xy(L-x-y)\,dy\,dx$$

9. (a) $\dfrac{5}{8L^2}$ (b) $c = \dfrac{1295}{2216L^2}$, $d = \dfrac{525}{4432L^4}$ (c) $\left(1 - \operatorname{sech}\dfrac{\sqrt{5}}{\sqrt{2}}\cosh\dfrac{\sqrt{5}x}{\sqrt{2}L}\right)(L^2-y^2)$

(d) $\dfrac{128L^2}{\pi^4}\displaystyle\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(-1)^{n+m}}{(2n-1)(2m-1)[(2n-1)^2+(2m-1)^2]}\cos\frac{(2n-1)\pi x}{2L}\cos\frac{(2m-1)\pi y}{2L}$

(e) $1.1111L^4$, $1.1231L^4$, $1.1171L^4$

10. (a) $e^{-2\sqrt{2}y/L}x(L-x)$ (b) $e^{-2\sqrt{3}y/L}x(L-x)$ (c) $e^{-\sqrt{10}y/L}x(L-x)$

11. $V_2(x,y) = (0.803488e^{-3.14162y/L} + 0.196512e^{-10.1060y/L})x(L-x)$
$\quad + 0.910465(e^{-3.14162y/L} - e^{-10.1060y/L})x^2(L-x)^2$

12. (b) $\dfrac{L^2}{4}e^{-\pi y/L}\sin\dfrac{\pi x}{L}$ (c) $\dfrac{L^2\pi}{12}e^{-\pi y/L}\sin\dfrac{\pi x}{L}$

13. (a) $Be^{-\sqrt{10}y/L}x(L-x)$ where $B = \dfrac{30}{L^5}\displaystyle\int_0^L g(x)x(L-x)\,dx$

(b) $(Be^{-\sqrt{10}y/L} + De^{-2\sqrt{7}y/L})x(L-x) - \dfrac{1}{8L}(7Be^{-\sqrt{10}y/L} + 16De^{-2\sqrt{7}y/L})x^2(L-x)$

$$B = \frac{160}{3L^5}\int_0^L g(x)x(L-x)\,dx,\ D = \frac{420}{L^6}\left[\frac{4L}{3}\int_0^L g(x)x(L-x)\,dx - \int_0^L g(x)x^2(L-x)\,dx\right]$$

15. (b) $V_1(x,y) = g(L/2)e^{-\pi y/L}\sin\dfrac{\pi x}{L}$

$$V_2(x,y) = \frac{1}{\sqrt{3}}\left\{[g(L/3)+g(2L/3)]e^{-\pi y/L}\sin\frac{\pi x}{L} + [g(L/3)-g(2L/3)]e^{-2\pi y/L}\sin\frac{2\pi x}{L}\right\}$$

(c) $c_1 = \dfrac{\pi}{2L}\displaystyle\int_0^L g(x)\,dx$

$$c_1 = \frac{L}{6\pi}\left[2\int_0^{L/2} g(x)\,dx + \int_{L/2}^L g(x)\,dx\right],\ c_2 = \frac{L}{6\pi}\left[4\int_0^{L/2} g(x)\,dx - \int_{L/2}^L g(x)\,dx\right]$$

(d) $c_1 = \dfrac{\pi}{2L}\displaystyle\int_0^L g(x)\,dx;\quad c_1 = \dfrac{\pi}{2L}\int_0^L g(x)\,dx,\ c_2 = \dfrac{\pi}{L^2}\int_0^L (L-2x)g(x)\,dx$

Exercises 15.6

2. (a) $U_1(x,y) = -\dfrac{3}{32}x(L^2-y^2)$

3. (a) $0.181k_1 + 0.204k_2$ (b) $0.128k_1 + 0.256k_2$
(c) $U_2(x, y) = \dfrac{9}{332}(9k_1 + 5k_2)x(1 - y^2) + \dfrac{3}{1328}(-29k_1 + 30k_2)x^2(1 - y^2)$
(d) $0.178k_1 + 0.203k_2$ (e) $Q_1 = (39k_1 - 30k_2)/44$, $Q_2 = (473k_1 + 120k_2)/664$

4. $U_1(x, y) = cx(L' - y)$ where $c = \dfrac{9}{LL'(L^2 + L'^2)}\left[L\int_0^{L'} f(y)(L' - y)\,dy - L'\int_0^L x\,g(x)\,dx - \int_0^L\int_0^{L'} x(L' - y)F(x, y)\,dy\,dx\right]$

5. $U_4(x, y) = (L^2 - x^2)(L^2 - y^2)\left[\dfrac{0.276821k}{L^2} + \dfrac{0.041015k}{L^4}(x^2 + y^2) + \dfrac{0.0781539kxy}{L^4}\right]$

6. $d = 8L'/171$, $f = 8L/171$

8. $d = L'^2/18$, $f = L/24$, $g = 0$

Exercises 15.7

1. $U_1(x, y) = cy(L - x)(L' - y)$ where $c = \dfrac{90}{LL'^3(10L^2 + 3L'^2 + 9LL'^2)}\left[L\int_0^{L'} y(L' - y)f(y)\,dy - \int_0^L\int_0^{L'} y(L - x)(L' - y)F(x, y)\,dy\,dx\right]$

2. $U_1(x, y) = \dfrac{(9 - 2L^2)xy}{6(2 + 3L)}$

3. (a) $0.0489k$ (b) $U_1(x, y) = \dfrac{15k}{19}xy(1 - x)$, $0.0987k$
(c) $U_3(x, y) = -\dfrac{30k}{127}xy(1 - x) + \dfrac{150k}{127}xy^2(1 - x)$, $0.0443k$

Exercises 15.8

2. (b) $c_1(t) = \left[\dfrac{16\sqrt{2L}\epsilon}{3k\pi^3}(1 - e^{-k\pi^2 t/L^2}) - \dfrac{\pi}{2\sqrt{2L}}e^{-k\pi^2 t/L^2}\right]^{-1}$

3. $U_1(r, t) = \dfrac{5}{4a^2}e^{-5kt/a^2}(a^2 - r^2)$

4. (a) $U_1(x, t) = \dfrac{20}{27}e^{-20t/27}\left(1 + x - \dfrac{x^2}{2}\right)$
(b) $U_2(x, t) = (0.5861e^{-0.7402t} + 2.444e^{-11.77t})\left(1 + x - \dfrac{x^2}{2}\right) + (0.1445e^{-0.7402t} - 2.291e^{-11.77t})\left(1 + x - \dfrac{x^3}{3}\right)$

5. (a) $U_1(x, t) = De^{-\pi^2 t/L^2}\sin\dfrac{\pi x}{L}$ where $D = \dfrac{2}{L}\int_0^L f(x)\sin\dfrac{\pi x}{L}\,dx$
(b) $U_1(x, t) = De^{-10t/L^2}x(L - x)$ where $D = \dfrac{30}{L^5}\int_0^L f(x)x(L - x)\,dx$

Exercises 16.2

3. $T_1(r) = \dfrac{37}{60} - \dfrac{7r}{20}, \quad T_2(r) = \begin{cases} 0.5256\left(\dfrac{r-1/2}{-1/2}\right) + 0.4568\left(\dfrac{r}{1/2}\right), & 0 \le r \le 1/2 \\ 0.4568\left(\dfrac{r-1}{-1/2}\right) + 0.2667\left(\dfrac{r-1/2}{1/2}\right), & 1/2 < r \le 1 \end{cases}$

5. (a) $V_1(x) = -2(x-2) + \dfrac{1}{3}(1 + 4\ln 2)(x-1)$

$V_2(x) = \begin{cases} -4(x-3/2) + 2(1.551)(x-1), & 1 \le x \le 3/2 \\ -2(1.551)(x-2) + 2(1.365)(x-3/2), & 3/2 < x \le 2 \end{cases}$

(b) $V_1(x) = 2(2x-3)(x-2) - 4(1.541)(x-1)(x-2) + 1.350(2x-3)(x-1)$

6. $2(4x-1)(2x-1) - 2.95826(8)x(2x-1) + 3.74831(2)x(4x-1)$, $0 \le x \le 1/2$
$3.74831(2)(4x-3)(x-1) - 4.33622(8)(2x-1)(x-1) + 4.70138(2x-1)(4x-3)$, $1/2 < x \le 1$

7.
$$\begin{pmatrix} \int_{x_1}^{x_2} [\alpha(\phi_1')^2 - \beta(\phi_1)^2]\,dx & \int_{x_1}^{x_2} (\alpha\phi_1'\phi_2' - \beta\phi_1\phi_2)\,dx \\ \int_{x_1}^{x_2} (\alpha\phi_2'\phi_1' - \beta\phi_1\phi_2)\,dx & \int_{x_1}^{x_2} [\alpha(\phi_2')^2 - \beta(\phi_2)^2]\,dx \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= \begin{pmatrix} \left\{\alpha Y_1'\phi_1\right\}_{x_1}^{x_2} \\ \left\{\alpha Y_1'\phi_2\right\}_{x_1}^{x_2} \end{pmatrix} + \begin{pmatrix} -\int_{x_1}^{x_2} F\phi_1\,dx \\ -\int_{x_1}^{x_2} F\phi_2\,dx \end{pmatrix}$$

8.
$$\begin{pmatrix} \int_{x_1}^{x_2} [\alpha(\phi_1')^2 - \beta(\phi_1)^2]\,dx & \int_{x_1}^{x_2} (\alpha\phi_1'\phi_2' - \beta\phi_1\phi_2)\,dx & \int_{x_1}^{x_2} (\alpha\phi_1'\phi_3' - \beta\phi_1\phi_3)\,dx \\ \int_{x_1}^{x_2} (\alpha\phi_1'\phi_2' - \beta\phi_1\phi_2)\,dx & \int_{x_1}^{x_2} [\alpha(\phi_2')^2 - \beta(\phi_2)^2]\,dx & \int_{x_1}^{x_2} (\alpha\phi_2'\phi_3' - \beta\phi_2\phi_3)\,dx \\ \int_{x_1}^{x_2} (\alpha\phi_1'\phi_3' - \beta\phi_1\phi_3)\,dx & \int_{x_1}^{x_2} (\alpha\phi_2'\phi_3' - \beta\phi_2\phi_3)\,dx & \int_{x_1}^{x_2} [\alpha(\phi_3')^2 - \beta(\phi_3)^2]\,dx \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$
$$= \begin{pmatrix} \left\{\alpha Y_1'\phi_1\right\}_{x_1}^{x_2} \\ \left\{\alpha Y_1'\phi_2\right\}_{x_1}^{x_2} \\ \left\{\alpha Y_1'\phi_3\right\}_{x_1}^{x_2} \end{pmatrix} + \begin{pmatrix} -\int_{x_1}^{x_2} F\phi_1\,dx \\ -\int_{x_1}^{x_2} F\phi_2\,dx \\ -\int_{x_1}^{x_2} F\phi_3\,dx \end{pmatrix}$$

9. (a) No (b) $R_2(r) = \begin{cases} (3-2r) + 1.680(2r-2), & 1 \le r \le 3/2 \\ 1.680(4-2r) + 2(2r-3), & 3/2 < r \le 2 \end{cases}$

Exercises 16.3

2. (a) $(1-\xi)/2$, $(1+\xi)/2$
3. (a) $-(9/16)(\xi+1/3)(\xi-1/3)(\xi-1)$, $(27/16)(\xi+1)(\xi-1/3)(\xi-1)$,
$-(27/16)(\xi+1)(\xi+1/3)(\xi-1)$, $(9/16)(\xi+1)(\xi+1/3)(\xi-1/3)$

Exercises 16.4

1. 5.01548 **2.** 3.11 **3.** (a) 3 (b) 2.60 (c) 1.49 **4.** (a) 3 (b) 2.59 (c) 1.49 **5.** (a) 6 (b) 5.86 (c) 4.86

Exercises 16.7

2. $\phi_1(\xi,\eta) = \dfrac{1}{2}(1-\xi-\eta)(2-3\xi-3\eta)(1-3\xi-3\eta)$, $\phi_2(\xi,\eta) = \dfrac{1}{9}\xi(3\xi-1)(3\xi-2)$,

$\phi_3(\xi,\eta) = \dfrac{1}{9}\eta(3\eta-1)(3\eta-2)$, $\phi_4(\xi,\eta) = \dfrac{9}{2}\xi(2-3\xi-3\eta)(1-\xi-\eta)$,
$\phi_5(\xi,\eta) = \dfrac{9}{4}\xi(3\xi-1)(1-\xi-\eta)$, $\phi_6(\xi,\eta) = \dfrac{9}{2}\xi\eta(3\xi-1)$,,
$\phi_7(\xi,\eta) = \dfrac{9}{2}\xi\eta(3\eta-1)$, $\phi_8(\xi,\eta) = \dfrac{9}{2}\eta(1-\xi-\eta)(2-3\xi-3\eta)$,
$\phi_9(\xi,\eta) = \dfrac{9}{2}\eta(3\eta-1)(1-\xi-\eta)$, $\phi_{10}(\xi,\eta) = 27\xi\eta(1-\xi-\eta)$

Exercises 16.8

1. $\phi_1(\xi,\eta) = \dfrac{1}{32}(1-\xi)(1-\eta)(2+3\xi+3\eta)(4+3\xi+3\eta)$,
$\phi_2(\xi,\eta) = \dfrac{1}{32}(1+\xi)(1-\eta)(2-3\xi+3\eta)(4-3\xi+3\eta)$,
$\phi_3(\xi,\eta) = \dfrac{1}{32}(1+\xi)(1+\eta)(2-3\xi-3\eta)(4-3\xi-3\eta)$,
$\phi_4(\xi,\eta) = \dfrac{1}{32}(1-\xi)(1+\eta)(2+3\xi-3\eta)(4+3\xi-3\eta)$,
$\phi_5(\xi,\eta) = \dfrac{9}{32}(1-\xi^2)(1-3\xi)(1-\eta)$, $\phi_6(\xi,\eta) = \dfrac{9}{32}(1-\xi^2)(1+3\xi)(1-3\eta)$,
$\phi_7(\xi,\eta) = \dfrac{9}{32}(1+\xi)(1-\eta^2)(1-3\eta)$, $\phi_8(\xi,\eta) = \dfrac{9}{32}(1+\xi)(1-\eta^2)(1+3\eta)$,
$\phi_9(\xi,\eta) = \dfrac{9}{32}(1-\xi^2)(1+\eta)(1-3\xi)$, $\phi_{10}(\xi,\eta) = \dfrac{9}{32}(1-\xi^2)(1+\eta)(1+3\xi)$,
$\phi_{11}(\xi,\eta) = \dfrac{9}{32}(1-\xi)(1-\eta^2)(1+3\eta)$, $\phi_{12}(\xi,\eta) = \dfrac{9}{32}(1-\xi)(1-\eta^2)(1-3\eta)$

APPENDIX F Problem Index

In this appendix, we have listed all examples and exercises from sections 4.2, 4.3, 6.2, 6.3, 6.4, 7.2, 7.3, 9.1, 9.2, 10.2, 10.4, 10.5, 11.4, 11.6, 11.7, and 12.4 that pertain to analytic solutions of physical applications of second- and fourth-order PDEs so that readers might be able to find problems of particular interest. Heat conduction problems, vibrations of strings, bars, and plates, and potential problems are classified first according to geometry:

One-dimensional
Rectangles
Boxes
Circles (including annuli)
Sectors (including quarter-circles and semicircles)
Spheres (including hemispheres)
Cylinders

Secondly, examples and exercises are sorted according to

Heat conduction
Vibration
Potential

The final sort is according to boundary conditions:

Dirichlet
Newmann
Robin
Mixed

At the end of the appendix, several specialized equations are referenced:

Vibrations of beams
Diffusion of Neutrons
Telegraphy equation
Heat conduction in thin wires
Oscillations of hanging cables
Helmholtz equation

One-dimensional Heat Conduction Problems in Rods

Dirichlet Boundary Conditions

1. Exercise 4.2–2
2. Exercise 4.2–3
3. Exerc
4. Exercise 4.2–5
5. Exercise 4.2–14
6. Exerc
7. Exercise 4.2–16
8. Example 4.5
9. Exam
10. Problem 4.78
11. Exercise 4.3–1
12. Exerc
13. Exercise 4.3–3
14. Exercise 4.3–4
15. Exerc
16. Exercise 4.3–10
17. Exercise 4.3–12
18. Exerc
19. Exercise 4.3–14
20. Exercise 4.3–16
21. Probl
22. Exercise 7.2–1
23. Exercise 7.2–3
24. Exerc
25. Exercise 7.2–8
26. Exercise 7.2–9
27. Exerc
28. Exercise 7.2–11
29. Example '`ex of infinite heat`'
30. Exerc
31. Exercise '`heat65`'
32. Exercise '`heat as GaussiaN`'
33. Exerc
34. Exercise '`xge0 exercisE`'
35. Exercise 11.6–1
36. Exerc
37. Example 10.13
38. Example 10.15
39. Exerc

40. Exercise 10.4–3
41. Exercise 10.4–4
42. Exerc
43. Exercise 10.4–7
44. Exercise 10.4–9
45. Exerc
46. Exercise 10.4–14
47. Exercise 10.4–15
48. Exerc
49. Exercise 13.7–5
50. Exercise 13.7–6
51. Exerc

Neumann Boundary Conditions

1. Example 4.2
2. Exercise 4.2–1
3. Exercise 4.2–6
4. Exercise 4.2–17
5. Exercise 4.3–6
6. Exercise 4.3–7
7. Exercise 4.3–8
8. Exercise 7.2–4
9. Exercise 7.2–7
10. Exercise 7.2–16
11. Exercise '`heat81`'
12. Exercise '`heat82`'
13. Example 11.21
14. Exercise 11.6–2
15. Exercise 11.6–4
16. Exercise 10.4–2
17. Exercise 10.4–6
18. Exercise 10.4–8
19. Exercise 13.7–8

Robin Boundary Conditions

1. Exercise 4.3–9
2. Exercise 7.2–17
3. Exercise 11.6–5
4. Exercise 11.6–6

Mixed Boundary Conditions

1. Exercise 4.2–7
2. Exercise 4.2–8
3. Exercise 4.2–9
4. Exercise 4.2–10
5. Problem 6.2
6. Exercise 6.2–1
7. Exercise 6.2–2
8. Exercise 6.2–3
9. Example 7.3
10. Exercise 7.2–5
11. Exercise 7.2–12
12. Exercise 7.2–14
13. Exercise 7.2–15
14. Exercise 10.4–10
15. Example 13.12
16. Exercise 13.7–9

One-dimensional Problems for Transverse Vibrations of Strings and Longitudinal Vibrations of Bars

Dirichlet Boundary Conditions

1. Problem 2.115
2. Exercise 2.7–1
3. Exercise 2.7–2
4. Exercise 2.7–3
5. Exercise 2.7–4
6. Exercise 2.7–5
7. Problem 2.166
8. Problem 2.169
9. Example 2.11
10. Example 2.12
11. Exercise 2.9–1
12. Exercise 2.9–2
13. Exercise 2.9–3
14. Exercise 2.9–4
15. Exercise 2.9–5
16. Exercise 2.9–6
17. Exercise 2.9–7
18. Exercise 2.9–8
19. Problem 2.171
20. Problem 2.173
21. Exercise 2.10–1
22. Exercise 2.10–2
23. Exercise 2.10–3
24. Exercise 2.10–4
25. Exercise 2.10–5
26. Problem 2.176
27. Exercise 2.11–2
28. Exercise 2.11–3
29. Exercise 2.11–4
30. Exercise 2.11–5
31. Exercise 2.11–6
32. Exercise 2.11–7
33. Exercise 2.11–8
34. Exercise 2.11–9
35. Exercise 2.11–10
36. Exercise 2.11–11
37. Problem 4.9
38. Exercise 4.2–18
39. Exercise 4.2–19
40. Exercise 4.2–20
41. Exercise 4.2–21
42. Exercise 4.2–22
43. Problem 4.35
44. Problem 4.58
45. Exercise 4.3–17
46. Exercise 4.3–18
47. Exercise 4.3–19
48. Exercise 4.3–21
49. Exercise 4.3–22
50. Exercise 4.3–23
51. Exercise 6.2–6
52. Exercise 6.2–8
53. Exercise 6.2–9
54. Exercise 6.2–12
55. Problem 7.4
56. Problem 7.18
57. Problem 7.47

Neumann Boundary Conditions

Robin Boundary Conditions

Mixed Boundary Conditions

Problems related to Rectangles

Heat Conduction Problems

Dirichlet Boundary Conditions

Neumann Boundary Conditions

Robin Boundary Conditions

Mixed Boundary Conditions

1. Exercise 4.2–37
2. Exercise 4.2–38
3. Exercise 4.2–41
4. Exercise 4.3–29
5. Exercise 6.3–5
6. Exercise 6.4–2
7. Exercise 6.4–3
8. Exercise 6.4–4
9. Exercise 6.4–10
10. Example 7.5
11. Exercise 7.3–2
12. Exercise 7.3–3
13. Exercise 11.4–7
14. Exercise 11.6–16
15. Exercise 11.6–17
16. Exercise 11.6–18
17. Exercise 11.6–19
18. Exercise 11.6–20
19. Exercise 13.7–13

Vibration Problems

Dirichlet Boundary Conditions

1. Exercise 4.2–39
2. Exercise 6.3–6
3. Problem 6.35
4. Exercise 6.4–11
5. Exercise 6.4–13
6. Exercise 6.5–9
7. Exercise 7.3–5
8. Exercise 7.3–6
9. Exercise 7.3–7
10. Exercise 7.3–8
11. Exercise 7.3–9
12. Exercise 7.3–10
13. Exercise 7.3–11
14. Exercise 13.8–9

Robin Boundary Conditions

1. Exercise 7.3–12

Mixed Boundary Conditions

1. Exercise 6.4–12
2. Exercise 6.5–10

Potential and Generic Problems

Dirichlet Boundary Conditions

1. Example 4.3
2. Exercise 4.2–30
3. Exercise 4.2
4. Exercise 4.2–32
5. Exerise 4.2–40
6. Exercise 4.3
7. Exercise 4.3–28
8. Exercise 6.3–1
9. Exercise 7.2
10. Exercise 7.2–49
11. Exercise 7.2–50
12. Exercise 7.2
13. Exercise 7.2–53
14. Example ‘`channel pot`’
15. Example ‘P
16. Exercise ‘`cl0sed form solN`’
17. Exercise 11.6–9
18. Exercise 11.
19. Exercise 11.6–21
20. Exercise 11.6–22
21. Exercise 11.
22. Exercise 11.6–24
23. Exercise 11.4–8
24. Exercise 11.
25. Example 13.6
26. Exercise 13.3–1
27. Exercise 13.
28. Exercise 13.3–3
29. Example 13.10
30. Exercise 13.

Neumann Boundary Conditions

1. Exercise 4.2–34
2. Exercise 4.2–35
3. Exercise 4.2–42
4. Exercise 4.3–30
5. Exercise 7.2–56
6. Example 13.8
7. Exercise 13.4–2
8. Exercise 13.4–3
9. Exercise 13.4–4
10. Exercise 13.6–10

Mixed Boundary Conditions

1. Exercise 4.2–33
2. Problem 6.29
3. Exercise 6.3–7
4. Exercise ‘`pot16`’
5. Example 11.22
6. Exercise 13.5–3

Problems Related to Boxes

Heat Conduction Problems

Mixed Boundary Conditions

Potential and Generic Problems

Dirichlet Boundary Conditions

Neumann Boundary Conditions

Problems Related to Circles (including annuli)

Heat Conduction Problems

Dirichlet Boundary Conditions

Neumann Boundary Conditions

Robin Boundary Conditions

Vibration Problems

Dirichlet Boundary Conditions

Potential and Generic Problems

Dirichlet Boundary Conditions

Neumann Boundary Conditions

Robin Boundary Conditions

Mixed Boundary Conditions

Problems Related to Sectors of Circles

(including quarter-circles and semi-circles)

Heat Conduction Problems

Dirichlet Boundary Conditions

Neumann Boundary Conditions

Mixed Boundary Conditions

Problems Related to Cylinders

Heat Conduction Problems

Dirichlet Boundary Conditions

Neumann Boundary Conditions

Robin Boundary Conditions

Mixed Boundary Conditions

Potential and Generic Problems

Dirichlet Boundary Conditions

Problems Related to Vibrations of Beams

Problems Related to Diffusion of Neutrons

Problems Related to the Telegraphy Equation

Problems Related to Heat Conduction in Thin Wires

Problems Related to Oscillations of Hanging Cables

Problems Related to the Helmholtz Equation