

الرياضيات والفيزياء العراقي

خاص بمنتدى

Partial Fraction Decomposition

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add:

Before we begin,
partial fraction
decomposition, let's
start with a review of
adding rational
expressions

$$\frac{2}{x-3} + \frac{1}{x+2}$$

$$\frac{\quad}{(x-3)(x+2)}$$

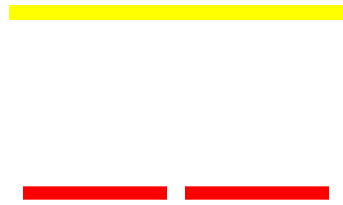
$$\frac{\quad}{(x-3)(x+2)}$$

$$\frac{3x + 1}{(x-3)(x+2)}$$

In partial fraction
decomposition, we
want to start with this
simplified rational
expression and work
back to find the two
rational expressions
that add to give us
this result.

Example 1 of Partial Fraction Decomposition

$$\frac{3x+1}{x^2 - x - 6} \Rightarrow$$



$$3x+1 =$$

let $x = -2$:

$$3(-2)+1 = A(-2+2) + B(-2-3)$$

$$-6+1 = \cancel{A(0)} + B(-5)$$

$$-5 = -5B$$

$$1 = B$$

let $x = 3$:

$$3(3)+1 = A(3+2) + B(3-3)$$

$$10 = A(5) + \cancel{B(0)}$$

$$10 = 5A$$

$$2 = A$$

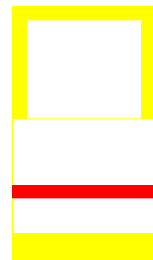
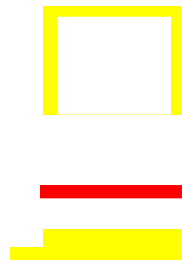
$$\frac{3x+1}{x^2 - x - 6} \Rightarrow \left[\frac{3x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \right] (x-3)(x+2)$$

$$1 = B \qquad 2 = A$$

$$\frac{3x+1}{(x-3)(x+2)} =$$

Example 2

$$\frac{1}{x^2 - 1} =$$



$$1 =$$

let $x = -1$:

$$1 = A(-1+1) + B(-1-1)$$

$$1 = \cancel{A(0)} + B(-2)$$

$$1 = -2B$$

$$-\frac{1}{2} = B$$

let $x = 1$:

$$1 = A(1+1) + B(1-1)$$

$$1 = A(2) + \cancel{B(0)}$$

$$1 = 2A$$

$$\frac{1}{2} = A$$

$$\frac{1}{x^2 - 1} \Rightarrow \left[\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \right] (x-1)(x+1)$$

$$-\frac{1}{2} = B \qquad \frac{1}{2} = A$$

$$\frac{1}{(x-1)(x+1)} =$$

$$\frac{1}{(x-1)(x+1)} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Example 3

$$\frac{x+1}{(\cancel{x+3})(\cancel{x^2+x+1})} = \frac{\boxed{}}{} + \frac{\boxed{}}{} + \frac{\boxed{}}{}$$

$$x+1 =$$

let $x = -3$:

$$-3+1 = A((-3)^2 + (-3)+1) + (B(-3)+C)(-3+3)$$

$$-2 = A(9-3+1) + (\cancel{3B+C})(0)$$

$$-2 = 7A$$

$$\frac{-2}{7} = A$$

$$\left[\frac{x+1}{(x+3)(x^2+x+1)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+x+1)} \right] (x+3)(x^2+x+1)^2$$

$$x+1 = A(x^2+x+1) + (Bx+C)(x+3)$$

let $x = 0$:

$$0+1 = A((0)^2 + (0)+1) + (\textcolor{red}{B(0)}+C)(0+3)$$

$$1 = A + 3C$$

$$1 = \frac{-2}{7} + 3C$$

$$\frac{-2}{7} = A$$

$$7 = -2 + 21C$$

$$9 = 21C$$

$$= \frac{9}{21} = C$$

$$\left[\frac{x+1}{(x+3)(x^2+x+1)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+x+1)} \right] (x+3)(x^2+x+1)^2$$

$$x+1 = A(x^2+x+1) + (Bx+C)(x+3)$$

let $x = 1$:

$$1+1 = A((1)^2 + (1)+1) + (B(1)+C)(1+3)$$

$$2 = 3A + 4B + 4C$$

$$2 = \frac{-2}{7} = A \quad \& \quad \frac{3}{7} = C$$

$$2 = \frac{-6}{7} + 4B + \frac{12}{7}$$

$$14 = -6 + 28B + 12$$

$$8 = 28B$$

$$\frac{8}{28} = B$$

$$\left[\frac{x+1}{(x+3)(x^2+x+1)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+x+1)} \right] (x+3)(x^2+x+1)^2$$

$$\frac{3}{7} = C \qquad \frac{-2}{7} = A \qquad \frac{2}{7} = B$$

$$\frac{x+1}{(x+3)(x^2+x+1)} =$$

$$\frac{x+1}{(x+3)(x^2+x+1)} =$$

Example 4

$$\frac{2x+1}{(\cancel{x+2})(\cancel{x-3})^2} = \frac{\boxed{}}{\boxed{}} + \frac{\boxed{}}{\boxed{}} + \frac{\boxed{}}{\boxed{}} + \frac{\boxed{}}{\boxed{}}$$

$$2x+1 =$$

let $x = 3$:

$$2(3)+1 = A(3-3)^2 + B(3+2)(3-3) + C(3+2)$$

$$6+1 = \cancel{A(0)^2} + \cancel{B(5)(0)} + C(5)$$

$$7 = 5C$$

$$\frac{7}{5} = C$$

$$\left[\frac{2x+1}{(x+2)(x-3)^2} = \frac{A}{(x+2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2} \right] (x+2)(x-3)^2$$

$$2x+1 = A(x-3)^2 + B(x+2)(x-3) + C(x+2)$$

let $x = -2$:

$$2(-2)+1 = A(-2-3)^2 + B(-2+2)(-2-3) + C(-2+2)$$

$$-4+1 = A(-5)^2 + \cancel{B(0)(-5)} + \cancel{C(0)}$$

$$-3 = 25A$$

$$-\frac{3}{25} = A$$

$$\left[\frac{2x+1}{(x+2)(x-3)^2} = \frac{A}{(x+2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2} \right] (x+2)(x-3)^2$$

$$2x+1 = A(x-3)^2 + B(x+2)(x-3) + C(x+2)$$

let $x = 0$:

$$2(0)+1 = A(0-3)^2 + B(0+2)(0-3) + C(0+2)$$

$$1 = A(-3)^2 + B(2)(-3) + C(2)$$

$$1 = 9A - 6B + 2C$$

$$1 = 9\left(\frac{-3}{25}\right) - 6B + 2\left(\frac{7}{5}\right)$$

$$\left[1 = -\frac{27}{25} - 6B + \frac{14}{5} \right] 25$$

$$25 = -27 - 150B + 70$$

$$-18 = -150B$$

$$\frac{-18}{-150} = B$$

$$\left[\frac{2x+1}{(x+2)(x-3)^2} = \frac{A}{(x+2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2} \right] (x+2)(x-3)^2$$

$$\frac{7}{5} = C$$

$$\frac{-3}{25} = A$$

$$\frac{3}{25} = B$$

$$\frac{2x+1}{(x+2)(x-3)^2} = \frac{\frac{-3}{25}}{(x+2)} + \frac{\frac{3}{25}}{(x-3)} + \frac{\frac{7}{5}}{(x-3)^2}$$

$$\frac{2x+1}{(x+2)(x-3)^2} =$$

Example 5

$$\frac{x+3}{x^2(x+1)} = \frac{\quad}{x^2} + \frac{\quad}{x+1}$$

$$x+3 =$$

let $x = 0$:

$$0+3 = A0(0+1) + B(0+1) + C \cdot 0^2$$

$$3 = \cancel{A(0)(1)} + B(1) + \cancel{C(0)}$$

$$3 = B$$

$$\left[\frac{x+3}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} \right] x^2(x+1)$$

$$x+3 = Ax(x+1) + B(x+1) + Cx^2$$

let $x = -1$:

$$-1+3 = A(-1)(-1+1) + B(-1+1) + C(-1)^2$$

$$2 = \cancel{A(-1)(0)} + \cancel{B(0)} + C(1)$$

$$2 = C$$

$$\left[\frac{x+3}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} \right] x^2(x+1)$$

$$x+3 = Ax(x+1) + B(x+1) + Cx^2$$

let $x = 1$:

$$1+3 = A(1)(1+1) + B(1+1) + C(1)^2$$

$$4 = A(1)(2) + B(2) + C(1)$$

$$4 = 2A + 2B + C$$

$$4 = 2A + 2(3) + (2)$$

$$4 = 2A + 6 + 2$$

$$-4 = 2A$$

$$-2 = A$$

$$\left[\frac{x+3}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} \right] x^2(x+1)$$

$$-2 = A \qquad 3 = B \qquad 2 = C$$

$$\frac{x+3}{x^2(x+1)} = \frac{-2}{x} + \frac{3}{x^2} + \frac{2}{(x+1)}$$

Example 6

$$\frac{3x+2}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x^2} + \frac{D}{(x-2)^2}$$

$$3x+2 =$$

let $x = 0$:

$$3(0)+2 = A(0)(0-2)^2 + B(0-2)^2 + C(0)^2(0-2) + D(0)^2$$

$$2 = \cancel{A(0)(-2)^2} + B(-2)^2 + \cancel{C(0)^2(-2)} + \cancel{D(0)^2}$$

$$2 = 4B$$

$$\frac{1}{2} = B$$

$$\left[\frac{3x+2}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} \right] x^2(x-2)^2$$

$$3x+2 = Ax(x-2)^2 + B(x-2)^2 + Cx^2(x-2) + Dx^2$$

let $x = 2$:

$$3(2)+2 = A(2)(2-2)^2 + B(2-2)^2 + C(2)^2(2-2) + D(2)^2$$

$$8 = \cancel{A(2)(0)^2} + \cancel{B(0)^2} + \cancel{C(2)^2(0)} + D(2)^2$$

$$8 = 4D$$

$$2 = D$$

$$\left[\frac{3x+2}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} \right] x^2(x-2)^2$$

$$3x+2 = Ax(x-2)^2 + B(x-2)^2 + Cx^2(x-2) + Dx^2$$

let $x = 1$:

$$3(1)+2 = A(1)(1-2)^2 + B(1-2)^2 + C(1)^2(1-2) + D(1)^2$$

$$5 = A(1)(-1)^2 + B(-1)^2 + C(1)^2(-1) + D(1)^2$$

$$5 = A + B - C + D \qquad B = \frac{1}{2}$$

$$5 = A + \frac{1}{2} - C + 2 \qquad D = 2$$

$$10 = 2A + 1 - 2C + 4$$

$$5 = 2A - 2C$$

$$\left[\frac{3x+2}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} \right] x^2(x-2)^2$$

$$3x+2 = Ax(x-2)^2 + B(x-2)^2 + Cx^2(x-2) + Dx^2$$

let $x = -1$:

$$3(-1)+2 = A(-1)(-1-2)^2 + B(-1-2)^2 + C(-1)^2(-1-2) + D(-1)^2$$

$$-3+2 = A(-1)(-3)^2 + B(-3)^2 + C(1)^2(-3) + D(-1)^2$$

$$-1 = -9A + 9B - 3C + D \qquad B = \frac{1}{2}$$

$$-1 = -9A + 9 \cdot \frac{1}{2} - 3C + 2 \qquad D = 2$$

$$-2 = -18A + 9 - 6C + 4$$

$$-15 = -18A - 6C$$

$$\left[\frac{3x+2}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} \right] x^2(x-2)^2$$

$$B = \frac{1}{2}$$

$$D = 2$$

$$\begin{aligned} 5 &= 2A - 2C \\ -15 &= -18A - 6C \end{aligned}$$

$$RREF \begin{bmatrix} 2 & -2 & 5 \\ -18 & -6 & -15 \end{bmatrix} \Rightarrow \begin{aligned} A &= \frac{5}{4} \\ C &= \frac{-5}{4} \end{aligned}$$

$$\frac{3x+2}{x^2(x-2)^2} =$$