Ripublic Of Iraq
Ministry Of Education
General Directorato Of Curricula

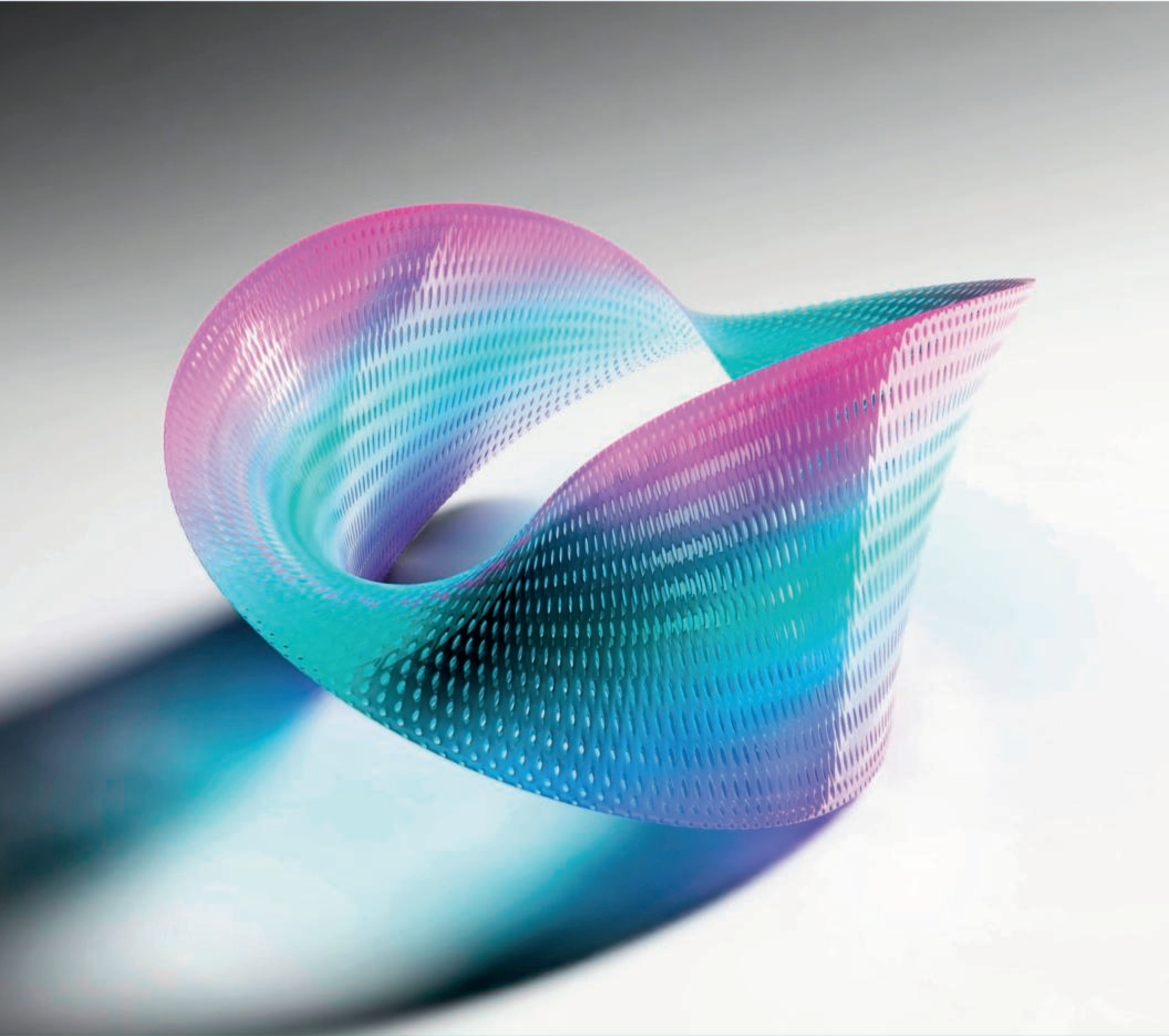
MATHEMATICS

5

SCIENTIFIC SECONDARY

5^{TH}

APPLIED BRANCH



Chapter 1

LOGARITHM

LOGARITHM

Consider the relation $a^x = N$. Imagine that we are asked to find one of the three numbers a, x or N given the other two numbers. Three examples of this are shown below.

Case	Solution	Method
$2^5 = p$	32	raise 2 to the 5th power
$p^3 = 27$	3	take the 3rd root of 27
$3^p = 5$	?	?

We can see that we cannot solve the last example with the algebra we have studied so far. We need to introduce a new concept: a **logarithm**.

A. BASIC CONCEPT

The **logarithm** of a number N to a base a is the power to which a must be raised in order to obtain N. We write this as $\log_a N$. In other words,

$$a^{\log_a N} = N \text{ where } a^x = N \text{ and } x = \log_a N.$$

This equation is called the **fundamental identity of logarithms**. In this equation, the base of the logarithm (a) is always positive and different from 1, and the number whose logarithm is taken (N) is positive. In other words, **negative numbers and zero do not have logarithms**.

Definition

logarithm, argument, base, exponential form, logarithmic form

For $a > 0$, $a \neq 1$ and $x > 0$, the real number y which is defined by

$$y = \log_a x \Leftrightarrow a^y = x$$

is called the **logarithm of x to the base a**. In this notation, x is called the **argument** of the logarithm.

We say that the equation $a^y = x$ is in **exponential form** and $\log_a x = y$ is the same equation in **logarithmic form**.

EXAMPLE 1 Write the equalities in logarithmic form.

a. $2^3 = 8$ b. $5^0 = 1$ c. $3^{-2} = \frac{1}{9}$

Solution By the definition of a logarithm, $a^y = x \Leftrightarrow y = \log_a x$. Therefore,

a. $2^3 = 8 \Leftrightarrow 3 = \log_2 8$. b. $5^0 = 1 \Leftrightarrow 0 = \log_5 1$.

c. $3^{-2} = \frac{1}{9} \Leftrightarrow -2 = \log_3 \frac{1}{9}$.

EXAMPLE 2 Write the equalities in exponential form.

a. $\log_{10} 100 = 2$ b. $\log_3 \frac{1}{27} = -3$ c. $\log_2 1 = 0$

Solution Again we use the definition $\log_a x = y \Leftrightarrow x = a^y$.

a. $\log_{10} 100 = 2 \Leftrightarrow 100 = 10^2$ b. $\log_3 \frac{1}{27} = -3 \Leftrightarrow \frac{1}{27} = 3^{-3}$

c. $\log_2 1 = 0 \Leftrightarrow 1 = 2^0$

EXAMPLE 3 Solve each equation for x.

a. $\log_x 27 = 3$ b. $\log_4 x = \frac{1}{2}$ c. $\log_4 16 = x$

Solution

a. $\log_x 27 = 3 \Leftrightarrow x^3 = 27 \Leftrightarrow x = 3$ b. $\log_4 x = \frac{1}{2} \Leftrightarrow 4^{\frac{1}{2}} = x \Leftrightarrow x = 2$

c. $\log_4 16 = x \Leftrightarrow 4^x = 16 \Leftrightarrow 4^x = 4^2 \Leftrightarrow x = 2$

EXAMPLE 4 Calculate the logarithms.

a. $\log_2 4$ b. $\log_3 \frac{1}{9}$ c. $\log_2(\log_3 9)$

Solution a. Let $\log_2 4 = y$.

Then $\log_2 4 = y \Leftrightarrow 2^y = 4 \Leftrightarrow 2^y = 2^2 \Leftrightarrow y = 2$, so $\log_2 4 = 2$.

b. Similarly, $\log_3 \frac{1}{9} = y \Leftrightarrow 3^y = \frac{1}{9} \Leftrightarrow 3^y = 3^{-2} \Leftrightarrow y = -2$.

c. Let $\log_3 9 = m$. Then $3^m = 9 \Leftrightarrow 3^m = 3^2 \Leftrightarrow m = 2$. So we need to calculate $\log_2 2$. Starting with $\log_2 2 = n$, we get $2^n = 2$ which gives us $n = 1$. Thus, $\log_2(\log_3 9) = 1$.

Remember:
$a^x = a^y \Leftrightarrow x = y$
by the bijective property of exponential functions.

EXAMPLE 5 Calculate the logarithms.

a. $\log_3 \frac{1}{3}$ b. $\log_{\frac{1}{3}} \sqrt[3]{81}$ c. $\log_a \sqrt[3]{a\sqrt{a}}$ d. $\log_3(\log_2(\log_9 81))$

Solution

a. By the definition of a logarithm, we can write $\log_3 \frac{1}{3} = y \Leftrightarrow 3^y = \frac{1}{3} \Leftrightarrow 3^y = 3^{-1} \Leftrightarrow y = -1$.

So $\log_3 \frac{1}{3} = -1$.

b. $\log_{\frac{1}{3}} \sqrt[3]{81} = y \Leftrightarrow (\frac{1}{3})^y = (81)^{\frac{1}{3}} \Leftrightarrow 3^{-y} = (3^4)^{\frac{1}{3}} \Leftrightarrow 3^{-y} = 3^{\frac{4}{3}} \Leftrightarrow y = -\frac{4}{3}$

c. $\log_a \sqrt[3]{a\sqrt{a}} = y \Leftrightarrow a^y = (a\sqrt{a})^{\frac{1}{3}} \Leftrightarrow a^y = (a \cdot a^{\frac{1}{2}})^{\frac{1}{3}} \Leftrightarrow a^y = (a^{\frac{3}{2}})^{\frac{1}{3}} \Leftrightarrow a^y = a^{\frac{1}{2}} \Leftrightarrow y = \frac{1}{2}.$

So $\log_a \sqrt[3]{a\sqrt{a}} = \frac{1}{2}.$

d. Starting from the innermost logarithm, we have

$\log_9 81 = x \Leftrightarrow 9^x = 81 \Leftrightarrow 9^x = 9^2 \Leftrightarrow x = 2.$

So we have to calculate $\log_3 (\log_2 2)$, and $\log_2 2 = y \Leftrightarrow 2^y = 2 \Leftrightarrow y = 1.$

So the given expression becomes $\log_3 1$, which is equal to zero:

$\log_3 1 = z \Leftrightarrow 3^z = 1 \Leftrightarrow z = 0.$ In conclusion, $\log_3(\log_2(\log_9 81)) = 0.$

Notice that in these examples we were able to find the desired logarithm by writing the argument as a rational power of the base. This is not always possible, however: many logarithms (for example: $\log_2 3$ and $\log_3 5$) are irrational, and cannot be calculated in this way.

EXAMPLE 6 Evaluate the expressions.

a. $2^{\log_2 8}$ b. $25^{\log_5 3}$ c. $3^{3\cdot\log_3 2}$

Solution a. By the fundamental identity of logarithms, $a^{\log_a N} = N$ and so $2^{\log_2 8}$ will be equal to 8. However, let us try to evaluate the expression in a different way:

Let $\log_2 8 = t$. Then we have to calculate 2^t.

By definition we have $\log_2 8 = t \Leftrightarrow 2^t = 8$. So $2^{\log_2 8} = 8$.

b. Let $\log_5 3 = t$. Then we have to calculate 25^t. By definition, $\log_5 3 = t \Leftrightarrow 5^t = 3$. So $25^t = (5^2)^t = (5^t)^2 = (3)^2 = 9$, i.e. $25^{\log_5 3} = 9$.

$(a^m)^n = (a^n)^m = a^{m\cdot n}$

c. In a similar way, let $\log_3 2 = t$ and let us calculate 3^{3t}. By definition, $\log_3 2 = t \Leftrightarrow 3^t = 2$, i.e. $3^{3t} = (3^t)^3 = 2^3 = 8$. So $2^{3\cdot\log_3 2} = 8$.

Check Yourself

1. Write the equalities in logarithmic form.

a. $2^4 = 16$ b. $10^3 = 1000$ c. $3^0 = 1$ d. $125^{\frac{1}{3}} = 5$ e. $3^{-3} = \frac{1}{27}$ f. $(2\sqrt{2})^{-\frac{2}{3}} = \frac{1}{2}$

2. Write the equalities in exponential form.

a. $\log_{10} 0.01 = -2$ b. $\log_{\frac{1}{2}} \frac{1}{16} = 4$ c. $\log_{10} 10000 = 4$

d. $\log_3 \frac{1}{81} = -4$ e. $\log_2 32 = 5$ f. $\log_{\frac{1}{5}} 125 = -3$

3. State whether each statement is true or false.

a. $\log_3 729 = 6$ b. $\log_{\frac{1}{2}} \sqrt[3]{4} = -\frac{2}{3}$ c. $\log_{10} \frac{1}{10\sqrt{10}} = -\frac{3}{2}$

d. $\log_{3\sqrt{3}} \frac{1}{3} = -\frac{3}{2}$ e. $\log_a \sqrt{a\sqrt{a\sqrt{a}}} = \frac{7}{8} \quad (a > 0,\ a \neq 1)$

4. Determine the logarithms of each set of numbers to the given base.

a. $27,\ 1,\ \frac{1}{9},\ \frac{1}{\sqrt[3]{3}},\ \sqrt[3]{9},\ \frac{9}{\sqrt[4]{3}}$ to base 3 b. $2,\ 4,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{32},\ 32,\ -64$ to base $\frac{1}{2}$

5. Solve for x.

a. $\log_x 4 = 2$ b. $\log_4 x = -\frac{1}{2}$ c. $\log_{25} 125 = x$

6. Calculate the logarithms.

a. $\log_5 25$ b. $\log_6 1296$ c. $\log_{49} \frac{1}{7}$ d. $\log_3(\log_2(\log_2 256))$

7. Evaluate the expressions.

a. $3^{-\log_3 4}$ b. $(2^{\log_2 5})^2$ c. $25^{-\log_5 10}$ d. $49^{\frac{1}{2}\cdot\log_7 \frac{1}{4}}$ e. $2^{2\log_2 5+\log_2 3}$

Answers

1. a. $\log_2 16 = 4$ b. $\log_{10} 1000 = 3$ c. $\log_3 1 = 0$

d. $\log_{125} 5 = \frac{1}{3}$ e. $\log_3 \frac{1}{27} = -3$ f. $\log_{2\sqrt{2}} \frac{1}{2} = -\frac{2}{3}$

2. a. $10^{-2} = 0.01$ b. $(\frac{1}{2})^4 = \frac{1}{16}$ c. $10^4 = 10000$

d. $3^{-4} = \frac{1}{81}$ e. $2^5 = 32$ f. $(\frac{1}{5})^{-3} = 125$

3. a. true b. true c. true d. false e. true

4. a. $3, 0, -2, -\frac{1}{3}, \frac{2}{3}, \frac{7}{4}$ b. $-1, -2, 0, 1, 2, 5, -5$, undefined

5. a. 2 b. $\frac{1}{2}$ c. $\frac{3}{2}$

6. a. 2 b. 4 c. $-\frac{1}{2}$ d. 1

7. a. $\frac{1}{4}$ b. 25 c. $\frac{1}{100}$ d. $\frac{1}{4}$ e. 75

B. TYPES OF LOGARITHM

1. Common Logarithms

Our counting system is based on the number 10. For this reason, a lot of logarithmic work uses the base 10. Logarithms to the base 10 are called **common logarithms**. We often write **log x** or to mean $\log_{10} x$. In this module, we will use $\log x$ to mean $\log_{10} x$.

Common logarithms are widely used in computation. Mathematicians have compiled extensive and highly accurate tables of common logarithms for use in these calculations. These tables and their use will be discussed later in this module.

2. Natural Logarithms

$e = 2.71828...$

Logarithms to the base e are called **natural logarithms** or **Euler logarithms**. We often write **ln x** to mean the natural logarithm $\log_e x$.

Natural logarithms are widely used in mathematical analysis in the study of limits, derivatives and integrals.

C. PROPERTIES OF LOGARITHMS

Property 1

If the argument and the base of a logarithm are equal, the logarithm is equal to 1. Conversely, if the logarithm is 1 then the argument and the base are equal:

$$\boxed{a = b \Leftrightarrow \log_a b = 1} \quad (a > 0,\ a \neq 1).\ b > 0$$

Proof By the fundamental identity of logarithms we have $a^{\log_a N} = N$. Setting $N = a$ gives us $a^{\log_a a} = a = a^1$, which gives us $\log_a a = 1$.

For example, $\log_3 3 = 1$, $\log 10 = 1$, $\ln e = 1$ and $\log_{\frac{1}{2}} \frac{1}{2} = 1$.

Property 2

The logarithm of 1 to any base is zero:

$$\boxed{\log_a 1 = 0}.$$

Proof $a^{\log_a 1} = 1 = a^0$. So $a^{\log_a 1} = a^0$, which gives us $\log_a 1 = 0$.

For example, $\log_3 1 = 0$, $\log_{\frac{1}{2}} 1 = 0$ and $\log_\pi 1 = 0$.

Property 3

The logarithm of the product of two or more positive numbers to a given base is equal to the sum of the logarithms of the numbers to that base:

$$\boxed{\log_a(x \cdot y) = \log_a x + \log_a y} \quad (x, y > 0).$$

Proof $a^{\log_a(x \cdot y)} = x \cdot y$. Substituting $x = a^{\log_a x}$ and $y = a^{\log_a y}$ gives us

$a^{\log_a(x \cdot y)} = a^{\log_a x} \cdot a^{\log_a y} = a^{\log_a x + \log_a y}$.

Comparing the exponents of the expressions on both sides gives us the required equation:

$$\log_a(x \cdot y) = \log_a x + \log_a y.$$

For example,

$\log_2 6 = \log_2(2 \cdot 3) = \log_2 2 + \log_2 3 = 1 + \log_2 3$

$\log_3 30 = \log_3(3 \cdot 10) = \log_3 3 + \log_3 10 = 1 + \log_3 10$

$\log_3 30 = \log_3(6 \cdot 5) = \log_3 6 + \log_3 5$

$\log_2 5 + \log_2 3 = \log_2(5 \cdot 3) = \log_2 15.$

Be careful!

$\log_a(x + y) \neq \log_a x + \log_a y$

Notice that we can generalize this property as follows:

$$\boxed{\log_a(x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_k) = \log_a x_1 + \log_a x_2 + \ldots + \log_a x_k} \ (x_1, x_2, x_3, \ldots, x_k > 0).$$

For example, we can write

$$\log_2 30 = \log_2(2 \cdot 3 \cdot 5) = \log_2 2 + \log_2 3 + \log_2 5 = 1 + \log_2 3 + \log_2 5.$$

EXAMPLE 7 Calculate $\log_4 2 + \log_4 8$.

Solution $\log_4 2 + \log_4 8 = \log_4(2 \cdot 8) = \log_4(4 \cdot 4) = \log_4 4 + \log_4 4 = 1 + 1 = 2$

EXAMPLE 8 Calculate $\log_2 3 + \log_2 5 + \log_2 \frac{1}{15}$.

Solution $\log_2 3 + \log_2 5 + \log_2 \frac{1}{15} = \log_2(3 \cdot 5 \cdot \frac{1}{15}) = \log_2 1 = 0$

Property 4 The logarithm of the power of a positive number is equal to the product of the power and the logarithm of the number.

$$\boxed{\log_a(x^m) = m \cdot \log_a x} \quad (m \in R,\ x > 0).$$

Be careful!

$(\log_a x)^m \neq m \cdot \log_a x$

Proof $x^m = a^{\log_a(x^m)}$. After substituting $x = a^{\log_a x}$ on the left side, we get $(a^{\log_a x})^m = a^{\log_a(x^m)}$, which gives us $a^{m \cdot \log_a x} = a^{\log_a(x^m)}$. Since the bases are the same on both sides, we can conclude $m \cdot \log_a x = \log_a(x^m)$.

For example,

$(a^m)^n = a^{m \cdot n}$

$$\log_2 8 = \log_2(2^3) = 3 \cdot \log_2 2 = 3 \cdot 1 = 3$$

$$\log_3 \frac{1}{243} = \log_3 \frac{1}{3^5} = \log_3(3^{-5}) = -5 \cdot \log_3 3 = -5 \cdot 1 = -5$$

$$\log_2 \sqrt{125} = \log_2 \sqrt{5^3} = \log_2 (5^{\frac{3}{2}}) = \frac{3}{2} \cdot \log_2 5.$$

$\sqrt[m]{x^n} = x^{\frac{n}{m}}$

$\frac{1}{x^n} = x^{-n}$

Note

This property gives us the following special cases:

4a. $\log_a \frac{1}{x^n} = -n \cdot \log_a x$ 4b. $\log_a \sqrt[m]{x^n} = \frac{n}{m} \cdot \log_a x.$

EXAMPLE 9 Write each sum as a single logarithm.

a. $(2 \cdot \log_3 a) + (3 \cdot \log_3 b) - \log_3 c$ b. $(\frac{1}{2} \cdot \log_2 a) + (3 \cdot \log_2 b) - (\frac{3}{2} \cdot \log_2 c)$

Solution We apply the property $\log_a(x^m) = m \cdot \log_a x$.

a. $(2 \cdot \log_3 a) + (3 \cdot \log_3 b) - \log_3 c = \log_3(a^2) + \log_3(b^3) + (-1) \cdot \log_3 c \Leftrightarrow$

$\log_3(a^2) + \log_3(b^3) + \log_3(c^{-1}) = \log_3(a^2 \cdot b^3 \cdot c^{-1}) = \log_3(\frac{a^2 \cdot b^3}{c})$

b. $(\frac{1}{2} \cdot \log_2 a) + (3 \cdot \log_2 b) - (\frac{3}{2} \cdot \log_2 c) = \log_2 a^{\frac{1}{2}} + \log_2 b^3 + (-\frac{3}{2}) \cdot \log_2 c \Leftrightarrow$

$\log_2 \sqrt{a} + \log_2 b^3 + \log_2 c^{(-\frac{3}{2})} = \log_2 \sqrt{a} + \log_2 b^3 + \log_2(\frac{1}{\sqrt{c^3}}) \Leftrightarrow$

$\log_2(\sqrt{a} \cdot b^3 \cdot \frac{1}{\sqrt{c^3}}) = \log_2(\frac{\sqrt{a} \cdot b^3}{\sqrt{c^3}})$

EXAMPLE 10 Calculate $\log_2 \sqrt[4]{2\cdot\sqrt{8\cdot\sqrt[3]{16}}}$.

Solution

$$\log_2 \sqrt[4]{2\cdot\sqrt{8\cdot\sqrt[3]{16}}} = \log_2\sqrt[4]{2\cdot\sqrt{8\cdot 16^{\frac{1}{3}}}} = \log_2\sqrt[4]{2\cdot\sqrt{2^3\cdot(2^4)^{\frac{1}{3}}}}$$

$$= \log_2\sqrt[4]{2\cdot\sqrt{2^{3+\frac{4}{3}}}} = \log_2\sqrt[4]{2\cdot\sqrt{2^{\frac{13}{3}}}} = \log_2\sqrt[4]{2\cdot(2^{\frac{13}{3}})^{\frac{1}{2}}} = \log_2\sqrt[4]{2^{1+\frac{13}{6}}}$$

$$= \log_2\sqrt[4]{2^{\frac{19}{6}}} = \log_2(2^{\frac{19}{24}}) = \frac{19}{24}\cdot\underbrace{\log_2 2}_{1} = \frac{19}{24}$$

Property 5

The logarithm of the quotient of two positive numbers is equal to the difference between the logarithms of the dividend and the divisor to the same base:

$$\boxed{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}\,.$$

$$\frac{\log_a x}{\log_a y} \neq \log_a x - \log_a y$$

Proof

$\frac{x}{y} = a^{\log_a\left(\frac{x}{y}\right)}$. If we substitute $x = a^{\log_a x}$ and $y = a^{\log_a y}$ on the left side, we obtain

$$\frac{a^{\log_a x}}{a^{\log_a y}} = a^{\log_a\left(\frac{x}{y}\right)} \Leftrightarrow a^{\log_a x - \log_a y} = a^{\log_a\left(\frac{x}{y}\right)} \Leftrightarrow \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

$$\frac{a^m}{a^n} = a^{m-n}$$

For example,

$$\log_2\frac{5}{3} = \log_2 5 - \log_2 3$$

$$\log_5(0.12) = \log_5\left(\frac{12}{100}\right) = \log_5\left(\frac{3}{25}\right) = \log_5 3 - \log_5(5^2) = \log_5 3 - 2$$

$$\log_2 10 + \log_2 4 - \log_2 5 = \log_2(10\cdot 4) - \log_2 5 = \log_2 40 - \log_2 5 = \log_2\frac{40}{5} = \log_2 8 = 3.$$

Notice that we can combine properties 4 and 5 to write expressions with addition and subtraction of logarithms as the logarithm of a single fraction. The addends form the numerator of the fraction and the subtrahends form the denominator, for example:

$$\log_a b + \log_a c - \log_a d + \log_a e - \log_a f = \log_a\left(\frac{b\cdot c\cdot e}{d\cdot f}\right).$$

As a numerical example, consider

$$\log_3 15 - \log_3 5 + \log_3 6 - \log_3 2 = \log_3\left(\frac{15\cdot 6}{5\cdot 2}\right) = \log_3 9 = \log_3(3^2) = 2.$$

Remember that this property only applies to logarithms with a common base.

EXAMPLE 11 Express $\log 30$ and $\log 3.\bar{3}$ in terms of p given $\log 3 = p$.

Solution Since $30 = 3 \cdot 10$, we get $\log 30 = \log(3 \cdot 10) = \underbrace{\log 3}_{p} + \underbrace{\log 10}_{1} = p + 1$.

Since $3.\bar{3} = \dfrac{10}{3}$, we have $\log 3.\bar{3} = \log\dfrac{10}{3} = \log 10 - \log 3 = 1 - p$.

EXAMPLE 12 Given $\log 300 = 2.47712$, calculate $\log(0.0027)$.

Solution

$\log(0.0027) = \log(\dfrac{27}{10^4}) = \log 27 - \log 10^4 = \log 3^3 - 4 \cdot \log 10$

$= (3 \cdot \log 3) - 4$ (1)

$\log 300 = \log(3 \cdot 100) = \log 3 + \log 10^2 = \log 3 + 2 \cdot \log 10 = 2 + \log 3$. So $\log 3 = \log 300 - 2$.

Using $\log 300 = 2.47712$, we get $\log 3 = 2.47712 - 2 = 0.47712$. (2)

Combining (1) and (2) gives us $\log(0.0027) = (3 \cdot 0.47712) - 4 = -2.56864$.

$\log 10 = \log_{10} 10 = 1$

EXAMPLE 13 Write each logarithm as a sum or difference of logarithms to base a.

a. $\log_a \dfrac{b^3 c^2}{d^4 e^5}$ b. $\log_a \dfrac{\sqrt[5]{(b+c)^2}}{(d-e)^3}$

Solution

a. $\log_a(\dfrac{b^3 \cdot c^2}{d^4 \cdot e^5}) = \log_a b^3 + \log_a c^2 - \log_a d^4 - \log_a e^5 = 3\log_a b + 2\log_a c - 4\log_a d - 5\log_a e$

b. We have $\log_a \dfrac{\sqrt[5]{(b+c)^2}}{(d-e)^3} = \log_a \sqrt[5]{(b+c)^2} - \log_a (d-e)^3 = \log_a (b+c)^{\frac{2}{5}} - 3\log_a(d-e)$

$$= \frac{2}{5}\log_a(b+c) - 3\log_a(d-e).$$

Notice that logarithms cannot be distributed over addition or subtraction, and also that logarithms enable us to perform simpler operations (addition and subtraction) instead of multiplication and division. This is why logarithms are so useful in computation.

EXERCISES

A. Basic Concept

1. Calculate the logarithms.

a. $\log_{1/3}\frac{1}{3}$ b. $\log_2\frac{1}{8}$ c. $\ln e$

d. $\log_{1/8} 1$ e. $\log_{\sqrt{2}} 2$ f. $\log\sqrt[5]{1000}$

g. $\log 1$ h. $\log(\ln e)$ i. $\ln\sqrt[3]{e}$

j. $\log_3(9\ln e^3)$ k. $\ln(\log 10^e)$ l. $\ln(\log 10)$

2. Solve each equation for x.

a. $3^x = 4$ b. $2^{x+1} = 3$ c. $3^{1-\frac{x}{2}} = 2$

d. $\sqrt{e^x} = 4$ e. $10^x = 5$ f. $10^{x-1} = 2$

3. Simplify the expressions.

a. $e^{\ln x}$ b. $10^{\log 3}$ c. $4^{\log_2 7}$

d. $5^{-\log_{25} 2}$ e. $27^{\log_{1/3} 4}$ f. $(x^{\log_3 5})^{\log_x 3}$

g. $x^{\log_x 3} + y^{\log_{1/y} 1/3} + z^{\log_z 1/3} - t^{\log_{1/t} 1/3}$

h. $-\log_2(\log_3\sqrt{\sqrt[4]{3}})$ i. $(\frac{16}{25})^{\log_{\frac{125}{64}} 3}$

B. Types of Logarithm

4. Calculate the logarithms, using $\log 2 = 0.30103$ and $\log 3 = 0.4771$.

a. $\log 18$ b. $\log 30$ c. $\log\frac{1}{5}$

5. Find the number of digits in each number if $\log 2 = 0.30103$ and $\log 3 = 0.4771$.

a. 2^{50} b. 9^{10} c. 27^9 d. 18^{20}

C. Properties of Logarithms

6. Write each expression as a single logarithm.

a. $\frac{1}{3}\log x - \log y + \log z^2$

b. $-\frac{1}{2}\log x + \frac{1}{2}\log y + \frac{1}{2}\log z$

7. Write each expression as the sum or difference of the logarithms of a, b and c.

a. $\log(a^3b^2c)$ b. $\log(\sqrt[3]{a}\sqrt{bc})$

8. Evaluate the expressions.

a. $\log_{24} 4 + \log_{24} 6$

b. $\log 8 + \log 25 + \log 5$

c. $\log_{1/2}\frac{1}{4} + \log_5 625 + \log_{1/3} 81 + \log_{49}\frac{1}{7}$

d. $\log_2 1000 - \log_2 125$

9. Calculate each logarithm in terms of the variable(s) provided, using the given relation(s).

a. $\log_2 3$; $\log_3 2 = a$ b. $\log 25$; $\log 2 = a$

c. $\log_7 21$; $\log_3 7 = p$ d. $\log_3 18$; $\log_3 12 = a$

e. $\log_{12} 60$; $\log_6 30 = a$ and $\log_{15} 24 = b$

10. $\log_x y = a$ is given. Express each logarithm in terms of a.

a. $\log_{x^3y^2} x^2y^3$ b. $\log_{x/y^2} x^3y^4$

11. Simplify the expressions.

a. $(\log_3 625\cdot\log_{1/5} 9) + (\log_4\frac{1}{125}\cdot\log_{1/25} 1024)$

b. $\log_{a^2} b^3\cdot\log_{b^3} c^4\cdot\log_{c^4} d^5\cdot\log_{d^5} a$

12. Find x in each case.

a. $\log_2 x = 3 - (2\cdot\log_2 3) + (3\cdot\log_2 5)$

b. $\log_3 x = 2 + (3\cdot\log_3 5) - (2\cdot\log_3 4)$

13. Prove each equality.

a. $\log_{x_1} x_2\cdot\log_{x_2} x_3\cdot\ldots\cdot\log_{x_n} x_1 = 1$

b. $x^{\log\frac{y}{z}}\cdot y^{\log\frac{z}{x}}\cdot z^{\log\frac{x}{y}} = 1$

14. Show that if $a^2 + b^2 = 7ab$ $(a, b > 0)$ then

$$\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b).$$

Chapter 2

SEQUENCE

INTRODUCTION...

An interesting unsolved problem in mathematics concerns the 'hailstone sequence', which is defined as follows: Start with any positive integer. If that number is odd, then multiply it by three and add one. If it is even, divide it by two. Then repeat. For example, starting with the number 10 we get the hailstone sequence 10, 5, 16, 8, 4, 2, 1, 4, 2, 1... . Some mathematicians have *conjectured* (guessed) that no matter what number you start with, you will always reach 1. This conjecture has been found true for all starting values up to 1,200,000,000,000. However, the conjecture, which is known as the 'Collatz Problem', '3n+1 Problem', or 'Syracuse Algorithm', still has not been proved true for all numbers.

Number sequences have been an interesting area for all mathematicians throughout history. Geometric sequences appear on Babylonian tablets dating back to 2100 BC. Arithmetic sequences were first found in the Ahmes Papyrus which is dated at 1550 BC. The reason behind the names 'arithmetic' and 'geometric' is that each term in a geometric (or arithmetic) sequence is the geometric (or arithmetic) mean of its successor and predecessor. If we think of a rectangle with side lengths x and y, then the geometric mean $\sqrt{xy}$ is the side length of a square that has the same area as this rectangle. Finding the dimensions of a square with the same area as a given rectangle was considered in those days as a very geometric problem. Although the arithmetic mean $(x + y)/2$ can also be interpreted geometrically (it is the length of the sides of a square having the same perimeter as the rectangle), lengths were viewed more as arithmetic, because it is easier to handle lengths by addition and subtraction, without having to think about two-dimensional concepts such as area. Although both problems involve arithmetic and can be interpreted geometrically, in ancient times one was viewed as much more geometric than the other, therefore the names.

Zeno (490-425 B.C.) was a mathematician whose paradoxes about motion puzzled mathematicians for centuries. They involved the sum of an infinite number of positive terms to a finite number. Zeno wasn't the only ancient mathematician to work on sequences. Several of the ancient Greek mathematicians used sequences to measure areas and volumes of shapes and regions. By using his reasoning technique called the 'method', Archimedes (287-212 B.C.) constructed several examples and tried to explain how infinite sums could have finite results. Among his many results was that the area under a parabolic arc is always two-thirds the base times the height.

The next major contributor to this area of mathematics was Fibonacci (1170-1240). He discovered a sequence of integers in which each number is equal to the sum of the preceding two numbers (1, 1, 2, 3, 5, 8, ...), and introduced it as a model of the breeding population of rabbits. This sequence has many remarkable properties and continues to find applications in many areas of modern mathematics and science. During this same period, Chinese astronomers developed numerical techniques to analyze their observation data and used the idea of finite differences to help analyze trends in their data.

Oresme (1325-1382) studied rates of change, such as velocity and acceleration, using sequences. Two hundred years later, Stevin (1548-1620) understood the physical and mathematical conceptions of acceleration due to gravity using series and sequences. During that time Galileo (1564-1642) applied mathematics to the sciences, especially astronomy. Based on his study of Archimedes, Galileo improved our understanding of hydrostatics. He developed equations for free-fall motion under gravity and the motion of the planets. Up until the middle of the 17th century, mathematicians developed and analyzed series of numbers.

Newton (1642-1727) and Leibnitz (1646-1716) developed several series representations for functions. Maclaurin (1698-1746), Euler (1707-1783), and Fourier (1768-1830) often used infinite series to develop new methods in mathematics. Sequences and series have become standard tools for approximating functions and calculating results in numerical computing.

The self-educated Indian mathematician Srinivasa Ramanujan (1887-1920) used sequences and power series to develop results in number theory. Ramanujan's work was theoretical and produced many important results used by mathematicians in the 20th century.

1 REAL NUMBER SEQUENCES

Real number sequences are strings of numbers. They play an important role in our everyday lives. For example, the following sequence:

$$20, 20.5, 21, 22, 23.4, 23.6, \ldots$$

gives the temperature measured in a city at midday for five consecutive days. It looks like the temperature is rising, but it is not possible to exactly predict the future temperature.

The sequence:

$$64, 32, 16, 8, \ldots$$

is the number of teams which play in each round of a tournament so that at the end of each game one team is eliminated and the other qualifies for the next round. Now we can easily predict the next numbers: 4, 2, and 1. Since there will be one champion, the sequence will end at 1, that is, the sequence has a finite number of terms. Sequences may be finite in number or infinite.

Look at the following sequence:

$$1000, 1100, 1210, \ldots$$

This is the total money owned by an investor at the end of each successive year. The capital increases by 10% every year. You can predict the next number in the sequence to be 1331. Each successive term here is 110% of, or 1.1 times, the previous term.

Can you recognize the pattern?

Real number sequences may follow an easily recognizable pattern or they may not. Recently a great deal of mathematical work has concentrated on deciding whether certain number sequences follow a pattern (that is, we can predict consecutive terms) or whether they are random (that is, we cannot predict consecutive terms). This work forms the basis of chaos theory, speech recognition, weather prediction and financial management, which are just a few examples of an almost endless list. In this book we will consider real number sequences which follow a pattern.

SEQUENCES

Definition

By the set of natural numbers we mean all positive integers and denote this set by $\mathbb{N}$. That is, $\mathbb{N} = \{1, 2, 3, ...\}$.

If someone asked you to list the squares of all the natural numbers, you might begin by writing

$$1, 4, 9, 16, 25, 36, ...$$

But you would soon realize that it is actually impossible to list all these numbers since there are an infinite number of them. However, we can represent this collection of numbers in several different ways.

A function is a relation between two sets A and B that assigns to each element of set A exactly one element of set B.

For example, we can also express the above list of numbers by writing

$$f(1), f(2), f(3), f(4), f(5), f(6), ..., f(n), ...$$

where $f(n) = n^2$. Here $f(1)$ is the first term, $f(2)$ is the second term, and so on. $f(n) = n^2$ is a **function** of n, defined in the set of natural numbers.

Definition

sequence

A function which is defined in the set of natural numbers is called a **sequence**.

However, we do not usually use functional notation to describe sequences. Instead, we denote the first term by a_1, the second term by a_2, and so on. So for the above list

$$a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, a_5 = 25, a_6 = 36, ..., a_n = n^2, ...$$

Here, a_1 is the first term,

a_2 is the second term,

a_3 is the third term,

⋮

a_n is the nth term, or the **general term**.

Since this is just a matter of notation, we can use another letter instead of the letter a. For example, we can also use b_n, c_n, d_n, etc. as the name for the general term of a sequence.

Notation

We denote a sequence by (a_n), where a_n is written inside brackets. We write the general term of a sequence as a_n, where a_n is written without brackets. For the above example, if we write the general term, we write $a_n = n^2$.

If we want to list the terms, we write $(a_n) = (1, 4, 9, 16, ..., n^2, ...)$.

Sometimes we can also use a shorthand way to write a sequence:

$(a_n) = (n^2 + 4n + 1)$ means the sequence (a_n) with general term $a_n = n^2 + 4n + 1$.

Note

An expression like $a_{2.6}$ is nonsense since we cannot talk about the 2.6th term of a sequence. Remember that a sequence is a function which is defined in the set of natural numbers, and 2.6 is not a natural number. Clearly, expressions like a_0, a_{-1} are also meaningless. We say that such terms are **undefined**.

Note

In a sequence, n should always be a natural number, but the value of a_n may be any real number depending on the formula for the general term of the sequence.

Example 1 Write the first five terms of the sequence with general term $a_n = \frac{1}{n}$.

Solution Since we are looking for the first five terms, we just recalculate the general term for $n = 1, 2, 3, 4, 5$, which gives $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

Example 2 Given the sequence with general term $a_n = \frac{4n-5}{2n}$, find a_5, a_{-2}, a_{100}.

Solution We just have to recalculate the formula for a_n choosing instead of n the numbers 5, –2, and 100. So $a_5 = \frac{3}{2}$, and $a_{100} = \frac{395}{200} = \frac{79}{40}$. Clearly, a_{-2} is undefined, since –2 is not a natural number.

Example 3 Find a suitable general term b_n for the sequence whose first four terms are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$.

Solution We need to find a pattern. Notice that the numerator of each fraction is equal to the term position and the denominator is one more than the term position, so we can write $b_n = \frac{n}{n+1}$.

Check Yourself

1. Write the first five terms of the sequence whose general term is $c_n = (-1)^n$.
2. Find a suitable general term a_n for the sequence whose first four terms are 2, 4, 6, 8.
3. Given the sequence with general term $b_n = 2n + 3$, find b_5, b_0, and b_{43}.

Answers

1. –1, 1, –1, 1, –1 2. $2n$ 3. 13, undefined, 89

Criteria for the Existence of a Sequence

If there is at least one natural number which makes the general term of a sequence undefined, then there is no such sequence.

Example 4 Is $a_n = \dfrac{2n+1}{n-2}$ a general term of a sequence? Why?

Solution No, because we cannot find a proper value for $n = 2$.

Example 5 Is $a_n = \sqrt{\dfrac{4-n}{2n+1}}$ a general term of a sequence? Why?

Solution Note that the expression $\sqrt{x}$ is only meaningful when $x \geq 0$. So we need $\dfrac{4-n}{2n+1} \geq 0$ to be true for any natural number n. If we solve this equation for n, the solution set is $(-\frac{1}{2}, 4]$, i.e. n is between $-\frac{1}{2}$ and 4, inclusive. When we take the natural numbers in this solution set, we get $\{1, 2, 3, 4\}$, which means that only a_1, a_2, a_3, a_4 are defined. So a_n is not the general term of a sequence.

Example 6 Is $a_n = \dfrac{n+1}{2n-1}$ a general term of a sequence? If yes, find $a_1 + a_2 + a_3$.

Solution $\dfrac{n+1}{2n-1}$ is not meaningful only when $n = \frac{1}{2} \notin \mathbb{N}$. Since a_n is defined for any natural number, it is the general term of a sequence. Choosing $n = 1, 2, 3$ we get $a_1 = 2$, $a_2 = 1$, $a_3 = 0.8$. So $a_1 + a_2 + a_3 = 3.8$.

Example 7 Given $b_n = 2n + 5$, find the term of the sequence (b_n) which is equal to

a. 25 b. 17 c. 96

Solution

a. $b_n = 25$
$2n + 5 = 25$
$n = 10$
10th term

b. $b_n = 17$
$2n + 5 = 17$
$n = 6$
6th term

c. $b_n = 96$
$2n + 5 = 96$
$n = 45.5 \notin \mathbb{N}$
not a term

THE FIBONACCI SEQUENCE AND THE GOLDEN RATIO

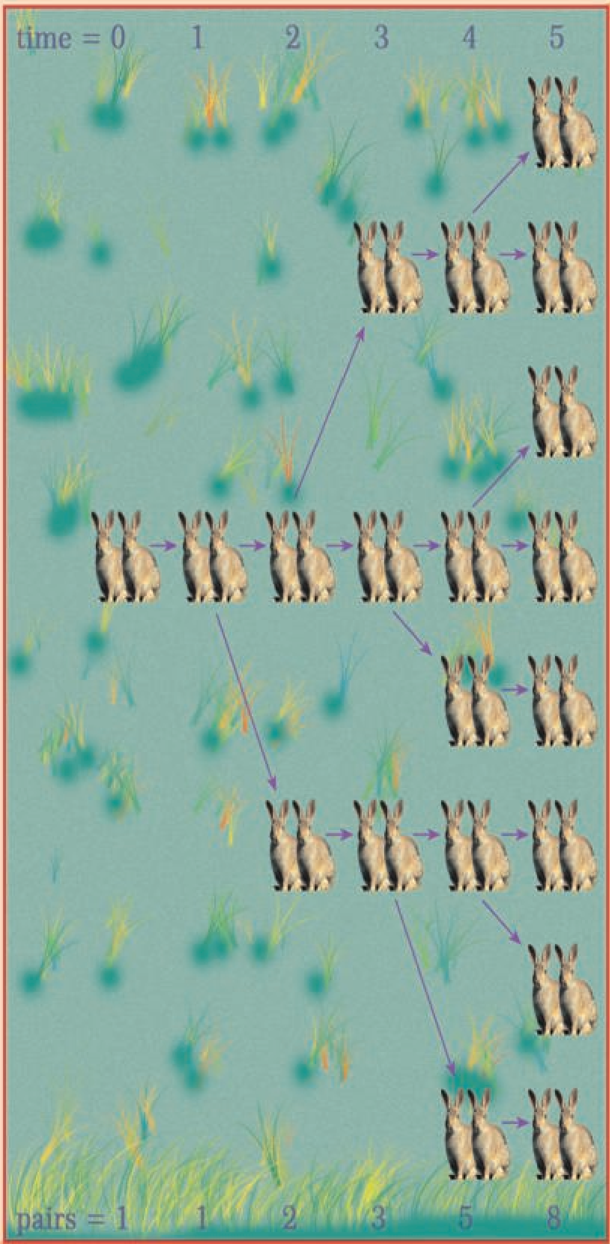
The sequence in the previous example is called the **Fibonacci sequence**, named after the 13th century Italian mathematician Fibonacci, who used it to solve a problem about the breeding of rabbits. Fibonacci considered the following problem:

Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age two months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month?

As a solution, Fibonacci found the following sequence:

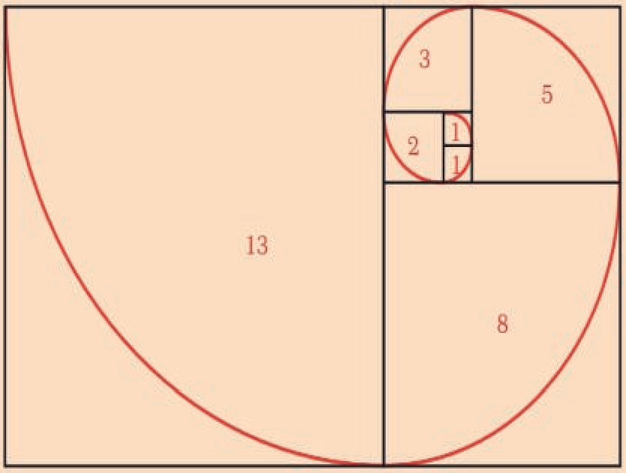
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

This sequence also occurs in numerous other aspects of the natural world.



The planets in our solar system are spaced in a Fibonacci sequence.

We can make a picture showing the Fibonacci numbers if we start with two small squares whose sides are each one unit long next to each other. Then we draw a square with side length two units (1 + 1 units) next to both of these. We can now draw a new square which touches the square with side one unit and the square with side two units, and therefore has side three units. Then we draw another square touching the two previous squares (side five units), and so on. We can continue adding squares around the picture, each new square having a side which is as long as the sum of the sides of the two previous squares. Now we can draw a spiral by connecting the quarter circles in each square, as shown on the next page. This is a spiral (the **Fibonacci Spiral**). A similar curve to this occurs in nature as the shape of a nautilus.



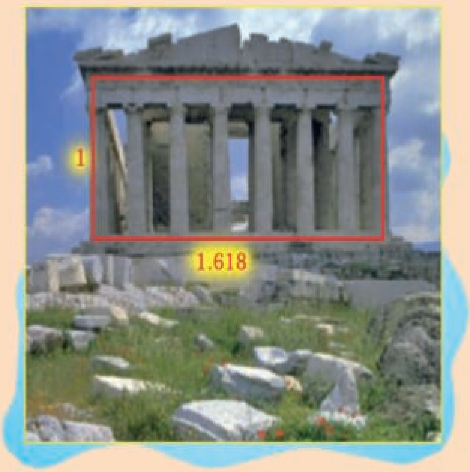
A nautilus has the same shape as the Fibonacci spiral.

The ratio of two successive Fibonacci numbers $\frac{f_{n+1}}{f_n}$ gets closer to the number $\frac{1+\sqrt{5}}{2} \approx 1.618$ as the value of n gets bigger. This number is a special number in mathematics and is known as the **golden ratio**.

The ancient Greeks also considered a line segment divided into two parts such that the ratio of the shorter part of length one unit to the longer part is the same as the ratio of the longer part to the whole segment.

This leads to the equation $\frac{1}{x} = \frac{x}{1+x}$ whose positive solution is $x = \frac{1+\sqrt{5}}{2}$. Thus, the segment shown is divided into the golden ratio!

A rectangle in which the ratio of one side to the other gives the golden ratio is called a **golden rectangle**. The Golden Rectangle is a unique and a very important shape in mathematics. It appears in nature and music, and is also often used in art and architecture. The Golden Rectangle is believed to be one of the most pleasing and beautiful shapes for the human eye.



The golden ratio is frequently used in architecture.

The ratio of the length of your arm to the length from the elbow down to the end of your hand is approximately equal to the golden ratio.

EXERCISES 1

Sequences

1. State whether each term is a general term of a sequence or not.

a. $3n - 76$ b. $\frac{n}{n+2}$ c. $\frac{2n+1}{2n-1}$

d. $\frac{4}{n^2-4}$ e. $\frac{13}{4}$ f. $(-1)^n \frac{1}{n^3}$

g. $\sqrt{n-5}$ h. $\sqrt{n^2+2n}$ i. $\sqrt{\frac{n^2-n-2}{n-2}}$

2. Find a suitable formula for the general terms of the sequences whose first few terms are given.

a. 1, 3, 5 b. –1, 3, –5

c. 0, 3, 8, 15 d. $-\frac{1}{5}, -\frac{8}{7}, -\frac{27}{9}$

e. 2, 6, 12, 20, 30

3. Find the stated terms for the sequence with the given general term.

a. $a_n = 2n + 3$, find the first three terms and a_{37}

b. $a_n = \frac{3n+1}{n+7}$, find the first three terms and a_{33}

c. $a_n = \sqrt{n^2+6n}$, find the first three terms and a_6

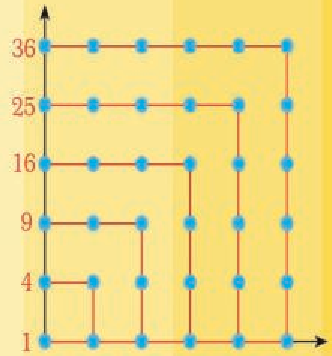
4. How many terms of the sequence with general term $a_n = n^2 - 6n - 16$ are negative?

5. How many terms of the sequence with general term $a_n = \frac{3n-7}{3n+5}$ are less than $\frac{1}{5}$?

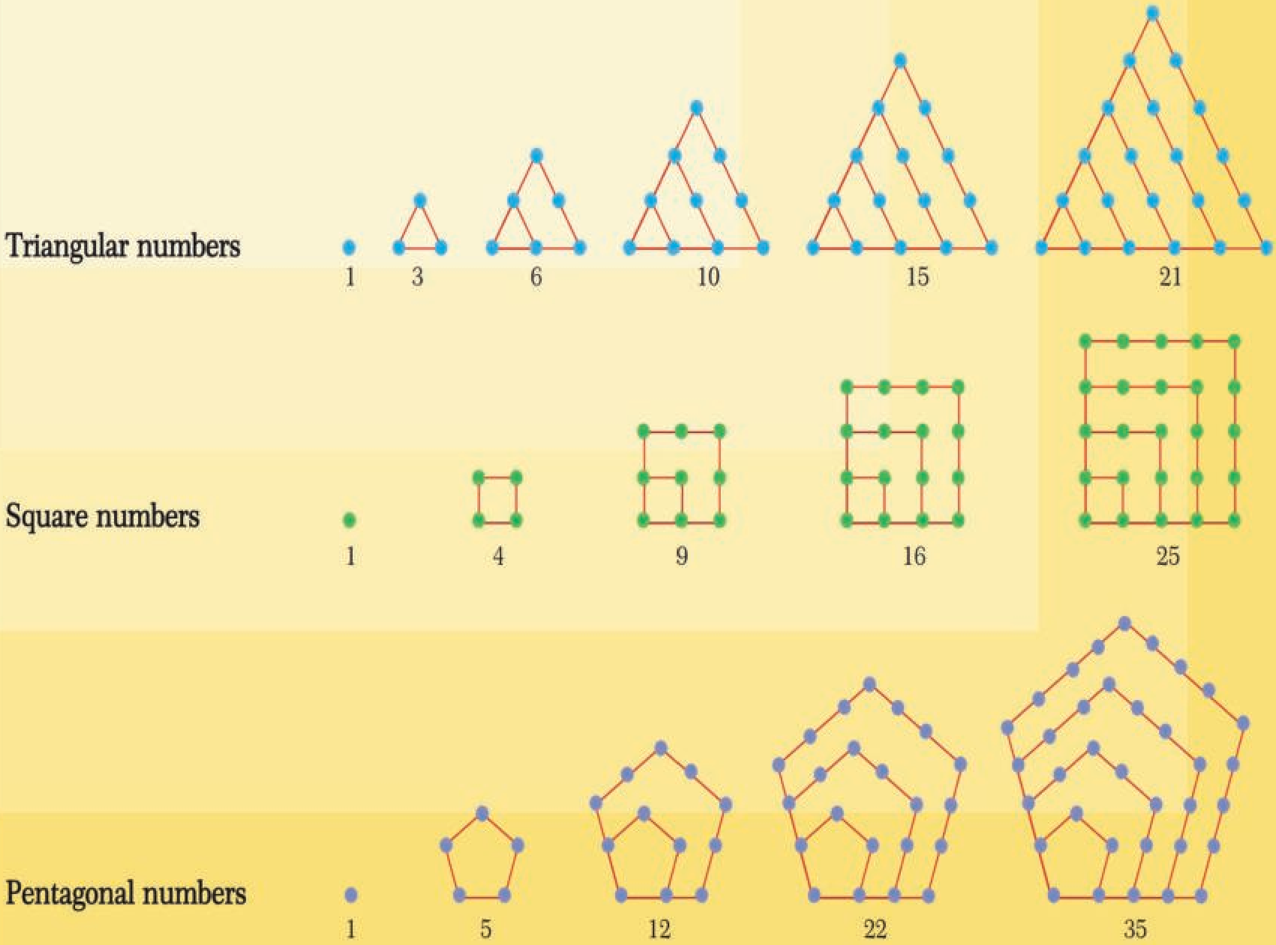
POLYGONAL NUMBERS

At the beginning of this book we looked at the sequence 1, 4, 9, 16, 25, 36, ... We call the numbers in this sequence **square numbers**. We can generate the square numbers by creating a sequence of nested squares like the one on the right. Starting from a common vertex, each square has sides one unit longer than the previous square. When we count the number of points in each successive square, we get the sequence of square numbers

(first square = 1 point, second square = 4 points, third square = 9 points, etc.).



Polygonal numbers are numbers which form sequences like the one above for different polygons. The Pythagoreans named these numbers after the polygons that defined them.



Polygonal numbers have many interesting relationships between them. For example, the sum of any two consecutive triangular numbers is a square number, and eight times any triangular number plus one is always a square number.

Can you find any more patterns? Can you find the general term for each set of polygonal numbers?

2 ARITHMETIC SEQUENCES

A. ARITHMETIC SEQUENCES

1. Definition

Let's look at the sequence 6, 10, 14, 18, ...

Obviously the difference between each term is equal to 4 and the sequence can be written as $a_{n+1} = a_n + 4$ where $a_1 = 6$.

For the sequence 23, 21, 19, ... the formula will be $a_{n+1} = a_n - 2$ where $a_1 = 23$.

In these examples, the difference between consecutive terms in each sequence is the same. We call sequences with this special property **arithmetic sequences**.

Definition

arithmetic sequence

If a sequence (a_n) has the same difference d between its consecutive terms, then it is called an **arithmetic sequence**.

In other words, (a_n) is arithmetic if $a_{n+1} = a_n + d$ such than $n \in \mathbb{N}$, $d \in \mathbb{R}$. We call d the **common difference** of the arithmetic sequence. In this book, from now on we will use a_n to denote general term of an arithmetic sequence and d (the first letter of the Latin word *differentia*, meaning difference) for the common difference.

If d is positive, we say the arithmetic sequence is **increasing**. If d is negative, we say the arithmetic sequence is **decreasing**. What can you say when d is zero?

EXAMPLE 8 State whether the following sequences are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. 7, 10, 13, 16, ... b. 3, –2, –7, –12, ... c. 1, 4, 9, 16, ... d. 6, 6, 6, 6, ...

Solution a. arithmetic, $d = 3$ b. arithmetic, $d = -5$ c. not arithmetic d. arithmetic, $d = 0$

EXAMPLE 9 State whether the sequences with the following general terms are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. $a_n = 4n - 3$ b. $a_n = 2^n$ c. $a_n = n^2 - n$ d. $a_n = \dfrac{n^2 + 5n + 4}{n + 4}$

Solution a. $a_{n+1} = 4(n + 1) - 3 = 4n + 1$, so the difference between each consecutive term is $a_{n+1} - a_n = (4n + 1) - (4n - 3) = 4$, which is constant. Therefore, (a_n) is an arithmetic sequence and $d = 4$.

b. $a_{n+1} = 2^{n+1}$, so the difference between each consecutive term is $a_{n+1} - a_n = 2^{n+1} - 2^n = 2^n$, which is not constant. Therefore, (a_n) is not an arithmetic sequence.

c. $a_{n+1} = (n + 1)^2 - (n + 1)$, so the difference between two consecutive terms is $a_{n+1} - a_n = [(n + 1)^2 - (n + 1)] - (n^2 - n) = 2n$, which is not constant. Therefore, (a_n) is not an arithmetic sequence.

d. By rewriting the general term we have $a_n = \dfrac{(n+4)(n+1)}{n+4}$. Since $n \neq -4$ (since we are talking about a sequence), we have $a_n = n + 1$. Therefore, $a_{n+1} = (n + 1) + 1$, and the difference between the consecutive terms is $a_{n+1} - a_n = 1$, which is constant. Therefore, (a_n) is an arithmetic sequence and $d = 1$.

With the help of the above example we can notice that if the formula for general term of a sequence gives us a linear function, then it is arithmetic.

Note

The general term of an arithmetic sequence is linear.

Arithmetic growth is linear.

2. General Term

Since arithmetic sequences have many applications, it is much better to express the general term directly, instead of recursively. The formula is derived as follows:

If (a_n) is arithmetic, then we only know that $a_{n+1} = a_n + d$. Let us write a few terms.

$a_1 = a_1$

$a_2 = a_1 + d$

$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$

$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$

$a_5 = a_1 + 4d$

.
.
.

$a_n = a_1 + (n - 1)d$

This is the general term of an arithmetic sequence.

GENERAL TERM FORMULA

The general term of an arithmetic sequence (a_n) with common difference d is

$$a_n = a_1+(n-1)d.$$

EXAMPLE 10 $-3, 2, 7$ are the first three terms of an arithmetic sequence (a_n). Find the twentieth term.

Solution We know that $a_1 = -3$ and $d = a_3 - a_2 = a_2 - a_1 = 5$. Using the general term formula,

$a_n = a_1 + (n-1)d$

$a_{20} = -3 + (20-1) \cdot 5 = 92.$

EXAMPLE 11 (a_n) is an arithmetic sequence with $a_1 = 4$, $a_8 = 25$. Find the common difference and a_{101}.

Solution Using the general term formula,

$a_n = a_1 + (n-1)d$

$a_8 = a_1 + 7d$

$25 = 4 + 7d$. So we have $d = 3$.

$a_{101} = a_1 + (101-1)d = 4 + 100 \cdot 3 = 304$

EXAMPLE 12 (a_n) is an arithmetic sequence with $a_1 = 3$ and common difference 4. Is 59 a term of this sequence?

Solution For 59 to be a term of the arithmetic sequence, it must satisfy the general term formula such that n is a natural number.

$a_n = a_1 + (n-1)d$

$59 = 3 + (n-1) \cdot 4$

$59 = 4n - 1$

$n = 15$

Since 15 is a natural number, 59 is the 15th term of this sequence.

EXAMPLE 13 Find the number of terms in the arithmetic sequence 1, 4, 7, ..., 91.

Solution Here we have a finite sequence. Using the general term formula,

$a_n = a_1 + (n-1)d$

$91 = 1 + (n-1) \cdot 3$

$n = 31$

Therefore, this sequence has 31 terms.

B. SUM OF THE TERMS OF AN ARITHMETIC SEQUENCE

1. Sum of the First n Terms

Let us consider an arithmetic sequence whose first few terms are 3, 7, 11, 15, 19.

The sum of the first term of this sequence is obviously 3. The sum of the first two terms is 10, the sum of the first three terms is 21, and so on. To write this in a more formal way, let us use S_n to denote **the sum of the first n terms**, i.e., $S_n = a_1 + a_2 + ... + a_n$. Now we can write:

$$S_1 = 3$$

$$S_2 = 3 + 7 = 10$$

$$S_3 = 3 + 7 + 11 = 21$$

$$S_4 = 3 + 7 + 11 + 15 = 36$$

$$S_5 = 3 + 7 + 11 + 15 + 19 = 55.$$

EXAMPLE 18 Given the arithmetic sequence with general term $a_n = 3n + 1$, find the sum of first three terms.

Solution $S_3 = a_1 + a_2 + a_3 = 4 + 7 + 10 = 21.$

How could we find S_{100} in the above example? Calculating terms and finding their sums takes time and effort for large sums. Since arithmetic sequences are of special interest and importance, we need a more efficient way of calculating the sums of arithmetic sequences. The following theorem meets our needs:

Theorem

The sum of the of first n terms of an arithmetic sequence (a_n) is $S_n = \frac{a_1 + a_n}{2} n$.

Proof $S_n = a_1 + a_2 + \dots + a_{n-1} + a_n$ or

$S_n = a_n + a_{n-1} + \dots + a_2 + a_1$.

Adding these equations side by side,

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \dots + (a_{n-1} + a_2) + (a_n + a_1)$$

$$2S_n = (a_1 + a_n) + (a_1 + d + a_n - d) + \dots + (a_n - d + a_1 + d) + (a_n + a_1)$$

$$2S_n = \underbrace{(a_1 + a_n) + (a_1 + a_n) + \dots + (a_1 + a_n) + (a_1 + a_n)}_{\text{n times}}$$

$$2S_n = (a_1 + a_n) \cdot n$$

$$S_n = \frac{a_1 + a_n}{2} \cdot n.$$

EXAMPLE 19 Given an arithmetic sequence with $a_1 = 2$ and $a_6 = 17$, find S_6.

Solution Using the sum formula,

$$S_6 = \frac{a_1 + a_6}{2}\ 6 = (2 + 17)\ 3 = 57.$$

EXAMPLE 20 Given an arithmetic sequence with $a_1 = -14$ and $d = 5$, find S_{27}.

Solution Using the sum formula,

$S_{27} = \frac{a_1 + a_{27}}{2} \cdot 27$ requires $a_{27} = a_1 + 26d = -14 + 26 \cdot 5 = 116$.

Therefore, $S_{27} = \frac{-14 + 116}{2} \cdot 27 = 1377$.

EXAMPLE 21 Given an arithmetic sequence with $a_1 = 56$ and $a_{11} = -14$, find S_{15}.

Solution Using the sum formula,

$S_{15} = \frac{a_1 + a_{15}}{2} \cdot 15$, so we need to find a_{15}. Let us calculate using a_{11}:

$a_{11} = a_1 + 10d$

$-14 = 56 + 10d$, so $d = -7$ and

$a_{15} = a_1 + 14d = 56 + 14 \cdot (-7) = -42$.

Therefore, $S_{15} = \frac{56-42}{2} \cdot 15 = 105$.

EXAMPLE 22 If $-5 + ... + 49 = 616$ is the sum of the terms of a finite arithmetic sequence, how many terms are there in the sequence?

Solution Let us convert the problem into algebraic language:

$a_1 = -5$, $a_p = 49$, and $S_p = 616$, and we need to find p.

Using the sum formula,

$S_p = \frac{a_1 + a_p}{2} \cdot p$, that is, $616 = \frac{-5+49}{2} \cdot p$, so $p = 28$. So 28 numbers were added.

Since $a_n = a_1 + (n-1)d$, we can also rewrite the sum formula as follows:

Check Yourself

1. Given an arithmetic sequence with $a_1 = 4$ and $a_{10} = 15$, find S_{10}
2. Given an arithmetic sequence with $a_{13} = 26$ and $d = -2$, find S_{13}.
3. Given an arithmetic sequence with $a_1 = 9$ and $S_8 = 121$, find d.
4. Find the sum of all the multiples of 3 between 20 and 50.

Answers

1. 95 2. 494 3. 1.75 4. 345

EXERCISES 2

A. Arithmetic Sequences

1. State whether the following sequences are arithmetic or not.

   a. $(a_n) = (n^2)$ b. $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots)$ c. $(a_n) = (4n+7)$

2. Find the formula for the general term a_n of the arithmetic sequence with the given common difference and first term.

   a. $d = 2, a_1 = 3$ b. $d = \sqrt{3}, a_1 = 1$

   c. $d = 0, a_1 = 0$ d. $d = -\frac{3}{2}, a_1 = -3$

   e. $d = -1, a_1 = 0$ f. $d = 7, a_1 = \sqrt{2}$

   g. $d = b + 3, a_1 = 2b + 7$

3. Find the common difference and the general term a_n of the arithmetic sequence with the given terms.

   a. $a_1 = 3, a_2 = 5$ b. $a_1 = 4, a_4 = 10$

   c. $a_5 = \sqrt{2}, a_8 = 6\sqrt{2}$ d. $a_{12} = -12, a_{24} = -24$

   e. $a_5 = 8, a_{37} = 8$ f. $a_6 = 6, a_{20} = -34$

   g. $a_3 = 1, a_5 = 2$

   h. $a_2 = 2x - y, a_8 = x + 2y$

4. Find the general term of the arithmetic sequence using the given data.

   a. $a_{n+1} = a_n + 7, a_1 = -2$

   b. $a_{17} = 41, d = 4$

5. Fill in the blanks to form an arithmetic sequence.

   a. _, _, _, 3, _, _, _, 32.

   b. $13, \underbrace{\_, \_, \ldots, \_}_{\text{seven terms}}, 45$

6. In an arithmetic sequence the first term is -1 and the common difference is 3. Is 27 a term of this sequence?

7. Given that the following sequences are arithmetic, find the missing value.

   a. $\frac{a_{12} + a_{20}}{2} = ?$ b. $a_6 = \frac{a_4 + ?}{2}$

B. Sum of the Terms of an Arithmetic Sequence

8. For each arithmetic sequence (a_n) find the missing value.

   a. $a_1 = -5,\ a_8 = 18,\ S_8 = ?$

   b. $a_1 = -3,\ a_7 = 27,\ S_{40} = ?$

   c. $a_1 = 7,\ S_{16} = 332,\ d = ?$

   d. $d = \frac{5}{3},\ S_{34} = 1173,\ a_1 = ?$

   e. $a_1 = 2,\ a_{n+1} = a_n - 2,\ S_{23} = ?$

   f. $a_1 = \frac{3}{2},\ d = \frac{1}{2},\ S_p = 1700,\ p = ?$

   g. $S_{100} = 10000,\ a_{100} = 199,\ a_{10} = ?$

   h. $a_n = -5n - 10,\ S_7 = ?$

   i. $a_1 = 5,\ a_p = 20,\ S_p = 250,\ p = ?$

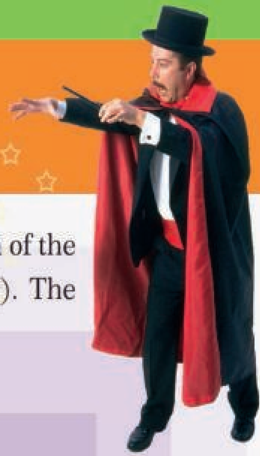
   j. $S_{60} = 3840,\ a_1 = 5,\ a_{61} = ?$

   k. $a_1 = 3,\ a_{10} - a_7 = -6,\ S_{20} = ?$

   l. $a_1 = 1,\ S_{22} - S_{18} = 238,\ a_7 = ?$

   m. $d = 5,\ S_{16} - S_{10} = 308,\ a_1 = ?$

MAGIC SQUARES

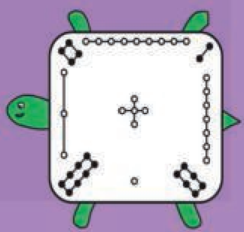


A **magic square** is an arrangement of natural numbers in a square matrix so that the sum of the numbers in each column, row, and diagonal is the same number (the **magic number**). The number of cells on one side of the square is called the **order** of the magic square.

Here is one of the earliest known magic squares:

4	9	2
3	5	7
8	1	6

It is a third order magic square constructed by using the numbers 1, 2, 3, ..., 9. Notice that the numbers in each row, column, and diagonal add up to the number 15, and 1, 2, 3, …,9 form an arithmetic sequence. This magic square was possibly constructed in 2200 B.C. in China. It is known as the **Lo-Shu** magic square.



The famous Lo-Shu is the oldest known magic square in the world. According to the legend, the figure above was found on the back of a turtle which came from the river Lo. The word 'Shu' means 'book', so 'Lo-Shu' means 'The book of the river Lo'.

Below is another magic square, this time of order four. Note that its elements are from the finite arithmetic sequence 7, 10, 13, 16, ..., 52, and the magic number is 118.

52	13	10	43
19	34	37	28
31	22	25	40
16	49	46	7

What kind of relation exists between the sequence and the magic number? Given any finite arithmetic sequence of n^2 terms is it always possible to construct a magic square? If the numbers do not form an arithmetic sequence, is it possible to construct a magic square?

Try constructing your own magic square of order three using the numbers 4,8,12, ...,36.

There are many unsolved puzzles concerning magic squares. The puzzle of Yang-Hui, which was solved in the year 2000, was one of them. According to the legend the 13th century Chinese mathematician Yang-Hui gave the emperor Sung his last magic square as a gift. This is Yang-Hui's square:

1668	198	1248
618	1038	1458
828	1878	408

+1

1669	199	1249
619	1039	1459
829	1879	409

The special property of Yang-Hui's square was that the square had elements of a finite arithmetic sequence with common difference 210 such that when 1 was added to each cell it would become another magic square with all elements prime numbers. But the emperor wanted the magic square to also give prime numbers when 1 was subtracted from each cell. He promised some land along the river to the mathematician if it was completed. Unfortunately, the life of Yang-Hui wasn't long enough to solve this puzzle. Below is the solution to the problem, calculated 725 years later:

372839669	241608569	267854789
189116129	294101009	399085889
320347229	346593449	215362349

−1

372839670	241608570	267854790
189116130	294101010	399085890
320347230	346593450	215362350

+1

372839671	241608571	267854791
189116131	294101011	399085891
320347231	346593451	215362351

3 GEOMETRIC SEQUENCES

A. GEOMETRIC SEQUENCES

1. Definition

In the previous section, we learned about arithmetic sequences, i.e. sequences whose consecutive terms have a common difference. In this chapter we will look at another type of sequence, called a **geometric sequence**. Geometric sequences play an important role in mathematics.

A sequence is called geometric if the ratio between each consecutive term is common. For example, look at the sequence 3, 6, 12, 24, 48, …

Obviously the ratio of each term to the previous term is equal to 2, so we can formulize the sequence as $b_{n+1} = b_n \cdot 2$. The consecutive terms of the sequence have a common ratio (2), so this sequence is geometric.

For the sequence 625, 125, 25, 5, 1, … the formula will be $b_{n+1} = b_n \cdot \frac{1}{5}$. The common ratio in this sequence is $\frac{1}{5}$.

Definition

geometric sequence

If a sequence (b_n) has the same ratio q between its consecutive terms, then it is called a **geometric sequence.**

In other words, (b_n) is geometric if $b_{n+1} = b_n \cdot q$ such that $n \in \mathbb{N}$, $q \in \mathbb{R}$. q is called the **common ratio** of the sequence. In this book, from now on we will use b_n to denote the general term of a geometric sequence, and q to denote the common ratio.

If $q > 1$, the geometric sequence is **increasing** when $b_1 > 0$ and **decreasing** when $b_1 < 0$.

If $0 < q < 1$, geometric sequence is **increasing** when $b_1 < 0$ and **decreasing** when $b_1 > 0$.

If $q < 0$, then the sequence is **not monotone**.

What can you say if $q = 1$? What about $q = 0$?

EXAMPLE 23 State whether the following sequences are geometric or not. If a sequence is geometric, find the common ratio.

a. 1, 2, 4, 8, …   b. 3, 3, 3, 3, …   c. 1, 4, 9, 16, …   d. $5, -1, \frac{1}{5}, -\frac{1}{25}, \ldots$

Solution a. geometric, $q = 2$ b. geometric, $q = 1$ c. not geometric d. geometric, $q = -\frac{1}{5}$

EXAMPLE 24 State whether the sequences with the given general terms are geometric or not. If a sequence is geometric, find the common ratio.

a. $b_n = 3^n$ b. $b_n = n^2 + 3$ c. $b_n = 3 \cdot 2^{n+3}$ d. $b_n = 3n + 5$

Solution a. $b_{n+1} = 3^{n+1}$, so the ratio between each consecutive term is $\frac{b_{n+1}}{b_n} = \frac{3^{n+1}}{3^n} = 3$, which is constant. So (b_n) is a geometric sequence and $q = 3$.

b. $b_{n+1} = (n + 1)^2 + 3$, so the ratio between each consecutive term is

$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2+3}{n^2+3} = \frac{n^2+2n+4}{n^2+3}$, which is not constant. So (b_n) is not a geometric sequence.

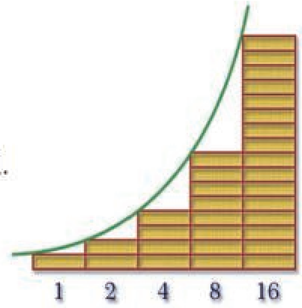
c. $b_{n+1} = 3 \cdot 2^{n+4}$, so the ratio between each consecutive term is $\frac{b_{n+1}}{b_n} = \frac{3 \cdot 2^{n+4}}{3 \cdot 2^{n+3}} = 2$, which is constant. So (b_n) is a geometric sequence and $q = 2$.

d. Since the general term has a linear form, this is an arithmetic sequence. It is not geometric.

With the help of the above example we can see that if the formula for the general term of a sequence gives us an exponential function with a linear exponent (a function with only one exponent variable), then it is geometric.

Note

The general term of a geometric sequence is exponential.



Geometric growth is exponential.

2. General Term

We have seen that for a geometric sequence, $b_{n+1} = b_n \cdot q$. This formula is defined recursively. If we want to make faster calculations, we need to express the general term of a geometric sequence more directly. The formula is derived as follows:

If (b_n) is geometric, then we only know that $b_{n+1} = b_n \cdot q$. Let us write a few terms.

$$b_1 = b_1$$
$$b_2 = b_1 \cdot q$$
$$b_3 = b_2 \cdot q = (b_1 \cdot q) \cdot q = b_1 \cdot q^2$$
$$b_4 = b_3 \cdot q = (b_1 \cdot q^2) \cdot q = b_1 \cdot q^3$$
$$b_5 = b_1 \cdot q^4$$
$$\vdots$$
$$b_n = b_1 \cdot q^{n-1}$$

This is the general term of a geometric sequence.

GENERAL TERM FORMULA

The general term of a geometric sequence (b_n) with common ratio q is

$$b_n = b_1 \cdot q^{n-1}$$

EXAMPLE 25 If 100, 50, 25 are the first three terms of a geometric sequence (b_n), find the sixth term.

Solution We can calculate the common ratio as $q = \frac{b_3}{b_2} = \frac{b_2}{b_1} = \frac{1}{2}$, so $b_1 = 100$, $q = \frac{1}{2}$.

Using the general term formula, $b_n = b_1 \cdot q^{n-1}$, so $b_6 = 100 \cdot (\frac{1}{2})^{6-1} = \frac{25}{8}$.

EXAMPLE 26 (b_n) is a geometric sequence with $b_1 = \frac{1}{3}$, $q = 3$. Find b_4.

Solution Using the general term formula,

$b_n = b_1 \cdot q^{n-1}$. Therefore, $b_4 = \frac{1}{3} \cdot 3^{4-1} = 9$.

EXAMPLE 27 (b_n) is a geometric sequence with $b_1 = -15$, $q = \frac{1}{5}$. Find the general term.

Solution Using the general term formula, $b_n = b_1 \cdot q^{n-1}$.

Therefore, $b_n = -15 \cdot \left(\frac{1}{5}\right)^{n-1} = -15 \cdot \left(\frac{1}{5}\right)^{n} \cdot \left(\frac{1}{5}\right)^{-1} = -75 \cdot \left(\frac{1}{5}\right)^{n}$.

How can you relate this building to a geometric sequence?

EXAMPLE 28 Consider the geometric sequence (b_n) with $b_1 = \frac{1}{9}$ and $q = 3$. Is 243 a term of this sequence?

Solution Using the general term formula,

$b_n = b_1 \cdot q^{n-1}$ and so $b_n = \frac{1}{9} \cdot 3^{n-1}$.

Now $243 = \frac{1}{9} \cdot \frac{3^n}{3}$, and so $3^n = 3^8$. Therefore, $n = 8$.

Since 8 is a natural number, 243 is the eighth term of this sequence.

EXAMPLE 29 In a monotone geometric sequence $b_1 \cdot b_5 = 12$, $\dfrac{b_2}{b_4} = 3$. Find b_2.

Solution $\dfrac{b_2}{b_4} = 3$, that is $\dfrac{b_1 \cdot q}{b_1 \cdot q^3} = 3$. So $q = \pm\dfrac{1}{\sqrt{3}}$.

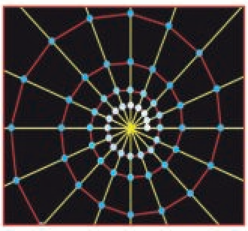
Since the sequence is monotone, we take $q = \dfrac{1}{\sqrt{3}}$.

$b_1 \cdot b_5 = 12$, that is $b_1 \cdot b_1 \cdot q^4 = 12$.

$b_1^2 \cdot \dfrac{1}{9} = 12$, that is $b_1 = 6\sqrt{3}$. So $b_2 = b_1 \cdot q = 6\sqrt{3} \cdot \dfrac{1}{\sqrt{3}} = 6$.

Why? Would the answer change if the sequence was not monotone? Why?

Check Yourself



1. Is the sequence with general term $b_n = \dfrac{1}{3} \cdot 4^{n+3}$ a geometric sequence? Why?
2. $\dfrac{3}{16}, \dfrac{3}{8}, \dfrac{3}{4}$ are the first three terms of a geometric sequence (b_n). Find the eighth term.
3. (b_n) is a non-monotone geometric sequence with $b_1 = \dfrac{1}{4}$, $b_7 = 16$. Find the common ratio of the sequence and b_4.
4. (b_n) with is a geometric sequence with $b_1 = -3$, $q = -2$. Is -96 a term of this sequence?

Answers

1. yes, because the general term formula is exponential 2. 24 3. $q = -2$; $b_4 = -2$ 4. no

B. SUM OF THE TERMS OF A GEOMETRIC SEQUENCE

1. Sum of the First n Terms

Let us consider the geometric sequence with first few terms 1, 2, 4, 8, 16.

The sum of the first term of this sequence is obviously 1. The sum of the first two terms is 3, the sum of the first three terms is 7, and so on. To write this in a more formal way, let us use S_n to denote **the sum of the first n terms**, i.e. $S_n = b_1 + b_2 + \dots + b_n$. Now,

$$S_1 = 1$$
$$S_2 = 1 + 2 = 3$$
$$S_3 = 1 + 2 + 4 = 7$$
$$S_4 = 1 + 2 + 4 + 8 = 15$$
$$S_5 = 1 + 2 + 4 + 8 + 16 = 31.$$

EXAMPLE 33 Given the geometric sequence with general term $b_n = 3\ (-2)^n$, find the sum of first three terms.

Solution $S_3 = b_1 + b_2 + b_3 = -6 + 12 - 24 = -18$

How could we find S_{100} in the previous example? Calculating terms and finding their sums takes time and effort for large sums. As geometric sequences grow very fast, we need a more efficient way of calculating these sums. The following theorem meets our needs:

Theorem

The sum of the first n terms of a geometric sequence (b_n) is $S_n = b_1 \cdot \frac{1-q^n}{1-q}$, $q \neq 1$.

Proof $S_n = b_1 + b_2 + b_3 + ... + b_{n-1} + b_n$

$$S_n = b_1 + b_1 \cdot q + b_1 \cdot q^2 + ... + b_1 \cdot q^{n-2} + b_1 \cdot q^{n-1} \quad (1)$$

$$q \cdot S_n = b_1 \cdot q + b_1 \cdot q^2 + b_1 \cdot q^3 + ... + b_1 \cdot q^{n-1} + b_1 \cdot q^n \quad (2)$$

Subtracting (2) from (1), we get

$$S_n - q \cdot S_n = b_1 - b_1 \cdot q^n$$

$$S_n = b_1 \cdot \frac{1-q^n}{1-q}$$

EXAMPLE 34 Given a geometric sequence with $b_1 = \frac{1}{81}$ and $q = 3$, find S_6.

Solution Using the sum formula,

$$S_n = b_1 \cdot \frac{1-q^n}{1-q}, \text{ so } S_6 = \frac{1}{81} \cdot \frac{1-3^6}{1-3} = \frac{364}{81}.$$

EXAMPLE 35 Given a geometric sequence with $S_6 = 3640$ and $q = 3$, find b_1.

Solution Using the sum formula,

$$S_6 = b_1 \cdot \frac{1-q^6}{1-q}, \text{ so } 3640 = b_1 \cdot \frac{1-3^6}{1-3}, \text{ and so } b_1 = 10.$$

EXAMPLE 36 Given a geometric sequence with $q = \frac{1}{3}$, $b_p = 5$ and $S_p = 1820$, find b_1.

Solution Using the sum formula,

$$S_p = b_1 \cdot \frac{1-q^p}{1-q} = \frac{b_1 - b_1 \cdot q^p}{1-q} = \frac{b_1 - b_{p+1}}{1-q} = \frac{b_1 - b_p \cdot q}{1-q}, \text{ so } 1820 = \frac{b_1 - 5 \cdot \frac{1}{3}}{1 - \frac{1}{3}}. \text{ Therefore, } b_1 = 1215.$$

EXERCISES

A. Geometric Sequences

1. State whether the following sequences are geometric or not.

a. $\left(2, -5, \frac{25}{2}, \ldots\right)$ b. $(b_n) = (4^{n^2-3})$

c. $(b_n) = (2n + 7)$

2. Find the general term of the geometric sequence with the given qualities.

a. $b_1 = 5, q = 2$ b. $b_1 = -3, q = \frac{1}{2}$

c. $b_1 = 1000, q = \frac{1}{10}$ d. $b_1 = \sqrt{3}, q = \sqrt{3}$

e. $b_1 = 4, b_4 = 32$ f. $b_1 = 3, b_5 = \frac{1}{27}$

g. $b_3 = 32, b_6 = \frac{1}{2}$ h. $b_5 = 5, b_{25} = 5$

i. $b_1 = 2, b_6 = 8\sqrt{2}$

3. Fill in the blanks to form a geometric sequence.

a. $3 - 2\sqrt{2}$, __, $3 + 2\sqrt{2}$

b. __, __, 36, __, 4

4. Find the general term of the geometric sequence with $b_4 = b_2 + 24$ and $b_2 + b_3 = 6$.

5. Write the first four terms of the non-monotone geometric sequence that is formed by inserting nine terms between –3 and –729.

6. Given a geometric sequence with $b_6 = 4b_4$ and $b_3 \cdot b_6 = 1152$, find b_1.

B. Sum of the Terms of a Geometric Sequence

7. For each geometric sequence (b_n) find the missing value.

a. $b_1 = -\frac{3}{2}, q = -2, S_7 = ?$

b. $b_2 = 6, b_7 = 192, S_{11} = ?$

c. $b_2 = 1, b_5 \cdot b_2 = 64 \cdot b_4 \cdot b_5, S_5 = ?$

d. $S_3 = 111, q^3 = 4, S_6 = ?$

8. The general term of a geometric sequence is $b_n = 3 \cdot 2^n$. Find S_{10}.

1. Which terms can be the general term of a sequence?

I. $\frac{n}{n-2}$ II. 3 III. $n^2 + 2n + 3$

IV. $\sqrt{7-n}$ V. 3^n VI. n^n

A) I, II, III, IV B) II, III, IV, VI

C) I, II, III, VI D) II, III, V, VI

E) III, IV, V, VI

2. Which of the following can be the general term of the sequence with the first four terms 3, 5, 7, 9?

A) $2n - 1$ B) $2n$ C) $2n + 1$

D) $n + 1$ E) $n^2 + 2$

3. Given $a_1 = 2$, and $a_{n+1} = \frac{2a_n + 5}{2}$ for $n \geq 1$, find a_{11}.

A) 27 B) 25 C) 22

D) $\frac{27}{2}$ E) $\frac{25}{2}$

4. How many terms of the sequence with general term $\frac{n^2 - 2n + 36}{n}$ are natural numbers?

A) 5 B) 6 C) 7 D) 8 E) 9

5. How many terms of the sequence with general term $a_n = (\frac{2n+1}{n+9})$ are less than $\frac{1}{3}$?

A) 0 B) 1 C) 2 D) 3 E) 4

6. Given $a_n = (\frac{3n^2 - 5n}{n+k-3})$ and $a_5 = 3$, find k.

A) 3 B) 5 C) $\frac{22}{3}$

D) $\frac{35}{3}$ E) $\frac{44}{3}$

7. How many of the following sequences are decreasing?

I. $(a_n) = (\frac{3n-5}{n+2})$ II. $(b_n) = (n-3)^2$

III. $(c_n) = (-1^n)$ IV. $(d_n) = (\frac{1}{n+1})$

V. $(e_n) = (\frac{(-1)^n}{3n+2})$

A) 1 B) 2 C) 3 D) 4 E) 5

8. What is the minimum value in the sequence formed by $a_n = (\frac{2n+3}{3n-7})$?

A) –1 B) –3 C) –2 D) –7 E) –8

9. Which one of the following is the general term of an arithmetic sequence?

A) $n^2 + 2n$ B) $4n + 5$ C) n^3 D) $2^n + 3$ E) 5^n

10. If $\frac{1}{3}$, a, b, c, $\frac{5}{8}$ are consecutive terms of an arithmetic sequence, find $a + b + c$.

A) $\frac{7}{24}$ B) $\frac{23}{24}$ C) $\frac{21}{16}$ D) $\frac{23}{16}$ E) $\frac{69}{49}$

11. (a_n) is an arithmetic sequence with $a_{11} = 8$ and $a_{20} = 35$. Find a_3.

A) –3 B) –6 C) –16 D) –22 E) –28

12. (a_n) is arithmetic sequence with $a_1 = 7$ and common difference $\frac{1}{3}$. Find the general term.

A) $3n + 4$ B) $\frac{n+7}{3}$ C) $\frac{n-4}{3}$ D) $\frac{n+4}{3}$ E) $\frac{n+20}{3}$

13. (a_n) is an arithmetic sequence such that $a_3 + a_4 = 23$ and $a_5 + a_4 = 37$. Find a_8.

A) 49 B) 47 C) 45 D) 44 E) 43

14. (a_n) is a finite arithmetic sequence with first term $\frac{1}{2}$, last term $\frac{1}{16}$, and sum 9. How many terms are there in this sequence?

A) 9 B) 16 C) 32 D) 48 E) 64

15. $x - 2$, $x + 8$, $3x + 2$ form an arithmetic sequence. Find x.

A) 12 B) 11 C) 10 D) 9 E) 8

16. (a_n) is an arithmetic sequence with $S_4 = 3(S_4 - S_7)$ and $a_1 = 1$. Find the common difference.

A) $-\frac{2}{51}$ B) $-\frac{13}{51}$ C) $\frac{2}{51}$ D) $\frac{13}{51}$ E) $\frac{15}{51}$