Outline for Solving Separable Equations

Problem Statement: A first order ordinary differential equation of the form

$$y' = f(x)g(y)$$

is called a ***separable equation***. The objective is to solve such equations, i.e., find a one parameter family of functions that define solutions of the given differential equation. If an initial condition is given, say $y(x_0) = y_0$, then the pair

$$y' = f(x)g(y),\ y(x_0) = y_0$$

is called an Initial Value Problem (IVP).

Step 1: Separate the variables.

This step is accomplished by recalling from calculus that $dy = y'dx$. If one multiplies both sides of the equation by dx, divides both sides by $g(y)$ and uses $dy = y'dx$, one obtains

$$\frac{1}{g(y)}dy = f(x)dx,$$

for $g(y) \neq 0$.

Step 2: Integrate both sides of the equation from Step 1.

Assuming both f and $\frac{1}{g}$ are continuous, this step is possible, at least theoretically. The actual computations depend on integration techniques from calculus. So if G and F are functions such that $\frac{d}{dy}G(y) = \frac{1}{g(y)}$ and $\frac{d}{dx}F(x) = f(x)$, then $G(y) = F(x) + C$, where C is a constant of integration. So the one-parameter family of functions that define solutions of the given equation is $G(y) = F(x) + C$. Note that y is defined <u>implicitly</u> by the equation. To express y <u>explicitly</u>, that is, $y = \phi(x)$, depends on your ability to solve the equation for y.

Step 3: **(only if an IVP is given)** Evaluate C using the initial condition.

If the initial condition is $y(x_0) = y_0$, then solve for C in the equation

$$G(y_0) = F(x_0) + C.$$

An Example

Problem Statement: Solve $y' = \dfrac{x^2}{y\left(1+x^3\right)}$.

Notice that $y \neq 0$ and $x \neq -1$ and $\dfrac{dy}{dx} = \dfrac{x^2}{y\left(1+x^3\right)} = \left(\dfrac{1}{y}\right)\left(\dfrac{x^2}{1+x^3}\right)$.

Step 1: Separate the variables.

$$ydy = \frac{x^2}{1+x^3}dx$$

Step 2: Integrate both sides of the equation from Step 1.

$$\int ydy = \int \frac{x^2}{1+x^3}dx = \frac{1}{3}\int \frac{du}{u} \qquad [u = 1+x^3, du = 3x^2dx]$$

$$\frac{1}{2}y^2 = \frac{1}{3}\ln|u| = \frac{1}{3}\ln\left|1+x^3\right| + C$$

So the one parameter family is

$$\boxed{\frac{1}{2}y^2 = \frac{1}{3}\ln\left|1+x^3\right| + C},$$

Or multiplying through by 6 and setting *K=6C*,

$$\boxed{3y^2 = 2\ln\left|1+x^3\right| + K}.$$

An Example

Problem Statement: Solve the IVP $y' = 2(1+x)\left(1+y^2\right), y(0) = 0$.

Step 1: Separate the variables.

$$\frac{dy}{1+y^2} = (2+2x)dx$$

Step 2: Integrate both sides of the equation from Step 1.

$$\int \frac{dy}{1+y^2} = \int (2+2x)dx$$

$$\tan^{-1} y = 2x + x^2 + C$$

Step 3: **(only if an IVP is given)** Evaluate C using the initial condition.

To solve the IVP, apply the initial condition $y(0) = 0$:

$$\tan^{-1} 0 = 2(0) + 0^2 + C = C$$

$$C = 0$$

So the solution is $\boxed{\tan^{-1} y = 2x + x^2}$, or, taking the tangent of both sides,

$$\boxed{y = \tan\left(2x + x^2\right)}.$$