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# MATHEMATICS

## Scientific Secondary 4

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## Introduction

This book is the first course of a series of modern math books for scientific high-school classes and it consists of seven chapters:

**Chapter one:** Mathematical logic

**Chapter two:** Equation and inequalities

**Chapter three**: Basic Skills of Exponent and Roots

**Chapter four:** Triangles

**Chapter five:** Geometric vectors

**Chapter six:** Coordinate Plane Geometry

**Chapter seven:** Statistics

We hope God help us to serve our Country and our Sons.

**Authors**

# 1 Chapter 1: Mathematical Logic

[1-1] Logical Statement
[1-2] Connective word if ............ then ............
[1-3] Connective if and only if
[1-4] Implication
[1-5] Open Statement
[1-6] Equivalent open statements
[1-7] Quantified Statements

Aims and Skills;

At the end of studying this chapter the student should be able to:

-Learn the truth value of the proposition and their negation and the compound statements.

- Learn the open statements and compound statements by knowing the connective symbols.
- Learn the equivalence of open statement.
- Learn the Quantified statements and their negations.

| *Term* | *Symbol* |
|---|---|
| Logical statement | |
| Connective word and | $\wedge$ |
| Or | $\vee$ |
| Connective if.... Then.. | $\longrightarrow$ |
| Connective word if and only if | $\leftrightarrow$ |
| Implication | $\Leftrightarrow$ , $\Rightarrow$ |
| Quantifying universe symbol | $\forall$ , $\exists$ |

# Mathematical logic

In order to obtain an accurate result for our problems in math we need a series of steps one related to other so math can be seen as a logical system, and writing math statements by symbols and formulas easy to use is said to be Mathematical Logic. According to this logic is not a theory but consider a language by all math scientists so they set agreements to explain the mathematical statements we use.

## [1-1] Logical statement

You have studied the logic in the third intermediate class last year mathematical logic is divided into two kinds of statements:

A) Non-statements

B) Statements

Statements are the type which can be considered either True or False and it's called logical statement.

| $\sim P$ | P |
|---|---|
| F | T |
| T | F |

(1-1)

If we symbolize the logical statement as P then the P is (True)(T) when the statement is correct and P is (False)(F) when the statement is wrong and the negation of P is True as shown in Table (1-1).

It,s useful to remind the truth table for conjunction ( $\wedge$ ), disjunction ( $\vee$ )

| P | Q | P $\vee$ Q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

(1-2)

| P | Q | P $\wedge$ Q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

(1-3)

## [1-2] Connective: If ... then

In this lesson, we will learn the connective " if ... Then" It's used to form compound statements.

Such as:

If the triangle ABC is Isosceles then the angles of its base are equal this compound consists of connecting the statement of The triangle ABC is Isosceles " with the statement " The angles of the base equal bu using the connective if …… then

Here the statement comes after `if` is called consequent while the statement as s whole "if ….then" is called condition.

So the statement (the triangle ABC is isosceles) is antecedent and the statement (The angles of the base are equal) is consequent.

**Let`s examine this example:**

The mother says: (if you pass the exam, then I will buy you a gift)

Let's study the following cases:

1) The son passes the exam and the mother buys him a gift.

2) The son passes the exam and the mother does not buy him a gift.

3) The son does not pass the exam and the mother buys him a gift.

4) The son does not pass the exam and the mother does not buy him a gift.

We accept the first third and fourth cases as true

…................... the case as false.

We denote the antecedent and consequent by p and q relatively and this compound statement is denoted as $P \longrightarrow Q$ .

and read ((if P then Q))

The truth table of the statement is:

| P | Q | $P \longrightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

(1-4)

**So $P \longrightarrow Q$ is false in only one case. If the antecedent is true and consequent is false**

**Example 1**

Consider the Truth value of the following statements:

1) If $\sqrt{2} < \sqrt{3}$ then $\sqrt{-2} \notin R$
2) If $7 + 5 = 12$ then $6 + 2 = 7$
3) If $7 + 5 = 11$ then $6 + 2 = 8$
4) If $1 = 0$ the $\sqrt{3}$ is a rational number

**Solution:**

1) True since antecedent is True and consequent is True.
2) False since antecedent is True and consequent is False.
3) True since antecedent is False and consequent is True.
4) True since antecedent is False and consequent is False

## [1-3] If and only if

We often use the expound statement:)

$(Q \to P) \wedge (P \to Q)$

For example, the triangles is equivalent if and only if the measure of its angles are equal also the angle measures of a triangle are equal if and only if it is equilateral

Such compound statements are called biconditional if we suppose p and q are two statements then the biconditional statement $(Q \to P) \wedge (P \to Q)$ is denoted by $P \leftrightarrow Q$

And read as P if and only if Q

The table (1-5) is the truth table of the statement $P \leftrightarrow Q$

| | | | | $P \leftrightarrow Q$ |
|---|---|---|---|---|
| P | Q | $P \longrightarrow Q$ | $Q \longrightarrow P$ | $(p \longrightarrow Q) \wedge (Q \longrightarrow p)$ |
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

**(1-5)**

**So $P \leftrightarrow Q$ is True in only two cases if the two compound statements same true together or two compound statements are false together**

**Example 2**

A) $X=-1, X=4 \leftrightarrow X^2-3X-4=0$

B) $X^5=-32 \leftrightarrow X=-2$

## [1-4] Implication

We will explain the implication by the following two cases:

The first case: One side implication denoted by $\Rightarrow$

Let's symbolize $x = 3$ by P and $x^2 = 9$ by Q

If $x = 3$ is True this implies that $x^2 = 9$

then $P \Rightarrow Q$

and $x^2 = 9$ if then $x = \pm 3$ then $Q \not\Rightarrow P$

The second case: Two sides implication denoted by $\Leftrightarrow$

Let's symbolize $x = 3$ by P and $x^3 = 27$ by Q

If $x = 3$ is True this implies that $x^3 = 27$

Then $P \Rightarrow Q$

And if $x^3 = 27$ is True, this implies that $x = 3$

Then $Q \Rightarrow P$

$Q \Rightarrow P \;\wedge\; P \Rightarrow Q$ mean that $Q \Leftrightarrow P$

## Example 3

Choose a suitable symbol ( $\Leftrightarrow$ , $\Rightarrow$ ) for the following statements.

A) $x^3 = 8$ , $x = 2$

B) $x > 5$ , $x > 2$

C) $X^2 \geq 0$ , $X \leq 0$

D) P: ABCD is a quadrilateral in which two diagonals bisect each other, ABCD is a parallelogram.

## Solution

A) $X^3=8 \Leftrightarrow X=2$

B) $X>5 \Rightarrow X>2$

C) $X \leq 0 \Rightarrow X^2 \geq 0$

D) $Q \Leftrightarrow P$

## Equivalent Statements

**Definition 1-1**

The statement P is said to be equivalent to statement Q if both have the same truth table and denoted by $Q \equiv \sim P$

### Example 4

Prove that $P \longrightarrow Q \equiv \sim P \vee Q$

**Solution**

We construct the following table:

| P | Q | $\sim P$ | $P \longrightarrow Q$ | $\sim P \vee Q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

$\equiv$

## [1-1] Exercises

**Q1**

Determine if the following statement is true or false and mention the reason:

A) 25 is divisible by 5 and 25 is divisible by 7.

B) 25 is divisible by 5 or 25 is divisible by 7.

C) 7 is not a prime number or 4 is a prime number.

D) Diagonals of the square are perpendicular or diagonals of the rectangle are perpendicular.

**Q2**

Use $\Rightarrow$ or $\Leftrightarrow$ to connect the statement in the following table to obtain True statements:

| Statement Q | Symbol | Statement P |
|---|---|---|
| A quadrilateral is a triangle | | The diagonals of the quadrilateral bisect each other |
| A quadrilateral is Rhombus | | The side of the quadrilateral are congruent |
| A quadrilateral is a rectangle | | The angles of the quadrilateral are right |
| $a=0 \vee b=0$ | | $a.b=0$ , $a,b \in R$ |
| $x^2 = 9$ | | $x = -3$ |
| A quadrilateral is a square | | The angles of the quadrilateral are right |
| $x = 5$ | | $x^2 = 25$ |
| $x = -5$ | | $x^3 = -125$ |
| ABC is a quadrilateral | | ABC is Isosceles Triangle |
| $(x-1)(x-2)=0$ | | $x=1 \vee x=2$ |

**Q3**

Prove that:

1) $P \longrightarrow Q \equiv \sim Q \longrightarrow \sim P$

2) $\sim(P \longrightarrow Q) \equiv P \wedge \sim Q$

**Q4**

If O is True, Q is True and S is False which of the following statements are true and which ones are false?

1) $(P \longrightarrow Q) \vee S$

2) $(P \longleftrightarrow S) \wedge P$

3) $(S \longrightarrow Q) \wedge P$

4) $(S \longleftrightarrow S) \vee S$

**Q5**

Chose the correct answer:

Consider that P and S are two statements in the followings:

1) $P \rightarrow \sim P$ is equivalent to

A) $P \rightarrow P$  B) $\sim P \rightarrow P$  C) $\sim P$  D) $\sim P \wedge P$

2) $S \longleftrightarrow S$ is a statement

A) Always True  B) True once only  C) Always False

D) False only once

3) The negation of the statement is: $\sim S \vee$ « $9 > 5 + 3$ »

A) $\sim S \vee$ « $9 < 5 + 3$ »  B) $\sim S \vee$ « $9 \geq 5 + 3$ »

C) $S \wedge$ « $9 \leq 5 + 3$ »  D) $\sim S \wedge$ « $9 \leq 5 + 3$ »

## [1-5] Open Sentences

We learned that logical statements can be classified either True or False but if we examine the following sentences:

A) x is an integer greater than 0 denoted by P(x)

B) $y+1=3$ which is denoted by Q(x)

C) a+b=6 such that a,b are integers which denoted by G(a,b)

D) ……… is one of Iraq Cities.

We find that it's impossible to consider any of the sentences as Logical statement. But if we substitute g instead of x in the sentences A it becomes (g is an integer greater than zero) and it's the s a True statements and if we substitute a value for (Y) in the sentence (B) to make it False statements and if we substitute 3 instead of a and b in the sentence C we obtain the statement (6=3+3) which is True write a suitable name in the blank of sentences D to obtain a True statements.

**Definition (1-2)**

1) The variable: is a symbol representing the value of a group of objects from the set of substituion.

2) Open sentence: is a sentence contains one or more variables changes to logical statements when a value is replaced instead of the variable from the set of substituion.

## [1-6] Equivalence of open sentences

Let P(X): 2X=4

Q(x): X-1=1

Suppose the substitution set for each is the set of integers (Z) we notice the solution set for P(x) is {2} and the solution set for Q(x) is {2} the open sentences Q(x) and P(x) are said to be equivalent since their solution set are equal.

**Example 5**

If $P(X):X=2$

$Q(X):X^2=4$

and the substitution set for each is the set of integers (Z) are P(x) and Q(x) equivalent?

**Solution**

Notice that the solution of P(x) is {2} and the solution set of Q(x) is {2,-2} since $\{2,-2\} \neq \{2\}$, P(x) and Q(x) are not equivalent.

**Definition (1-3)**

The negation of open sentences P(x) is `not P(x)` or any open sentence equivalent it denoted by : $P(x)$.

**Example 6**

Suppose that the substitution set for each of the following is the set of integers (Z):

| P(x) Open Sentences | It's Negation ~P(x) |
|---|---|
| $x^2-4=0$ | $x^2-4\neq 0$ |
| x is even number | x isn't even number |
| $x=4$ and $x+1\neq 6$ | $x\neq 4$ and $x+1=6$ |

## Exercises 1-2

**Q1)** Write the solution set for each of the following open statements:

| Open sentences | Substitution set |
|---|---|
| **A)** $x > 3$ | N |
| **B)** $X^2-11X+30=0$ | {10,6,5,3} |
| **C)** $(X-1)(X-\frac{3}{5})(X-30)=0$ | Z |
| **D)** $(X-1)(X-5)=0$ و $X>4$ | N |
| **E)** x is not dvisible by 4 | {10,8,6,4,2} |
| **F)** $X+5 \geq 0$ | Z |

**Q2)** There exist a pair of open sentences in each of the following. Determine which pairs represent equivalent open sentences with substitution set Z.

**A)** $X-3=3$ , $3X-5=X+7$ **B)** $x = 2$ , $x^2 = 4$

**C)** $X=3$ , $X^2=9$ $X=-3$

**D)** $X+1=0$ , $(X+1)(2X+1)=0$ **E)** $X^2-6X+5=0$ , $(X-1)(X-5)=0$

**F)** $x$ is greater than -1 less than 1 , $x = 0$

**G)** $(X-1)(X-2)=0$ , $3 > X \geq 0$

**Q3)** Negate each of the followings and find the solution set for the negated sentence using the substitution set {1,2,3,4,5}

**A)** $2x = 4$ **B)** $x + 4 = 7$ **C)** $(X-3)(X-4)=0$

**D)** $X+2=4$ $X^2 \neq 9$ **E)** $X-1=4$ $X^2=16$

**Q4)** if x and y are elements of teh set {0,1,2,3,4,5,6,7,8,9} , write the soluton set for each of the open sentences in the form of ordered pair

A) $X-Y=3$ B) $x + y = 5$

## [1-7]Quantified Proposition

### [1-7-1] Universal Quantified Statement and Existential Quantified Statement

Mathematical logic often uses symbols instead of words, in this lesson we will learn two symbols:

First: If we want to state that every element of set A makes f(x) a True statement we say it for all an in A such that F(a) is True statements

Or for all a $\in$ A such that f (a) is True statement. The symbol $\forall$ is called the universal quantifier and the statement $\forall a \in A$ , F(a) is True then it's called universal quantified statement

For example

$(x+1)^2 = x^2+2x+1$ is True for every Natural Number to be substituted for x, and it can be written as $\forall x \in N$ such that $(x+1)^2 = x^2+2x+1$.

Second: If we want to state that some of the elements in the set A make G(x) a true statement then we say:

There exists on an element in A makes G(x) a True statement and denoted by $\exists b \in A$ such that G(b) is a True statement the symbol $\exists$ is called the Existential Quantifier and the statement $\exists b \in A$ , G(b) is Existential Quantified Statement.

For instance: If we want to state the equation x+1=2 has a solution in an integer (Z):

Then we say $\exists x \in Z$ such that x+1=2, and read it as:

`There exists an element $x \in Z$ such that the equation x+1=2 is True`

### [1-7-2] Negation of Quantified Statements

When negation the Quantified Statements we consider the following:

Every statement can only and only be true or false

- For Example, If we want to Negate the statement

Every perpendicular drawn from the center of the circle bisect the chord

We say:

There exist at least one chord drawn on the circle such that the perpendicular line which passes through the center bisects it.

- If we want to prove the mistake of:

Every Natural Number divisible by 2 is divisible by 6,

It's enough to mention the correct statement:

There exist at least a Natural Number divisible by 2 and not divisible by 6.

- If we want to negate:

There exist at least one Right triangle doesn't satisfy Pythagorean theorem

We say Every Right triangle satisfy the Pythagorean theorem.

From the previous examples we conclude:

$\sim$ [P(x) is that $\forall$ x $\in$ X] $\equiv \sim$ P(x) such that $\exists$ x $\in$ X

$\sim$ [P(x) is that $\exists$ x $\in$ X] $\equiv \sim$ P(x) such that $\forall$ x $\in$ X

**Example 7**

Negate each of the following:

1) $\forall$ x such that P(x) is :

P(x): if x is a Natural Number then $x>0$

2) $\exists$ x such that P(x) is;

P(x): x is positive even number

3) $P \vee [\exists x \in R: x+3 \geq 5]$

**Solution**

1) $\sim [P(x)$ that is $\forall x] \equiv \sim P(x)$ that is $\exists x \sim$

$\sim P(x) : \exists x$ a natural number such that $x \leq 0$

By words: there exist a natural number less than or equal to zero.

2) $\sim P(x)$ that is $\forall x] \equiv \sim P(x)$ that is $\forall x$

$\sim P(x)$ " $\forall x$ even numbers x is not positive.

By words: for every even number x, x is not positive.

3) $\sim \ P \wedge \ ( X+3 < 5 : \forall \ X \in R )$

## [1-7-3] Tautology

Let's consider the logical statements P, if all the logical possibilities of this statements are true then P represents a Tautology.

**Example 8**

Let P be a statements does $p \vee \sim p$ represent a Tautology?

**Solution**

| P | ~ P | P ∨ ~ P |
|---|---|---|
| T | F | T |
| F | T | T |

Then P represents Tautology

**Note:** **If all the truth values are False then it represents a (Contradiction).**

## Exercises 1-3

**Q1)** Negate each of the following without using `Not True`:

A) All the similar triangles are Isosceles.
B) Some of the similar triangles are non-congruent.
C) If the triangle is Right angle, then it is Isosceles.
D) Some equations have no solution.
E) Every four-side figure is a rectangle.
F) Q: $\forall X \in N : X^2 = 25$
G) $( \forall X \in R : X < 8 ) \wedge P$

**Q2)** Determine the truth value for each of the following:

A) $\forall$ x such that p(x) is:
P(x): if x is a natural number then $x^2 = x$

B) $\exists$ x such as P(x) is:
P(x): x is natural number $x^2 = x$

C) $\forall$ x such that P(x) is:
P(x): if x is negative then $x^2$ is positive.

D) Q, P are two statements: $Q \wedge P \rightarrow Q$ is a Tautology

E) P is a statement then $\sim P \wedge P$ is a contraction.

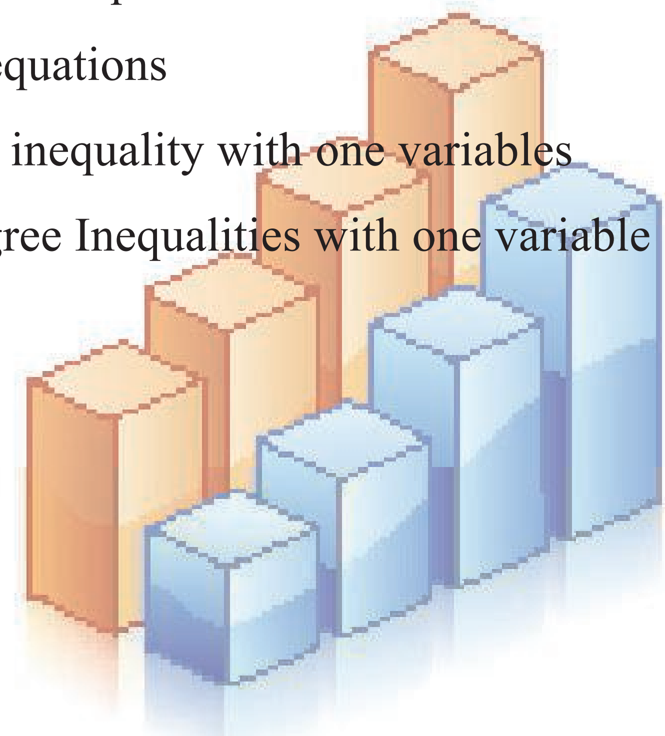
F) P and Q are two statements $( P \leftrightarrow Q ) \leftrightarrow ( P \leftrightarrow Q )$ is a Tautology.

# 2 Chapter 2: Equations and inequalities



Aims and Skills

- Learning the Absolute value
- Solving Absolute value equation
- Solving System of equations
- Solving first-degree inequality with one variables
- Solving Second-degree Inequalities with one variable



## [2-1] Absolute Value

**Definition (2-15)**

The absolute value of the real number x denoted by |X| is defined as:

$$|x| = \begin{cases} X, & \forall\ X > 0 \\ 0, & X = 0 \\ -X, & \forall\ X < 0 \end{cases}$$

**Example 1**

Solve each of the following Absolute value by definition:

A) $\left|3-\sqrt{10}\right|$ B) $|X-3|$ such that $X \in R$

**Solution**

Since

A) $3=\sqrt{9}<\sqrt{10}$ $\therefore \left|3-\sqrt{10}\right| = \sqrt{10}-3 > 0$

$\therefore \left|3-\sqrt{10}\right| = \sqrt{10}-3 > 0$

B) $$|X-3| = \begin{cases} X-3, & \forall\ X > 3 \\ 0, & X = 3 \\ -X+3, & \forall\ X < 3 \end{cases}$$

The following properties of Absolute value can be concluded from the definition:

1) $\forall\ X \in R$ then $|X| \geq 0$

2) $\forall\ X \in R$ then $|-X| = |X|$

3) $\forall\ X \in R$ then $-|X| \leq X \leq |X|$

4) $\forall\ X \in R$ then $|X|^2 = X^2$

5) $\forall\ X \in R$ then $|X.Y| = |X|.|Y|$ $\left|\frac{X}{Y}\right| = \frac{|X|}{|Y|}$ such then $Y \neq 0$

6) $\forall\ X,Y \in R$ then $Y| \leq |X| + |Y|$

7) $\forall\ a > 0\ X \in R$ if $|X| \leq a$ then $a \leq X \leq a$

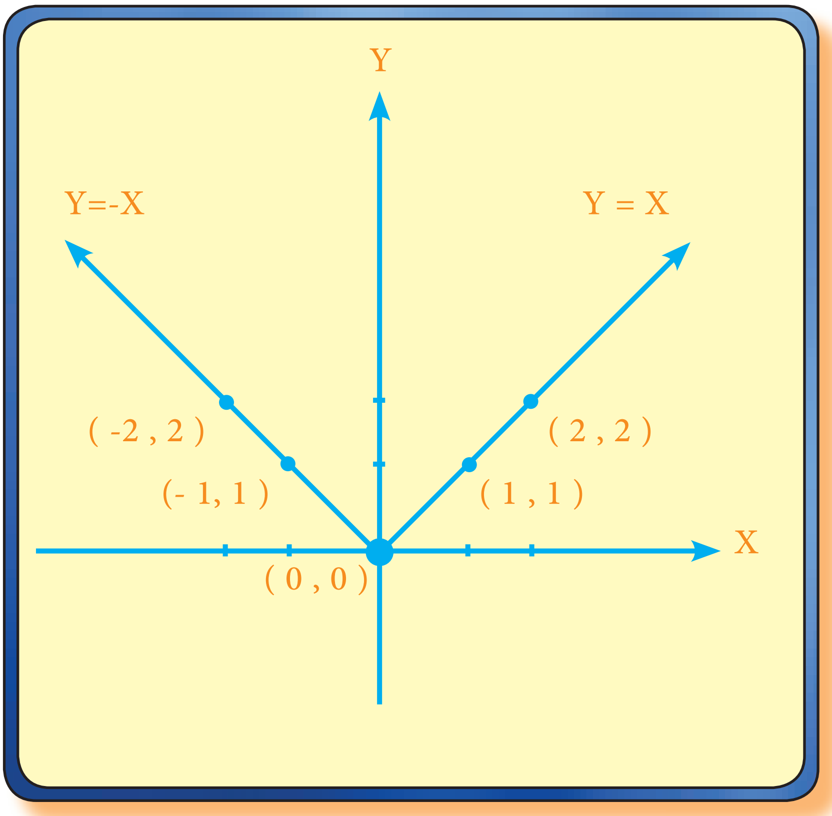
## Example 2

Graph $y = |x|$

$$Y = \begin{cases} X , & X>0 \\ 0 , & X=0 \\ -X , & X<0 \end{cases}$$

## Solution

According to definition (2,15):




Y=| X |

**First:** Graph the line

$x \geq 0$ , Y = X

| X | Y | ( X , Y ) |
|---|---|---|
| 0 | 0 | ( 0, 0 ) |
| 1 | 1 | ( 1 , 1 ) |
| 2 | 2 | ( 2 , 2 ) |

**Second:**

X<0 , Y=-X :

| X | Y | ( X , Y ) |
|---|---|---|
| 0 | 0 | ( 0 , 0 ) gap |
| -1 | 1 | ( -1 , 1 ) |
| -2 | 2 | ( -2 , 2 ) |

## Example 3

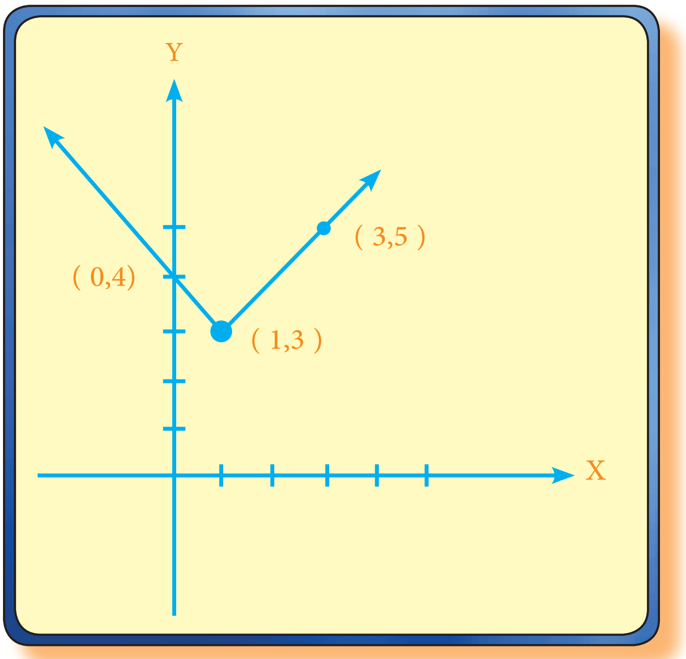
Graph $Y=|X-1|+3$

## Solution

By the definiton (2-15)

$$Y=\begin{cases}(X-1)+3\ , & \forall X\geq 1\\ (-X+1)+3\ , & \forall X<1\end{cases}$$

$$\therefore\ Y=\begin{cases}X+2\ , & \forall X\geq 1\\ -X+4\ , & \forall X<1\end{cases}$$




$Y=|X-1|+3$

**First:** graph straight line:

$\forall X\geq 1\ ,\ Y=X+2$

| X | Y | ( Y , X ) |
|---|---|---|
| 1 | 3 | ( 1 , 3 ) |
| 3 | 5 | ( 3 , 5 ) |

**Second:** Graph the straight line: $\forall X<1\ ,\ Y=-X+4$

| X | Y | ( X, Y ) |
|---|---|---|
| 1 | 3 | ( 1 , 3 ) gap |
| 0 | 4 | ( 0 , 4 ) |

**Example 4**

Find the solution set of the equation $|3x+6| = 9$ such that $X \in R$:

**Solution**

From the definition of the absolute value we conclude that:

$$|3X+6| \begin{cases} 3X+6 & \text{such then} \quad -2 \leq X \quad 0 \leq 3X+6 \\ -(3X+6) & \text{such then} \quad -2 > X \quad 0 > 3X+6 \end{cases}$$

This equation is equivalent to the system:

$$\begin{cases} 3X+6=9 & \textit{the substituon set is} \quad \{X: X \geq -2\} \\ -3X-6=9 & \textit{the substituon set is} \quad \{X: X < -2\} \end{cases}$$

We can consider this a system of equations with two variables x,y where the cofficient of y = 0. And the solution set of this system is:

$S_1 = \{1\}$ ، $S_2 = \{-5\}$ then the solution set is

$S = S_1 \cup S_2 = \{1, -5\}$

## Example 5

Find the solution set of the equation $\forall X \in R$ ، $X^2 |X| - 8 = 0$ .

## Solution

By the definition of the absolute value $X^2 |X| - 8 = 0$ is equivalent to system:

$X^3 - 8 = 0 , \forall X \geq 0 \Rightarrow X^3 = 8 \Rightarrow X = 2$

$S_1 = \{ 2\}$

$- X^3 - 8 = 0 , \forall X < 0 \Rightarrow X^3 = - 8 \Rightarrow X = -2$

$S_2 = \{ -2\}$

$S = S_1 \cup S_2 = \{ 2 , -2\}$

## Example 6

Find the solution set for $\forall X \in R$ ، $X^2 + |X| - 12 = 0$ .

## Solution

By the definition of absolute value $X^2 + |X| - 12 = 0$ is equivalent to

$X^2 + X - 12 = 0 , \forall X \geq 0 \Rightarrow ( X + 4 )( X - 3 ) = 0$

Either x = -4 (neglected) why? or x = 3

Then $S_1 = \{ 3\}$

$X^2 - X - 12 = 0 , \forall X < 0 \Rightarrow ( X - 4 )( X + 3 ) = 0$

Either x = -4 (negleded) why? or x = -3

Then:

$S_2 = \{ -3\}$

$S = S_1 \cup S_2 = \{ 3 , -3\}$

## [2-3] Solving System of Equations with two variables

You have learned to solve two first degree equations with two variables graphically if $S_1$ is the solution of the first equation and $S_2$ is a solution of the second equation then $S = S_1 \cap S_2$ if the two equations are related by the connective "and" and if the connective is "or" the solution is $S = S_1 \cup S_2$

**Example 7**

Solve the following system by analytically and graphically in R

$X - 2Y = 5$

$2X + Y = 0$

**Solution**

By analytically

We multiply the 2nd equation by 2

$$X - 2Y = 5$$

$$4X + 2Y = 0 \quad \text{we add the equations}$$

---

$5X=5 \Rightarrow X = 1$

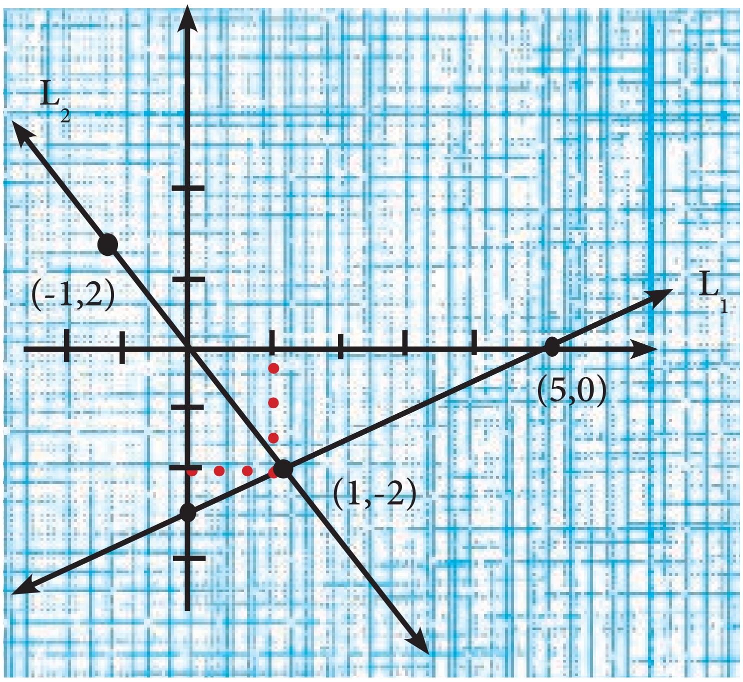
Substitute in equation (1)

$1 - 2Y = 5$

$\Rightarrow Y = -2$

S.S = {1,-2} and it represents the intersection point of two lines.

Graphically: The first line $X - 2Y = 5 : L$




$(L_1)$: x-2y

| X | Y | ( X , Y ) |
|---|---|---|
| 0 | -5/2 | ( 0 , -5/2 ) |
| 1 | -2 | ( 1 , -2 ) |
| 5 | 0 | ( 5 , 0 ) |

$(L_2)$: 2x+y

| X | Y | X, Y |
|---|---|---|
| 0 | 0 | (0,0 ) |
| 1 | -2 | (1 ,-2) |
| -1 | 2 | (-1 ,2) |

**Example 8**

Solve the following system of equations in R:

$X - Y = 1$

$X^2 + Y^2 = 13$

**Solution**

This system can be solved by a substituent such that we find x in terms of y or the opposite then

$x = y + 1$ will be substituted in the second equation

$(1+y)^2 + y^2 = 13 \Rightarrow 2y^2 + 2y - 12 = 0$ (dividing by 2)

$y^2 + y - 6 = 0 \Rightarrow (y+3)(y-2)=0$

$y + 3 = 0 \Rightarrow y = -3 \Rightarrow x = -2 \Rightarrow (-2,3)$

$y - 2 = 0 \Rightarrow y = 2 \Rightarrow x = 3 \Rightarrow (3,2)$

$\therefore S=\{(-2,-3),(3,2)\}$

**Example 9**

Solve the following system by elimination in R

$2x^2 - 3y^2 = -46\ ,\ x^2 + y^2 = 17$

**Solution**

Since both equations same degree it can be solved by substitution and elimination (up to student)

$x^2 + y^2 = 17$ We multiply the first equation by 3

$2x^2 - 3y^2 = -46$

$3x^2 + 3y^2 = 51$ by adding

$2x^2 - 3y^2 = -46$

$5x^2 = 5 \Rightarrow x^2 = 1 \Rightarrow x = \mp 1$

$x=1 \Rightarrow (1)^2 + y^2 = 17 \Rightarrow y^2 = 16 \Rightarrow y = \mp 4 \Rightarrow (1,4),(1,-4)$

$x=-1 \Rightarrow (-1)^2 + y^2 = 17 \Rightarrow y^2 = 16 \Rightarrow y = \mp 4 \Rightarrow (-1,4),(-1,-4)$

$S = \{(1,4),(1,-4),(-1,4),(-1,-4)\}$

Summary

1) if two equations are the same degree (first on second degree) the system can be solved by

*elimination *substitution

2) if one of the equation is first degree and the other second degree the system can be solved by substitution

## [2-4] Intervals

Let $a , b \in R$ , $a < b$

**1)** The real Numbers Set is the set of to be :

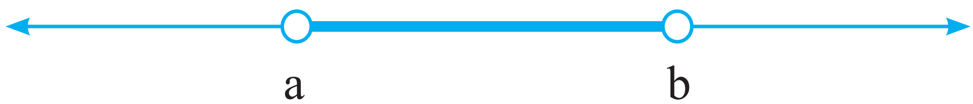
$\{X : X \in R , a \leq X \leq b\}$

The closed interval from a to b and denoted by [a,b] and represented on number line as the figure (2,1) , we denoted the start point of the line segment by (a) and the end point by (b) notice that endpoints of the interval [a,b] are finite


(2-1)

**2)** The set $(a,b) = \{X : X \in R , a < X < b\}$ is said to be the open interval from (a) to (b) and represented on the number line as in the figure (2-2)

Notice that $b \notin (a , b)$ , $a \notin (a , b)$ in this case and shown by empty circles



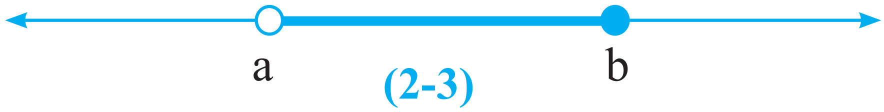

(2-2)

3) And both of the sets

$(a, b] = \{X : X \in R, a < X \leq b\}$

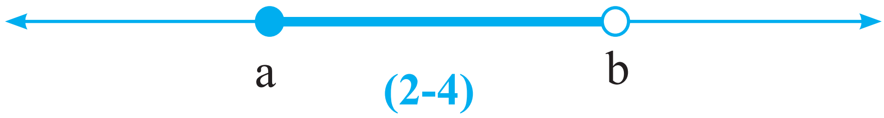
$[a, b) = \{X : X \in R, a \leq X < b\}$

Said to be half-open intervals, such that a<b and the first set is represented on the number line as in the figure (2-3)




(2-3)

And the second set represented on the number line as in the figure (2,4)

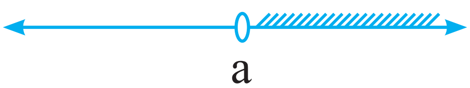



(2-4)

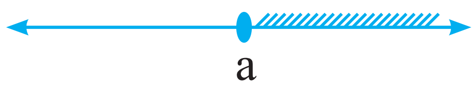
4) The set of real numbers greater than or equal to (a) is;

$\{X : X \in R, X \geq a\}$

is represented on the figure (2-5) and the set $\{X : X \in R, X > a\}$ represented on the figure (2-6)




(2-6)




(2-5)

Note: The sets (4) and (5) are infinite numerical sets called (Rays)

5) The set of real numbers less than or equal to (a) is, $\{X : X \in R, X \leq a\}$

Represented on the figure (2-7) and the set $\{X : X \in R, X < a\}$ represented on the figure (2-8)

(2-8)

(2-7)

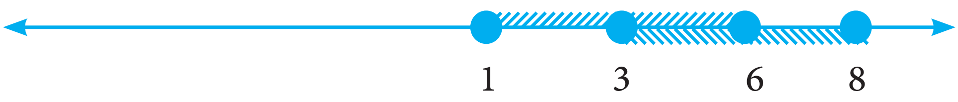
## Example 1

Let x=[1,6] , y=[3,8] represent the following on the number line

1) $X \cap Y$    2) $X \cup Y$

3) $X - Y$    4) $Y - X$

## Solution




1 ) $X \cap Y = [3, 6]$

2 ) $X \cup Y = [1, 8]$

3 ) $X - Y = [1, 3)$

4 ) $Y - X = (6, 8]$

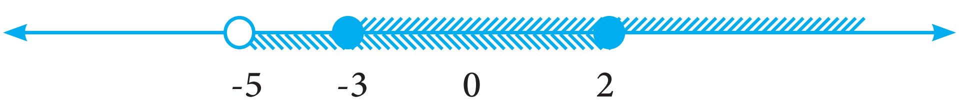
## Example 2

Represent the following on the number line

1) $\{X : X \geq -3\} \cup (-5, 2]$

2) $\{X : X \geq -3\} \cap (-5, 2]$

## Solution




$\therefore$ 1 ) $\{X : X \geq -3\} \cup (-5, 2] = \{X : X > -5\}$

2 ) $\{X : X \geq -3\} \cap (-5, 2] = [-3, 2]$

## [2-5] Solving first degree inequalities in one variable

The inequality contains x as a variable denoted as f(x)<g(x) such that f(x), g(x) are two open sentences is said to be Inequality in One Variable.

As you know from your previous study the solution set of the inequality is the set of values you substitute instead of x which makes the statement true equivalent inequalities defined similarly to equivalent equations.

**Definiton (2-16)**

The inequality f(x)<g(x) is equivalent to the inequality h(x)<I(x) if both of them have the same solution set

We will consider inequalities f(x), g(x) which have more than one term

**Example 1**

Solve the inequality 3x+1<x+5 on R and represent on a number line

**Solution**

$3X + 1 < X + 5$

$3X + 1 + (-X) < X + 5 + (-X)$

$\Rightarrow \quad 2X + 1 < 5$ *Inequality properties*

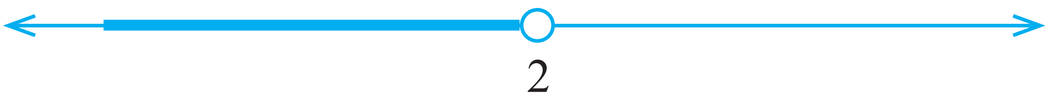
$\Rightarrow \quad 2X + 1 + (-1) < 5 + (-1)$

$\Rightarrow \quad 2X < 4$ *Inequality properties*

$\Rightarrow (2X)(\frac{1}{2}) < 4(\frac{1}{2})$

$\Rightarrow \quad X < 2$

The solution set $= \{X : X \in R ، X < 2\}$




If we connect two inequalities by the connective “and” then the value of “x” which satisfies this system must belong to the solution set of first inequality $S_1$ and the second Inequality $S_2$ which is $S = S_1 \cap S_2$ this means the solution set of the system is $S = S_1 \cap S_2$ we conclude from this the solution set of the system of inequalities related by connective “or” is $S = S_1 \cap S_2$

**Example 2**

Solve the inequality $5x + 11 < 1$ and $2x + 3 < 6$ in R, and represent on a number line.

**Solution**

The solution set of the first inequality is $S_1 = \{ X : X < -2 \}$

The solution set of the second inequality $S_2 = \{X:X < \frac{3}{2}\}$

The solution set of the system is: S

$$S = S_1 \cap S_2 = \{X : X < -2\} \cap \{X : X < \frac{3}{2}\}$$

$$S = \{X : < -2 \text{ and } \frac{3}{2} > X\}$$


The common elements between $S_1$ , $S_2$ are the same $S_1$

$$S_1 \cap S_2 = S_1 = \{X : X < -2 ، X \in R\}$$

**Example 3**

Solve the previous example using the connective "or" instead of "and" find the solution set and represent on the number line

**Solution**

The solution set of the system $5x + 11 < 1$ or $2x + 3 < 6$ is

$$S_2 \cup S_1 = \{X : X < \frac{3}{2} \quad X < -2\}$$

$$S = \{X : X \in R \ , \ X < \frac{3}{2}\}$$


Note that the common elements of $S_1$ or $S_2$ is the element of $S_2$

**Example 4**

Find the solution set of the inequality $|X-2| > 5$ in R

**Solution**

$$|X-2| = \begin{cases} X-2, & \forall \; X \geq 2 \\ 2-X, & \forall \; X < 2 \end{cases}$$

$2-X > 5$ or $X-2 > 5 \Leftrightarrow |X-2| > 5$

Then we find the solution set is

$S_1 \cup S_2 = \{X : X \in R ، X > 7\} \cup \{X : X \in R ، X < -3\}$

**Example 5**

Solve the inequality $|x+1| \leq 2$ such that $x \in R$

**Solution**

Note that this inequality can be solved according to property 7 from page 52

Then $|x+1| \leq 2 \Rightarrow -2 \leq x+1 \leq 2$

Adding (-1 to the inequality

$$-2+(-1) \leq x+1+(-1) \leq 2+(-1)$$

$$-3 \leq x \leq 1$$

$$\therefore s = [-3, 1]$$

# [2-6] Solving Second Degree Inequalities in one Variable

## Theorem

Let (a) be a positive real number then

1) The solution set of the inequality $X^2 \leq a^2$ is the interval [-a,a]

2) The solution set of the inequality $X^2 < a^2$ is the interval (-a,a)

**Proof 2:**

$(X - a)(X + a) < 0 \Leftarrow X^2 - a^2 < 0 \Leftarrow X^2 < a^2$

If we have $a.b < 0$ then $\begin{cases} \text{either } b < 0 \text{ and } a > 0 \\ \text{or } \quad b > 0 \text{ and } a < 0 \end{cases}$

Similarly

$[(X - a) < 0 \quad (X + \quad ) > 0]$ or $[(X - a) > 0 \quad (X + a) < 0]$

$\Longrightarrow [X < a \quad X > -a]$ or $[X > a$ and $X < -a]$

$\Longrightarrow (-a, a) \cup \phi = (-a, a)$

We can prove part 1 in the theorem similarly (left to student)

**Example 6**

If $X^2 < 9$ then the solution set of the inequality is (-3,3) and

If $X^2 \leq 9$ then the solution set of the inequality is [-3,3]

But if we have $X^2 > 9$ then the solution set of the inequality is $R / X^2 \leq 9$ which means $SR / (-3,3)$

And the solution set of the inequality $X^2 \geq 9$ is $R / X^2 < 9$ which means $SR / (-3,3)$

**Example 7**

Find the solutions set off the inequality $5 \leq |2X+5| < 7$

**Solution**

$$|2X+5| = \begin{cases} 2X+5, & \forall \; X \geq \frac{-5}{2} \\ -(2X+5), & \forall \; X < \frac{-5}{2} \end{cases}$$

The inequality $7 > |2X+5| \geq 5$ is equivalent to the system

$[\; 7 > -(2X+5) \geq 5 \;]$ or $[\; 7 > 2X+5 \geq 5 \;]$

$\Rightarrow [12 > -2X \geq 10]$ or $[2 > 2X \geq 0]$

$\Rightarrow \quad [-6 < X \leq -5]$ or $[\; 1 > X \geq 0]$

The solution set is $(-6,-5] \cup [0,1)$

Summary

In order to solve first degree inequality in one variable :

Define the absolute value if exist

Use the properties of real numbers such as:

*additive inverse* → *adding* → *identity element of addition* (0) →

multiplicative inverse → *adding* → *identity element of multiplication* (1)

After these steps we find the solution set of the inequality in R

## Exercises (2-4)

**Q1)** If $A = [-2, 5)$, $B = \{ X : X \geq 1 \}$

Then find $A \cup B$ ، $A \cap B$ ، $A - B$ ، $B - A$

**Q2)**

**A)** Graph the function $Y = |X + 2| - 5$

**B)** $y = 3 - |x + 1|$

**Q3)** Find the solution set of each of the following equations and check your answer

**A)** $|4X + 3| = 1$

**B)** $X|X| + 4 = 0$

**C)** $X^2 - 2|X| - 15 = 0$

**D)** $|X^2 + 4| = 29$

**E)** $x|x + 2| = 3$

**F)** $|2x + 1| = x$

**Q4)** Find the solution set of the following systems of equations

**A)** $2X + Y = 4$ ، $X - Y = -1$ *graphically*

**B)** $4X + 3Y = 17$ ، $2X + 3Y = 13$ *elimination*

**C)** $X - Y = 1$ ، $5X^2 + 2Y^2 = 53$

**D)** $2X^2 - Y^2 = 34$ ، $3X^2 + 2Y^2 = 107$

**Q5)** Find the solution set of the following inequalities

**A)** $|X - 6| \leq 1$

**B)** $2 \leq |X + 1| \leq 4$

**C)** $-9 < |2X - 3| - 12 \leq -3$

**D)** $2X^2 \leq 8$

**E)** $3X^2 - 27 > 0$

# 3 Chapter 3: Roots and Exponent



Aims and skills

At the and of this chapter the student will acknowledge:

- The rules of exponent in integers
- How to solve exercises including exponents
- How to solve basic exponent equations
- The Roots
- How to solve exercises including roots
- The conjugate numbers
- The real function and how to find the domain
- The properties of the exponential function
- Representing some real functions

| Term | Mathematical Symbol |
|---|---|
| Exponent | $a^x$ |
| Root | $\sqrt[n]{\ }$ |
| Exponential Function | $f_a(x) = a^x$ |
| Conjugate Numbers | $\sqrt{a} \pm \sqrt{b}$ |

Mathematics was known in three sections

1) Accounting
2) Completing and balancing
3) Geometry

It becomes in this perfect and complete form at the and of the eighteenth and the beginning of the nineteenth century. Arab and Muslim Mathematicians discovered a lot of relations between the three sections some of these examples are:

Omar Ibn Ibrahim Al Khayyâm (1044-1122) born and died Nishapur in Persia, he is most notable for this work on geometry “A commentary on the difficulties concerning the postulates of Euclid’s element’ and <<maqalah fi al cabr>>.

Muhammad ibn Musa al Khawarizmi (781-850) born in Khwarezm the moved the Baghdad his most important work was “ The Compendious book and calculating by Completion and Balancing “ and he discovered Algebraic methods to solve first and second-degree equations in one or two variables.

You have studied the exponents and roots in secondary classes and you learned the power of a number if the exponent is a natural number also you have learned the square roots for a positive number and the properties of it.

## [3-1] Exponent of Integers

### Indices

**Definition (3-1)**

If $a \in R, n \in Z$

1) $n = a \times a \times \dots \times a$ (multiplied by itself n-times)

2) Special case $a^0 = 1$

3) $a^{-n} = (a^{-1})^n$ , $a^{-1} = \frac{1}{a}$ , $a \neq 0$

### The properties of exponents:

$\forall\, a, b \in R, \forall\, n, m \in Z$ (Z is the set of integers), $a \neq 0$ , $b \neq 0$ then:

1) $a^n \times a^m = a^{m+n}$ (if bases are same exponents will be added )

2) $a^{-n} = \frac{1}{a^n}$

3) $\frac{a^m}{a^n} = a^{m-n}$ (in a division, if bases are same powers will be subtracted )

4) $(a^m)^n = a^{mn}$ (power rule)

5) $(a \cdot b)^n = a^n \cdot b^n$

6) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Note: $a^n$ is said to be the n-th power of a, a is called the base and n is called power and we say a is raised to the power of n.

## Roots

**Definition (3-2)**

If $a \in R , n \in N , n > 1$ then every real number x satisfies the equation $x^n = a$ is said to be the nth root of the number a and its denoted by $\sqrt[n]{a}$ or $\frac{1}{a^n}$

From the definition, we conclude the following:

1) $\forall n \in N , n > 1 , \sqrt[n]{0} = 0$

2) If (n) is a natural number and (a) is a positive real number then the both of $X = -\sqrt[n]{a}$ , $X = \sqrt[n]{a}$ satisfies $X^n = a$ the equation

3) If (n) is even a natural number and (a) is a negative real number then there is no real number satisfies the equation

4) If (n) is an odd natural number and (a) is a real number then there is only one real number satisfy the equation

**Theorem (3-1)**

If $a , b \in R , n \in N , n > 1$ then

1) $\sqrt[n]{a.b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ (such that $b \geq 0 , a \geq 0$ if (n) is even number)

2) $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ $\begin{cases} 0 < b , 0 \leq a & \text{if "} n \text{" is even number} \\ b \in R/\{0\} , a \in R & \text{if "} n \text{" is odd number} \end{cases}$

## Example 1

Find the value of $\dfrac{8^{-3} \times 18^{2}}{81 \times 16^{-2}}$

$(-a)^m = a^m$ if m is even.

$-a^m$ if m is odd

example:

$(-1)^{25} = -1$

$(-1)^{64} = 1$

## Solution

$$\frac{8^{-3} \times 18^{2}}{81 \times 16^{-2}} = \frac{(2^{3})^{-3} \times (3^{2} \times 2)^{2}}{3^{4} \times (2^{4})^{-2}} = \frac{2^{-9} \times 2^{2} \times 3^{4}}{3^{4} \times 2^{-8}}$$

$$3^{4-4} \times 2^{-9+2+8} = 3^{0} \times 2^{1} = 1 \times 2 = 2$$

## Example 2

If $m ، n \in Z$ then proof

$$\frac{125 \times 15^{m-2} \times 25^{m+n}}{75^{m} \times 5^{2n+m}} = \frac{5}{9}$$

## Solution

left side $= \dfrac{125 \times 15^{m-2} \times 25^{m+n}}{75^{m} \times 5^{2n+m}} = \dfrac{5^{3} \times (5 \times 3)^{m-2} \times (5^{2})^{m+n}}{(3 \times 5^{2})^{m} \times 5^{2n+m}}$

$$= \frac{5^{3} \times 5^{m-2} \times 3^{m-2} \times 5^{2m+2n}}{3^{m} \times 5^{2m} \times 5^{2n+m}}$$

$$= 5^{3+m-2+2m+2n-2m-2n-m} \times 3^{m-2-m}$$

right side $= 5 \times 3^{-2} = 5 \times \dfrac{1}{3^{2}} = \dfrac{5}{9} =$

## Exercise 3-1

**Q1)** Find the result of the following:

**A)** $\sqrt{\sqrt[3]{64}}$ **B)** $16 + (16)^{-1}$ **C)** $(3)^{-1}+(2)^{-1}$ **D)** $(9)^{0}+(8)^{0}$

**E)** $(a \neq 0)$ ، $3a^{0}$ **F)** $(\sqrt{27}\,)^{\frac{5}{3}}$ **G)** $\dfrac{10^{3} \times 4^{7}}{10^{-5} \times 2^{5}}$ **H)** $\dfrac{2^{-3} \times 4^{-5}}{6^{-1} \times 3^{3}}$

**I)** $(\sqrt[5]{-32}\,)^{-3}$ **J)** $(a+b \neq 0)$ ،$(a + b)^{0}$ **K)** $(3a)^{0}$

**Q2)** Write the following in the simplest form:

**A)** $\sqrt{(\dfrac{3}{4})^{2}\ \dfrac{20a^{3}}{45a}}$ **B)** $(-a)^{4}\ [\dfrac{(-a)^{3}\ \sqrt[6]{729}}{3a}]^{2}$

**C)** $\dfrac{3x^{-5} \cdot y^{2}}{2^{-1}\ y^{-2}}$ ، $x \neq 0$ **D)** $c \neq 0$ ، $\sqrt{25\,b^{2}\,c^{-8}}$

**Q3)** Write the followings such that the denominator becomes 1 without using roots use exponents:

**A)** $\sqrt[5]{x}$ **B)** $\dfrac{1}{b^{5}}$ ، $b \neq 0$ **C)** $\dfrac{b\,c}{d}$ ، $d \neq 0$

**D)** $\sqrt[3]{x} \times \sqrt[4]{x}$ ، $x \geq 0$ **E)** $\dfrac{1}{b^{2} + c^{2}}$ **F)** $\dfrac{4b^{2}}{b^{2}\,c^{-3}}$ ، $b \neq 0$

**Q4)** If   and m is an even integer, which one of the following is correct?

**A)** $a^{m} > 0$    **B)** $a^{m} < 0$

**C)** $a^{m} \geq 0$    **D)** $a^{m} \leq 0$

**Q5)** If   and a is negative number, and m is an odd integer. Which of the followings is correct?

**A)** $a^{m} > 0$    **B)** $a^{m} < 0$

**C)** $a^{m} \geq 0$    **D)** $a^{m} \leq 0$

**Q6)** Prove that:

**A)** $a^{(x-y)z} \cdot a^{(z-x)y} \cdot a^{(y-z)x} = 1$    **B)** $\left[x^{n^2-1} \div x^{n-1}\right]^{\frac{1}{n}} = x^{n-1}$

**Q7)** Prove that:

$$\frac{1}{1 + a^{c-b}} + \frac{1}{1 + a^{b-c}} = 1$$

**Q8)** Prove that:

$$\frac{5 \times 3^{2n} - 4 \times 3^{2n-1}}{2 \times 3^{2n+1} - 3^{2n}} = \frac{11}{15}$$

**Q9)** Simplify the following:

$$\frac{6^{4n-1} \times 27^{2n}}{2^{n+1} \times 8^{n-1} \times 9^{n+2}} , \frac{3^{2+n} + 3^{n+1}}{3^{n} - 3^{n-1}}$$

**Q10)** Prove that:

$$\left[\frac{\left(9^{n+\frac{1}{4}}\right) \times \sqrt{3 \times 3^{n}}}{3\sqrt{3^{-n}}}\right]^{\frac{1}{n}} = 27$$

## [3-2] Solving Basic Exponential Equations

The exponential equations contain a variable in the exponent to solve such kind of equations follow the steps

**1)** In any equation: (( if bases are equal then the exponents are equal to if the base $\neq 1$))

Such that: $a^x = a^y \Rightarrow x = y \, , a \neq 1$

**2)** If $x^n = y^n$ then $x = y$ n is odd

$x = \mp y$ if n is even

**3)** If , $0 = m = n \Leftarrow x^n = y^m$

Notice that the solution of the following equations:

**A)**

$$(x+2)^{-\frac{3}{5}} = \frac{1}{\sqrt[5]{27}} \Rightarrow (x+2)^{-\frac{3}{5}} = 3^{-\frac{3}{5}}$$

$$x + 2 = 3 \Rightarrow x = 1 \Rightarrow$$

$$= \{1\}$$

**B)**

$$x^{\frac{2}{3}} = 3^{-2}$$

$$x^{\frac{2}{3}} = \frac{1}{3^2}$$

$$(x^{\frac{1}{3}})^2 = (\frac{1}{3})^2 \quad \textit{Taking the root of both sides}$$

$$x^{\frac{1}{3}} = \pm\frac{1}{3} \quad \textit{Cubing both sides}$$

$$(x^{\frac{1}{3}})^3 = \pm(\frac{1}{3})^3$$

$$x = \pm\frac{1}{3^3}$$

$$x = \pm\frac{1}{27}$$

$$\{\pm\frac{1}{27}\} =$$

**Example 3**

Solve the equation $2^{x^2-2x+1} = 4^{x+3}$

**Solution**

*we make the same bases in both sides of the equation*

$$2^{x^2-2x+1} = 2^{2(x+3)}$$

$$x^2 - 2x + 1 = 2x + 6$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0 \Longrightarrow x = 5 \text{ ، } x = -1$$

This equation is said to be exponential equation since the exponents are variable.

**Example 4**

Solve the equation $3^{2x+1} - 4\times3^{x+2} = -81$

**Solution**

$$3^{2x} \times 3 - 4 \times 3^{x} \times 3^{2} + 81 = 0 \div 3$$

$$3^{2x} - 12 \times 3^{x} + 27 = 0$$

$$(3^{x} - 3)(3^{x} - 9) = 0$$

$$3^{x} = 9 \Longrightarrow 3^{x} = 3^{2} \Longrightarrow x = 2$$

$$3^{x} = 3 \Longrightarrow x = 1$$

Solution set = { 1 , 2 }

## Example 5

Find the value of x if:

**A)** $(x-1)^6 = 2^6$ **B)** $(x+3)^5 = {}^{5}4$ **C)** $3^{x-1} = 5^{x-1}$

## Solution

**A)** By applying the note 3

$$3^{x-1} = 5^{x-1} \Rightarrow x-1=0 \Rightarrow x=1$$

**B)** By applying the note 2

$$(x+3)^5 = 4^5 \Rightarrow x+3=4 \Rightarrow x=1$$

**C)** By applying the note 2

$$(x-1)^6 = 2^6 \Rightarrow x-1 = \mp 2 \Rightarrow x=3$$

$$x=-1$$

## Example 6

Solve the equation in R such that

$$8^{\frac{x}{2}} + 8^{\frac{x}{2}+\frac{1}{3}} + 8^{\frac{x}{2}+\frac{2}{3}} = 14$$

$$8^{\frac{x}{2}} + 8^{\frac{x}{2}} \times 8^{\frac{1}{3}} + 8^{\frac{x}{2}} \times 8^{\frac{2}{3}} = 14$$

$$8^{\frac{x}{2}}\left(1 + 8^{\frac{1}{3}} + 8^{\frac{2}{3}}\right) = 14$$

$$8^{\frac{x}{2}}(1+2+4) = 14$$

$$8^{\frac{x}{2}} \times 7 = 14 \Rightarrow 8^{\frac{x}{2}} = 2 \Rightarrow \left(2^3\right)^{\frac{x}{2}} = 2 \Rightarrow 2^{3\frac{x}{2}} = 2^1 \Rightarrow \frac{3x}{2} = 1 \Rightarrow x = \frac{2}{3}$$

## Exercises (3-2)

**Q1)** Solve each of the following equations:

**A)** $(x+2)^{\frac{1}{2}} = 3$

**B)** $(\sqrt[5]{243})^2 = (x^{-\frac{1}{2}})^2$

**C)** $\sqrt[5]{x^3} = \frac{1}{27}$

**D)** $6^{x^2-3x-2} = 36$

**E)** $-6\times5^x+25^x+5 = 0$

**F)** $10^{(x-4)(x-5)} = 100$

**G)** $5\,(5^x+5^{-x})=26$

**H)** $2^{2x+3}-57 = 65(2^x-1)$

**I)** $3^{(x^2+5x+4)} = 27^{(-x-4)}$

**Q2)** Solve the equation in R such that

$$3^{x+1} \times 9^{x} - 9^{\frac{1}{2}} \times 3^{\frac{3}{x}} = 0$$

**Q3)** Solve each of the following equations:

$$\frac{(243)^{x-1}\times(27)^{x-2}}{(729)^{\frac{1}{2}x}} = 81$$

**Q4)** Find the value of $x \in R$ if:

**A)**

$$3^{x^2-1}+3^{x^2}+3^{x^2+1} = 39$$

**B)**

$$\frac{4^x+4(2^x)+3}{4^x+2^x} = 25$$

## [3-3]Squre Roots and Operations on Square Roots

Some of the roots are quantities can never be calculated exactly such as: $\sqrt[5]{61}$ ، $\sqrt[3]{10}$ ، $\sqrt{2}$

This kind of square roots is called the surds square roots, we will study some properties to help us simplify them.

**Properties**

1. $\sqrt[n]{x} \times \sqrt[n]{y} = \sqrt[n]{xy}$ and vice versa

For example:

$$\sqrt[5]{6} \times \sqrt[5]{12} = \sqrt[5]{72}$$

$$\sqrt[4]{5} \quad \sqrt[4]{3} \quad \sqrt[4]{x^3} = \sqrt[4]{15x^3}$$

$$\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}$$ and vice versa such that $y \neq 0$

For example:

$$\frac{\sqrt{21}}{\sqrt{3}} = \sqrt{\frac{21}{3}} = \sqrt{7}$$

$$\sqrt[3]{\frac{3x}{2y}} = \frac{\sqrt[3]{3x}}{\sqrt[3]{2y}}$$

**Example 7**

Order the following square root ascendingly:

$\sqrt[6]{147}$ ، $\sqrt{5}$ ، $\sqrt[3]{12}$

**Solution**

$$\sqrt[3]{12} = \sqrt[6]{12^2} = \sqrt[6]{144}$$

$$\sqrt{5} = \sqrt[6]{5^3} = \sqrt[6]{125}$$

$$\sqrt[6]{147} = \sqrt[6]{147}$$

The order is:

$\sqrt{5}$ ، $\sqrt[3]{12}$ ، $\sqrt[6]{147}$

## [3-4] Conjugate Numbers $\sqrt{a}$ m$\sqrt{b}$

The conjugate is the number multiplied by the irrational value to convert it to a rational value.

The conjugate of $2\sqrt{3}$ is $\sqrt{3}$ since $2\sqrt{3} \times \sqrt{3} = 2 \times 3 = 6$

And the conjugate of $\sqrt[3]{3}$ is $\sqrt[3]{3^2}$ such that

And conjugate of $5 - \sqrt{6}$ is $5 + \sqrt{6}$ since their product is:

$$(5 - \sqrt{6})(5 + \sqrt{6}) = 25 - 6 = 19$$

And the conjugate of $3\sqrt{2} - 2\sqrt{5}$ is $3\sqrt{2} + 2\sqrt{5}$ since

$$(3\sqrt{2} - 2\sqrt{5})(3\sqrt{2} + 2\sqrt{5}) = 9 \times 2 - 4 \times 5 = -2$$

and the conjugate of $\sqrt[3]{5^2} + \sqrt[3]{5} + 1$ is $\sqrt[3]{5} - 1$ since

$$(\sqrt[3]{5} - 1)(\sqrt[3]{25} + \sqrt[3]{5} + 1) = \sqrt[3]{125} - 1 = 5 - 1 = 4$$

(difference of two cubes)

**Example 8**

Simplify the denominator to become a rational quantity:

$$\frac{1}{\sqrt{2} - 1} + \frac{1}{\sqrt{3} + \sqrt{2}} + \frac{1}{2 + \sqrt{3}}$$

**Solution**

$$\frac{1}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} + \frac{1}{\sqrt{3} + \sqrt{2}} \times \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} + \frac{1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}}$$

$$= \frac{\sqrt{2} + 1}{2 - 1} + \frac{\sqrt{3} - \sqrt{2}}{3 - 2} + \frac{2 - \sqrt{3}}{4 - 3}$$

$$= \sqrt{2} + 1 + \sqrt{3} - \sqrt{2} + 2 - \sqrt{3} = 3$$

## [3-5] Real Functions

We studied the function in the previous years and now we will explain the concept of function which it has domain and codomain are nonempty sets belong to R expressed as $\exists\, y \in B$ ، $\forall\, x \in A$ $f: A \to B$ (y is unique element), Such that $A,B \subseteq R$ ، $y = f(x)$

### [3-5-1] Finding the domain of Real Functions

we will study: polynomial functions, fractional functions, Root functions, exponential functions such that the domain varies from a type to other.

***The polynomial function:** is expressed as $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$

$a_n, a_{n-1}, ... a_0 \in R$

It also includes the linear fnction such that $f(x) = 3x - 1$

square function: $g(x) = x^2 - 5x + 9$ , and the cubic function:

$h(x) = x^3 + 2x^2 + x - 1$

The domain in these cases is R

***Fractional Function:** to find the domain of this type of functions we make the denominator equal to zero and find x the domain becomes then R\{possible values of x}

For example:

1) $f(x) = \frac{2x - 1}{x + 5}$ we make $x + 5 = 0 \Rightarrow x = -5$ then the domain is $R\backslash\{-5\}$

2) $g(x) = \frac{2}{x^2 - 4}$ $x^2 - 4 = 0 \Rightarrow x = \mp 2$ the domain is $R\backslash\{\mp 2\}$

***The root function:** to find the domain of this function we find the possible values of x the makes the inside of root greater or equal to "zero" , for example:

1) $f(x) = \sqrt{x-7}$ , $x-7 \geq 0 \Rightarrow x \geq 7$ $\{x \in R : x \geq 7\}$

2) $g(x) = \sqrt{3x+5}$, $3x + 5 \geq 0 \Rightarrow x \geq \frac{-5}{3}$ $\{x \in R : x \geq \frac{-5}{3}\}$

***The exponential function:** $f_a(x) = a^x$ such that $x \in R$ ، $a \in R^+\backslash\{1\}$ such that a is the base, x is the exponent, for example:

$f_{\frac{1}{2}}(x) = (\frac{1}{2})^x$, $h_{\sqrt{5}}(x) = (\sqrt{5})^x$, $g_3(x) = 3^x$, $f_2(x) = 2^x$

**Note:** $f(x) = 1^x = 1$ this is a constant function, so we rule out a=1 in the exponential function.

## [3-5-2] Representing Real Function in the Coordinate Plane

**First:** representing the linear function $f(x) = ax + b$ such that $a \neq 0$ , $a,b \in R$

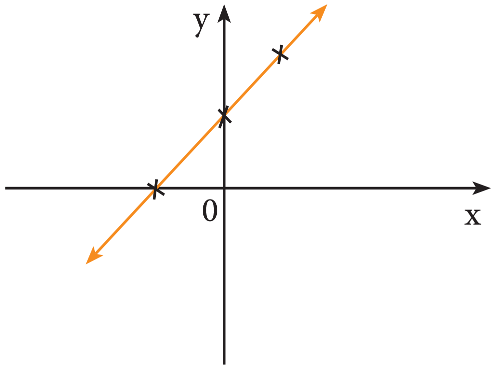
For example:

$f(x) = 2x + 3$ , $\forall\ x \in R$

| x | 1 | 0 | -1 |
|---|---|---|---|
| y | 5 | 3 | 1 |

Note that this function represents a straight line.

$y = 2x + 3$



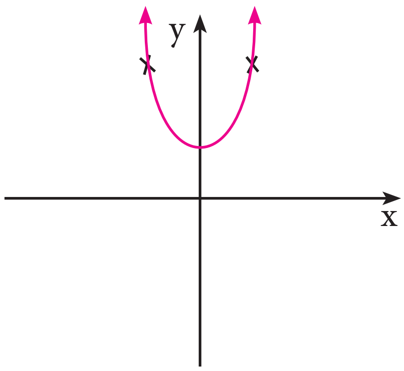

**Second:** representing the second-degree function $f(x) = ax^2 + b$ such that $a,b \in R$, $a \neq 0$

Example: represent the function

$f(x) = 2x^2 + 3$ when $a > 0, b \geq 0$

| x | -1 | 0 | 1 |
|---|---|---|---|
| y | 5 | 3 | 5 |

Note that the function represents $\cup$ on the upper half of the coordinate plane.

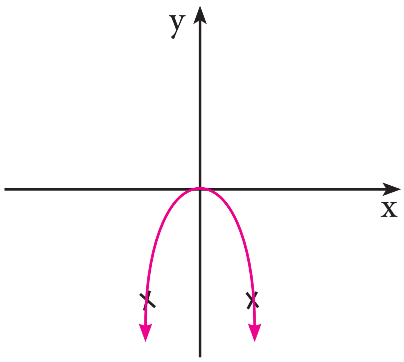



Example: represent the function

$f(x) = -4x^2$ when $a < 0$

| x | 1 | 0 | -1 |
|---|---|---|---|
| y | -4 | 0 | -4 |

Note that the shape of the function is $\cap$ on the upper half of the coordinate plane.




**Third:** Representing the third-degree functions

$a,b \in R$ , $a \neq 0$, $f(x) = ax^3 + b$

Example: represent the function

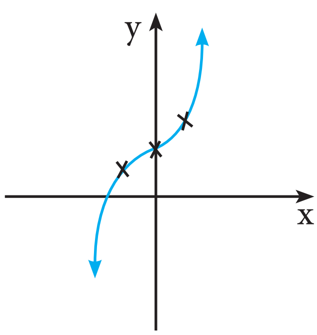
$f(x) = x^3 + 2$

| x | 1 | 0 | -1 |
|---|---|---|---|
| y | 3 | 2 | 1 |

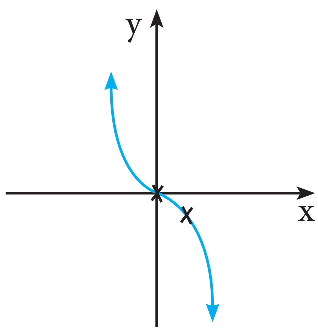
Example: Represent the function

$f(x) = -x^3$

| x | 1 | 0 | -1 |
|---|---|---|---|
| y | -1 | 0 | 1 |




$f(x) = x^3 + 2$




$f(x) = -x^3$

Fourth: representing the exponential function $f_a(x)=a^x$

Example:

A) Find the values of the function $f(x) = 2^x$ for reach $x=-3,-2,-1.0,1,2,3$ the use them to graph the same part of the function.

B) Find the method by the help of the previous function to graph some part of the function: $f_{\frac{1}{2}}(x)$ on the same figure

Solution:

A) $f(x) = 2^x$

| x | 3 | 2 | 1 | 0 | -1 | -2 | -3 |
|---|---|---|---|---|---|---|---|
| $2^x$ | 8 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

B) $g(x) = f_{\frac{1}{2}}(x) = (\frac{1}{2})^x = (2^{-1})^x = 2^{-x} = f(-x)$

Let's assume $R_y$ is a reflection by the y-axis

$R_y : (x, y) = (-x, y)$ then the image of $(x, 2^x) = (-x, 2^x)$ we find a curve of the function $g(x)=(\frac{1}{2})^x$ from the curve $f(x)=2^x$ by reflection by the y-axis, as shown in the figure (3-1).

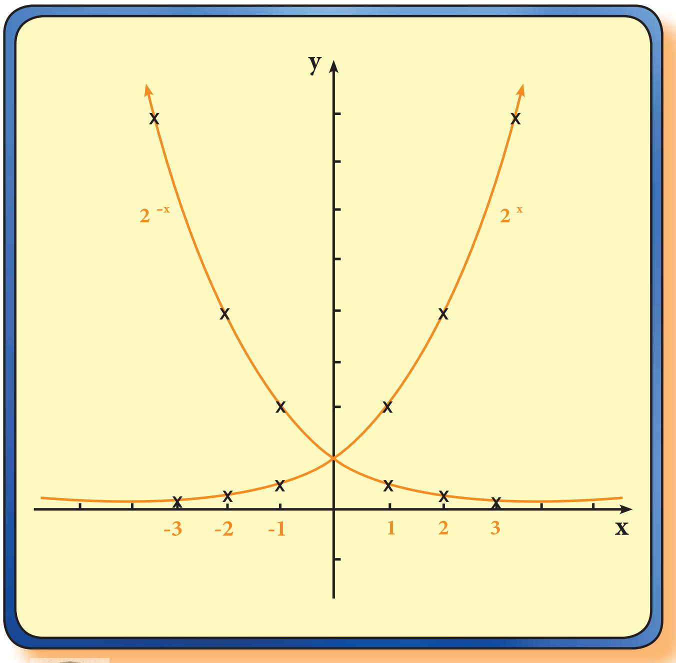



**Figure (3-1)**

## Some properties of the exponential function $f(x) = a^x$

**1.** If we graph the curves of the functions $2^x ، 3^x ، 4^x ، 5^x ، ......$

and the functions $(\frac{1}{2})^x ، (\frac{1}{3})^x ، (\frac{1}{4})^x ، (\frac{1}{5})^x ، ......$

We will find two groups of the curves:

First: When $a > 1$ such that the value of the function $a^x$ increases when x increases.

Second: when $1 < a < 0$ such that the value of function $a^x$ decreases when x increase.

We represented the graph of six of these functions on the figure (3-2) (a part of each curve), in which three of them $a > 1$ and three of them $1 < a < 0$ and then chose the value of the last three a the reverse of first three, we note **all curves pass through the point (0,1)**

**2.** If we consider the curve of any of the exponential functions $a^x, a \neq 0$, we find its domain is R.

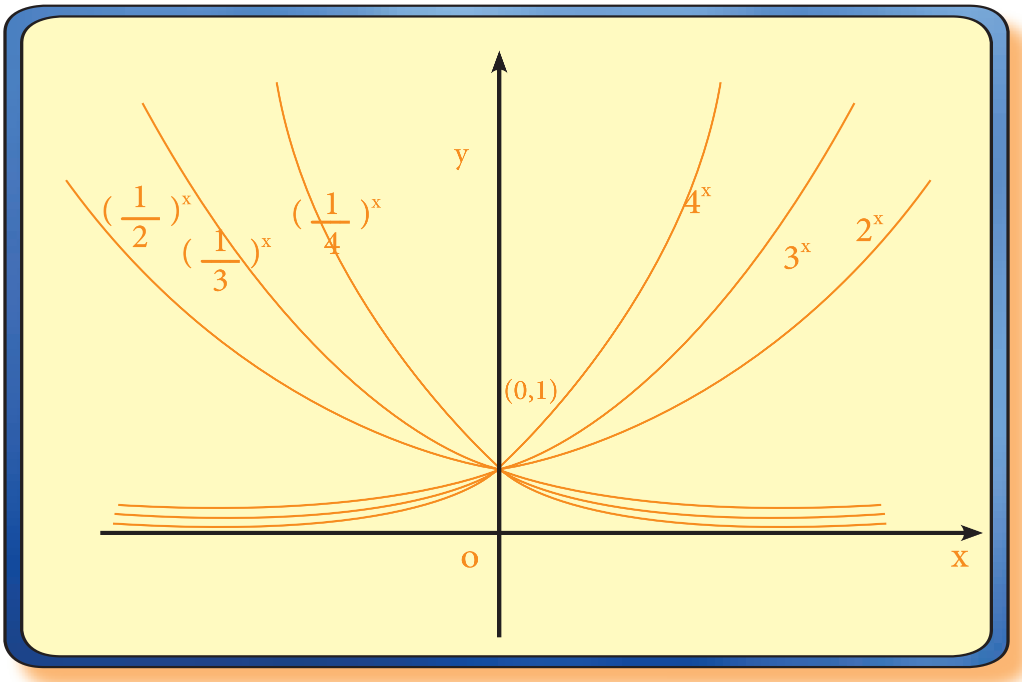



**Figure (3-2)**

## Exercises (3-3)

**Q1)** **A)** Simplify $\dfrac{\sqrt{a^2-b^2}}{a+b}\left[\dfrac{\sqrt{a+b}}{\sqrt{a-b}}-\dfrac{\sqrt{a-b}}{\sqrt{a+b}}\right]^{-1}$

**B)** if $y=\sqrt[3]{4}-\sqrt[3]{2}+1$ , $x=\sqrt[3]{2}+1$ prove that $x\,y=3$

**Q2)** Represent the functions graphically:

a) $f(x)=-4x^2+5$  b) $f(x)=x-8$  c) $f(x)=2-x^3$

**Q3)** Find the domain of the function:

a) $f(x)=x^2-5+9$  b) $f(x)=\dfrac{x-1}{x+9}$  c) $f(x)=\sqrt{x-9}$

d) $f(x)=\sqrt{3-5x}$  e) $f(x)=\dfrac{1}{x^2-9}$

**Q4)** Find the value of the following such that the denominator becomes a rational quantity:

**A)** $\dfrac{3}{a-b}\cdot\sqrt{\dfrac{2x}{a-b}}\div\sqrt{\dfrac{18x^3}{(a-b)^5}}$  Ans : $\dfrac{a-b}{x}$

**B)** $\dfrac{\sqrt{\frac{3}{2}}-\sqrt{\frac{8}{27}}}{\sqrt{2}\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)}$  Ans : $\dfrac{5}{24}$  **C)** $\sqrt{\dfrac{\sqrt{5}+1}{\sqrt{5}-1}}-\sqrt{\dfrac{\sqrt{5}-1}{\sqrt{5}+1}}$  Ans : 1

**5)** Prove that:

**A)** $\dfrac{-15}{x-6+\sqrt{x}}+\dfrac{3}{\sqrt{x}-2}-\dfrac{3}{\sqrt{x}+3}=0$

**B)** Find x in the form of $a+\sqrt{3b}$ İf $x+\sqrt{3x}=8$

**6)** Graph a part of the curve for the function $y=\left(\dfrac{1}{5}\right)^x$

# 4 Chapter 4: Trigonometry

[4-1] Directed angle in standard position

[4-2] The degree mesure and the radian measure of angles

[4-3] The relation between the radian and the degree measure

[4-4] Trigonometric ratios for acute angles and some basic relations

[4-5] Trigonometric ratios for special angles

[4-6] Unit circle and trigonometric point

[4-7] Circular applications

[4-8] Using the calculator to find the values of circular applictions

[4-9] Solution of Right Angled Triangle

Aims and Skills

- Learning the Absolute value
- Solving Absolute value equation
- Solving System of equations
- Solving first-degree inequality with one variables
- Solving Second-degree Inequalities with one variable

| ***Terms*** | ***Symbol or Mathematical relations*** |
|---|---|
| Directed angle | $(\overrightarrow{BA}, \overrightarrow{BC})$ |
| The degree mesure and the radian measure | D° ، Q |
| Sine x | Sin x |
| Cosine x | Cos x |
| Tangent x | Tan x |
| Triangle point | ( Cos x , Sin x ) |

***Introduction:*** The Muslims have a big role in collecting what was spread of the trigonometry from the greek books. The Babylonians, the Egyptians, the Indians, the Chinese and the Greeks had clear signs in this area of study.and from the Muslim scientists that contributed in this area:

AL- Beronni (362-440 h)=( 973-1048 ): he is Abo AL-rayhan mohammed bin ahmed Alfalaki. he is an arab with persian origins, he was born in Kath Bokharizm and died in Ghazna in Afghanistan and has a theory to find the earth's perimeter in his book "Al- Istirlab" and it icalled AL-Beronni's formula and states that $\frac{b \text{ Cos } x}{a - \text{Cos } x}$ where r:is half of the earth's diameter, a:the height of the tallest mountain, b:the observed height, x:the angle of the slope of the horizon.

AL-Bozjani: :(328-388 h) = (940-988) he is Mohammed bin Mohammed Yahya bin Ismael bin Al-Abbas abo AL-Wafaa, born in the city of Bozjan and in 959 he mmoved to Baghdad. he is thw first to put the triangular ratio and use it to slove mathemaical equations and he made the following formulas:

$$\text{Sin } x = 2\text{Sin } \frac{x}{2} \text{ Cos } \frac{x}{2} \text{ ، } 2\text{Sin}^2 \frac{x}{2} = 1 - \text{Cos } x$$

$$\text{Sin}(x+y) = \sqrt{\text{Sin}^2 x - \text{Sin}^2 x \text{ Sin} y} + \sqrt{\text{Sin}^2 y - \text{Sin}^2 y \text{ Sin}^2 x}$$

$$\text{Tan} x = \frac{\text{Sin } x}{\text{Cos } x} \text{ ، } \text{Cot} x = \frac{\text{Cos } x}{\text{Sin } x}$$

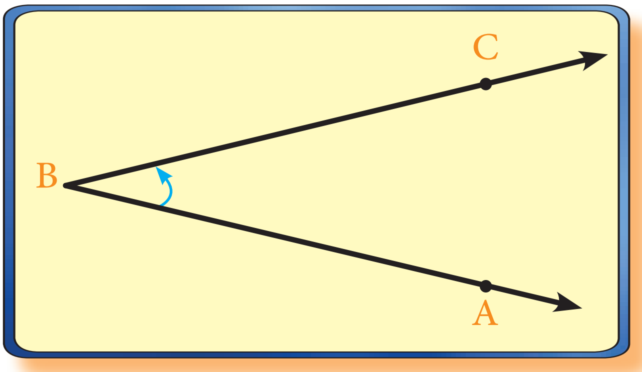
$$\text{Sec} x = \sqrt{1 + \text{Tan}^2 x} \quad \text{، } \quad \text{cec} x = \sqrt{1 + \text{Cot}^2 x}$$

*Al-Kashi (899 h) = (1492 ): He is Ghayath AL-deen Jamsheed bin Masood bin Mahmood bin Al-Kashi. he was born in the city of Kashan in Iran and he died in Samarkand, he is considered as an excellent mathematician as the pi (π) was found by him in his book (Al-resala Al mohita ) where he gave 15 decimals for 2π , 2π = 6.2831857179586 And one of his books is called (Miftah Al-Hesab) that contains: Calculus, Trigonometry,Areas etc... And the field of Trigonometry has developed in the 17th century by the scotttish scintist JOhn Nappier(1550-1617) and the field of trigonometry has many uses in area, geography, physics and navigation. In this chapterwe are going to study the basics of trigonometry.

## [4-1] Directed angle in standard position

**Definition ( 4 - 1)**

Directed Angle : If the two rays $\overrightarrow{BA}$ , $\overrightarrow{BC}$ have a common initial point that is B then the ordere pair ( $\overrightarrow{BA}$,$\overrightarrow{BC}$ ) s called the directed angle that;s initial side is $\overrightarrow{BA}$ and it's terminal side is BC and it's head is the point B and is written as ($\overrightarrow{BA}$,$\overrightarrow{BC}$ ) or < $\overrightarrow{ABC}$ .




**Definition ( 4 - 2 )**

Directed angle in standar position : In a coordiante plane, if we have a directed angle that's head is on the origin point and it's initial side is on the positive side of the X axis, it is called a directed angle in a standard position as in figure 4-1.

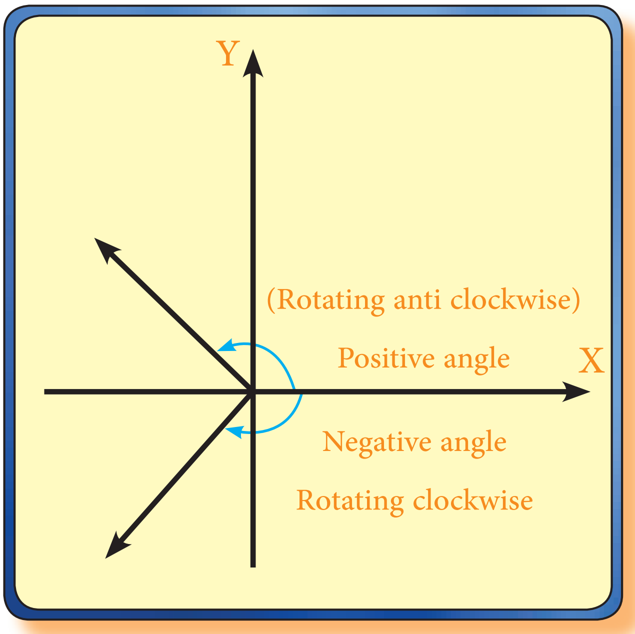



Figure ( 4 - 2 )

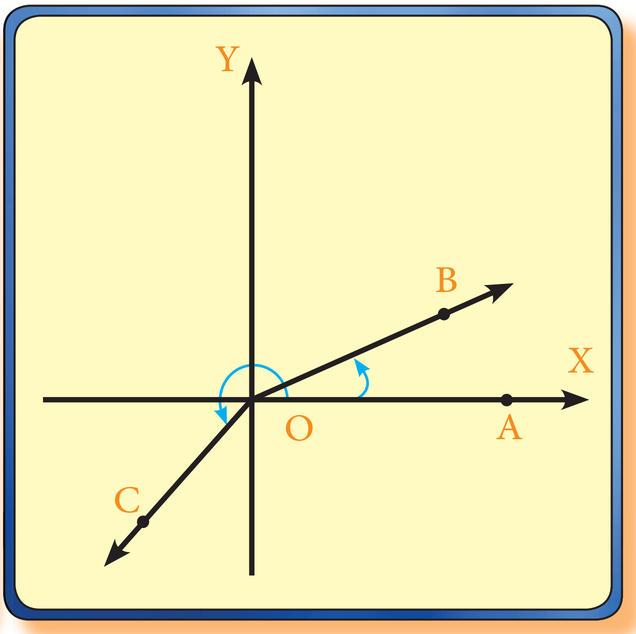



Figure ( 4 - 1 )

## [ 4 - 2 ] The degree mesure and the radian measure of angles

Degree Measure: We leanred from our previous studies that if we made the circles as 360 equal parts we will have 360 equal arcs and each arc is opposing a central angle in that circle and the measure of that angle is called the degree measure and is denoted by $1^\circ$

$1^\circ$=60 minutes= 60', 1'=60 seconds = 360"

Radian Measure: There is another system for measuring the angle and it is called the radian measure of angles.

**Definition (4 - 3)**

The unit of measuring the angles by radian measure ia the angle that is half diametric and it's the measure of the angles that's vertix is on the circle's center and is opposed by an arc tha's length is equal to radius of the circle.

In figure(4-3) if we assumed hat the length of the arc opposing to the central angle AOB is equal to L the unit of the length of half the diameter =r units and if L=r , m<AOB =1 the angle of the radius.

and if L= 2r as in figure (4-4) then

m<AOB by radian measure =2

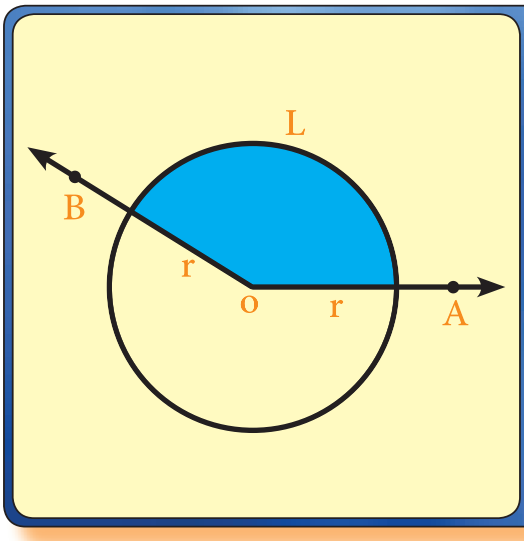


Figure ( 4 - 4 )

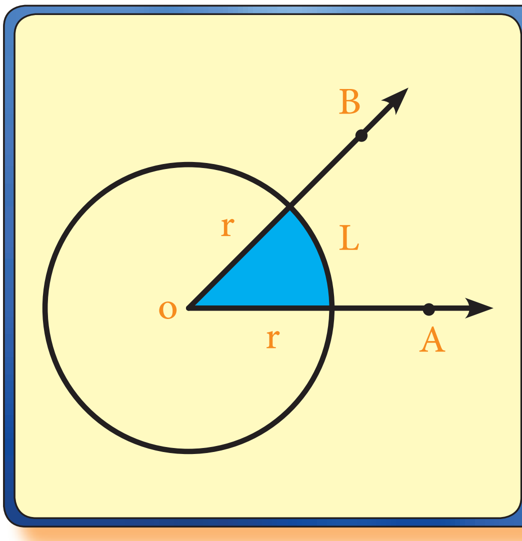


Figure ( 4 - 3 )

And from definition(4-3) we can know that the length of the arc of the circle that's radius is r: L= | Q | . r , | Q | is the measure of the central angle opposite to that arc measured by radian.

$$|Q| = \frac{\text{Length of the arc}}{\text{radius}} = \frac{L}{r}$$

## *[4-3] The relation between the radian and the degree measure*

We have previously learned that the circumference of the circle = $2\pi r$

$$|Q| = \frac{L}{r} = \frac{2\pi r}{r} = 2\pi$$

$2\pi$ the angle of the radius = 360

$\pi$ angle of the radius =180

$$\frac{180^\circ}{\pi} = \text{radian angle } 1 \Leftarrow$$

$$\text{radian angle } \frac{\pi}{^\circ 180} = 1^\circ \Leftarrow$$

In general : A) If the measure of a directed angle =Q radian angle

Then Q is a radian angle $= \dfrac{D^\circ \times \pi}{180^\circ}$

B) If the measure of a directed angle = $^\circ$D

Then $D^\circ = (\, Q \times \dfrac{180^\circ}{\pi}\,)$ radian angle :

We conclude : $\dfrac{Q}{D^\circ} = \dfrac{\pi}{180^\circ}$

We use this relation to convert from the radian measure to the degree measure and vice versa.

### *Example 1*

If < $\overrightarrow{AOB}$ is in standard position and opposite to an arc that's length is 10 cm in a circle that has a radius of 12cm

A) Find m< $\overrightarrow{AOB}$ by radians measure where :

$0 \leq A\,O\,B > m \leq 2\pi$ note that the center of the circle is the origin point.

B) Find by radians m< $\overrightarrow{AOB}$ where:

$-2\pi < A\,O\,B > m \leq 0$

### *Solution :*

L = 10 cm ، r = 12 cm

A ) $\therefore$ Radian angle $|Q| = \dfrac{5}{6} = \dfrac{L}{r} = \dfrac{10}{12} = 833\ .0$

B) In this case the measure of the angle will be negative:

$$|Q| = \frac{L}{r} = \frac{10}{12} = \frac{5}{6} = 833\ .0$$

$\therefore$ Q = - 833 .0 Radians because the angle is negative.

### *Example 2*

If < $\overrightarrow{AOB}$ was in it's standard position and it's measure was $\dfrac{3\pi}{4}$ what's it's measure in degrees?

### *Solution*

$$\frac{Q}{D^\circ} = \frac{\pi}{180^\circ}$$

$$\frac{\frac{3\pi}{4}}{D^\circ} = \frac{\pi}{180^\circ} \Rightarrow D^\circ = 180^\circ \times \frac{3}{4} = 135^\circ$$

*Example 3*

Convert : A ) $^\circ 45$ to radians

B ) $2.6\pi$ to de.....

*Solution :*

A ) Radians $\frac{\pi}{4} = Q \Leftarrow \frac{Q}{45} = \frac{\pi}{180^\circ} \Leftarrow \frac{\pi}{180^\circ} = \frac{Q}{D^\circ} \because$

B ) $468^\circ = 2.6 \times 180^\circ = D^\circ \Leftarrow \frac{2.6\pi}{D^\circ} = \frac{\pi}{180^\circ} \Leftarrow \frac{\pi}{180^\circ} = \frac{Q}{D^\circ}$

*Example 4*

If the measure of a central angle is $60^\circ$ what is the length of the arc that it opposite to, knowing that the length of the radius is 9 cm.

*Solution :*

Radian $\frac{1}{3}\pi = Q \Leftarrow \frac{\pi}{180^\circ} = \frac{Q}{60^\circ} \Leftarrow \frac{\pi}{180^\circ} = \frac{Q}{D^\circ} \because$

$|Q| = \frac{L}{r} \because$

$\frac{\pi}{3} = \frac{L}{9} \Longrightarrow L = 3\pi = 3 \times 3.142$

$= 9.426$ cm

*Example 5*

The length of the arc of a central angle is $21\frac{1}{4}$ cm and the length of its radius is 20 cm, what is the degree measure of that central angle?

***Solution :***

$$\text{Radians } \frac{17}{16} = \frac{21\frac{1}{4}}{20} = |Q| \Leftarrow \frac{L}{r} = |Q|$$

$$\frac{\pi}{180^\circ} = \frac{\frac{17}{16}}{D^\circ} \Leftarrow \frac{\pi}{180^\circ} = \frac{Q}{D^\circ}$$

$$D^\circ = \frac{17}{16} \times 180^\circ \times \frac{7}{22} = 60.85^\circ$$

***Example 6***

In a right angle triangle, the difference betweeen its two acute angles is 0.44 radians, what is the measure of each angle in degrees?

***Solution :***

$$\frac{0.44}{D^\circ} = \frac{\pi}{180^\circ} \Leftarrow \frac{Q}{D^\circ} = \frac{\pi}{180^\circ} \because$$

$$D^\circ = \frac{0.44 \times 180}{\pi} = \frac{0.44 \times 180}{3.14} = 25.2^\circ \therefore$$

We assume that the degree measures of the two acute angles are A,B

$$A + B = 90^\circ \;\ldots\ldots\; 1$$

$$A - B = 25.2^\circ \;\ldots\ldots\; 2$$

ADD THEM

$$2A = 115.2$$

$$\therefore A = 57.6^\circ$$

$$B = 32.4^\circ$$

**Conclusion**

The relation between the degree and the radian measures is : $\frac{Q}{D^o} = \frac{\pi}{180^o}$

The relation beetween the central angle Q and the length of the arc L and the radius of their circle r is: $|Q| = \frac{L}{r}$

## Excercise (4 - 1)

### Q1 /

Convert each ofthe following degrees to radians:

$300^\circ$ ، $120^\circ$ ، $30^\circ$

### Q2 /

Convert each of the follwing radians to degrees:

$\frac{1}{3}$ ، $\frac{5\pi}{6}$ ، $\frac{3\pi}{5}$

### Q3 /

The measure of a central angle in a circle is $\frac{5}{6}$ radians opposite to an arc that's length is 25cm find the radius of the circle.

Ans/ 30cm

### Q4 /

What is the length of the arc that opposite to the central angle that's measure is $135^\circ$ in a circle that has a radius of 8cm?

Ans/ 18.857

### Q5 /

The sum of two angles is $\frac{\pi}{4}$ radians and the differnce between them is $9^\circ$ find the measure of these two angles in degrees.

Ans/ $27^\circ$ ، $18^\circ$

### Q6 /

Draw $< \overrightarrow{AOB}$ in its standard position if it's measure is $\frac{5\pi}{4}$ then find its measure in degrees.

## [4-4]Trigonometric ratios for acute angles and some basic relations

Figure (4-5) represents a right triangle at C

Let m < ABC = Q

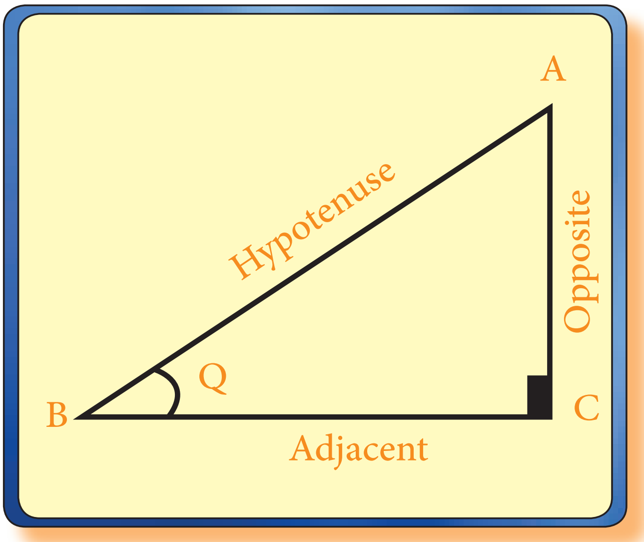



Figure ( 4 - 5 )

**Definition (4 - 4)**

We call the number that represents the ratio as follows:

1. The ratio $\frac{AC}{AB}$ is called Sine of the acute angle (Q)

And it is written as Sin Q $= \frac{AC}{AB} = \frac{OPP.}{HYP.}$

2. The ration $\frac{BC}{AB}$ is called the Cosine of the acute angle Q

And it is written as Cos Q $= \frac{BC}{AB} = \frac{ADJ.}{HYP.}$

3. The ratio $\frac{AC}{BC}$ is called the Tangent of the acute angle Q

And it's written as tan Q $= \frac{AC}{BC} = \frac{OPP.}{ADJ.}$

And from Figure (4-5) we get $(AC)^2+(BC)^2 = (AB)^2$ (Pythagorean theorem)

When we divide both sides by $(AB)^2$ we get:

$$\left(\frac{AC}{AB}\right)^2 = \left(\frac{BC}{AB}\right)^2 + \left(\frac{AB}{AB}\right)^2$$

And from definition (4-4)

$$\text{Sin}^2\, Q + \text{Cos}^2\, Q = 1$$

And from definition (4-4) we also get:

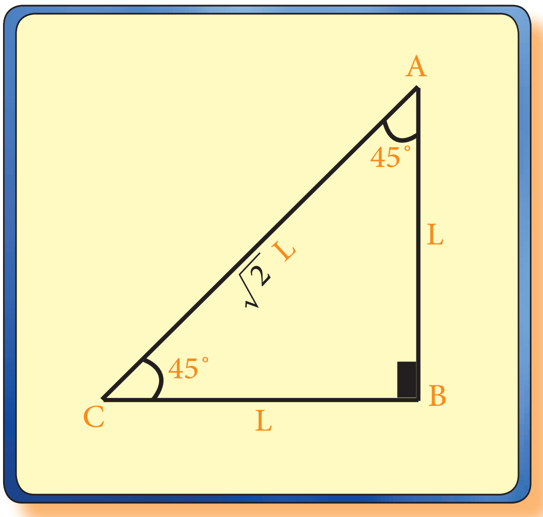
$\tan Q = \dfrac{AC}{BC}$ and by dividing both sides by (AB) we get:

$$\tan Q = \frac{\frac{AC}{AB}}{\frac{BC}{AB}}$$

$$\therefore \quad \tan Q = \frac{\text{Sin}\, Q}{\text{Cos}\, Q}$$

## 1) A 45° Angle

We draw a triangle that is right at B. And one of it's angles is (45°). So the other angles is (45°) also.




$\therefore$ AB = BC = L

Pythagorean: $(AC)^2 = (AB)^2 + (BC)^2$

$(AC)^2 = L^2 + L^2 = 2L^2$

$\therefore$ AC = $\sqrt{2}\,L$

$\text{Sin } 45° = \dfrac{L}{\sqrt{2}L} = \dfrac{1}{\sqrt{2}} \Rightarrow$ $\boxed{\text{Sin } 45° = \dfrac{1}{\sqrt{2}}}$

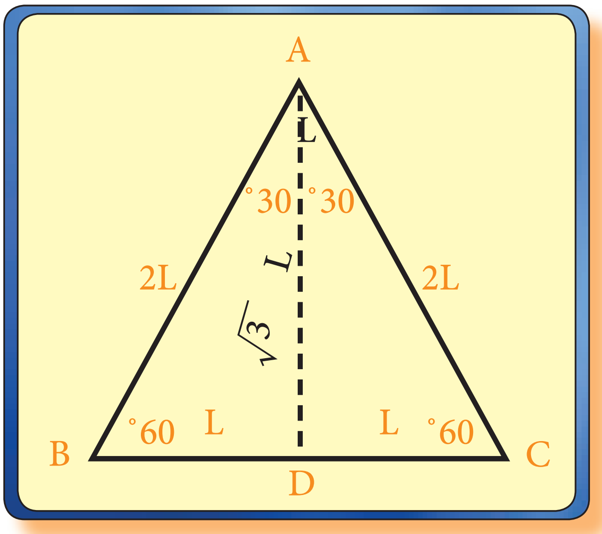
$\text{Cos } 45° = \dfrac{L}{\sqrt{2}L} = \dfrac{1}{\sqrt{2}} \Rightarrow$ $\boxed{\text{Cos } 45° = \dfrac{1}{\sqrt{2}}}$

$\tan 45° = \dfrac{L}{L} = 1 \Rightarrow$ $\boxed{\tan 45° = 1}$

## 2) A 30°,60° Angle

We draw an equalteral triangle with the length of one side = 2L

And the measure of all its angles=60°




We draw $\overline{BC} \perp \overline{AD}$ as in the figure.

$\therefore$ L= DB = CD unit

And m$\sphericalangle$ BAD = 30°

Using the Pythagorean theorem we get:

AD = $\sqrt{3}$ L

$\text{Sin } 30° = \dfrac{L}{2L} = \dfrac{1}{2} \Rightarrow$ $\boxed{\text{Sin } 30° = \dfrac{1}{2}}$

$\sin 60° = \dfrac{\sqrt{3}L}{2L} = \dfrac{\sqrt{3}}{2} \Rightarrow$ $\boxed{\sin 60° = \dfrac{\sqrt{3}}{2}}$

$$\text{Cos } 30^\circ = \frac{\sqrt{3}L}{2L} = \frac{\sqrt{3}}{2} \Rightarrow \boxed{\text{Cos } 30^\circ = \frac{\sqrt{3}}{2}}$$

$$\text{Cos } 60^\circ = \frac{L}{2L} = \frac{1}{2} \Rightarrow \boxed{\text{Cos } 60^\circ = \frac{1}{2}}$$

$$\tan 30^\circ = \frac{L}{\sqrt{3}L} = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\tan 30^\circ = \frac{1}{\sqrt{3}}}$$

$$\tan 60^\circ = \frac{\sqrt{3}L}{L} = \sqrt{3} \Rightarrow \boxed{\tan 60^\circ = \sqrt{3}}$$

Notice that: $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$

Also $\cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$

Meaning that the sine of one of the angles is equal to the consine of the other angle. And in general if Q was an Acute angle then we can say (90°-Q):

**Note :**

The angles 30°,60°, are complementary because

$90^\circ = 30^0 + 60^\circ$

$\sin(90^\circ - Q) = \cos Q$

$\cos(90^\circ - Q) = \sin Q$

## Conclusion

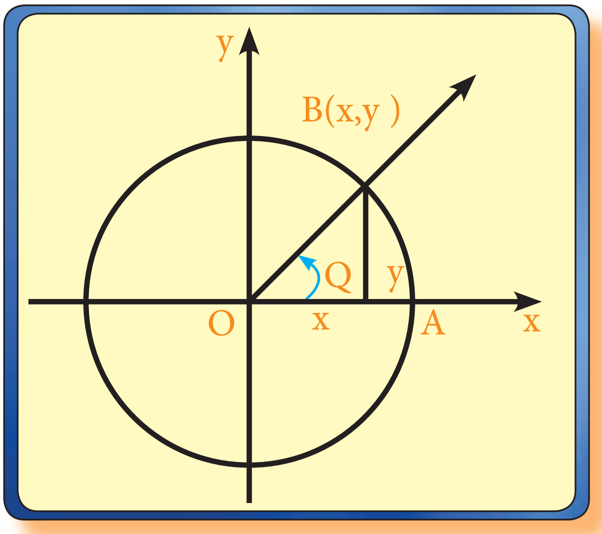
* $\sin Q = \frac{\text{Opposite}}{\text{Hypotenuse}}$, $\cos Q = \frac{\text{Adjacent}}{\text{Hypotenuse}}$, $\tan Q = \frac{\text{Opposite}}{\text{Adjacent}}$
* $\text{Sin}^2 Q + \cos^2 Q = 1$ , $\tan Q = \frac{\sin Q}{\cos Q}$
* $\sin(90^o - Q) = \cos Q$ , $\cos(90^o - Q) = \sin Q$
* $\sin 30^o = \cos 60^o = \frac{1}{2}$ , $\cos 30^o = \sin 60^o = \frac{\sqrt{3}}{2}$
* $\sin 45^o = \cos 45^o = \frac{1}{\sqrt{2}}$

## [4-6] Unit circle and trigonometric point

### Definition (4 - 5)

Unit circle: Is a cicle thats center is the origin point and the length of its radius is equal to 1 length unit.

The trigonometric point for an angle in the figure Q= m $\sphericalangle \overrightarrow{BOA}$, A direct angle in standard poition, B is the point of intersection between the terminal side $\overrightarrow{OB}$ with the unit circle. We assume that B(x,y)




$$\sin Q = \frac{y}{1} \Rightarrow \sin Q = Y$$

$$\cos Q = \frac{x}{1} \Rightarrow \cos Q = X$$

$$\therefore B(x,y) = (\cos Q, \sin Q)$$

**Note:**
By using the unit circle and the conversions in the plane we can find the following trigonometric ratios:

$\sin(180^\circ - Q) = \sin Q$

$\cos(180^\circ - Q) = -\cos Q$

$\tan(180^\circ - Q) = -\tan Q$

### Definition (4 - 6)

The trigonometric point for the directed angle in standard position is the intersection point between the terminal side of the angle with the unit circle.

Notice that point B is a trigonometric point for angle $\overrightarrow{AOB}$, we can see that every diected angle Q in standard position is a trigonometric point(x,y) and sinQ=y, cosQ=x .

***Example 7***

Find sinQ, cosQ, tanQ if you know that Q=0,90,180

***Solution :***

We know that 0˚,90˚, 180˚ are on the terminal side for each of them on one of the axis. As in figure (4-6) so:

$(\cos 0 , \sin 0) = (1 , 0) \Longrightarrow \cos 0^\circ = 1$

$\sin 0^\circ = 0$

$\therefore \tan 0^\circ = \dfrac{\sin^\circ 0}{\cos^\circ 0} = \dfrac{0}{1} = 0 \Longrightarrow \tan 0^\circ = 0$

$(\cos 90^\circ , \sin 90^\circ) = (0,1)$ ✳

$\Longrightarrow \cos 90^\circ = 0$ ، $\sin 90^\circ = 1$

But $\dfrac{\sin 90^\circ}{\cos 90^\circ} = \tan 90^\circ$ undefined

$(\cos 180^\circ , \sin 180^\circ) = (-1, 0)$ ✳

$\Longrightarrow \cos 180^\circ = -1$ ، $\sin 180^\circ = 0$

$\Longrightarrow \tan 180^\circ = 0$

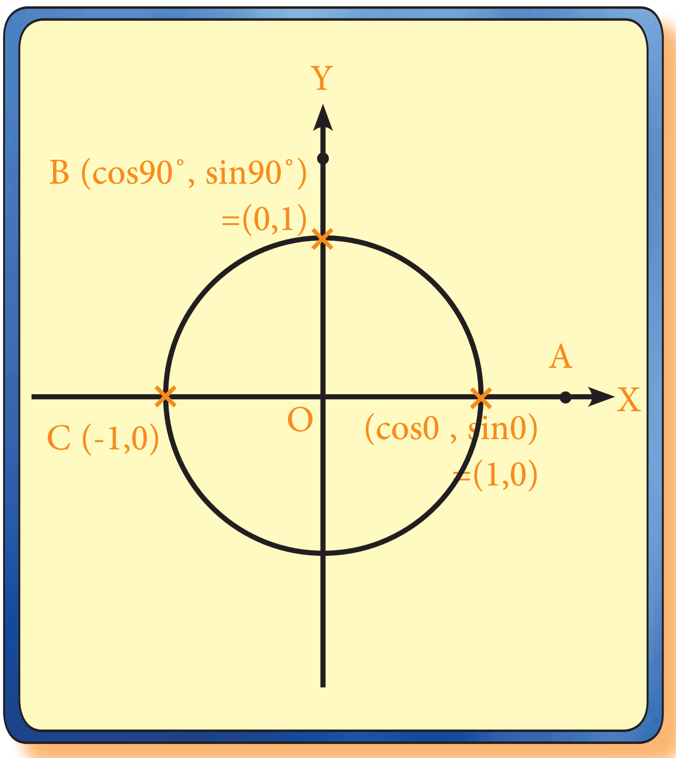



Figure (4 - 6)

# [4-7] Circular applications

## [4-7-1] Angles of elevation and depression

We can find the heights and dimensions when we can measure the angle that we are seeing the mat. If the observer stands in point A and looks at point C that is over the horizon of point A, then the angle that is between the line from the eye of the observer to point C and the horizon of A is called ( angle of elevation ) for ex: the angle $\measuredangle$ CAB in figure (4-7).

And if the eye of the observer is at C and he looks at A below the horizon of C, then the angle formed between the line from the eye of the observer to point A and the horizion C is called (angle of depression) for ex: angle $\measuredangle$ ACD in figure (4-7)

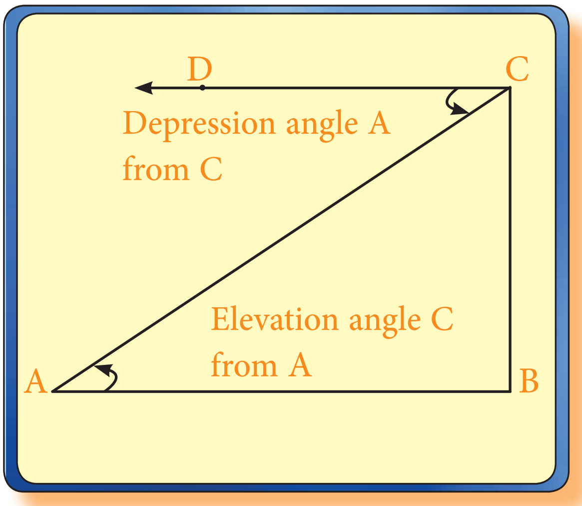


Figure (4 - 7)

***Example 8***

The length of the string of a kite is 30m, if the angle that the string is makin with the earth (the horizon) is 45°. Find the height of the kite.

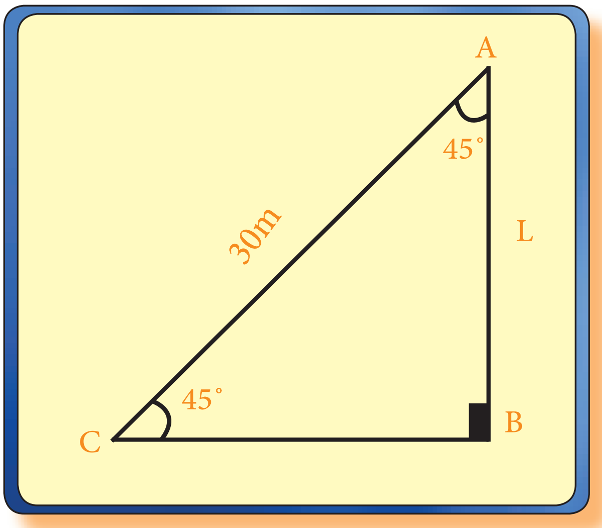


Figure (4 - 8)

***Solution :***

We assume that the height = L length unit

the triangle ABC is a right triangle at B

$$\therefore \ \sin 45^\circ = \frac{\text{OPP.}}{\text{Hyp.}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{L}{30}$$

$$\therefore \quad L = \frac{30}{\sqrt{2}} = 21.21\text{m}$$

***Example 9***

An observer found that the angle of elevation of a minaret from a point on the ground that is 8 m away from its base is 60˚ what is the height of the minaret?

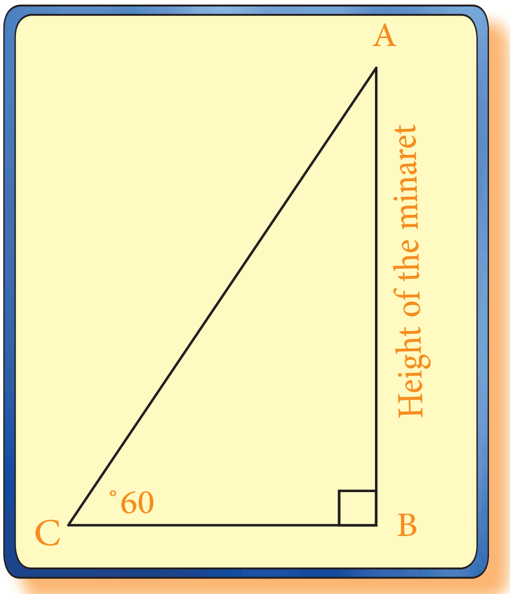



FIGURE (4 - 9)

***Solution :***

$\Delta$ A B C has a right angle at B:

$$\tan 60^\circ = \frac{OPP.}{ADJ.}$$

$$\sqrt{3} = \frac{AB}{8}$$

$\therefore$ Height of the minaret $AB = 8\sqrt{3}$

***Example 10***

The height of a mountain 2350m, and observer found from its top that the depression angle on the ground is 70˚. What is the distance between the point and the observer? Sin 70˚= 0.9396 .

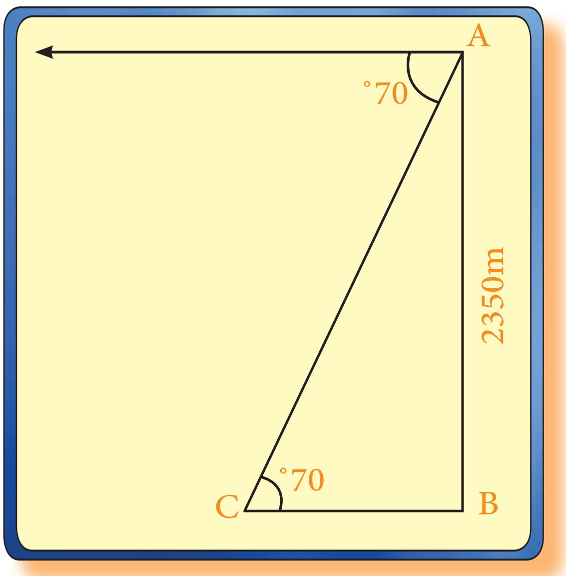



Figure (4 - 10)

***Solution :***

Elevation angle= Depression angle

$\Delta$ ABC has a right angle at B

$$\sin 70^\circ = \frac{AB}{AC}$$

$$0.9396 = \frac{2350}{AC}$$

$$\therefore \ AC = \frac{2350}{0.9396} \cong 2500m$$

## *Example 11*

From the roof of a house height of 7 meters, observer found a monitor that the angle of the height of the highest building in front of 60 and the angle of low base 30 very dimension between the observer and building and the height of building.

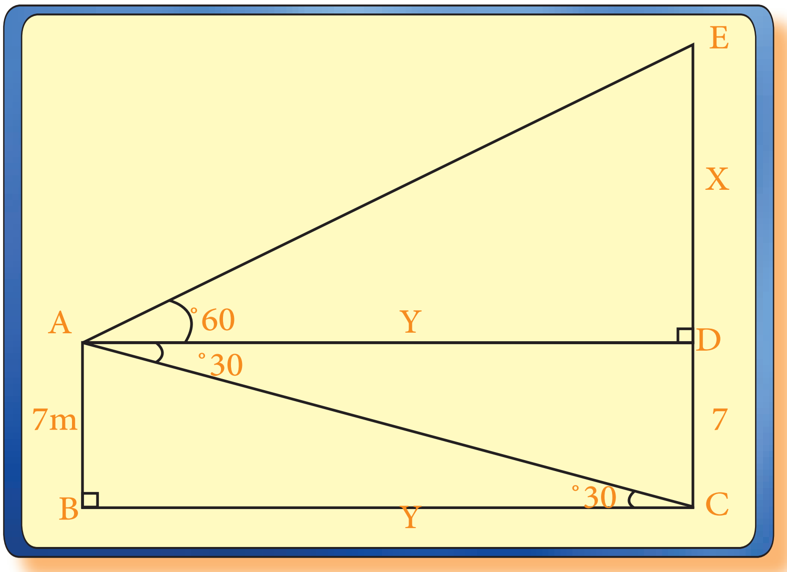



Figure (4 - 11)

*Solution :*

$\sphericalangle DAC = \sphericalangle ACB$

$\Delta$ ABC right angled in B:

$\tan 30^\circ = \frac{7}{Y}$

Distance between the observer and building

$\frac{1}{\sqrt{3}} = \frac{7}{Y} \Rightarrow Y = 7\sqrt{3}$

$\Delta$ E A D in the right angle D:

$\tan 60^\circ = \frac{X}{Y}$

$\sqrt{3} = \frac{X}{7\sqrt{3}} \Rightarrow X = 21\text{ m}$

$\therefore$ The height of building = X + 7

= 21 + 7 = 28m

*Example 12*

The observer Watched that the angle of the height of the balloon is 30 and when the observer went in the horizontal level towards the balloon distance of 1000 meters saw that the angle of elevation is 45 find the height balloon to the nearest meter.

*Solution*

$\Delta$ ABC right angle in B:

$$\tan 45^\circ = \frac{x}{y}$$

$$1 = \frac{x}{y}$$

$$\therefore\ x = y \ldots\ldots \text{(1)}$$

$$\tan 30^\circ = \frac{x}{y + 1000} \ldots\ldots \text{(2)}$$

$$\frac{1}{\sqrt{3}} = \frac{y}{y + 1000} \Rightarrow \sqrt{3}\,y = y + 1000$$

$$1.7\,y - y = 1000$$

$$y = \frac{1000}{0.7} = 1428.6$$

$\therefore$ The height of the balloon x = 1429

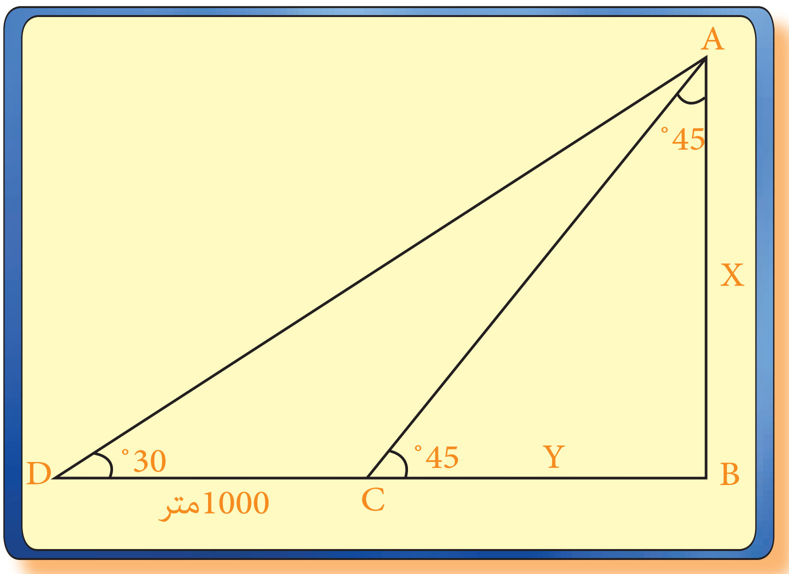



Figure (4 - 12)

## [4-7-2] Circular Sector:

**Definition (4 - 7)**

A circular sector is a part of the circl's surface that is defined by an arc of the circle and two radiuses passing through both ends of the arc

In figure (4-13) $\sphericalangle$ AOB is called a Ccentral angle by the angle of the smaller sector that has an angle with the measure of less than 180˚ .

Area of the circular sector = $\frac{1}{2}$ Length of the arc $\overset{\frown}{AB}$ x r

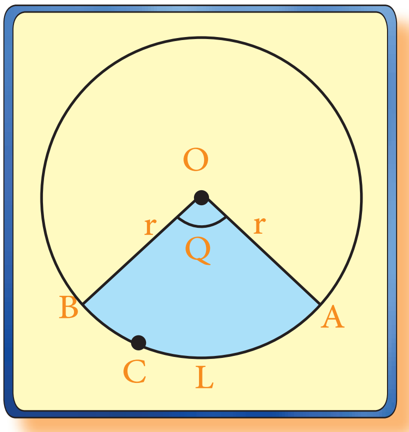



Figure (4-13)

① ........ Area of the circular sector = $\frac{1}{2}$ Lr

If we assume that the measure of the central angle by radians=Q

$Q = \frac{L}{r} \Rightarrow L = Q\,r$

And by substituting in the first equation:

② ........ Area of the circular sector = $\frac{1}{2}$ Q $r^2$

**Note:**

Perimeter of the circula sector

= r + r + L

= 2 r + L

Where L is the length of the arc of the circular sector and r is the length of the radius

**Corollary1:** If we assume that the surface of a circle is a sector with an angle of $2\pi$

$\therefore$ Area of the circle $= \frac{1}{2}(2\pi) \times r^2 = \pi r^2$

**Corollary2:**

$$\frac{Q}{2\pi} = \frac{\frac{1}{2}\,Q\,r^2}{\pi r^2} = \frac{\text{Area of circular sector}}{\text{Area of surface of circle}}$$

$\therefore \frac{D^0}{360^0} = \frac{Q}{2\pi}$ Where D˚ is the measure of the central angle of the sector in degrees

$$\therefore \frac{\text{Area of circular sector}}{\text{Area of surface of circle}} = \frac{D^0}{360^0} = \frac{Q}{2\pi}$$

$\therefore$ Area of the circular sector = $\frac{\text{The measeure of its degree angle}}{360}$ x area of the circle

*Example 13*

Find the area of the circular sector if the measure of its angle is 60° and the length of the radius is 8cm.

*Solution :*

$\because$ Area of the sector $= \frac{1}{2} Q\ r^2$

$$\frac{1}{2} \times \frac{\pi\ 60}{180} \times 64 =$$

$$\frac{1}{2} \times \frac{3.14}{3} \times 64 = 33.49\ cm^2$$

Another solution : area of the circular sector $= \frac{D^\circ}{360^\circ} \times$ the area of its circle

$$\pi \times 64 \times \frac{60^\circ}{360^\circ} =$$

$$64 \times 3.14 \times \frac{1}{6} = 33.49\ cm^2$$

*Example 14*

The erea of a circular sector is $15cm^2$ and the length of its arc is 6cm. Find the length of the radius, perimeter of the circle, the measure of its angle in Degree

*Solution :*

1- $\frac{1}{2} L\ r$ = Area of circular sector

$$15 = \frac{1}{2} \times 6 \times r \Rightarrow r = 5$$

2- Perimeter of circular sector $= 2r + L$

$$cm\ 16 = 6 + 5 \times 2 =$$

3- Radians $1.2 = \frac{6}{5} = Q \Leftarrow \frac{L}{r} = |Q| \because$

$$\frac{3.14}{180^\circ} = \frac{1.2}{D^\circ} \Leftarrow \frac{\pi}{180^\circ} = \frac{Q}{D^\circ}$$

$$68.7898^\circ = \frac{180^\circ \times 1.2}{3.14} = D^\circ \therefore$$

## [4-7-3] Circular Segment :

**Definition (4 - 8)**

A circular segment is a part of the circle that is dfned by an arc and a chord passing through both ends of th arc. <AOB is a central angle as in Figure 4-14

The angle of the smaller segment and is angle is less than 180°

To find the segment`s area:

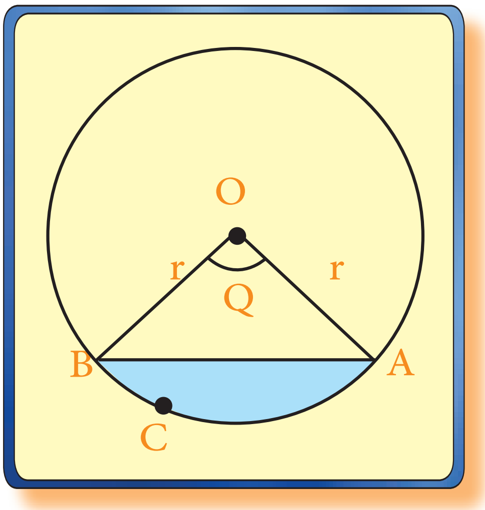



Figure (4-14)

We assume that Q is the radian measure of the smaller segment

$$\therefore \text{Area of segment ABC} = \text{Area of sector } (O\overset{\frown}{ACB}) - \text{Area of } \Delta \text{ OAB}$$

$$\because (O\overset{\frown}{ACB})\text{`s sector`s area} = \frac{1}{2} Q r^2$$

$$\text{OAB } \Delta\text{`s area} = \frac{1}{2} \times OA \times OB \times \sin Q$$

$$\therefore \text{OAB } \Delta = \frac{1}{2} \times r \times r \sin Q$$

$$\therefore \text{AC`s area} = \frac{1}{2} Q r^2 - \frac{1}{2} r^2 \sin Q$$

$$\text{ACB Segment's area} = \frac{1}{2} r^2 (Q - \sin Q)$$

Where Q is the measure of the segment's angle in radians , r is the length of the circle's radius

*Example 15*

Find the area of the segment that has a radius of 12cm and the measure ofits angleis $30^{\circ}$

*Solution :*

$$Q = 0.5236 \Leftarrow \frac{Q}{30^{\circ}} = \frac{\pi}{180^{\circ}} \Leftarrow \frac{Q}{D^{\circ}} = \frac{\pi}{180}$$

$\because$ The Area of the segment $= \frac{1}{2} r^{2} (Q - \sin 30°)$

$$= (-0.5\ 0.5236) \times 144 \times \frac{1}{2}$$

$\therefore$ Area of the segment $= (0.0236) \times 144 \times \frac{1}{2} = 1.7\text{cm}^{2}$

*Example 16*

O is the center of a circle that's radius is 6cm, a 6cm chord was drawn in it, estimate the area of the smaller segment in $\text{cm}^2$

*Solution :*

$\Delta$ AOB is equilateral

$m\sphericalangle AOB = 60^{\circ}$

$$1.047 = \frac{22}{21} = \frac{\pi}{3} = Q \Leftarrow \frac{Q}{60^{\circ}} = \frac{\pi}{180^{\circ}} \Leftarrow \frac{Q}{D^{\circ}} = \frac{\pi}{180^{\circ}}$$

$\frac{1}{2} r^{2} (Q - \sin Q)$ =Area of the segment

$\frac{1}{2} \times 36\ (1.047 - \sin 60^{\circ})$ =

$18\ (1.047 - 0.865)$ =

$3.276\text{cm}^{2} = 18\ (0.182)$ =

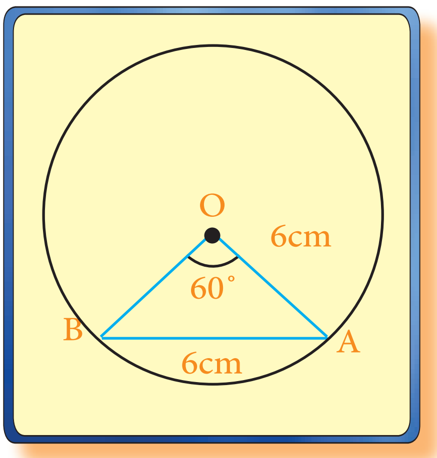



Figure 4-15

## Exercises ( 4 - 2 )

**Q1 /**

A person stood on top of a tower and saw two trees collinear with the tower's base, if the depreseeion angle of the first tree is $70^{o}$ and the depression angle of the second tree is $50^{o}$, find the distance between the two trees kowing that that height of the tower is 30m, $\tan 50^{o}=1.2$, $\tan 70^{o}=2.8$. Ans/ 14.28cm

**Q2 /**

From a point that is 50m away from the base of a tower, it was found that the elevation angle of its peak is $30^{o}$ , what is the height of the tower.

Ans/ 28.9cm

**Q3 /**

Find the area of a circular sector that's arc's length is 8cm and the length of the radius of it's circle is 3.2cm. Ans/ $12.8cm^2$

**Q4 /**

Find the area of a circular sector, if the mesure of its angle is $100^{o}$, the length of the radius is 10cm. Ans/ $87.3cm^2$

**Q5 /**

The area of a circular segment is 37.68cm2 and the length of the radius of its circle is 6cm, find the length of its arc. Ans/ 12.56cm

**Q6 /**

Half the perimeter of a circle is 10cm. Find the area of the circular sector in it that's angle is 45o Ans/ $3.98cm^2$

**Q7 /**

Find the area of a cicrular segment thats angle is 60o and the measur of the radius of its circle is 8cm. Ans/ $5.81cm^2$

## [4-8] Using the calculator to find the values of circular applictions

You knew in section [4-2] that an angle has two methods to calculate that are: the

degree measure and the radian mesaure and a calculator uses both of these ways. we can see the degree measure on top of a calculator which is denoted by (DEG) which stands for (Degree).

And the Radian measure is denoted by (RAD) that stands for RADIAN.

and these two symbols appear on top of the screen after pressing DRG⟶, First press shows DEG and the second shows RAD and vice versa.There are also special

keys for Traingular ratios too.

key (sin) stands for (Sine)

Key (cos) stands for (Cosine)

Key (tan) stands for (tangent)

How to use the calculator:

1.We choose the degree measure system (DEG) or the radian measure (RAD) by (DRG)

2.We type in the measure of the angle acoording to the system we chose

3.We press the keys of the trigonometric rations

The following examples will explain it:

***Example 17*** (1) sin $30^\circ$ (2) cos $120^\circ$ (3) tan $350^\circ$

Find

***Solution :***

NOTE:

$\sin(-Q) = -\sin Q$ ،

$\cos(-Q) = \cos Q$ ،

$\tan(-Q) = -\tan Q$

(1)Degree measure: We press until DEG shows on top of the screen.

* type in 3.

*press sin and you'll get the result 0.5

2) Degreee measure: press until you can see DEG on top of the screen

*type in 120

*press ( cos) and you'll get the result~0.5

3) Degree measure: We press until we can see DEG on top of the screen.

*type in 350 then press-tan so the answer will be close to -0.17630

***Example 18***

Find $\tan \frac{7\pi}{5}$ (3) ، $\cos(-3\pi)$ (2) ، $\sin \frac{5\pi}{4}$ (1)

***Solution :***

*Radian measure: press until RAD shows up.

* Press o the key that's usually found on the keys 2ndf or INV and its usually in color other than black(yellow, red, etc..)

* Press the key: π calculatios Ratio = Answer

(1) $\sin \frac{5\pi}{4}$

*press until RAD shows up.

*Press 2ndf then π = 3.141592564 times 5 = 15.70796327

Divided by 4 = 3.3926990817 then sin =0.707106781

(2) cos (-3 π)

It is known that cosQ= cos(-Q) ( we remove the negatvie sign)

*press until RAD shows up.

*Press 2ndf then π =3.141592564 multiply by 3 = 9.42477961

Then cos =-1

(3) $\tan \frac{7\pi}{5}$

*press until RAD shows up.

*Press 2nd f then π =3.131592654 multiply by 7= 21.9114858

Divide by 5 = 4.398229715 then press tan = 3.07763537

Exercise :

Find the followigs using a calculator

(1) $\sin(\frac{\pi}{6})$ (2) $\cos(-400°)$ (3) $\tan(-15°)$

(4) $\tan(-36°)$ (5) $\cos\frac{2\pi}{3}$ (6) $\tan\frac{8\pi}{5}$

Solution :

(1) 0.5

(2) 0.766044443

(3) 0.267949192

(4) - 0.588

(5) - 0.5

(6) -3.077683537

## [4-9] Solution of Right Angled Triangle :

Each angle consists of 6 elements( 3 sides and 3 angles) and solving the triangle means finding the unknown values of the elements.

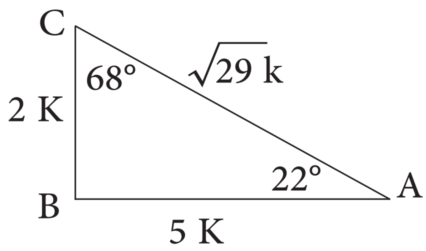
*Example 19*

If $\tan 22^\circ = 0.4$ find:

(1) $\sin 22^\circ$ ، $\cos 22^\circ$

(2) $\cos 68^\circ$، $\sin 68^\circ$

*Solution :*




$\tan 22^\circ = \frac{\text{opposite}}{\text{Adjacent}} = \frac{4}{10} = \frac{2}{5}$

$\therefore$ Opposite = 2k

$\therefore$ Adjacent = 5k

Pythagorean theorem $(AB)^2 + (BC)^2 = (AC)^2$

$4K^2 + 25K^2 = (Ac)^2$

$AC = \sqrt{29}\,K$

$\sin 22^\circ = \frac{BC}{AC} = \frac{2k}{\sqrt{29}k} = \frac{2}{\sqrt{29}}$ (1)

$\cos 22^\circ = \frac{AB}{AC} = \frac{5k}{\sqrt{29}k} = \frac{5}{\sqrt{29}}$

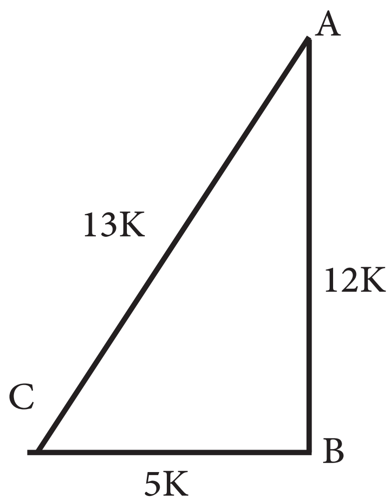
$\sin 68^\circ = \sin(90^\circ - 22^\circ) = \cos 22^\circ = \frac{5}{\sqrt{29}}$ (2)

$\cos 68^\circ = \cos(90^\circ - 22^\circ) = \sin 22^\circ = \frac{2}{\sqrt{29}}$

## *Example 20*

If you knew that $\cos C = \frac{5}{13}$ in triangle ABC that is right angle at B.. Find sinA, tanC, cosA.

## *Solution :*




We draw ABC that is right at B:

$\cos C = \frac{\text{opposite}}{\text{Hypotenuse}} = \frac{5k}{13k}$

$\therefore$ (Pythagorean) $(AC)^2 = (AB)^2 + (BC)^2$

$\therefore 169\ K^2 = (AB)^2 + 25\ K^2$

$\therefore (AB)^2 = 144\ K^2 \Longrightarrow AB = 12K$

$\tan C = \frac{12k}{5k} = \frac{12}{5}$

$\sin A = \frac{5k}{13k} = \frac{5}{13}$

$\cos A = \frac{12k}{13k} = \frac{12}{13}$

## *Example 21*

ABC is a triangle that has a right angle at A, AC=24cm, AB=7cm find:

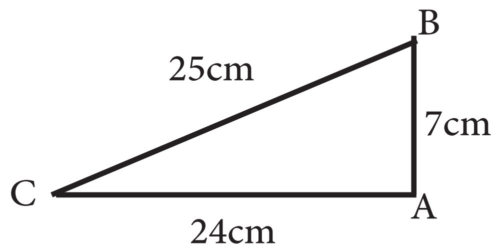
sin C, sin B, tan C, cos B

## *Solution :*

$(BC)^2 = (AB)^2 + (AC)^2$

$(BC)^2 = (7)^2 + (24)^2 = 49 + 576 = 625$

$\therefore BC = 25$ cm

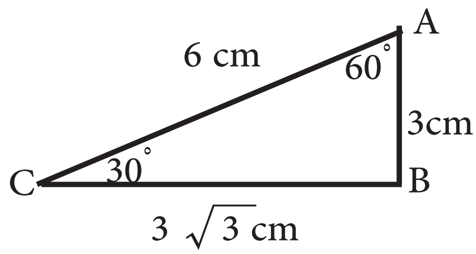


$\therefore \quad \sin C = \frac{7}{25} \quad , \quad \sin B = \frac{24}{25}$

$\tan C = \frac{7}{24} \quad , \quad \cos B = \frac{7}{25}$

***Example 22***

Solve for the triangle ABC that has a right angle at B. If you knew that AB = 3cm, AC = 6cm




***Solution :***

$(AC)^2 = (AB)^2 + (BC)^2$

$36 = 9 + (BC)^2$

$BC = 3\sqrt{3}$

We found the lengths of the sides, now we will find the measures of the angles.

$\tan C = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow C = 30°$

$m < A = 90° - 30° = 60°$

## Conclusion

In the solution for a right triangle we use:

*Trigonometric ratios sinQ, cosQ, tanQ

*We use the pythagorean theorem

and depending on the nature of the question

## Excercises ( 4 - 3 )

**Q1 /**

ABC is a right triangle at B, sinC = $\frac{8}{17}$ Find sinA , tanC, cosC

**Q2 /**

ABC is a right triangle at C, BC= 24cm, AB=25. Find $\sin^2 B + \cos^2 B$ using the given information

**Q3 /**

If cosQ = $\frac{4}{5}$ Find tan Q, sin Q

**Q4 /**

The length of a ladder that's anchored on a horizontal ground is 10 m. it's other end is on a vertical wall. if the angle between the ledder and ground is $30^{\circ}$. What is the distance between it`s top end and the ground, and the distance between it`s bottom end and the wall. Use 3 =$\sqrt{1.73}$

**Q5 /**

ABC is a right triangle at C, m<CAB =60o, AB=20 cm. Find its area.

**Q6 /**

(A) $\frac{3}{4}\tan^2 30^{\circ} + 2\sin 60^{\circ} + 3\tan 45^{\circ} + \cos^2 30^{\circ} - \tan 60^{\circ}$

(B) $\cos^2 45^{\circ} \sin 60^{\circ} \tan 60^{\circ} \cos^2 30^{\circ}$.

(C) sin 120° ، cos 135°، tan 150°.

**Q7 /**

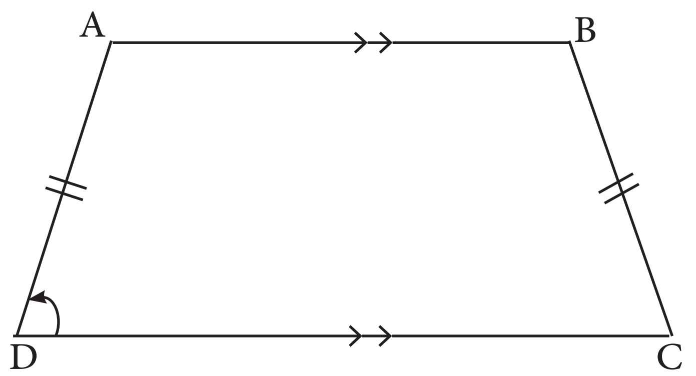
In the figure:

ABCD is a trapezoid where

AD = BC (iscosceles)

DC =2 0 cm, AB =14cm,

AD = 6cm, Find m < CDA



# Chapter 5: Vectors

5

[5-1] Concept of vectors (geometric and algebraic)
[5-2] Conorlical Vector
[5-3] The Vector Length And Its Direction
[5-4] The addition of vectors and multiply it by real number
[5-5] Giving the vector in terms of unity in the coordinate plane

Aims and skills

At the and of this chapter the student will acknowledge:

- Learning the Vector geometrically
- Learning the Vector algebraically
- Learning the Conorlical Vector
- Finding the Length of Conorlical Vector
- Finding the Direction of Conorlical Vector
- Finding the addition of vectors
- Finding the multiply of the Vector by real number
- Learning the unit Vector
- put the Vector by unit Vectors

| ***Terms*** | ***Symbol or Mathematical relations*** |
|---|---|
| Vector a | $\overrightarrow{a} = (x , y)$ |
| Vector Length a | $\lVert\overrightarrow{a}\rVert = \sqrt{x^2 + y^2}$ |
| Zero Vector | $\overrightarrow{0} = (0 , 0)$ |
| unit Vector $u_1 , u_2$ | $\overrightarrow{u_1} = (1 , 0) , \overrightarrow{u_2} = (0 , 1)$ |

## [5-1] Concept of vectors (geometric and algebraic)

***Introduction :*** Some physical and mathematical quantities such as length, mass, time, size, distance, etc. It is determined by a number indicating only its quantity, and such quantities are called numerical quantities or non-vector Quantities. Other amounts such as power, speed and displacement will be trend plus the amount is necessary to fully determine such quantities are called vector quantities. Originated an idea originally intended in the mechanics to represent

The force, speed, displacement, etc., and used the piece A straight line from a point such as A is called the starting point to another point such as the B called point Finish to represent the vector and usually denotes the vector with the symbol A B where the arrow means that the object is directed From A to B. The vector may be represented by a single letter such as a (with its beginning and end) there are two ways to study vectors


(1) geometric

(2) algebraic

In our study in this chapter we focus on algebraic part and using geometric part to Illustration

***Basic concepts:*** Vectors geometrically means a line segment directed as we said above AB, CD, EF are different vectors figure (5-1)

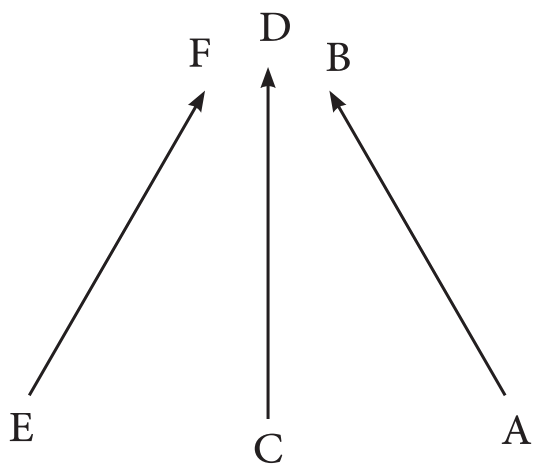



Figure ( 5 - 1 )

***Parallel Vectors:*** If the two segments are parallel, parallel vectors may have the same direction or may be in the opposite direction. Figure (5-2) shows that $\overrightarrow{AB}$ parallel to $\overrightarrow{CD}$ and has the same direction

However, $\overrightarrow{AB}$ is parallel to $\overrightarrow{EF}$ as they are in opposite direction

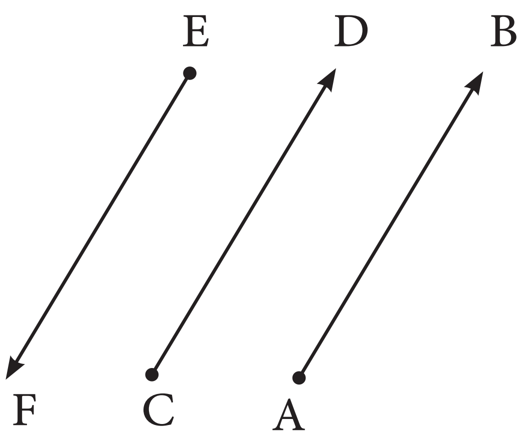



Figure ( 5 - 2 )

***Equivalent vectors:*** if they have same length and same direction

## [5-2] Conorlical Vector

For each vector in the plane there is a single vector whose equivalent starts from the point of origin (0, 0), so instead of Dealing from an infinite number of vectors equal in length and direction, we will take the vector equalizer Which begins with the origin point representing all of them, is called the vector that begins with the origin point of the vector Standard or restricted vector. The rest with non-point vector are called free vector.

Notice that:

$\overrightarrow{OF}$، $\overrightarrow{OE}$ two conorlical vectors

While $\overrightarrow{CD}$, $\overrightarrow{AB}$ are free vectors

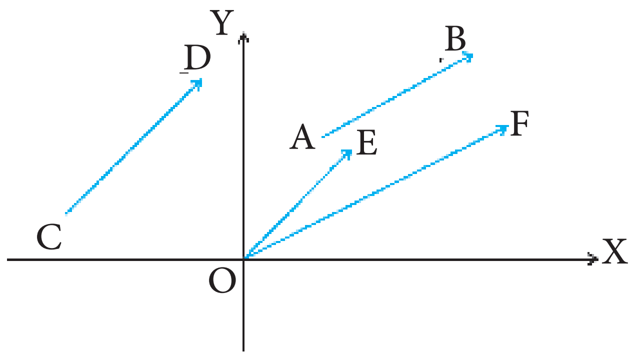



Figure ( 5 - 3 )

### 5-2-1 Vectors and its representations

We represented the pair (4, 3) with a point at the parallel coordinate and each pair of real numbers. We can represent it by one point. The two pairs (3, 5), (2, 3) are represented by points C and B Respectively. Its origin is the point of origin and the end of ordered pairs known Oriented segments $\overrightarrow{OC}$, $\overrightarrow{OB}$, $\overrightarrow{OA}$.

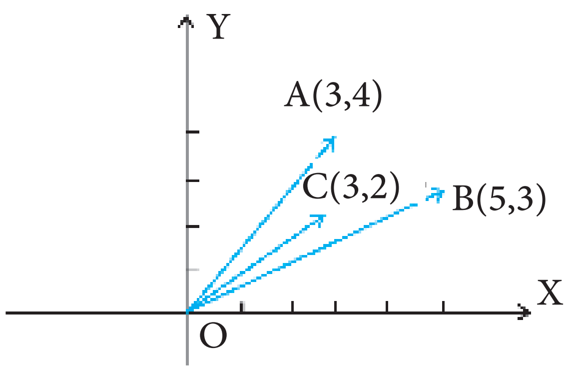



Figure ( 5 - 4 )

The ordered pairs (3.4), (5.3), (3.2).On this basis we will represent the vector with a pair of real numbers

we write $\overrightarrow{OA} = \overrightarrow{A} = \overrightarrow{(x , y)}$

Because we will limit our study to vectors Conorlical only, so all start with the point of origin Only the final point is mentioned.

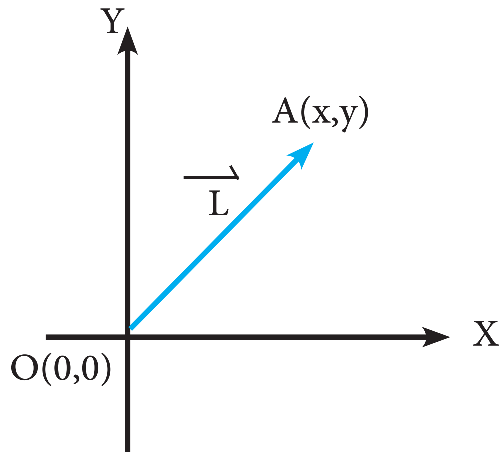


Figure ( 5 - 5 )

## [5-3] The Vector Length And Its Direction

### 5-3-1 The Vector Length

Is the distance between the vectors starting point and the end point AB is the length of $\overrightarrow{AB}$ and symbolizes by $||\overrightarrow{AB}||$

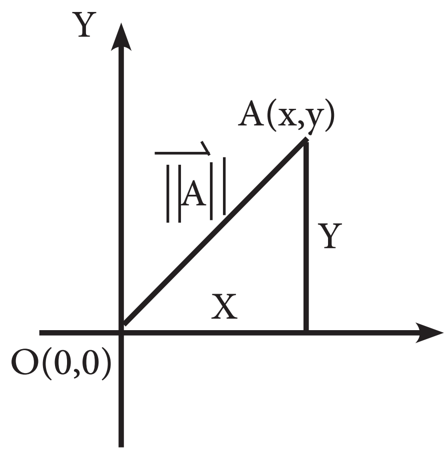


Figure ( 5 - 6 )

**Definition (5-1)**

if $\overrightarrow{A}$ was a vector in which $\overrightarrow{A}$=(x,y) so that

$$||\overrightarrow{OA}|| = ||\overrightarrow{A}|| = OA = \sqrt{x^2 + y^2}$$

Notice that the figure (5-6)

*Example 1*

Find the length for each of the following :

$$( -12 , -9 ) ،\ \left( \frac{\sqrt{2}}{10} , \frac{7\sqrt{2}}{10} \right) ،\ (3,4)$$

*Solution*

Length vector ( 3,4 ) is $\sqrt{( 3 )^2 + ( 4 )^2} = \sqrt{9 + 16} = 5$

Length vector $\left(\frac{\sqrt{2}}{10}\right) , \left(\frac{7\sqrt{2}}{10}\right)$ is $\sqrt{\left(\frac{\sqrt{2}}{10}\right)^2 , \left(\frac{7\sqrt{2}}{10}\right)^2} = \sqrt{\left(\frac{2}{100}\right) , \left(\frac{98}{100}\right)} = \sqrt{\frac{100}{100}} = 1$

$$\sqrt{(-12)^2 + (-9)^2} = \sqrt{144 + 81} = \sqrt{225} = 15$$

## Definition (5-2)

Zero Vector : The vector (0,0) is called the zero vector because its starting and ending point Is the origin point. and symbolizis by $\vec{0}$, and length $\vec{0} = || \vec{0} ||$ .

## Definition (5-3)

Equals vectors: The vector (x1, y1) and (x2, y2) are said to be equal if and only if $x_1 = x_2$ ، $y_1 = y_2$

## Definition (5-4)

Vector direction: The angle made by the vector with the positive direction of the X- axis

## [5-3-2] The direction of the vector

If $\overrightarrow{A}=(x,y)$ was vector $\overrightarrow{A}$ is known as the measured angle Q where $0 \leq Q < 2\pi$ measured In the opposite direction of the clockwise direction from positive the X- axis to the vector $\overrightarrow{A}$

Note that the direction of zero vector is undefined

$$\cos Q = \frac{\sqrt{x}}{x^2+y^2} \quad , \quad \sin Q = \frac{\sqrt{y}}{x^2+y^2}$$

***Example 2***

Find length and direction of $\overrightarrow{OB} = (\sqrt{3}, -1)$

***Solution :***

$$\| \overrightarrow{OB} \| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$$

Assume that Q equals the measurement of the angle that Vector $\overrightarrow{OB}$ determines with the positive direction of the X- axis

So $\cos Q = \frac{\sqrt{3}}{2}$

$\sin Q = \frac{-1}{\sqrt{2}}$

And from the figure ( 5- 7) we can see that Q on the fourth quadrant.

And the direction of vector is

$$2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

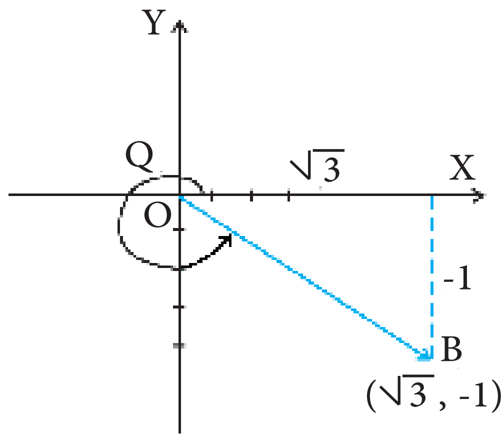



Figure ( 7 - 5 )

*Example 3*

Find the direction of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

*Solution :*

Assume that Q is equal to vector angle measurement $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

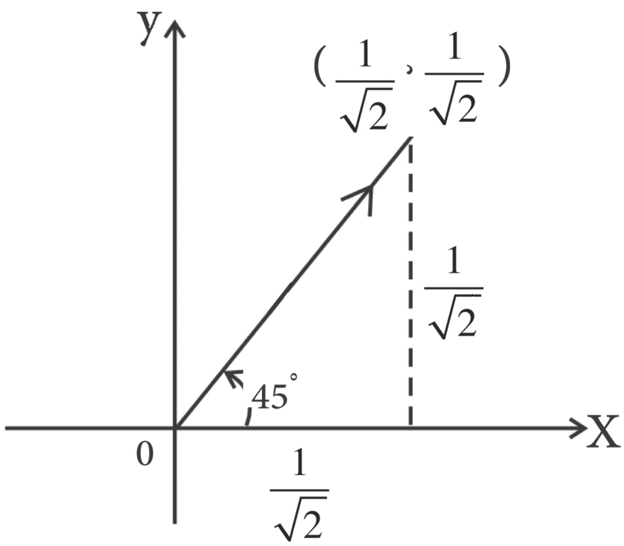



Figure (5 - 8)

$$\cos Q = \frac{\frac{1}{\sqrt{2}}}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\sin Q = \frac{\frac{1}{\sqrt{2}}}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}} = \frac{1}{\sqrt{2}}$$

From the figure (5-8) we noticed that

Q is on the first quadrant $\frac{\pi}{4}$

***Example 4***

Find the vector if its length 5 and its direction $\frac{\pi}{6}$

***Solution :***

Suppose that a(x,y)

$$\cos Q = \frac{x}{\|\vec{a}\|} \Rightarrow \cos\frac{\pi}{6} = \frac{x}{5} \Rightarrow \frac{\sqrt{3}}{2} = \frac{x}{5}$$

$$\therefore x = \frac{5\sqrt{3}}{2}$$

$$\sin Q = \frac{y}{\|\vec{a}\|} \Rightarrow \sin\frac{\pi}{6} = \frac{y}{5} \Rightarrow \frac{1}{2} = \frac{y}{5}$$

$$\therefore y = \frac{5}{2}$$

$\therefore$ the vector is $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$

**Conclusion**

1)The length $\vec{A}(x,y)$ equal $\|\vec{A}\|$ so $\|\vec{A}\| = \sqrt{x^2 + y^2}$

2) To find the direction $\vec{A}(x,y)$ we use $\cos\theta = \frac{x}{\|\vec{A}\|}$ , $\sin\theta = \frac{y}{\|\vec{A}\|}$

## Exercise ( 5 - 1 )

**Q1 /**

Find the length and the direction of the following vector and draw directed line segments which referred to:

A) (-2 , 2 ) ، B) (-3 , 0) ، C ) ( 1 , $\sqrt{3}$ )

D) ( 0 , 6) ، E) ($\sqrt{3}$ , -1 ) ، E) ( -3 , -3 ) ، G) ( 0, -8 )

**Q2 /**

Find the vector with length and direction as the following:

A) $\|\overrightarrow{B}\| = 2$ ، $Q = \frac{\pi}{6}$

B) $\|\overrightarrow{B}\| = \sqrt{2}$ ، $Q = \frac{\pi}{4}$

C) $\|\overrightarrow{B}\| = 4$ ، $Q = \pi$

D) $\|\overrightarrow{B}\| = 3$ ، $Q = \frac{3\pi}{2}$

E) $\|\overrightarrow{B}\| = 4$ ، $Q = \frac{2\pi}{3}$

## [5-4] The addition of vectors and multiply it by real number

### The addition of vectors(5-4-1)

To add two vectors such as $\vec{A}, \vec{B}$ we geometrically draw one of them and from the other end point of the vector Which begins with the starting point of the first vector and ends at the end point of the second vector is the sum of the vectors, check the fig(5-9). The sum of two vectors is found in a Parallelogram way, representing the total diameter Parallelogram in which two vectors are adjacent as in a fig(5-10):

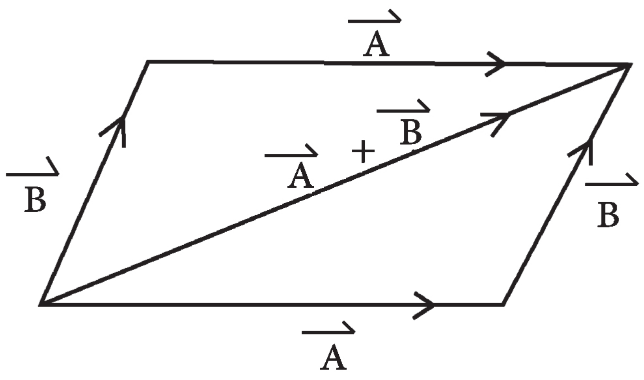



Figure (5-9)

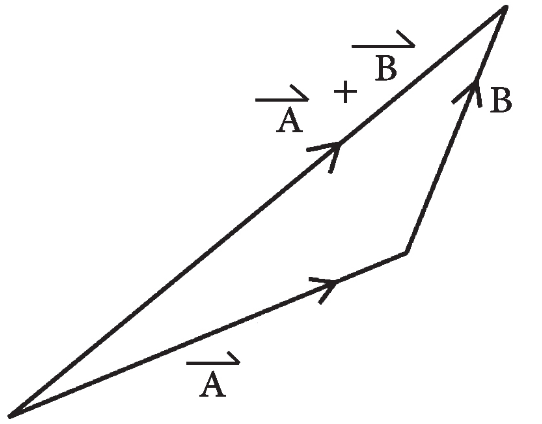



Figure (5-10)

Two victors may be located on one straight line, then they are said to be on one straightness, as in the vectors, $\vec{A}, \vec{C}$ while $\vec{A}$ and $\vec{B}$ are opposite in direction as in fig (5-11).

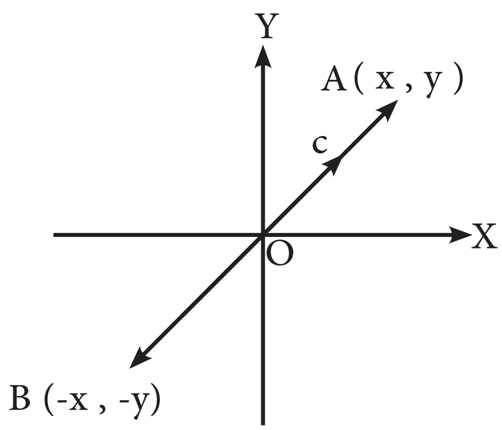



Figure (5-11)

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are on one straightness, they are equal in length and opposite in direction, It was

if $\overrightarrow{A} = (x, y)$ so $\overrightarrow{B} = (-x , -y)$

Notes that $\|\overrightarrow{A}\| = \|\overrightarrow{B}\| = \sqrt{x^2+y^2}$

The vector A is denoted by the symbol $-\overrightarrow{A}$

## Definition (5-5)

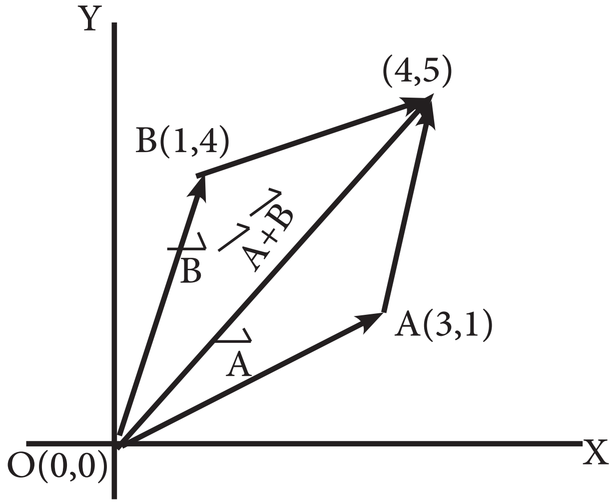
If $\overrightarrow{A} = (x_1, y_1)$ ، $\overrightarrow{B} = (x_2, y_2)$ then

$\overrightarrow{A} + \overrightarrow{B} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

***Example 5***

If $\overrightarrow{A} = (3 , 1)$ ، $\overrightarrow{B} = (1, 4)$ find $\overrightarrow{A} + \overrightarrow{B}$

***Solution :***




الشكل (5 - 12)

$\overrightarrow{A} + \overrightarrow{B} = (3 ,1) + (1 , 4) = (4 , 5)$

This can be illustrated geometrically as in Figure (5-12)

Note that $\overrightarrow{A} + \overrightarrow{B}$ represents the parallelogram diameter of the vector $\overrightarrow{A}, \overrightarrow{B}$

***Example 6 :***

If $\vec{A} = (-4, 3)$ ، $\vec{B} = (5, -2)$ find $\vec{A} + \vec{B}$ .

***Solution :***

$\vec{A} + \vec{B} = (-4, 3) + (5, -2) = (1, 1)$

## *5-4-2 the Vector addition properties*

(1) Closure Property: If both $\vec{A}$ and $\vec{B}$ are vectors then, $\vec{A} + \vec{B}$ is also vectors

(2) Associative property: If $\vec{A}, \vec{B}, \vec{C}$ are vectors then $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

(3) Commutative Property: If $\vec{A}$ and $\vec{B}$ were a vector then $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

(4) Additive Identity: The zero vector is the Identity element in addition of the vector this means that if $\vec{A}$ is any vector,

$\vec{A} + (0,0) = (0,0) + \vec{A} = \vec{A}$

(5)Additive Inverse: : If $\vec{A}$ is any vector, another vector is $-\vec{A} = \vec{B}$ such that

$\vec{A} + (\vec{B}) = (\vec{B}) + \vec{A} = (0,0)$

(6) Elimination property: if $\vec{A}, \vec{B}$ are vectors and $\vec{A} + \vec{B} = \vec{A} + \vec{C}$ then $\vec{B} = \vec{C}$

***Example 7***

Find the additive inverse for the vector ( -2 , 3)

***Solution :***

The additive inverse of (-2, 3) is (2,-3) because

$(-2, 3) + (2, -3) = \big(-2 + 2,\ 3 + (-3)\big) = (0, 0)$

## *5-4-3 Multiplying a vector by real number*

**Definition (5 - 6)**

if $\overrightarrow{A}$=(x,y) and K is a real number then $K\overrightarrow{A}=\overrightarrow{A}K$=(Kx,Ky)

that can be illustrated geometrically

Suppose that $\overrightarrow{A}$=(x,y) and $K\overrightarrow{A}$ is a Straight vector on $\overrightarrow{A}$ and length equal K || $\overrightarrow{A}$ ||

means k times

As long as the length of vector $\overrightarrow{A}$ when it is k<0 and has the same vector direction

Figure (5-13) (If k <0) is negative, the vector $\overrightarrow{A}$ K is located on the straightness of $\overrightarrow{A}$ and its length equal K || $\overrightarrow{A}$ ||

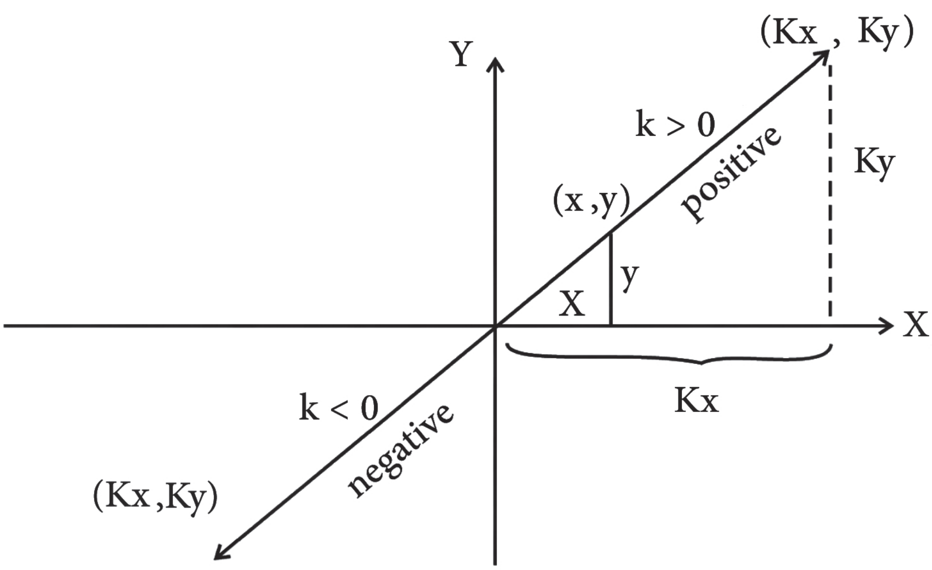



Figure (5 - 13)

*Example 8*

If $\overrightarrow{C}$ = (3, -1) then find $2\overrightarrow{C}$ ، $\overrightarrow{C}$، $-3\overrightarrow{C}$

*Solution*

$2\overrightarrow{C}$ = 2(3,-1) = (6 , -2 )

$\frac{1}{2}\overrightarrow{C} = \frac{1}{2}(3,-1) = (\frac{3}{2} , \frac{-1}{2})$

$-3\overrightarrow{C}$ = -3 (3 , -1) = (-9,3)

*Example 9*

If $\overrightarrow{A}$ = (3 , -2) ، $\overrightarrow{B}$ = (4 , 3) and K = 3 ، L = -2 then find

(1) $\overrightarrow{A} + \overrightarrow{B}$ (2) K $\overrightarrow{A}$ (3) L $\overrightarrow{B}$ (4) K $\overrightarrow{A}$ + L $\overrightarrow{B}$

*Solution :*

(1) $\overrightarrow{A}+\overrightarrow{B}$ = ( 3 + 4 , -2 + 3) = (7 , 1)

(2) K $\overrightarrow{A}$ = 3 (3 , -2) = (9 , -6)

(3) L $\overrightarrow{B}$ = -2 (4 , 3) = ( -8 , -6 )

(4) K $\overrightarrow{A}$ + L B = (9 , - 6) + ( -8 , -6 )

= (1 , -12)

## *(5-4-4) The of vector multiplication by real numbe properties*

(1) Distribution property For each $\overrightarrow{A}$, $\overrightarrow{B}$ are vectors, K is real number

($\overrightarrow{A}$ K+ $\overrightarrow{B}$ K) = ($\overrightarrow{A}$+ $\overrightarrow{B}$) K

(2) Associative property: For each A vector and both K, L∈R

So (K x L) $\overrightarrow{A}$ = K(L A ) =L (K$\overrightarrow{A}$) ∈ R

(3) Elimination property: For each A, B vector, $R \in K$ where $K \neq 0$

If $\overrightarrow{KA} = \overrightarrow{KB}$ so that $\overrightarrow{A} = \overrightarrow{B}$ and vice versa

$$1 \times \overrightarrow{A} = \overrightarrow{A} \times 1 = \overrightarrow{A} \qquad (4)$$

$$0 \times \overrightarrow{A} = \overrightarrow{A} \times 0 = 0 \qquad (5)$$

## *[5-4-5] Subtraction of vectors :*

**Definition (5 - 7)**

If $\overrightarrow{A}$ , $\overrightarrow{B}$ were a vectors so that $\overrightarrow{A}$ $-\overrightarrow{B}$ can define as $\overrightarrow{A} + (-\overrightarrow{B})$

***Example 10***

If $\overrightarrow{A} = (3 , 4)$ ، $\overrightarrow{B} = (-1 , 3)$ find $\overrightarrow{A} - \overrightarrow{B}$

***Solution***

$\overrightarrow{A} - \overrightarrow{B} = \overrightarrow{A} + (-\overrightarrow{B}) = (3 , 4) + (1 , -3) = ( 4 , 1)$

We can show that on the coordinate which means $\overrightarrow{A}$ and $-\overrightarrow{B}$ represent Parallelogram diameter for $\overrightarrow{A}$ and for negative vector $\overrightarrow{B}$ .

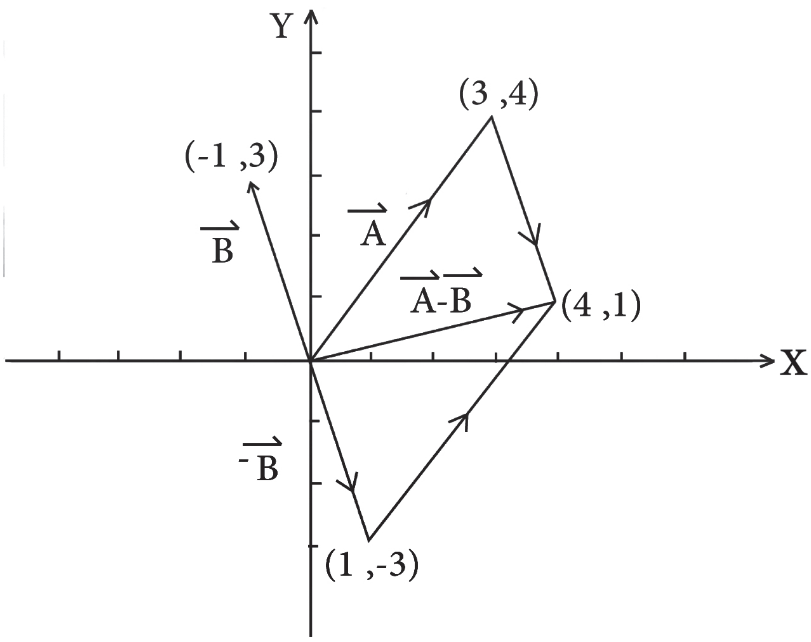



Figure (5 - 14)

*Example 11*

If $\overrightarrow{A} = (2 , 3)$ ، $\overrightarrow{B} = (-2 , -1)$ ، K = 2 ، L = -1 find

$K\overrightarrow{A} - L\overrightarrow{B}$ ( 2 )  $\overrightarrow{A} - \overrightarrow{B}$ (1)

*Solution :*

(1) $\overrightarrow{A} - \overrightarrow{B} = (2 , 3)-(-2 , -1)$

$= (2 , 3)+(2 , 1)=(4 , 4)$

Figure (5 - 15) shows that:

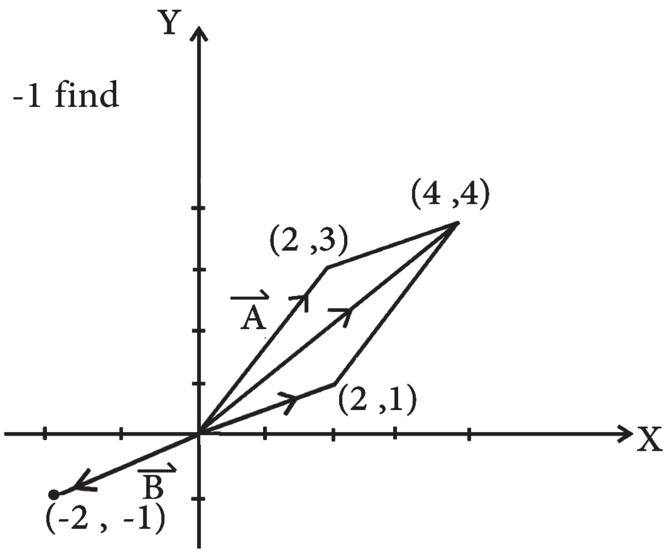



Figure ( 5 - 15 )

(2) $K\overrightarrow{A} - L\overrightarrow{B} = 2\ (2 ,3) - (-1)(-2,-1)$

$= (4 , 6) + (-2,- 1)$

$= (2 , 5)$

This is illustrated in the figure below:

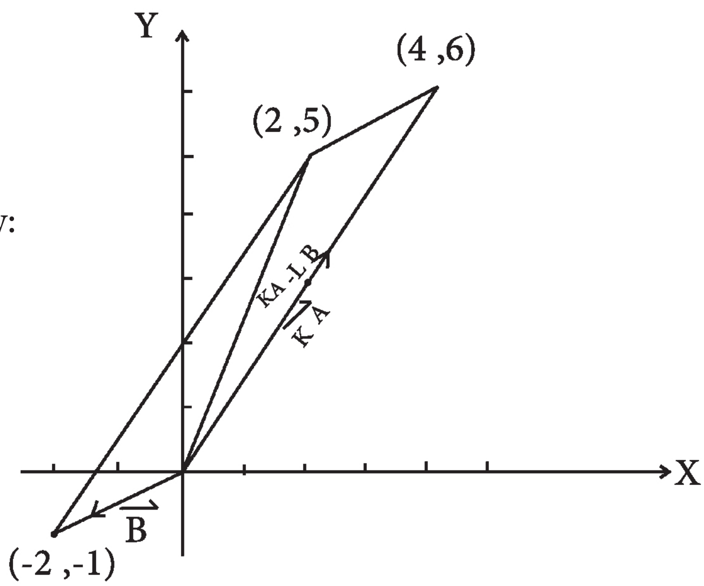



Figure ( 5- 16 )

We draw $\overrightarrow{A}$ then we extend it as long as we get $2\overrightarrow{A}$, then we draw and then find $-\overrightarrow{B}$

then $-1 \times -\overrightarrow{B} = \overrightarrow{B}$ we will return to $\overrightarrow{B}$ again and $2\overrightarrow{A} + \overrightarrow{B}$

as we did in the previous example.

## 5-5 Giving the vector in terms of unity in the coordinate plane

### [ 5 - 5 - 1 ] : Unit Vector

**Definition (8 - 5)**

(1)Unit vector U1: Is the straight-pointed object whose beginning is the point of origin and the length of one unit and its direction is the positive direction of the X- axis and symbolizes by $\overrightarrow{U_1}=(1,0)$

(2) Unit vector U2: Is the straight-pointed object whose beginning is the point of origin and the length of one unit and its direction is the positive direction of the Y-axis and symbolizes $\overrightarrow{U_2}=(0,1)$ .

If $\overrightarrow{C} = (x, y)$ so $\overrightarrow{C} = (x , 0) + (0 , y)$

$\overrightarrow{C} = x (1 , 0) + y ( 0 , 1)$

and this represents vector C in the terms of $\overrightarrow{U_1}\ \overrightarrow{U_2}$ $C = x \overrightarrow{U_1} + y \overrightarrow{U_2}$

and we can write the vectors ( 9 , 0 ) ، (-3 , 0) ، ( 0 , -2) ، ( 0 , 6) in terms of $\overrightarrow{U_1}\ \overrightarrow{U_2}$ as following

$(9,0)= 9\overrightarrow{U_1}$ ، $(-3 , 0) = -3\overrightarrow{U_1}$ ، $(0 ,-2) = -2\overrightarrow{U_2}$ ، $(0 , 6) = 6 \overrightarrow{U_2}$

*Example 12*

If $\overrightarrow{A} = (4 , 7)$ ، $\overrightarrow{B} = (-5 , 3)$ find $\overrightarrow{A} + \overrightarrow{B}$ and show the result in the terms of Units Vector

*Solution*

$\overrightarrow{A} + \overrightarrow{B} = ( 4 , 7 ) + ( -5 , 3 ) = ( -1, 10 ) = -(1, 0 ) + 10(0 , 1) = - \overrightarrow{U_1} + 10 \overrightarrow{U_2}$

Based on that we can write any vector by $\vec{U}_1$ ، $\vec{U}_2$ as the following examples:

$(2, 5) = 2\vec{U}_1 + 5\vec{U}_2$

$(-4, 2) = -4\vec{U}_1 + 2\vec{U}_2$

$(-2 , -3) = -2\vec{U}_1 - 3\vec{U}_2$

if the vector written as two unit vectors we can find the order pair wich represent it, example

If $\vec{A} = 4\vec{U}_1 + 5\vec{U}_2$ then $\vec{A} = (4,5)$

$\vec{B} = -2\vec{U}_1 + 3\vec{U}_2$ so $\vec{B} = (-2 , 3)$ and soon.

### *Example 13*

إذا كان $\vec{B} = 2\vec{U}_1 + \vec{U}_2$ ، $\vec{A} = \vec{U}_1 - 3\vec{U}_2$ جد $\vec{A} + \vec{B}$

### *Solution*

$$\vec{A} + \vec{B} = (\vec{U}_1 - 3\vec{U}_2) + (2\vec{U}_1 + \vec{U}_2) = \vec{U}_1(1+2) + \vec{U}_2(-3+1) = 3\vec{U}_1 - 2\vec{U}_2$$

$$=(3,-2)$$

### *Example 14*

If $\vec{A} = (5 , -3)$ and $\vec{B} = (-3,4)$ and K= 2 ، L = 3, find $K\vec{A} - L\vec{B}$ then write it unit vectors in the terms of

### *Solution*

$$K\vec{A} - L\vec{B} = 2(5, -3) - 3(-3, 4)$$

$$= (10,-6) + (9 , -12)$$

$$= (19 ,-18)$$

$$= 19\vec{U}_1 - 18\vec{U}_2$$

## Excercise ( 5 - 2 )

**Q1 /**

Find the value the direction for each of the following with a graph

$(-2 , -2)$ ، $(3, 0)$ ، $\sqrt{3}\ \overrightarrow{U}_1 + \overrightarrow{U}_2$ ، $-\overrightarrow{U}_1 - 2\overrightarrow{U}_2$

**Q2 /**

Simplify the following

$4(1 ,-1)$ ، $2(1 , -1)$ ، $-7(1,5)$ ، $3 (2,-1 )+ 4(-1 , 5)$ ، $7(3\overrightarrow{U}_1 + 2\overrightarrow{U}_2)$ ، $-4(2\overrightarrow{U}_1 - \overrightarrow{U}_2)$

**Q3 /**

Represent each of the following vectors in the terms of Units Vectors $\overrightarrow{U}_1$ ، $\overrightarrow{U}_2$

$(3 , 2)$ ، $(-1 ,4)$ ، $(-3 , -5)$ ، $(0 , -1)$ ، $( 5 , 3)$ ، $(2 , 0)$

**Q4 /**

if $\overrightarrow{E}$ = x, y where x, y $\in$ R and $\overrightarrow{A}$ is a vector so that

$\overrightarrow{A} + \overrightarrow{E} = \overrightarrow{E} + \overrightarrow{A} = \overrightarrow{A}$ prove that $\overrightarrow{E} = (0 , 0)$

**Q5 /**

If $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A} = (0,0)$ prove that $\overrightarrow{A} = -\overrightarrow{B}$

**Q6 /**

If $\overrightarrow{A} = (\sqrt{3} , 1)$ ، $\overrightarrow{B} = (\sqrt{2}, \sqrt{3})$ ، K = 3 ، L = -2 then find each of the following

$K\overrightarrow{B}$ ، $L\overrightarrow{A}$ ، $\overrightarrow{A} + \overrightarrow{B}$ ، $K\overrightarrow{A} + \overrightarrow{B}$ ، $K\overrightarrow{A} - \overrightarrow{B}$ ، $K\overrightarrow{A} + L\overrightarrow{B}$

$K\overrightarrow{A} - L\overrightarrow{B}$ ، $K(\overrightarrow{A} + \overrightarrow{B})$ ، $(L + K)\overrightarrow{A}$ ، $(L + K)(\overrightarrow{A} + \overrightarrow{B})$ ،

$K(L\overrightarrow{A} + K\overrightarrow{B})$ ، $KL(\overrightarrow{A} - \overrightarrow{B})$

**Q7 /**

Solve the question 6 by representing each of the vector in the terms of Unit Vectors $\overrightarrow{U}_1$ ، $\overrightarrow{U}_2$

**Q8 /**

Represent each of the vector in the terms of Unit Vectors $\overrightarrow{U}_1$ ، $\overrightarrow{U}_2$

A) the length 3 and direction $\frac{\pi}{3}$

B) the length 10 and direction $\frac{\pi}{6}$

C) the length 5 and direction $\frac{\pi}{4}$

D) the length $\frac{3}{4}$ and direction $\pi$

**Q9 /**

If $\overrightarrow{A} = (5 , 2)$ ، $\overrightarrow{B} = (2 ,-4)$ find x when $2\overrightarrow{A} + 3x = 5\overrightarrow{B}$

# 6 Chapter 6 : Analytical Geometry

[6 - 1] Coordinate plane
[6 -2] Distance Between Two Points
[6 - 3] Coordinates of a partition point
[6 - 4] Slope of The Line
[6 - 5] Parallel Condition
[6 - 6] Perpendicular Condition
[6 - 7] Equation of The Line
[6 - 8] The distance of a point from a given line

Aims and skills
At the and of this chapter the student will acknowledge:
- Learning the coordinate plane
- Finding distance between two points
- Finding the midpoint of line segment
- Finding of coordinates of a partition point
- Learning the first degree equation with two variables
- Learning the slope of the Line
- Finding a first degree equation with two variables
- Distinguish between the parallel and the perpendicular lines by their slopes
- Finding the distance of a point from a given line

| *Terms* | *Symbol or Mathematical relations* |
|---|---|
| Distance Between Two Points | $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ |
| Coordinates of a partition point $\frac{n_1}{n_2}$ | $\left(\frac{n_1 x_2 + n_2 x_1}{n_1 + n_2}, \frac{n_1 y_2 + n_2 y_1}{n_1 + n_2}\right)$ |
| Parallel Lines $L_1, L_2$ | $\overleftrightarrow{L_1} // \overleftrightarrow{L_2} \Longleftrightarrow m_1 = m_2$ |
| Perpendicular Lines $L_1, L_2$ | $\overleftrightarrow{L_1} \perp \overleftrightarrow{L_2} \Longleftrightarrow m_1 \times m_2 = -1$ |
| Equation of The Line | $ax + by + c = 0$ |
| distance between a point and line | $D = \frac{\lvert ax_1 + by_1 + c \rvert}{\sqrt{a^2 + b^2}}$ |

## [Coordinate plane [6-1

If we draw two perpendicular lines $\overleftrightarrow{x\acute{x}}$، $\overleftrightarrow{y\acute{y}}$ that cross each other at point (O) and we represented the real numbers ith R and we assumed that O represens the origin Figure (6-1)1

By doing that we have we have constructed a coordinate plane, we call the $\overleftrightarrow{x\acute{x}}$ , the x-axis and the $\overleftrightarrow{y\acute{y}}$, the y- axis. when we take any point in this plane, for example point A and we draw two lines, the first on the x-axis and the second on the y-axis to make points $\overline{AB}$, and $\overline{AC}$ (see figure 6-1)

When we write A(3,2)as an ordered pair of real numbers the x-axis comes first and the y-axis comes second. in this chapter we will assume that the x and y axis are perpendicular and that the length unit used in one of the axises is the .same that it used in the other

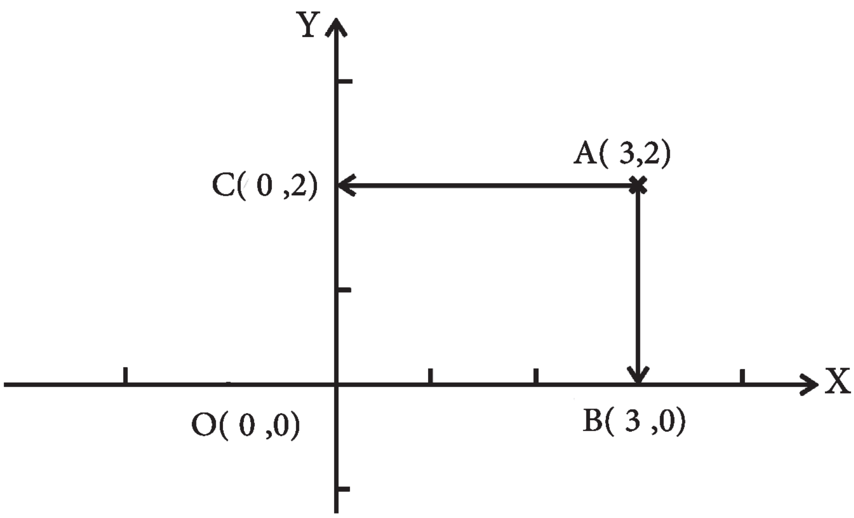



Figure (6-1)1

## [6 -2] Distance Between Two Points

If we knew the coordinates of two points belonging to a plane ,the distance between can be calculated in the following way

let two points A $(x_1 , y_1)$ ، B $( x_2 , y_2)$

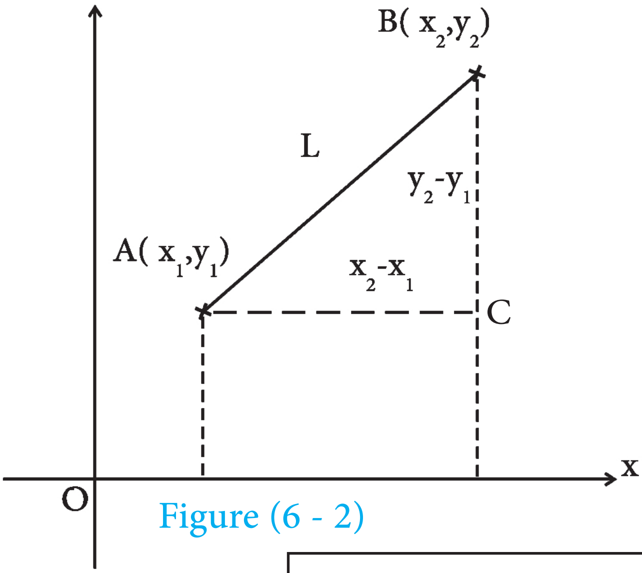



Figure (6 - 2)

Δ A B C : is a right angle at C

$L^2 = (AC)^2 \quad ( BC)^2$

$L = \sqrt{( x_2 - x_1 )^2 + (y_2 - y_1)^2}$

distance between two points formula

Or by using the $\overrightarrow{AB} = \overrightarrow{A} - \overrightarrow{B}$ formula

$\overrightarrow{AB} = (x_2 , y_2) - ( x_1 , y_1)$

$= (x_2 - x_1 ، y_2 - y_1)$

| Distance between two points $\|\overrightarrow{AB}\| = \sqrt{( x_2 - x_1 )^2 + (y_2 - y_1)^2}$ |
|---|

***Example 1 :***

prove that the points A (-2 , 7) ، B (-3, 4) ، C (1, 16) .belong to a single line

***Solution :***

First way :

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (-3 , 4) - (-2, 7) = (-1 , -3)$

$\overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = (1 , 16) - (-2 , 7) = (3 , 9) = -3 (-1 ,-3)$

$\therefore \overrightarrow{AC} = -3\overrightarrow{AB}$

$\therefore$ points A,B,C belong to a same single straight line

Second way :

$AB = \sqrt{(-2+3)^2 + (7-4)^2} = \sqrt{1+9} = 10$

$BC = \sqrt{(-3-1)^2 + (4-16)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10}$

$AC = \sqrt{(-2-1)^2 + (7-16)^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$

BC = AB + AC

the points A,B,C must belong to a single straight line,because if they weren't they would have made a triangle and the sum of two sides would be bigger than .the third side

***Example 2***

prove that the triangle which has the following vertices A (1 , 1) ، B (2 , 2) ، C (5 , -1) is a right triangle.

***Solution***

$AB = \sqrt{(2-1)^2+(2-1)^2} = \sqrt{1+1} = \sqrt{2}$

$AC = \sqrt{(5-1)^2+(-1-1)^2} = \sqrt{16+4} = \sqrt{20}$

$BC = \sqrt{(5-2)^2+(-1-2)^2} = \sqrt{9+9} = \sqrt{18}$

$\because AC^2 = AB^2 + BC^2$

$(\sqrt{20})^2 = (\sqrt{2})^2 + (\sqrt{18})^2$

ABC triangle is a right angle

$\because$ at B

Example 3

prove that the points A(-3, -1) ، B (1 , -4) ، C (10, -5) ، D (6 , -2) represent the vertices of a parallelogram.

Solution

$AB = \sqrt{(-3-1)^2 + (-1+4)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$BC = \sqrt{(1-10)^2 + (-4+5)^2} = \sqrt{81+1} = \sqrt{82}$

$CD = \sqrt{(10-6)^2 + (-5+2)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$AD = \sqrt{(6+3)^2 + (-2+1)^2} = \sqrt{81+1} = \sqrt{82}$

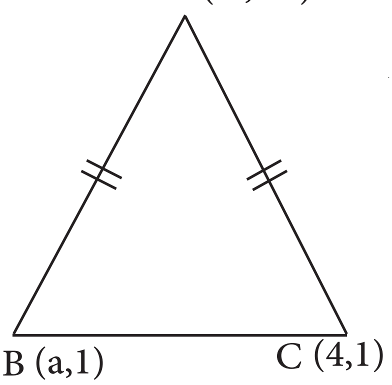
Where AB=CD,BC=AD

figure ABCD represents a parallelogram

Example 4

if the points C(4, 1) ، B(a, 1) ، A(3, 2a) are the heads of an isosceles triangle where AB=AC find a,a $\in$

*Solution*




given AB=AC

$\Rightarrow \sqrt{(3-a)^2 + (2a-1)^2} = \sqrt{(3-4)^2 + (2a-1)^2}$

$\Rightarrow (3-a)^2 + (2a-1)^2 = 1 + (2a-1)^2$ squaring both sides

taking the square root of both sides

$\Rightarrow (3-a)^2 = 1$

$\Rightarrow 3-a = \pm 1$

$: 3-a = 1 \Rightarrow a = 2$

or: $3-a = -1 \Rightarrow a = 4$ neglected(find out why)

## Exercises (6-1)

**Q1 /**

Find the distance between every pair of the following points

A) (4 , 3) ، (0 , 0).    B) (4 , 6) ، (2 , 1) .

C) (5 - , -3 ) ، ( -1 , 5 )    D) (-2 , 3) ، (-1 , 4).

**Q2 /**

Find the area of the triangle that's heads are on the points

A(5 , 7) ، B( 1, 10) ، C (-3 , -8).

**Q3 /**

the vertices of a square shape are A(4 , -3) ، B( 7 , 10) ، C (-8 ,2) ، D (-1 , -5)

find the length of its diameter.

**Q4 /**

prove that the points. A(-2 , 5) ،B (3 , 3) ،C (-4 , 2) are the vertices of a parallelogram.

**Q5 /**

If the points A (2 , 3) ، B (-1 , -1) ، C (3 , -4) are three vertices of a parallelogram ABCD

Find the coordinates of D.

**Q6 /**

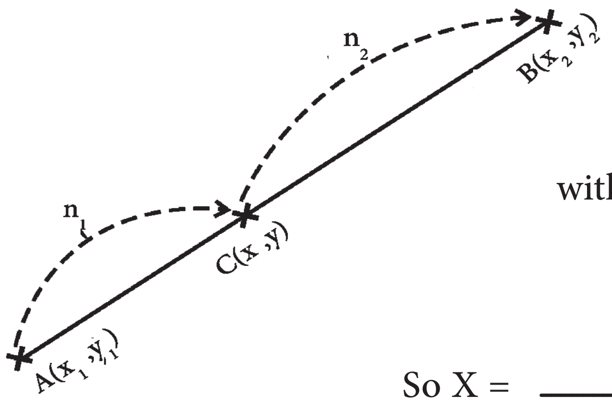
prove that the triangle that has the vertices A (2 , 3) ، B (-1 , -1) ، C (3 , -4) is an isosceles triangle.

**Q7 /**

Prove that the points (-3,-4),(6,8),(0,0 are on a single straight line.

## Coordinates of a partition point (6-3)

Partition of a line segment is used to find the coordinates of a point that is located between two points in a given ration.




$A = (x_1 , y_1)$ ، $B = (x_2 , y_2)$

It is wanted to find C that divides $\overline{AB}$ with a ration of n : n so, we say $C = (x , y)$

$$\frac{AC}{CB} = \frac{n_1}{n_2}$$

$$\text{So } X = \frac{n_1x_2+n_2x_1}{n_1+n_2} \quad \text{also } Y = \frac{n_1y_2+n_2y_1}{n_1+n_2}$$

$$\text{Partition point C } \left(\frac{n_1x_2+n_2x_1}{n_1+n_2} , \frac{n_1y_2+n_2y_1}{n_1+n_2}\right)$$

*Example 4 :*

Find the coordinates of the point that divides the line segment that connect the points A (4 , -3 ) ، B (-5, 0) with a ratio of $\frac{1}{2}$

*Solution 4 :*

$$x = \frac{n_1x_2+n_2x_1}{n_1+n_2} = \frac{1\ (-5)+2(4)}{1+2} = \frac{-5+8}{3} = 1$$

$$y = \frac{n_1y_2+n_2y_1}{n_1+n_2} = \frac{1\ (0)+2(-3)}{1+2} = \frac{-6}{3} = -2$$

$\therefore$ coordinates of partition point(1,-2)

We assume that M is the partition point of line AB

Where as A $(x_1, y_1)$ ، B $((x_2, y_2)$

So

$$M \text{ (mid point)} = \left( \frac{x_1+x_2}{2} , \frac{y_1+y_2}{2} \right)$$

And to prove we have to make n=n1=n 2and substitute it in the previous equation.

***Example 5***

If point C was in the middle of $\overline{AB}$ where A(-3 , 2) ، B (7 , -8)

find the coordinates of point C

***Solution***

$$C = \left( \frac{x_1+x_2}{2} , \frac{y_1+y_2}{2} \right)$$

$$= \left( \frac{-3+7}{2} , \frac{2+(-8)}{2} \right)$$

C = ( 2 , -3 )

## Summary

Point C(x,y) divides the line segment which connect points A(x ,y),B(x ,y) with the ratio

$\frac{n_1}{n_2}$ is : $C\left(\frac{n_1 x_2 + n_2 x_1}{n_1 + n_2} , \frac{n_1 y_2 + n_2 y_1}{n_1 + n_2}\right)$

Coordinates of the mid points are $\left(\frac{x_1 + x_2}{2} , \frac{y_1 + y_2}{2}\right)$

## Exercises (6-2)

**Q1 /**

Find the coordinates of the point that divides the line segment AB where A (1 , 3) ، B (4 , 6) with a ration of $\frac{2}{1}$

**Q2 /**

Find the coordinates of the point that halves the line segment AB where A (2 , -4) ، B (-3 , -6)

**Q3 /**

Find the coordinates of point C that divides the line segment AB with a ratio of $\frac{5}{3}$ where A (2 , 1) ، B (1, -3)

**Q4 /**

Find the coordinates of point C which is 3 times as far from as it is from B where B (4 , -4) , A(2 , 6)

**Q5 /**

Find the coordinates of the mid-points of the sides of triangle A B C where: A(4 , 0) ، B (5 , 2) ، C (2 , -3) then find the lengths of the lines connecting from the heads of the triangle to the mid-points on the opposite side.

**Q6 /**

Show that the diameters of the tetragonal shape with the vertices (-5,-8),(-3,-3),(3,-1),(-1,-2) halve each other.

## [6 - 4] Slope of The Line

### Definition (6-1)

If $A(x_1,y_1), B(x_2,y_2)$ then

Slope of the line $AB = \frac{y_2 - y_1}{x_2 - x_1}$ , $x_1 \neq x_2$

Note:

1) $y_2 - y_1 = 0$ then the slope of $\overleftrightarrow{AB}$ = is zero

Which means that $\overleftrightarrow{AB}$//Y-axis

Which means that $\overleftrightarrow{AB}$ //to the X-axis.

The slope of the X-axis= slope of the line parallel to it = 0

02) If the slope of $x_2 - x_1 = 0$ then the slope of $\overleftrightarrow{AB}$ is undefined

meaning that the slope of the Y-axis = the slope of the line parallel to it and it is undefined.

3) If Q is the measure of the positive angle that $\overleftrightarrow{AB}$ makes with the positive direction for the X-axis then the slope of $\overleftrightarrow{AB}$ is equal to tanQ where Q belongs to [0,180)/{90}

*Example 6*

Find the slope of the line passing through the two points A (2 , 3) ، B (5 , 1)

*Solution*

$$m\ \overleftrightarrow{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 3}{5 - 2} = \frac{-2}{3}$$

## *[5 - 6] Parallel Condition*

Two parallel lines have the same slope meaning $\overleftrightarrow{L} = \overleftrightarrow{L}$ if and only if $m_1 = m_2$.

***Example 7***

Prove that points A(4,3),B(2,1),C(1,0) belong to a same line.

***Solution***

$$m\,\overline{AB} = \frac{1-3}{2-4} = \frac{-2}{-2} = 1$$

$$m\,\overline{BC} = \frac{0-3}{1-2} = \frac{-1}{-1} = 1$$

$$\because\ m\,\overline{AB} = m\overline{BC}$$

$\therefore$ C ، B ، A belong to a same line

## *[6 - 6] Perpendicular Condition*

If two lines are perpendicular then their slope product is equal to =-1 as $\overleftrightarrow{L_1} \perp$ s$\overleftrightarrow{L_2}$ if and only if $-1 = m_1 \times m_2$

or $m_1 = \frac{-1}{m_2}$ which means that the slope of one of them is equal to the inverse of the other one by inversing the sign , for example if the slope of a line was equal to $\frac{-3}{4}$ ,then the slope of any line parallel to it will be equal to $\frac{-3}{4}$ and the slope of any line perpendicular on it will be $\frac{4}{3}$

## *Example 8*

By using slope prove that the triangle which has the vertices A(3 , -1), B(10 , 4) , C(5 , 11) is a right angle triangle in B

## *Solution*

$$m\overline{AB} = \frac{4-(-1)}{10-3} = \frac{5}{7}\ , \qquad m\overline{BC} = \frac{11-4}{5-10} = \frac{7}{-5}$$

$$\therefore\ m\overline{AB} \times m\overline{BC} = \frac{5}{7} \times \frac{7}{-5} = -1$$

$$\therefore\ \overline{AB} \perp \overline{BC}$$

$\therefore$ $\Delta$ ABC right angle in B .

## *Example 9*

If the points A(0 , b), B(-1 , 2) , C(-2 , b-4) are in a straightness then find the value of $b \in R$ .

## *Solution*

Note

$$m = \frac{\Delta y}{\Delta x}$$

$\therefore$ A,B,C On a straightness

$$\Rightarrow m\overline{AB} = m\overline{BC}$$

$$\Rightarrow \frac{2-b}{-1-o} = \frac{(b-4)-2}{-2-1}$$

$$\frac{2-b}{-1} = \frac{b-6}{-1} \Rightarrow 2-b = b-6$$

$$\Rightarrow b = 4$$

## Exercise (6 - 3)

**Q1 /**

(1)Find the slope of the line passing through points (0,-2) , (2,0)

(2) Prove that the points (2,3),(-1,4),(-7,6) are on one straight line

(3) If points A(2,3) , (-3,h) find the value of h if m $\overline{AB} = \frac{1}{2}$

(4) ABC is a triangle, it's vertices are A(1,6), B(-2,-8), C(7,-2) find the slope of the middle line for the triangle passing through B

**Q2 /**

for each of the following questions there are four answers, only one of them is right, choose the right answer

1)If $\overleftrightarrow{H}$ ، $\overleftrightarrow{L} \perp \overleftrightarrow{H}$ passes through points (1,5),(2,3) the slope of L equals

A) $\frac{-2}{3}$ B) $\frac{2}{3}$ C) 2 D) $\frac{1}{2}$

1)If $\overleftrightarrow{H}$ ، $\overleftrightarrow{L}$ // $\overleftrightarrow{H}$ passes through points (-3,2),(3,-2) the slope of $\overleftrightarrow{L}$ is equal to

A) $\frac{-2}{3}$ B) $\frac{2}{3}$ C) $\frac{-3}{2}$ D) $\frac{3}{2}$

2) If the line L $\in$ (4 , 3) ، $\overleftrightarrow{(x , 6)}$ and H $\in$ (3,-1),(5,-1) if L // H then the value of X is A)-3 B)3 C)1 D) none of the above

## Q3 /

1)By using the slope prove that the points A(5,2),B(2,1)C(2,-2) are the vertices of a triangle

2)Let A(-1,5),B(5,1),C(6,-2),D(0,2) prove that the figure ABCD is a parallelogram

3)Let A(5,2),B(2,-1),C(-1,2),D(2,5) prove that figure ABCD is a square

4)ABC is a triangle it's vertices are on points A(2,4), B (6,0), C(-2,-3) find

A) slope of the line drawn from A to $\overline{BC}$

B) The slope of the line drawn from B and parallel to $\overline{AC}$

5) Prove that the quadrilateral shape with the vertices A(-2,2),B(2,2),C(4,2),D(2,4) is a trapezoid

# Equation of The Line [6-7]

If (x,y) is any point on any line then the relation between x and y is called the equation of the line

The standard equation of the line is $ax + by + c = 0$

we can graphically represent the line intersecting both axis by making $x=0$

$$\therefore\ y = \frac{-c}{b}$$

$$y = 0 \Longrightarrow x = \frac{-c}{a}$$

1) When $b=0$ , $ax+c=0$ represents the equation of the line parallel to the Y axis and when $x=0$ it represents the equation of the Y axis

2) When $a=0$, $by+c$ will represent the equation of the line parallel to the X axis and $y=0$ represents the equation of the X axis

3) When $c=0$ , $ax+by=0$ represents the equation of a line passing through the origin point

## *How to find the equation of line*

1. If you know two points

the equation of the line where $A(x_1, y_1)$ ، $B(x_2, y_2)$

Let $C(x,y) \in \overleftrightarrow{AB}$ so the equation will be

The formula for finding the equation of the line for 2 points $\boxed{\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}}$

If you knew one point and the slope

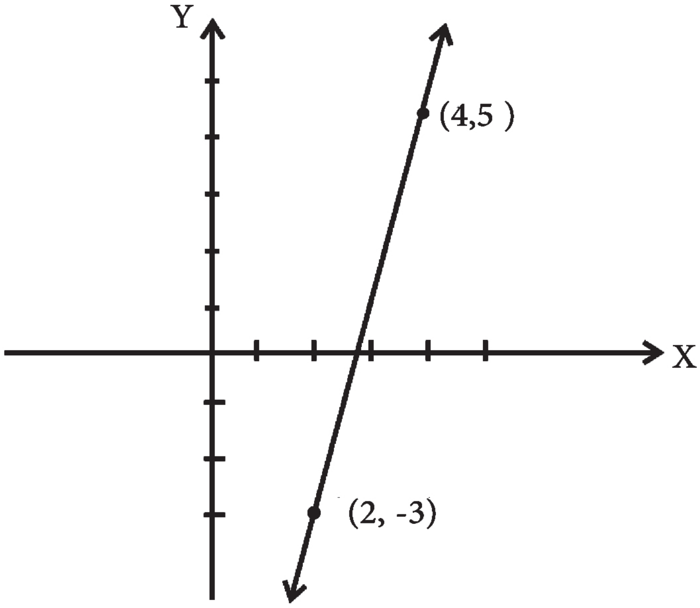
From the previous formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Formula of finding the equation using a point and slope $y - y_1 = m(x - x_1)$

### *Example 9*

Find the equation of the line passing through points (2 , -3) ، (4 , 5).

### *Solution*




$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y + 3}{x - 2} = \frac{5 + 3}{4 - 2}$$

$$\frac{y + 3}{x - 2} = \frac{4}{1}$$

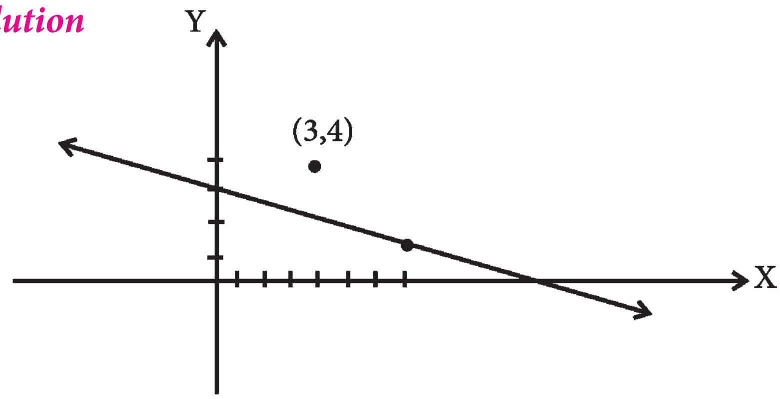
$$y+3 = 4x - 8$$

$\therefore$ $4x - y - 11 = 0$ ......... equation of the line .

### *Example 10*

Find the equation of the line passing through points (7,1) , (0,3) and does the point (3,4) belong it

### *Solution*




$$\frac{y - 1}{x - 7} = \frac{3 - 1}{0 - 7}$$

Equation of the line ....... $2x + 7y - 21 = 0$

to make sure that point (3,4)we have to substitute x=3 , y=4 in the line equation.

$2( 3) + 7( 4 ) - 21 = 0$

$6 + 28 - 21 = 0$

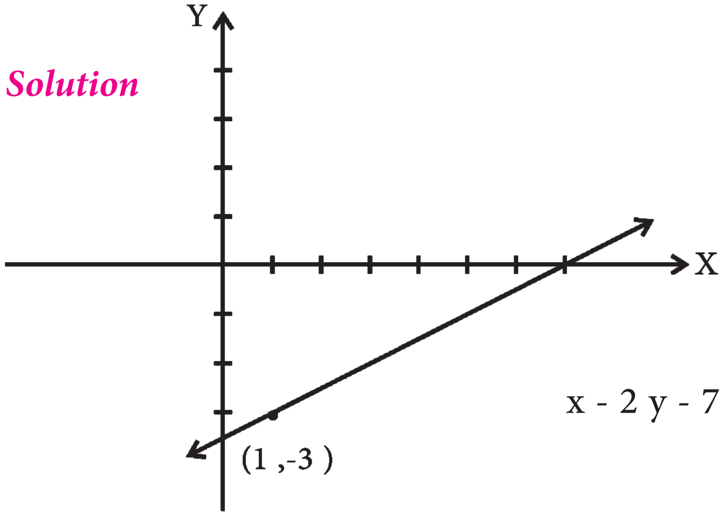
$\therefore\ 0 \neq 13$

$\therefore$ Point (3,4) does not belong to the line

***Example 11***

Find the equation of the line passingthrough point (1,-3) and with a slope of $\frac{1}{2}$.

***Solution***



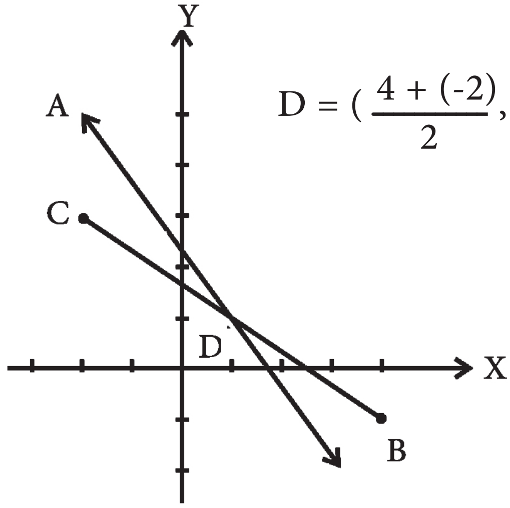

$y - y_1 = m (x - x_1)$

$x - 2y - 7 = 0$ . . . . . . . . equation of the line

***Example 12***

Find the equation of the line passing through point A(1,-3) and the mid point of the line that's ends are at points B(4,-1), C(-2,3)

***Solution:***




$D = \left(\frac{4 + (-2)}{2}, \frac{-1 + 3}{2}\right) = (1,1) \Leftarrow$ Let D be the mid-point of $\overline{BC}$

$\frac{y - 5}{x + 2} = \frac{1 - 5}{1 + 2}$ :The equation of $\overleftrightarrow{AD}$ is

$3y - 15 = -4x - 8\ \therefore$

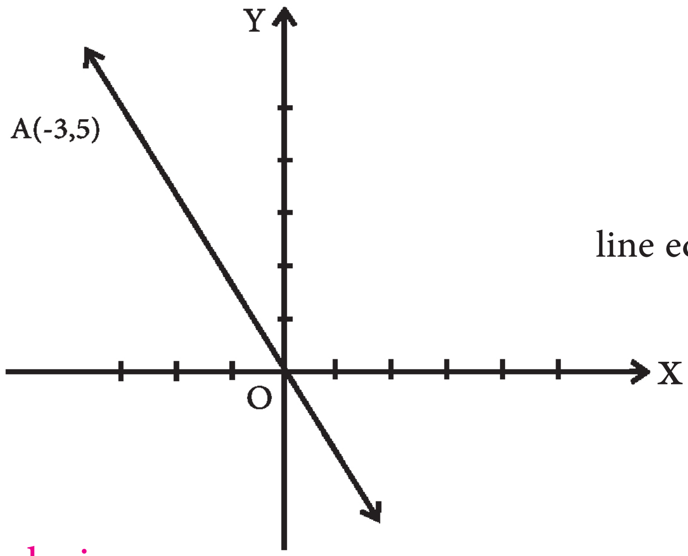
Line equation . . . . . . . . $4x + 3y - 7 = 0$

## *Example 13*

Find the line equation for the line passing through

the origin point and point (-3,5)

## *Solution*




. O (0 ,0) ، A (-3 ,5 )

line equation is $\frac{y - 0}{x - 0} = \frac{5 - 0}{-3 - 0}$

$$\frac{y}{x} = \frac{5}{-3}$$

equation of the line $5x + 3y = 0$

## Conclusion

We can find the slope of the line from

its equation

we assume that $ax + by + c = 0$

$\therefore$ Slope of the line = $\frac{-a}{b}$ by inversing the sign by having x,y on only

one side of the equation

and $b = 0$

the slope of the line $\frac{-x}{y}$

So, the slope of the line $\frac{-a}{b}$

## *Example 14*

Find the the slope and the Y axis of the line that's equation:

$3x - 4y - 12 = 0$

## *Solution*

$$m = \frac{-a}{b} = \frac{-3}{-4} = \frac{3}{4}$$

so we get : $-4y - 12 = 0 \Rightarrow y = -3$

## *Example 15*

Find the equation of the line that makes an angle 150 with the positive side of the X axis and passes through point (-4, 1).

## *Solution*

The slope of the line

$m = \tan 150^\circ$

$= \tan (180^\circ - 30^\circ)$

$= -\tan 30^\circ$

$\because y - y_1 = m (x - x_1)$

$\therefore \quad y + 4 = \frac{-1}{\sqrt{3}} (x - 1)$

$x + \sqrt{3}\, y + 4\sqrt{3} - 1 = 0$ . . . . . . . . . the line of the equation.

*Example 13*

Find the equation of the line that passes through point(-2,1) and is perpendicular to the line that's equation is $2x - 3y -7 = 0$

Solution

From the line $2x - 3y -7 = 0$

The slope of the line $m = \frac{-a}{b} = \frac{-2}{-3} = \frac{2}{3}$

the slope of the required line $= \frac{-3}{2}$ ( because it's perpendicular to it )

$y - y1 = m (x - x_1)$

$y - 1 = \frac{-3}{2} \quad x+2$

$3 x + 2 y + 4 = 0$ ..... the equation of the required line

Summary

1)The slope of the line passing through points $(x_1,y_1),(x_2,y_2)$ is $m=\frac{y_2-y_1}{x_2-x_1}$

2)The slope of the line ax+by+c=0 is $m= \frac{-a}{b}$

3)The slope of the line that makes the angle $\theta$ with the X axis is $m = \tan\theta$

4)The slope of the line passing through points $(x_1,y_1),(x_2,y_2)$ is $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

5)The equation of the line passing through point $(x_1,y_1)$ m is the slope

$y-y_1=m(x-x_1)$

## Exercises (6-4)

### Q1 /

1.Find the equation of the line with $m \frac{1}{2}$ and passes through (0,-4)

2. Find the equation of the line that is parallel to the X axis and passing through point (2,-1)

3.Find the equation of the line parallel to the Y axis and passing through point (2,-1).

4.Find the equation of the line passing through points(-1,3), (5,-1)

5.Find the equation of the line L passing through point (2,-1) and parallel to $L_1$ which has the slope of $\frac{2}{3}$

6.Find the equation of the line passing through point (0,-2) which is perpendicular on the line that has a slope of $\frac{-3}{5}$

7.Find the line equation passing through point (3,-4) and perpendicular to the line passing through points(2,-2),(0,3)

8.Let A(4,-2),B(1,2) find the equation of the line that bisects $\overline{AB}$

### Q2 /

1.Find the equation of the line that has the slope of -3 and splits a positive part of the Y axis that length is 7 units

2.Find the equation of the line that has a slope of 2 and splits a negative part of the X axis that's length is 6 units

3. Find the slope and the dissected part of the Y axis for the following lines

A. $\overleftrightarrow{L1}$: 2x-3y+5=0

B. $\overleftrightarrow{L2}$: 8 y = 4x + 16

C. $\overleftrightarrow{L3}$: 3 y = -4

Find the equation of the line passing through points (2,-5) and parrallel to the line that has the equation: 2x-y+3=0

Find the equation of line L that bisects a negative part of the Y axis and its length is 4 units and perpendicular on the line that has the equation 2y= 4x-1

Let $\overleftrightarrow{L}$ be a line, it's equation is: x+y-2=0 Find it's slope and

it's intersection point with the Y-axis then draw $\overleftrightarrow{L}$

Find the equation of the line L passing through point (2,-2) and

perpendicular on the line that has an equation x+y=0 then find the

intersection point of L with the other two axes

8. The line $\overleftrightarrow{L}$ : 2x-y=3 and the line $\overleftrightarrow{H}$ : 3x + 6x=-3

A.Prove that $\overleftrightarrow{L} \perp \overleftrightarrow{H}$

B. Find algebraically the intersection point of L, H

9.Find the equation of the line that makes 135 with the positive side of the X axis

and that is passing through the origin point

10.The line L : 27 = ax + 1 passes through point (1,2) find

A. The value of

B. The slope of the line L

C. It's Y intercept.

# *The distance of a point from a given line [6-8]*

## Definition 6-2

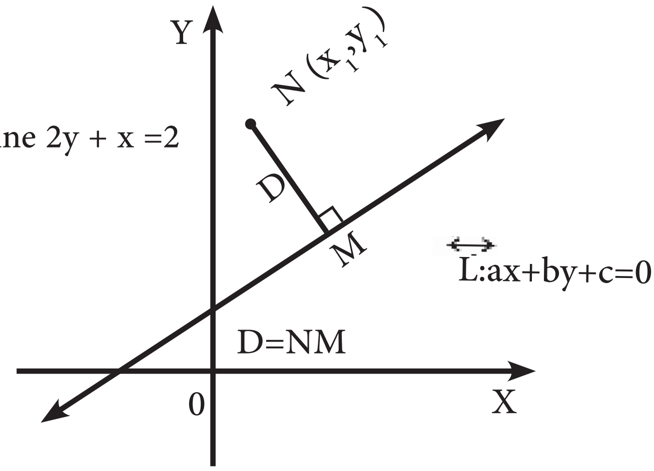
If the line L : ax + by + c = 0

and point N (x , y) is known, then the distance after point N on line L as the vertical distance D between point N and line L and it is given in the following equation.

$$D = \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$$

### *Example 17*

Find the distance from point A(1,3) to the line 2y + x =2




### *Solution*

We put the equation in the following way x +2y -2=0,

a=1 , b=2, c=-2

$$= \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}} = \frac{|(1)(1)+(2)(3)-2|}{\sqrt{(1)^2+(2)^2}}$$

$$D = \frac{5}{\sqrt{5}} = \sqrt{5} \quad \text{unit}$$

We can find the distance between two parallel points

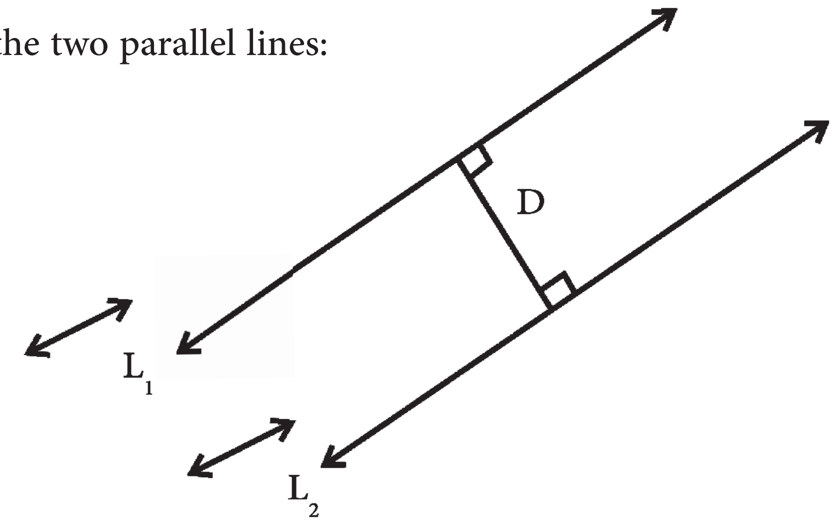
$$\overleftrightarrow{L_1}: a_1 x + b_1 y + c_1 = 0 ، \overleftrightarrow{L_2}: a_2 x + b_2 y + c_2 = 0$$

Distance between $\overleftrightarrow{L_1}$ ، $\overleftrightarrow{L_2} = \frac{|C_2 - C_1|}{\sqrt{a^2+b^2}}$

*Example 18*

Find the distance between the two parallel lines:

$\overleftrightarrow{L_1}: x - 3y = 1$، $\overleftrightarrow{L_2}: x - 3y = 4$




*Solution*

The distance between two parallel lines is the distance between any point belonging to a line and the other line.

As in $L_1$: $y = 0 \Longrightarrow x = 1$

$\therefore$ Point (1,0)

$$\therefore D = \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\therefore D = \frac{|(1)(1) - 3(0) - 4|}{\sqrt{1 + 9}} = \frac{3}{\sqrt{10}}$$

Another solution according to the result

$$D = \frac{|4-1|}{\sqrt{1 + 9}} = \frac{3}{\sqrt{10}}$$

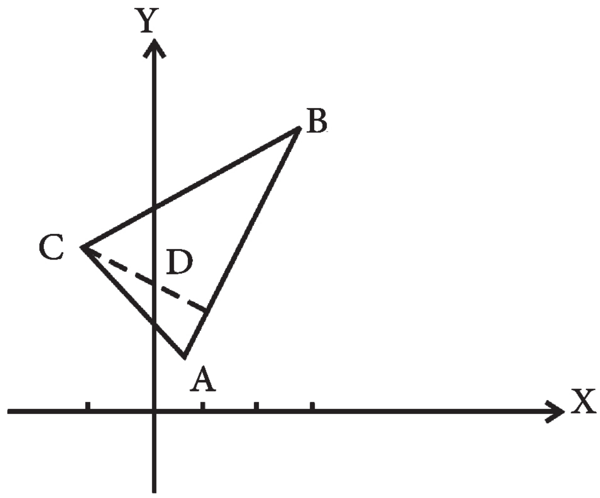
*Example 19*

Find the area of the triangle that's vertices are on points (1,2) ، B (3 ,5) ، C ( 1 ,3)

*Solution*

We find the equation of one of the triangle's

sides and let the line AB be:




$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y - 2}{x - 1} = \frac{5 - 2}{3 - 1} \Rightarrow \frac{y - 2}{x - 1} = \frac{3}{2}$$

$$\therefore 3x - 2y + 1 = 0$$

And now the distance between point C(-1,3) from line AB represents the height

of Δ ABC

$$D = \frac{|3(-1) - 2(3) + 1|}{\sqrt{9 + 4}} = \frac{8}{\sqrt{13}} \text{ unit}$$

$$AB = \sqrt{(3-1)^2 + (5-2)^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$\text{Area } \Delta = \frac{1}{2} (AB) \,.\, D$$

$$= \frac{1}{2} \times (\sqrt{13}) \,.\, \frac{8}{\sqrt{13}} = 4 \text{ unit}^2$$

## Exercises (6-5)

**Q1 /**

Write T in front of the right sentences and F in front of the false sentences from the following

1. The distance between the origin point and line: $y = 3$ is 3 units

2.The distance between the origin point and line: $y =-5$ is 5 units

3.The distance between the origin point and line $x =-5$ is 5 units

4.The distance between the two parallel lines $y = 4$ and $y=-1$ is 3 units

**Q2 /**

1.Find the distance between point (-2,1) and line $6x + 8y - 21 = 0$

2.Find the distance between the origin point and the line that has the slope of $m = \frac{1}{3}$ and passes through a positive side of of the x-axis that's length is 4 units

3.Find the distance between the two parallel lines:

$\overleftrightarrow{L1}: 8x - 6y + 4 = 0$

$\overleftrightarrow{L2}: 4x - 3y - 1 = 0$

4.Find the distance between point (0,-2) and the line that passes through points A (1,-1)، B (3, 5).

5.Find the area of the triangle ABC where A(-4,6), B(-3,-1), C(5,-2)

# Chapter 7 : Statistics

7

[7 - 1] Measures of Central Tendency
[7 -2] Arithmatic Mean
[7 - 3] Median
[7 - 4] Mode
[7 - 5] Measures of Variation

Aims and skills

At the and of this chapter the student will acknowledge:

- Learning Arithmatic Mean
- Finding Arithmatic Mean
- Learning Median
- Finding Median
- Learning Mode
- Finding Mode
- Learning Standard Variation
- Finding Standard Variation
- Learning Correlation coefficient
- Finding Correlation coefficient

| ***Terms*** | ***Symbol or Mathematical relations*** |
|---|---|
| Mean | $\bar{X}$ |
| Median | ME |
| Mode | MO |
| Rang | R |
| Standard Variation | S |
| Correlation coefficient | r |

## *[ 7 - 1 ] Measures of Central Tendency*

We took in the previous studies ways of gathering data and present them graphically and tabularly, and now we want to find a scale that is more representative of the study subject, meaning that we want to get one value representing all the other values. The average income in a country represents all the incomes in a country meaning that it represents the general level of the income in that country and one of the characteristics of data, is that it has a tendency to concentrate on a single average value and these values that the data concentrates on are called medians or the measures of central tendency

And we will take the most important measures of central tendency with a little more details after you have taken them in your middle school studies. here are the subjects we are going to learn values of central tendency

-Arithmetic mean

-Median

-Mode

And these three differ from each other from the idea and their method

of calculation and each one of them has it's specialties and has it's flaws.

There are some cases where we use on of the measurements and not the

other.

## [ 7 - 2 ] Arithmatic Mean

**Definition ( 7 – 1)**

The arithmetic mean of a set of numbers is defined as the value that, if replaced with any of the other values in the set, the sum of the new set would be equal to the sum of the original set

And so the arithmetic mean is equal to the sum of the values divided by their number

Calculation Method:

First way

### 1)If the statistics(data) are unclassified

$$\text{Arithmetic mean} = \frac{\text{sum of the values}}{\text{their number}}$$

Meaning that : $\overline{X} = \dfrac{x_1 + x_2 + x_3 + \dots\dots\dots + x_n}{n}$

*Example 1*

If the ages of 5 people are:5,8,9,11,12. Find the arithmetic mean of their ages.

*Solution*

$$\overline{X} = \frac{x_1 + x_2 + x_3 + \dots\dots\dots + x_n}{n} = \frac{12 + 11 + 9 + 8 + 5}{5}$$

$$\frac{45}{5} = 9 \text{ Years}$$

## 2) If the statistics(data) are classified

If the statistical values are classified in a repeating order, the following formula can be used;

Arithmetic mean The sum of the products of every group and the number of their repetition

the number of repetitions

$$\overline{X} = \frac{x_1 f_1 + x_2 f_2 + x_3 f_3 + \dots\dots\dots + x_n f_n}{f_1 + f_2 + \dots\dots + f_n}$$

***Example 2***

Let's assume that there are 3 people aged 8, 5 people aged 9, 4 people aged 11 and two people aged 12 as in the following table.

| Age | 8 | 9 | 11 | 12 |
|---|---|---|---|---|
| Number of People | 3 | 5 | 4 | 2 |

(This table is without groups)so the number of (age) represents the group find the arithmetic mean of their ages.

| Age (x) | frequency (f) | age x frequency (fx) |
|---|---|---|
| 8 | 3 | $8 \times 3 = 24$ |
| 9 | 5 | $9 \times 5 = 45$ |
| 11 | 4 | $11 \times 4 = 44$ |
| 12 | 2 | $12 \times 2 = 24$ |
| sum | 14 | 137 |

***Solution***

we symbolized the age with x and the repetitions with f, the solution steps can be shown in the table

$$\therefore \overline{X} = \frac{137}{14}$$

$= 9.786$ years

(arithmetic mean of age)

### *Example 3*

The following table shows the distribution of 100 people according

to their weight groups.It is wanted to find the arithmetic mean of their weights.

| Weight Groups | 30- | 40- | 50- | 60- | 70- | 80-90 | Sum |
|---|---|---|---|---|---|---|---|
| Number of People | 9 | 15 | 22 | 25 | 18 | 11 | 100 |

### *Solution*

the median of the first weight group : $35 = \dfrac{30 + 40}{2}$

The median of the second weight group 35 + 10 = 45........ and so on.

The solution steps are

1- finding the medians of the groups

2- multiply the median of group with its frequency

3- find the mean

| Weight Group | frequency | Centers | F × X |
|---|---|---|---|
| 30- | 9 | 35 | 315 |
| 40- | 15 | 45 | 675 |
| 50- | 22 | 55 | 1210 |
| 60- | 25 | 65 | 1625 |
| 70- | 18 | 75 | 1350 |
| 80- 90 | 11 | 85 | 935 |
| Sum | 100 | | 6110 |

$$\overline{X} = \frac{x_1 f_1 + x_2 f_2 + x_3 f_3 + .......... + x_n f_n}{f_1 + f_2 + ........ + f_n}$$

$$\overline{X} = \frac{6110}{100}$$

$\overline{X} =$ 61.1 Kilogram

*Example*

Find the mean of the following table :

| Groups | 8- | 10- | 12- | 14- | 16- | 18-20 | sum |
|---|---|---|---|---|---|---|---|
| frequency | 5 | 15 | 20 | 10 | 6 | 4 | 60 |

| Groups | frequency f | Centers | f × x |
|---|---|---|---|
| 8- | 5 | 9 | 45 |
| 10- | 15 | 11 | 165 |
| 12- | 20 | 13 | 260 |
| 14- | 10 | 15 | 150 |
| 16- | 6 | 17 | 102 |
| 18- 20 | 4 | 19 | 76 |
| sum | 60 | | 798 |

*Solution*

$$\overline{X} = \frac{798}{60}$$

$$\overline{X} = 13.3$$

Arithmetic mean

## Second Way

## The hypothetical mean or the deviation :

This way counts on using one of the values (C enters) as a hypothetical mean then find the deviation of every group on that hypothetical mean and then we :use the law

$$\text{The arithmetic mean=hypothetical mean} + \frac{\text{The sum ( deviations of the medians times the frequency}}{\text{Number of frequency}}$$

Hypothetical mean = $\overline{X_0}$ $\overline{X} = X_0 + \frac{\Sigma f.E}{\Sigma f}$

Number of frequency = $\Sigma f$ $= X - X_0$ ,frequency of the group = f

## *Example 5*

The following table shows the ages of 100 university students. Find the arithmetic mean of the ages using the hypothetical mean.

## *Solution*

| ages | 18 | 20 | 22 | 24 | 26 | 28-30 | Sum |
|---|---|---|---|---|---|---|---|
| number of student | 20 | 44 | 18 | 13 | 3 | 2 | 100 |

1 - We take the C enters of the groups

2 - We choose the hypothetical mean (X) from the groups and let 21 that corresponds to the most frequency

3 - We take the deviation of every median of the hypothetical mean ( deviation = median of the group

hypotheticalmean $E = X - X_0$

4 - We find the product of every group (f) X the deviation of its hypothetical mean

5 - We find the sum of the frequency and the whole sum, we write the previous information in the table

| Age Groups | Number of students frequency f | Median of the group (X) | Deviation $E = X - \overline{X}_0$ | f.E |
|---|---|---|---|---|
| 18- | 20 | 19 | 19 - 21 = -2 | 20 × -2 = -40 |
| 20- | 44 | $21 = \overline{X}_0$ | 21 - 21 = 0 | 44 × 0 = 0 |
| 22- | 18 | 23 | 23 - 21 = 2 | 18 × 2 = 36 |
| 24- | 13 | 25 | 25 - 21 = 4 | 13 × 4 = 52 |
| 26- | 3 | 27 | 27 - 21 = 6 | 3 × 6 = 18 |
| 28- 30 | 2 | 29 | 29 - 21 = 8 | 2 × 8 = 16 |
| Sum | 100 | | | 82 |

$$\overline{X} = \overline{X}_0 + \frac{\Sigma f.E}{\Sigma f}$$

$$\overline{X} = 21 + \frac{82}{100} = 21 + 0.82$$

$$\overline{X} = 21.82$$

## The privileges and flaws of the Arithmetic mean

Privileges

1) easy to calculate.

2) All the values are used in it's calculations.

Flaws

1) It is affected by abnormal values and very small or very large values

2) It can't be calculated graphically

## *[7 - 3] Median*

**Definition (7-2)**

The median of a group of values is the value that is in the middle of the group after ordering them ascendingly or descendingly so that the number of values smaller than it is equal to the number of values bigger than it.

### How to find the median:

### 1)Unclassified statistics(data)

We order the values ascendingly or descendingly then we take the value that is in the middle for it to be the median
Assuming that the number of values is odd
But if the number of values is even then we take the two values that are in the middle so that the median is equal to the sum of the two values divided by two

### *Example 6*

Find the median of the weights of some students that are :52 kg, 58 kg, 50 kg, 63 kg, 55, kg.

### *Solution*

We order the numbers ascendingly: 50,52,55,58,63. we notice that the value that is in the middle is the third one in order

Median = 55

### *Example 7*

Find the median of the following weights of some students: 52 kg, 58 kg, 50 kg, 63 kg, 57 kg, 55 kg.

### *Solution*

We order the values ascendingly: 50, 52 ، 55 ، 57 ، 58 ، 63

We notice that there are two values in the middle that are 55, 57

The order of the first one $= \frac{n}{2} = \frac{6}{2} = 3$ (third)

The order of the second one $= \frac{n}{2} + 1 = 3 + 1 = 4$ (fourth)

Median $= \frac{\text{third + fourth}}{2} = \frac{57 + 55}{2} = 56$

## 2) Classified statistics (data)

The median can be found in classified data with groups: and the steps are as follows

1 ) We make the table of the ascending combined frequency from the frequency table

2) Calculation of the order of the median $= \dfrac{\text{group of the Frequency}}{2}$

Identifying the group that has the median from the ascending combined frequency table

and it is called the median group and it is the group that corresponds to the first bigger

frequency or that is equal to the order of the median

Median = The lowest possible median group + $\dfrac{\text{Order of the median - the ascending combined frequency of the group before the median}}{\text{The median group}}$

$$ME = L + \frac{\frac{\sum f}{2} - fb}{fm} . w$$

where ME= The median, fb= the ascending combined frequency of the group before the median,

fm = the frequency of the median, W=length of the group, L=lowest possible value of the median

***Example 8***

Find the median of the weights in the table:

| The ascending combined frequency | frequency of number of people | Weight Group |
|---|---|---|
| 9 | 9 | 30- |
| 24 | 15 | 40- |
| 46 | 22 | 50- |
| 71 | 25 | 60- |
| 89 | 18 | 70- |
| 100 | 11 | 80 - 90 |
| | 100 | SUM |

***Solution***

The order of the median $= \dfrac{100}{2} = 50$

$\therefore$ The median group = ( 70 - 60 )

$$ME = L + \frac{\frac{\sum f}{2} - fb}{fm} . W$$

$$ME = 60 + \frac{50 - 46}{25} \times 10$$

$$= 60 + \frac{8}{5} \quad \Rightarrow \quad ME = 60 + 1.6 = 61.6$$

## The privileges and the flaws of the median:

Privileges

(1) It is not affected by abnormal and very large or very small values

2)It can be calculated graphically

Flaws

1)Not all the values are used in the calculation

2) In case of the classified statistics with groups, it is solved by estimation

## *[7-4] Mode*

**Definition (7 - 3)**

The mode of a group is defined as the value that is most repeated and symbolized by MO

### *Calculating mode :*

### *Unclassified statistics(data)*

***Example 9***

What is the mode of the following set of number

A)4,2,4,7,3,4,9,7,4

***Solution***

Mode= 4 because it is more repeated

B)6,5,1,8,6,5,10,18

Mode=6,5 because they are most repeated

C)12,11,10,7,3,4,5,8

Mode= No Mode

## Classified statistics(data)

### A) Differences way ( pearson's way)

Mode = the lowest possible group mode + $\frac{d_1}{d_1 + d_2}$ × The length of the group mode Where d = the frequency of the group mode - the frequency of the group before it

$d_2$ = the frequency of the group mode - the frequency of the group after it

If the mode frequency is the biggest frequency of the table. and the group modec corresponding to the most frequency.

***Example 10***

Calculate the mode from the table:

***Solution***

| | frequency | groups |
|---|---|---|
| | 9 | 30- |
| | 15 | 40- |
| Previous frequency → | 22 | 50- |
| Mode frequency → | 25 | 60- |
| Following frequency → | 18 | 70- |
| | 11 | 80-90 |

$d_1 = 25 - 22 = 3$

$d_2 = 25 - 18 = 7$

The length of group mode = 70-60 = 10

Mode=The lowest possible group mode+ $\frac{d1}{d1+ d2}$ ×The length of the group

$$\text{Mode} = 60 + \frac{3}{3+7} \times 10$$

Mode = 60 + 3

Mode = 63

### B) The Method of Moments

1) In this way we draw a lever and we make the frequency of the mode group a force acting on one of the ends of the lever and the frequency of the group after the mode group a force acting on the other end of the lever and the length of the lever = the length of the group

1)We assume that the fixed point that represents the distance of the mode from one of the ends=x

3)We apply the lever formula (force x it's spindle=the resistance x it's spindle

4)We take the value of x and add it to the lowest value of the mode group and we will get the mode

*Example 11*

Find the mode from the following table:

| groups | 40- | 50- | 60- | 70- | 80- | 90 -100 |
|---|---|---|---|---|---|---|
| frequency | 6 | 38 | 59 | 37 | 8 | 2 |

*Solution*

Mode = 70-60

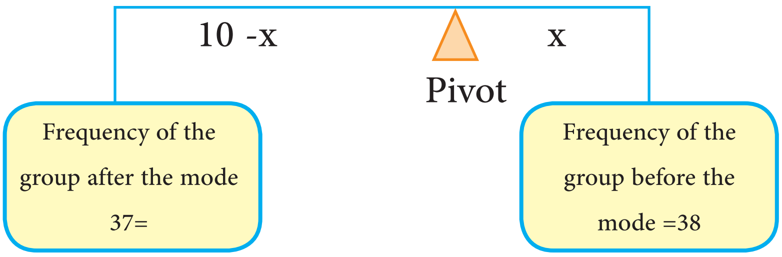
length of the lever = length of the group = 10

The force x it's spindle = The resistance x it's spindle

$(10 - x)\ (37) = x\ (38)$

$370 - 37x = 38x$

Length of the lever= length of the group =10




$75x = 370$

$x = \dfrac{370}{75} = 4.9 \therefore$

$\text{Mode} = 60 + 4.9 = 64.9 \therefore$

## The privileges and the flaws of the mode:

Privileges :

Easy to calculate

It is not affected by abnormal and very large or very small values

Flaws :

1)In the case of classified data with groups, it is calculated using estimation

2)It can not be found if there aren't values that are repeated more than the others

3)There might be more than one mode if there are values that are repeated equally

## Exercises (7-1)

**Q1 /** Define the arithmetic mean,median and the mode

**Q2 /** The following data represents the ages of a group of student:

17,18,17,15,16,18,16,17,15,19. find

A) The arithmetic mean B)Median C)The mode

**Q3 /** If the arithmetic mean of the monthly income of 5 people is (40000) Iraqi dinars. what is the total of their incomes?

**Q4 /** The following table shows the temperatures in a city during 90 days in the summer

| Temprature groups | 20- | 24- | 28- | 32- | 36- | 40- | 44-48 | sum |
|---|---|---|---|---|---|---|---|---|
| Number of days | 8 | 10 | 18 | 23 | 15 | 9 | 7 | 90 |

Find A) The arithmetic mean B)Median C)The mode

**Q5 /** The following table shows the wages of 60 teachers at a school. fine the arithmetic mean of their wages

| Wage in thousands of dinar | 150- | 160- | 170- | 180- | 190- | 200-210 |
|---|---|---|---|---|---|---|
| Number of teachers | 5 | 10 | 15 | 20 | 7 | 3 |

**Q6 /** The following table shows the daily profits of some shops in a city. find the arithmetic mean of their profits

| Daily profit in thousands of dinars | 4- | 8- | 12- | 16- | 20- | 24-28 |
|---|---|---|---|---|---|---|
| Number of shops | 8 | 10 | 15 | 20 | 12 | 6 |

## *Measures of Variation [7-5]*

For every set of numbers, there is an arithmetic mean and the numbers of the set might be distributed close to it or far away from it. If the numbers are distributed close to their arithmetic mean , their measures of variation are low, and if the numbers are distributed far away from their arithmetic mean, their measures of variation are high

Ex: The arithmetic mean of the numbers 30,40,50,60,70 is 50 and the arithmetic mean of the numbers 10, 20 ,90,100,30 is also 50 We can see that the measure of variation in the first set of numbers is low while the measure of variation in the second set of numbers is high

### Measures of variations:

The measures of variation that we are going to study are:

1- Range

2- Standard deviation

***[7-5-1] Range: is the difference between the largest and the smallest value***

The range is not an important measure of variability because it deals with only two of the values of the set. These two values are the largest and the smallest values of the set . So it is greatly affected by the changes in these two values

A)Unclassified Data:

***Example 12***

What is the range for the following st of numbers 98, 24، 68 ، 35، 12

***Solution :***

R = 98 - 12 = 86

## B) Classified Data:

*Example 13*

Find the range:

| Groups | 5- | 15- | 25- | 35- | 45-55 |
|---|---|---|---|---|---|
| frequency | 3 | 8 | 15 | 14 | 7 |

*Solution*

Range = highest value of the last group - the lowest value of the first group

R = 55 - 5 = 50

## *[7-5-2] Standard deviation*

The standard deviation is one of the mostly used measures of variation. if we have n from the terms $x_1, x_2, \ldots, x_n$ and their arithmetic mean is $\overline{x}$. then these terms are close to each other if they are close to their arithmetic mean $\overline{x}$, meaning that their deviation from $\overline{x}$ is small

So the deviation of the terms from their arithmetic mean can be used to measure variation, and that can be done by taking the median of these deviations

**Definition(7-4)**

**Standard deviation**

it is the positive square root of the mean for the squared deviations for the values of the distribution expression off their arithmetic mean and it is denoted by S

$$S = \sqrt{\frac{\sum x^2}{n} - (\overline{x})^2}$$

| x | $x^2$ |
|---|---|
| 1 | 1 |
| 3 | 9 |
| 5 | 25 |
| 7 | 49 |
| 9 | 81 |
| 25 | 165 |

Sum

***Example 14***

Find the standard deviation for the following values:

1، 3، 5، 7، 9

***Solution***

$$\overline{X} = \frac{x_1 + x_2 ... + x_n}{n} = \frac{25}{5} = 5$$

$$S = \sqrt{\frac{\sum x^2}{n} - (\overline{x})^2}$$

v

$$S = \sqrt{\frac{165}{5} - 25} = \sqrt{33 - 25}$$

Standard deviation .... $\sqrt{8} = 2\sqrt{2}$

**Note**

When a constant number is subtracted from all the values , the value of the standard deviation remains constant as in example 15

***Example 15***

Subtract 1 from the numbers [1,3,5,7,9] then find the standard deviation of the new values. Compare you answer with example 14, what do you observe

***Solution***

| x | $x^2$ |
|---|---|
| 0 | 0 |
| 2 | 4 |
| 4 | 16 |
| 6 | 36 |
| 8 | 64 |
| 20 | 120 |

Sum

The numbers 1 ، 3، 5 ، 7 ، 9

we subtract 1 : 0 ، 2، 4 ، 6 ، 8

$$\overline{X} = \frac{8 + 6 + 4 + 2 + 0}{5} = \frac{20}{5} = 4$$

$$S = \sqrt{\frac{\sum x^2}{n} - (\overline{x})^2}$$

$$S = \sqrt{\frac{120}{5} - 16} = \sqrt{24 - 16}$$

Standard deviation .... $\sqrt{8} = 2\sqrt{2}$

We observe that the standard deviation didn't change

## Standard Degree :

### Definition (7-4)

Standard degree is defined as the quotient of the deviation of the that variable subtracted from the arithmetic mean of that set and the standard deviation

Meaning that : $SD = \frac{X - \overline{X}}{S}$

## Correlation [ 7 - 5 - 3 ]

### Definition (7-5)

Correlation: It is the mathematical equation between two variables so that if one of them changes to a given direction, the other one tends to change to a given direction too. if the change occurs in only one way it is called a positive correlation. and if the change occurs in two opposite ways, it is called a negative correlation

## Correlation coefficient (r) between the variables x,y

We calculate the correlation coefficient r

$$r = \frac{\frac{\Sigma xy}{n} - \overline{x}\,\overline{y}}{S_x S_y}$$

Where x = The arithmetic mean of the variable x

y= The arithmetic mean of the variable y

S = The standard deviation of the variable x

S = The standard deviation of the variable y

## Some of the properties of (r)

r is positive in the case of positive correlation

r=1 in an absolute direct correlation

r is negative in an inverse correlation

r=-1 in an absolute inverse correlation

r=0 in the lack of correlation

we can observe that the value of correlation coefficient belong to [-1,1] and the close r the value of r is from +1 or -1, the stronger the correlation between the two variables and the closer it is to 0 the closer it is to the lack of correlation

## *Example 16*

Find the correlation coefficient between the variables x,y for x=5

| x | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| y | 2 | 4 | 6 | 8 | 10 |

Then show it's type

| x | y | $x^2$ | $y^2$ | x y |
|---|---|---|---|---|
| 1 | 2 | 1 | 4 | 2 |
| 2 | 4 | 4 | 16 | 8 |
| 3 | 6 | 9 | 36 | 18 |
| 4 | 8 | 16 | 64 | 32 |
| 5 | 10 | 25 | 100 | 50 |
| 15 | 30 | 55 | 220 | 110 |

Sum

### *Solution*

$$\overline{X} = \frac{15}{5} = 3$$

$$\overline{Y} = \frac{30}{5} = 6$$

$$S_x = \sqrt{\frac{55}{5} - 9} = \sqrt{2}$$

$$S_y = \sqrt{\frac{220}{5} - 36} = \sqrt{8} = 2\sqrt{2}$$

$$r = \frac{\frac{\Sigma x y}{n} - \overline{x}\,\overline{y}}{S_x \; S_y} = \frac{\frac{110}{5} - (3)(6)}{(\sqrt{2})(2\sqrt{2})}$$

$$r = \frac{22 - 18}{4} = \frac{4}{4} = 1$$

$\therefore$ The type of correlation is direct

$$SD = \frac{x - \overline{x}}{Sx}$$

$$\therefore SD = \frac{5 - 3}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

*Example 18*

Find the correlation coefficient an between x,y and show it's type

| x | -5 | -2 | 1 | 4 | 7 |
|---|---|---|---|---|---|
| y | 9 | 6 | 3 | 0 | -3 |

*Solution*

| x | y | $x^2$ | $y^2$ | xy |
|---|---|---|---|---|
| 5 - | 9 | 25 | 81 | –45 |
| –2 | 6 | 4 | 36 | –12 |
| 1 | 3 | 1 | 9 | 3 |
| 4 | 0 | 16 | 0 | 0 |
| 7 | –3 | 49 | 9 | –21 |
| 5 | 15 | 95 | 135 | –75 |

Sum

$$\overline{X} = \frac{5}{5} = 1$$

$$\overline{Y} = \frac{15}{5} = 3$$

$$S_x = \sqrt{\frac{95}{5} - (1)^2} = \sqrt{19 - 1} = \sqrt{18} = 3\sqrt{2}$$

$$S_y = \sqrt{\frac{135}{5} - (1)^2} = \sqrt{27 - 9} = \sqrt{18} = 3\sqrt{2}$$

$$r = \frac{\frac{\sum xy}{n} - \overline{x}\,\overline{y}}{S_x\ S_y} = \frac{\frac{-75}{5} - (1)(3)}{3\sqrt{2} \times 3\sqrt{2}}$$

type= absolute inverse correlation

$$r = \frac{-15 - 3}{(9)(2)} = \frac{-18}{(18)} = -1$$

## Exercises 7-2

**Q1 /**

A)Find the range for the following numbers: 3,0,8,7,9,12

B) Find the range from the following table

| Groups | 20 – | 22 – | 24 – | 26 – | 28 – | 30 - 32 |
|---|---|---|---|---|---|---|
| frequency | 5 | 10 | 20 | 10 | 5 | 2 |

**Q2 /** Define the standard deviation then find the standard deviation for the following numbers: 2,4,6,8,10

**Q3 /** Find the standard deviation for the numbers:5,7,1,2,6,3. then add 5 to all the values and prove that adding 5 does not affect the value of the standard deviation but affects the value of arithmetic mean

**Q4 /** Find the correlation coefficient between x,y then show it's type.

| x | 1 | 2 | 3 |
|---|---|---|---|
| y | 2 | 4 | 6 |

**Q5 /** If you multiply all the values of x in the previous question by 4, you'll get a new table. find the new correlation coefficient .

| x | 4 | 8 | 12 |
|---|---|---|---|
| y | 2 | 4 | 6 |

**Q6 /** Find the correlation coefficient between x,y and show it

| x | 13- | 9- | 5- | 1- | 3 |
|---|---|---|---|---|---|
| y | +3 | +1 | -1 | -3 | -5 |

# Contents