CHALLENGE PROBLEMS

CHAPTER 2

A Click here for answers. S Click here for solutions.

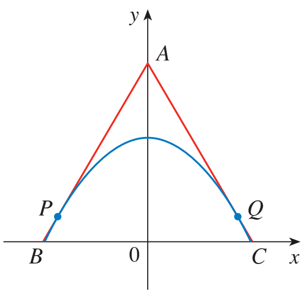


FIGURE FOR PROBLEM 1

1. Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x-axis and the tangent lines at P and Q is an equilateral triangle.

2. Find the point where the curves $y = x^3 - 3x + 4$ and $y = 3(x^2 - x)$ are tangent to each other, that is, have a common tangent line. Illustrate by sketching both curves and the common tangent.

3. Suppose f is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

(a) Find $f(0)$. (b) Find $f'(0)$. (c) Find $f'(x)$.

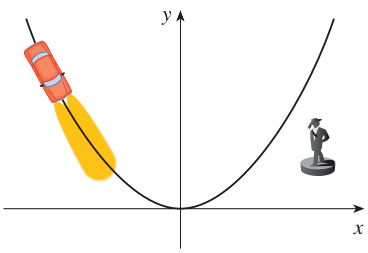


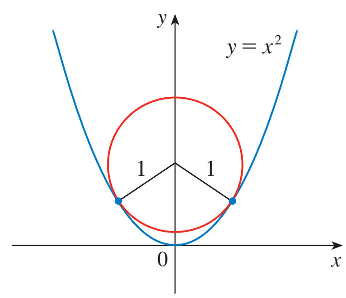
FIGURE FOR PROBLEM 4

4. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin (see the figure). The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue?

5. Prove that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

6. Find the nth derivative of the function $f(x) = x^n/(1 - x)$.

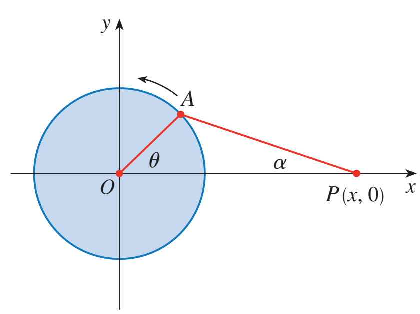
7. The figure shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



8. If f is differentiable at a, where $a > 0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

9. The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.
(a) Find the angular velocity of the connecting rod, $d\alpha/dt$, in radians per second, when $\theta = \pi/3$.
(b) Express the distance $x = |OP|$ in terms of θ.
(c) Find an expression for the velocity of the pin P in terms of θ.



10. Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y = x^2$ and they intersect at a point P. Another tangent line T is drawn at a point between P_1 and P_2; it intersects T_1 at Q_1 and T_2 at Q_2. Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$

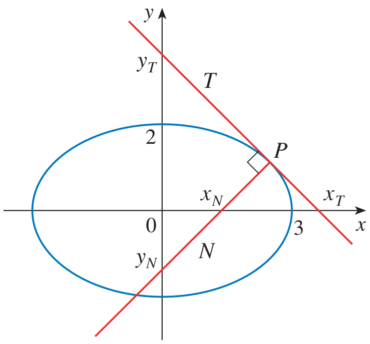


FIGURE FOR PROBLEM 11

11. Let T and N be the tangent and normal lines to the ellipse $x^2/9 + y^2/4 = 1$ at any point P on the ellipse in the first quadrant. Let x_T and y_T be the x- and y-intercepts of T and x_N and y_N be the intercepts of N. As P moves along the ellipse in the first quadrant (but not on the axes), what values can x_T, y_T, x_N, and y_N take on? First try to guess the answers just by looking at the figure. Then use calculus to solve the problem and see how good your intuition is.

12. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin(3 + x)^2 - \sin 9}{x}$.

13. (a) Use the identity for $\tan(x - y)$ (see Equation 14b in Appendix A) to show that if two lines L_1 and L_2 intersect at an angle α, then

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

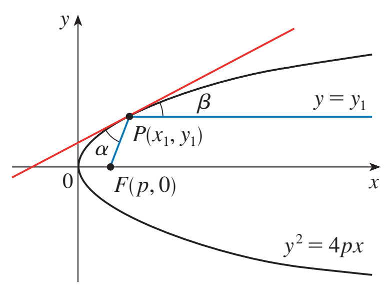
where m_1 and m_2 are the slopes of L_1 and L_2, respectively.

(b) The **angle between the curves** C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.

(i) $y = x^2$ and $y = (x - 2)^2$

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$

14. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP, and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the x-axis. This explains why *paraboloids*, the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)



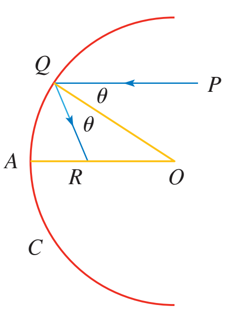


FIGURE FOR PROBLEM 15

15. Suppose that we replace the parabolic mirror of Problem 14 by a spherical mirror. Although the mirror has no focus, we can show the existence of an *approximate* focus. In the figure, C is a semicircle with center O. A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that $\angle PQO = \angle OQR$ (the angle of incidence is equal to the angle of reflection). What happens to the point R as P is taken closer and closer to the axis?

16. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

17. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}$.

18. Given an ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a \neq b$, find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.

19. Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

20. Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0.

21. A *lattice point* in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope $\frac{2}{5}$ intersects some of these circles.

22. A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

23. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is πrl, where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of 2 cm^3/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?

CAS **24.** (a) The cubic function $f(x) = x(x - 2)(x - 6)$ has three distinct zeros: 0, 2, and 6. Graph f and its tangent lines at the *average* of each pair of zeros. What do you notice?

(b) Suppose the cubic function $f(x) = (x - a)(x - b)(x - c)$ has three distinct zeros: a, b, and c. Prove, with the help of a computer algebra system, that a tangent line drawn at the average of the zeros a and b intersects the graph of f at the third zero.

ANSWERS

S Solutions

1. $\left(\pm\sqrt{3}/2, \frac{1}{4}\right)$ **3.** (a) 0 (b) 1 (c) $f'(x) = x^2 + 1$ **7.** $\left(0, \frac{5}{4}\right)$

9. (a) $4\pi\sqrt{3}/\sqrt{11}$ rad/s (b) $40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$ cm (c) $-480\pi \sin\theta \left(1 + \cos\theta/\sqrt{8 + \cos^2\theta}\right)$ cm/s

11. $x_T \in (3, \infty)$, $y_T \in (2, \infty)$, $x_N \in \left(0, \frac{5}{3}\right)$, $y_N \in \left(-\frac{5}{2}, 0\right)$ **13.** (b) (i) 53° (or 127°) (ii) 63° (or 117°)

15. R approaches the midpoint of the radius AO.

17. $-\sin a$ **19.** $(1, -2)$, $(-1, 0)$ **21.** $\sqrt{29}/58$ **23.** $2 + \frac{375}{128}\pi \approx 11.204 \text{ cm}^3/\text{min}$

SOLUTIONS

E Exercises

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \quad \Rightarrow$ $a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

3. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0+0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \quad \Rightarrow \quad f(0) = 2f(0)$.
Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\left[f(0) + f(h) + 0^2 h + 0h^2\right] - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{f(h)}{h} = \lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$

(c)
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[f(x) + f(h) + x^2 h + xh^2\right] - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(h) + x^2 h + xh^2}{h} = \lim_{h \to 0} \left[\frac{f(h)}{h} + x^2 + xh\right] = 1 + x^2 \end{aligned}$$

5. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned} \tfrac{d}{dx}\left(\sin^4 x + \cos^4 x\right) &= 4\sin^3 x\cos x - 4\cos^3 x \sin x = 4\sin x\cos x\left(\sin^2 x - \cos^2 x\right) \\ &= -4\sin x\cos x\cos 2x = -2\sin 2x\cos 2x = -\sin 4x = \sin(-4x) \\ &= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \quad \text{when } n = 1 \end{aligned}$$

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right) = 4^{k-1}\cos\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned} \tfrac{d^{k+1}}{dx^{k+1}}\left(\sin^4 x + \cos^4 x\right) &= \tfrac{d}{dx}\left[\tfrac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right)\right] = \tfrac{d}{dx}\left[4^{k-1}\cos\left(4x + k\tfrac{\pi}{2}\right)\right] \\ &= -4^{k-1}\sin\left(4x + k\tfrac{\pi}{2}\right) \cdot \tfrac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\left(4x + k\tfrac{\pi}{2}\right) \\ &= 4^k\sin\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right) = 4^k\cos\left(4x + (k+1)\tfrac{\pi}{2}\right) \end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$\sin^4 x + \cos^4 x = \left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{4}(1 - \cos 4x) = \frac{3}{4} + \frac{1}{4}\cos 4x.$

Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = \dfrac{d^n}{dx^n}\left(\frac{3}{4} + \frac{1}{4}\cos 4x\right) = \frac{1}{4} \cdot 4^n\cos\left(4x + n\frac{\pi}{2}\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$.

7. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $\left(\pm x_0, x_0^2\right)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $\left(\pm x_0, x_0^2\right)$ respectively, so the normals have the equations $y - x_0^2 = -\dfrac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \dfrac{1}{2x_0}(x + x_0)$. The common

y-intercept is $x_0^2+\frac{1}{2}$. We want to find the value of x_0 for which the distance from $\left(0, x_0^2+\frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is $(x_0-0)^2+\left[x_0^2-\left(x_0^2+\frac{1}{2}\right)\right]^2=x_0^2+\frac{1}{4}=1 \quad\Leftrightarrow\quad x_0=\pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2+\frac{1}{2}=\frac{5}{4}$, so the center of the circle is at $\left(0,\frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0,a)$. Then the equation of the circle is $x^2+(y-a)^2=1$. Solving with the equation of the parabola, $y=x^2$, we get $x^2+\left(x^2-a\right)^2=1 \quad\Leftrightarrow\quad x^2+x^4-2ax^2+a^2=1$ $\Leftrightarrow\quad x^4+(1-2a)x^2+a^2-1=0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1-2a)^2-4\left(a^2-1\right)=0 \quad\Leftrightarrow$ $1-4a+4a^2-4a^2+4=0 \quad\Leftrightarrow\quad 4a=5$, so $a=\frac{5}{4}$. The center of the circle is $\left(0,\frac{5}{4}\right)$.

9. We can assume without loss of generality that $\theta=0$ at time $t=0$, so that $\theta=12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $=360\cdot(2\pi\text{ rad})/(60\text{ s})=12\pi$ rad/s.] Then the position of A as a function of time is $A=(40\cos\theta,40\sin\theta)=(40\cos 12\pi t,40\sin 12\pi t)$, so $\sin\alpha=\dfrac{y}{1.2\text{ m}}=\dfrac{40\sin\theta}{120}=\dfrac{\sin\theta}{3}=\frac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha\cdot\dfrac{d\alpha}{dt}=\frac{1}{3}\cdot 12\pi\cdot\cos 12\pi t=4\pi\cos\theta$. When

$\theta=\frac{\pi}{3}$, we have $\sin\alpha=\frac{1}{3}\sin\theta=\frac{\sqrt{3}}{6}$, so $\cos\alpha=\sqrt{1-\left(\frac{\sqrt{3}}{6}\right)^2}=\sqrt{\frac{11}{12}}$ and

$$\frac{d\alpha}{dt}=\frac{4\pi\cos\frac{\pi}{3}}{\cos\alpha}=\frac{2\pi}{\sqrt{11/12}}=\frac{4\pi\sqrt{3}}{\sqrt{11}}\approx 6.56\text{ rad/s}.$$

(b) By the Law of Cosines, $|AP|^2=|OA|^2+|OP|^2-2\,|OA|\,|OP|\cos\theta \quad\Rightarrow$

$120^2=40^2+|OP|^2-2\cdot 40\,|OP|\cos\theta \quad\Rightarrow\quad |OP|^2-(80\cos\theta)\,|OP|-12{,}800=0 \quad\Rightarrow$

$$\begin{aligned}|OP|&=\tfrac{1}{2}\left(80\cos\theta\pm\sqrt{6400\cos^2\theta+51{,}200}\right)=40\cos\theta\pm 40\sqrt{\cos^2\theta+8}\\&=40\left(\cos\theta+\sqrt{8+\cos^2\theta}\right)\text{ cm}\quad[\text{since }|OP|>0]\end{aligned}$$

As a check, note that $|OP|=160$ cm when $\theta=0$ and $|OP|=80\sqrt{2}$ cm when $\theta=\frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x=40\left(\cos\theta+\sqrt{8+\cos^2\theta}\right)$, so

$$\frac{dx}{dt}=\frac{dx}{d\theta}\frac{d\theta}{dt}=40\left(-\sin\theta-\frac{2\cos\theta\sin\theta}{2\sqrt{8+\cos^2\theta}}\right)\cdot 12\pi=-480\pi\sin\theta\left(1+\frac{\cos\theta}{\sqrt{8+\cos^2\theta}}\right)\text{ cm/s}.$$

In particular, $dx/dt=0$ cm/s when $\theta=0$ and $dx/dt=-480\pi$ cm/s when $\theta=\frac{\pi}{2}$.

11. It seems from the figure that as P approaches the point $(0,2)$ from the right, $x_T\to\infty$ and $y_T\to 2^+$. As P approaches the point $(3,0)$ from the left, it appears that $x_T\to 3^+$ and $y_T\to\infty$. So we guess that $x_T\in(3,\infty)$ and $y_T\in(2,\infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N\in(0,3)$, and $y_N\in(-\infty,0)$ or $(-2,0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1 \quad\Rightarrow\quad \dfrac{2x}{9}+\dfrac{2y}{4}y'=0$, so $y'=-\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0,y_0) on the ellipse, an equation of the tangent line is $y-y_0=-\dfrac{4}{9}\dfrac{x_0}{y_0}(x-x_0)$ or $4x_0x+9y_0y=4x_0^2+9y_0^2$. This can be written as

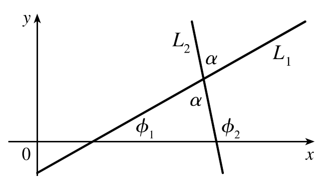
$\dfrac{x_0 x}{9} + \dfrac{y_0 y}{4} = \dfrac{x_0^2}{9} + \dfrac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0 x}{9} + \dfrac{y_0 y}{4} = 1$.

Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0 x_T}{9} = 1 \quad \Leftrightarrow \quad x_T = \dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0 y_T}{4} = 1 \quad \Leftrightarrow \quad y_T = \dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0, y_0)} = \dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x_N - x_0) \quad \Rightarrow \quad x_N = -\dfrac{4x_0}{9} + x_0 = \dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(0 - x_0) \quad \Rightarrow \quad y_N = -\dfrac{9y_0}{4} + y_0 = -\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

13. (a)

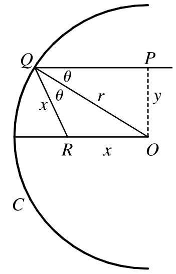


If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1 = \tan\,\phi_1$ and $m_2 = \tan\,\phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180^\circ - \phi_2) = 180^\circ$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x - y)$, we have $\tan\alpha = \tan(\phi_2 - \phi_1) = \dfrac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2 \tan\phi_1}$ and so $\tan\,\alpha = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.

(b) (i) The parabolas intersect when $x^2 = (x-2)^2 \quad \Rightarrow \quad x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent to $y = x^2$ at $(1, 1)$ is $m_1 = 2(1) = 2$. If $y = (x-2)^2$, then $y' = 2(x-2)$, so the slope of the tangent to $y = (x-2)^2$ at $(1, 1)$ is $m_2 = 2(1-2) = -2$. Therefore, $\tan\alpha = \dfrac{m_2 - m_1}{1 + m_1 m_2} = \dfrac{-2-2}{1+2(-2)} = \frac{4}{3}$ and so $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53^\circ$ (or 127°).

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + \left(x^2 - 3\right) + 3 = 0 \quad \Leftrightarrow \quad 2x(x-2) = 0 \quad \Rightarrow \quad x = 0$ or 2, but 0 is extraneous. If $x = 2$, then $y = \pm 1$. If $x^2 - y^2 = 3$ then $2x - 2yy' = 0 \quad \Rightarrow \quad y' = x/y$ and $x^2 - 4x + y^2 + 3 = 0 \quad \Rightarrow \quad 2x - 4 + 2yy' = 0 \quad \Rightarrow \quad y' = \dfrac{2-x}{y}$. At $(2, 1)$ the slopes are $m_1 = 2$ and $m_2 = 0$, so $\tan\alpha = \frac{0-2}{1+2\cdot 0} = -2 \quad \Rightarrow \quad \alpha \approx 117^\circ$. At $(2, -1)$ the slopes are $m_1 = -2$ and $m_2 = 0$, so $\tan\alpha = \frac{0-(-2)}{1+(-2)(0)} = 2 \quad \Rightarrow \quad \alpha \approx 63^\circ$ (or 117°).

15. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \dfrac{r^2}{2r\cos\theta} = \dfrac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \dfrac{r}{2\cos 0} = \dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.

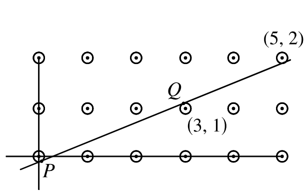


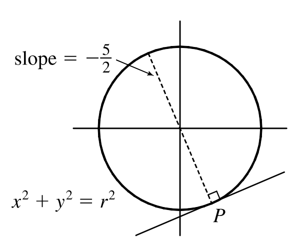
17.
$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2}
&= \lim_{x\to 0} \frac{\sin a\cos 2x + \cos a\sin 2x - 2\sin a\cos x - 2\cos a\sin x + \sin a}{x^2} \\
&= \lim_{x\to 0} \frac{\sin a\,(\cos 2x - 2\cos x + 1) + \cos a\,(\sin 2x - 2\sin x)}{x^2} \\
&= \lim_{x\to 0} \frac{\sin a\,(2\cos^2 x - 1 - 2\cos x + 1) + \cos a\,(2\sin x\cos x - 2\sin x)}{x^2} \\
&= \lim_{x\to 0} \frac{\sin a\,(2\cos x)(\cos x - 1) + \cos a\,(2\sin x)(\cos x - 1)}{x^2} \\
&= \lim_{x\to 0} \frac{2(\cos x - 1)[\sin a\cos x + \cos a\sin x](\cos x + 1)}{x^2(\cos x + 1)} \\
&= \lim_{x\to 0} \frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x + 1)} = -2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \frac{\sin(a+x)}{\cos x + 1} = -2(1)^2\frac{\sin(a+0)}{\cos 0 + 1} = -\sin a
\end{aligned}$$

19. $y = x^4 - 2x^2 - x \quad\Rightarrow\quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad\Rightarrow\quad (a-b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad\Rightarrow$ $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad\Rightarrow\quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = \left(a^2 + ab + b^2\right) - \left(a^2 + b^2\right) = 1 - \frac{2}{3} = \frac{1}{3}$, so $b = \dfrac{1}{3a}$. Hence, $a^2 + \dfrac{1}{9a^2} = \dfrac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad\Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = \left(3a^2 - 1\right)^2$. So $3a^2 - 1 = 0 \quad\Rightarrow\quad a^2 = \frac{1}{3} \quad\Rightarrow\quad b^2 = \dfrac{1}{9a^2} = \frac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

21.



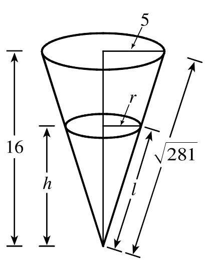


Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3,1)$ and the circles centered at $(0,0)$ and $(5,2)$. To find P, the point at which the line is tangent to the circle at $(0,0)$, we simultaneously solve $x^2+y^2=r^2$ and $y=-\frac{5}{2}x \quad\Rightarrow\quad x^2+\frac{25}{4}x^2=r^2 \quad\Rightarrow\quad x^2=\frac{4}{29}r^2 \quad\Rightarrow$ $x=\frac{2}{\sqrt{29}}r$, $y=-\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x-3)^2+(y-1)^2=r^2$ and $y-1=-\frac{5}{2}(x-3)$. As above, we get $x=3-\frac{2}{\sqrt{29}}r$, $y=1+\frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so

$$m_{PQ}=\frac{1+\frac{5}{\sqrt{29}}r-\left(-\frac{5}{\sqrt{29}}r\right)}{3-\frac{2}{\sqrt{29}}r-\frac{2}{\sqrt{29}}r}=\frac{1+\frac{10}{\sqrt{29}}r}{3-\frac{4}{\sqrt{29}}r}=\frac{\sqrt{29}+10r}{3\sqrt{29}-4r}=\frac{2}{5} \quad\Rightarrow\quad 5\sqrt{29}+50r=6\sqrt{29}-8r \quad\Leftrightarrow$$

$58r=\sqrt{29} \quad\Leftrightarrow\quad r=\frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r=\frac{\sqrt{29}}{58}\approx 0.093$.

23.



By similar triangles, $\frac{r}{5}=\frac{h}{16} \quad\Rightarrow\quad r=\frac{5h}{16}$. The volume of the cone is $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{5h}{16}\right)^2h=\frac{25\pi}{768}h^3$, so $\frac{dV}{dt}=\frac{25\pi}{256}h^2\,\frac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\frac{dV}{dt}=2-k\pi rl$.

Equating the two expressions for $\frac{dV}{dt}$ and substituting $h=10$, $\frac{dh}{dt}=-0.3$, $r=\frac{5(10)}{16}=\frac{25}{8}$, and $\frac{l}{\sqrt{281}}=\frac{10}{16}$ $\Leftrightarrow\quad l=\frac{5}{8}\sqrt{281}$, we get $\frac{25\pi}{256}(10)^2(-0.3)=2-k\pi\frac{25}{8}\cdot\frac{5}{8}\sqrt{281} \quad\Leftrightarrow\quad \frac{125k\pi\sqrt{281}}{64}=2+\frac{750\pi}{256}$. Solving for k gives us $k=\frac{256+375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\frac{dV}{dt}=0$. $k\pi rl=\frac{256+375\pi}{250\pi\sqrt{281}}\cdot\pi\cdot\frac{25}{8}\cdot\frac{5\sqrt{281}}{8}=\frac{256+375\pi}{128}\approx 11.204\text{ cm}^3/\text{min}$.